ADDENDA TO "FOUNDATIONS OF GARSIDE THEORY"

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Abstract. This text consists of additions to the book "Foundations of Garside Theory", EMS Tracts in Mathematics, vol. 22 (2015)—see introduction and table of contents in arXiv:1309.0796—namely skipped proofs and solutions to selected exercises.

Chapter I: Examples

Skipped proofs

(none)

Solution to selected exercises

(none)

Chapter II: Preliminaries

Skipped proofs

Proposition II.1.18 (**collapsing invertible elements**).— *For $\mathcal{C}$ a left-cancellative category, the following conditions are equivalent:*

(i) *The equivalence relation $=^{\times}$ is compatible with composition, in the sense that, if g_1g_2 is defined and $g'_i =^{\times} g_i$ holds for $i = 1, 2$, then $g'_1g'_2$ is defined and $g'_1g'_2 =^{\times} g_1g_2$ holds;*

(ii) *The family $\mathcal{C}^{\times}(x, y)$ is empty for $x \neq y$ and, for all g, g' sharing the same source, $g \,{}^{\times}\!\!=^{\times} g'$ implies $g =^{\times} g'$;*

(iii) *The family $\mathcal{C}^{\times}(x, y)$ is empty for $x \neq y$ and we have*

$$\text{(II.119)} \qquad \forall x, y \in \mathcal{O}bj(\mathcal{C})\ \forall g \in \mathcal{C}(x, y)\ \forall \epsilon \in \mathcal{C}^{\times}(x, x)\ \exists \epsilon' \in \mathcal{C}^{\times}(y, y)\ (\epsilon g = g\epsilon').$$

When the above conditions are satisfied, the equivalence relation ${}^{\times}\!\!=^{\times}$ is compatible with composition, and the quotient-category $\mathcal{C}/{}^{\times}\!\!=^{\times}$ has no nontrivial invertible element.

Proof. Assume (i). Let $\epsilon \in \mathcal{C}^\times(x,y)$. Then $\epsilon =^\times 1_x$ and $1_y =^\times 1_y$ are satisfied, and $\epsilon\, 1_y$ is defined. By (i), $1_x\, 1_y$ must be defined as well, which is possible only for $x = y$. Let now $g \in \mathcal{C}(x,y)$ and $\epsilon \in \mathcal{C}^\times(x,x)$. Then $\epsilon =^\times 1_x$ and $g =^\times g$ are satisfied, and $\epsilon\, g$ is defined. By (i), we must have $g =^\times \epsilon g$, that is, there must exist ϵ' in $\mathcal{C}(y,y)$ satisfying $\epsilon g = g\epsilon'$. So (i) implies (iii).

Assume now (ii). Let $g \in \mathcal{C}(x,y)$ and $\epsilon \in \mathcal{C}^\times(x,x)$. Then $g \;{}^\times\!\!=^\times \epsilon g$ holds, as we can write $\epsilon\, g = \epsilon g\, 1_y$. By (ii), we deduce $g =^\times \epsilon g$, so, as above, there must exist ϵ' in $\mathcal{C}(y,y)$ satisfying $\epsilon g = g\epsilon'$. So (ii) implies (iii).

Assume now (iii). Let $g_1 \in \mathcal{C}(x,y)$, $g_2 \in \mathcal{C}(y,z)$, and assume $g'_1 =^\times g_1$ and $g'_2 =^\times g_2$. By assumption, there exists ϵ_i in $\mathcal{C}^\times$ satisfying $g'_i = g_i\epsilon_i$ for $i = 1, 2$. Applying (II.1.19) to g_2 and ϵ_1, we deduce that there exists ϵ'_1 in $\mathcal{C}^\times(z,z)$ satisfying $\epsilon_1 g_2 = g_2\epsilon'_1$. We deduce $g'_1g'_2 = g_1g_2\epsilon'_1\epsilon_2$, whence $g'_1g'_2 =^\times g_1g_2$. So (iii) implies (i).

Next, assume (iii) again, and $g' \;{}^\times\!\!=^\times g$. By definition, there exist ϵ, ϵ' satisfying $\epsilon g' = g\epsilon'$. By (iii), there exists an invertible element ϵ'' satisfying $\epsilon g' = g'\epsilon''$, and we deduce $g' = g\,\epsilon'\epsilon''^{-1}$, whence $g' =^\times g$. So (iii) implies (ii).

Finally, assume that (i), (ii), and (iii) are satisfied, we have $g_1 \;{}^\times\!\!=^\times g'_1$ and $g_2 \;{}^\times\!\!=^\times g'_2$, and g_1g_2 and $g'_1g'_2$ exist. By assumption, there exist for $i = 1, 2$ invertible elements ϵ_i, ϵ'_i satisfying $\epsilon'_i g_i = g'_i\epsilon_i$. Let x and y be the source and target of g_2. By construction, $\epsilon_1^{-1}\epsilon'_2$ belongs to $\mathcal{C}^\times(x,x)$. Applying (iii), we deduce the existence of ϵ in $\mathcal{C}^\times(y,y)$ satisfying $\epsilon_1^{-1}\epsilon'_2g_2 = g_2\epsilon$. Then we obtain

$$\epsilon'_1\, g_1g_2 = \epsilon'_1\, g_1g_2\,\epsilon = \epsilon'_1g_1\epsilon_1^{-1}\,\epsilon'_2g_2 = g'_1g'_2\,\epsilon_2,$$

which shows that $g_1g_2 \;{}^\times\!\!=^\times g'_1g'_2$ is true. So ${}^\times\!\!=^\times$ is a congruence, and there exists a well defined quotient category $\mathcal{C}/{}^\times\!\!=^\times$, obtained from $\mathcal{C}$ by identifying elements that are ${}^\times\!\!=^\times$-equivalent. In the current case, according to (ii), distinct objects of $\mathcal{C}$ are never connected by invertible elements, so the collapsing only involves the elements. Then, by construction, the category $\mathcal{C}/{}^\times\!\!=^\times$ has no nontrivial invertible element. ☐

Solution to selected exercises

Exercice 4 (atom).— *Assume that $\mathcal{C}$ is a left-cancellative category. Show that, for $n \geqslant 1$, every element g of $\mathcal{C}$ satisfying $\mathrm{ht}(g) = n$ admits a decomposition into a product of n atoms.*

Solution. By Proposition II.2.48, there exists a decomposition $(g_1, ..., g_n)$ of g consisting of n non-invertible entries. The assumption that g_i is non-invertible implies $\mathrm{ht}(g_i) \geqslant 1$ for every i. Hence we must have $\mathrm{ht}(g_i) = 1$ for each i, so each factor g_i is an atom. Hence g admits a decomposition as a product of n atoms.

Exercice 5 (unique right-mcm).— *Assume that $\mathcal{C}$ is a left-cancellative category that admits right-mcms. Assume thatf, g are elements of $\mathcal{C}$ that admit a common right-multiple and any two right-mcms of f and g are $=^\times$-equivalent. Show that every right-mcm of f and g is a right-lcm of f and g.*

Solution. Let h be a right-mcm of f and g, and $\hat{h}$ be a common right-multiple of f and g. Then $\hat{h}$ is a right-multiple of some right-mcm h' of f and g. By assumption, $h' =^\times h$ holds, hence $\hat{h}$ is a right-multiple of h as well. Hence h is a right-lcm of f and g.

Exercice 6 (**right-gcd to right-mcm**).— *Assume that $\mathcal{C}$ is a cancellative category that admits right-gcds, f, g are elements of $\mathcal{C}$ and h is a common right-multiple of f and g. Show that there exists a right-mcm h_0 of f and g such that every common right-multiple of f and g that left-divide h is a right-multiple of h_0.*

Solution. (See Figure 1.) Write $h = f\hat{g} = g\hat{f}$, and let $\hat{h}$ be a right-gcd of $\hat{f}$ and $\hat{g}$. By definition $\hat{h}$ right-divides $\hat{f}$ and $\hat{g}$, so there exist f', g' satisfying $\hat{f} = f'\hat{h}$ and $\hat{g} = g'\hat{h}$. Then we have $fg'\hat{h} = f\hat{g} = g\hat{f} = gf'\hat{h}$, whence $fg' = gf'$ by right-cancelling $\hat{h}$. Assume $fg'' = gf'' \preccurlyeq h$, say $h = fg''h''$. By left-cancelling f, we deduce $\hat{g} = g''h''$ and, similarly, $\hat{f} = f''h''$. So h'' is a common right-divisor of $\hat{f}$ and $\hat{g}$, hence it is a right-divisor of $\hat{h}$, that is, there exists h' satisfying $\hat{h} = h'h''$. This implies $fg''h'' = f\hat{g}$, whence $g''h'' = \hat{g}$ by left-cancelling f, and, finally, $h'' = fg'' \preccurlyeq f\hat{g} = h$. So every common right-multiple of f and g that left-divides h is a right-multiple of fg'.

Now assume $fg'' = gf'' \preccurlyeq fg'$. A fortiori, we have $fg'' = gf'' \preccurlyeq h$, so the above result implies $fg' \preccurlyeq fg''$, whence $fg'' =^{\times} fg'$. So fg' is a right-mcm of f and g.

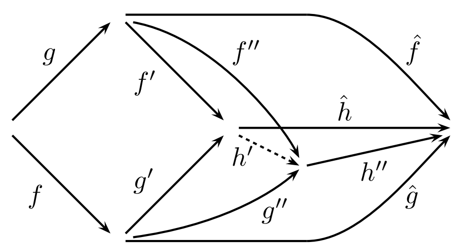


FIGURE 1. Solution to Exercise 6

Exercice 8 (**conditional right-lcm**).— *Assume that $\mathcal{C}$ is a left-cancellative category.* (i) *Show that every left-gcd of fg_1 and fg_2 (if any) is of the form fg where g is a left-gcd of g_1 and g_2.* (ii) *Assume moreover that $\mathcal{C}$ admits conditional right-lcms. Show that, if g is a left-gcd of g_1 and g_2 and fg is defined, then fg is a left-gcd of fg_1 and fg_2.*

Solution. (i) Since f left-divides fg_1 and fg_2, it left-divides every left-gcd of fg_1 and fg_2, so the latter can be written fg. Now assume that fg is a left-gcd of fg_1 and fg_2. As $\mathcal{C}$ is left-cancellative, $fg \preccurlyeq fg_i$ implies $g \preccurlyeq g_i$. Next, assume that h left-divides g_1 and g_2. Then fh left-divides fg_1 and fg_2, implying $fh \preccurlyeq fg$, whence $h \preccurlyeq g$. So g is a left-gcd of g_1 and g_2.

(ii) It is clear that fg is a common left-divisor of fg_1 and fg_2. Conversely, assume that h is a common left-divisor of fg_1 and fg_2. By assumption, f and h admit fg_1 as a common right-multiple, so they admit a right-lcm, say fh'. By assumption, we have $fh' \preccurlyeq fg_1$ and $fh' \preccurlyeq fg_2$, whence $h' \preccurlyeq g_1$ and $h' \preccurlyeq g_2$ by left-cancelling f. This in turn implies $h' \preccurlyeq g$ since g is a left-gcd of g_1 and g_2. Hence we deduce $h \preccurlyeq fh' \preccurlyeq fg$, which shows that fg is a left-gcd of fg_1 and fg_2.

Exercice 9 (**left-coprime**).— *Assume that $\mathcal{C}$ is a left-cancellative category. Say that two elements f, g of $\mathcal{C}$ sharing the same source x are* left-coprime *if 1_x is a left-gcd of f and g. Assume that g_1, g_2 are elements of $\mathcal{C}$ sharing the same source, and fg_1 and fg_2 are defined. Consider the properties* (i) *The elements g_1 and g_2*

are left-coprime; (ii) *The element f is a left-gcd for fg_1 and fg_2. Show that* (ii) *implies* (i) *and that, if $\mathcal{C}$ admits conditional right-lcms,* (i) *implies* (ii). *[Hint: Use Exercise 8.]*

Solution. If g is a non-invertible common left-divisor of g_1 and g_2, then fg is a non-invertible common left-divisor of fg_1 and fg_2, so clearly (ii) implies (i).

Conversely, assume that $\mathcal{C}$ admits conditional right-lcms and g_1, g_2 are left-coprime. By definition, 1_x is a left-gcd of g_1 and g_2. Hence, by Exercise 8, f is a left-gcd of fg_1 and fg_2. So (i) implies (ii) in this case.

Exercice 10 (**subgroupoid**).— *Let $M = \langle \mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d} \mid \mathtt{ab} = \mathtt{bc} = \mathtt{cd} = \mathtt{da} \rangle^+$ and $\Delta = \mathtt{ab}$.* (i) *Check that M is a Garside monoid with Garside element Δ.* (ii) *Let M_1 (resp. M_2) be the submonoid of M generated by* a *and* c *(resp.* b *and* c*). Show that M_1 and M_2 are free monoids of rank 2 with intersection reduced to $\{1\}$.* (iii) *Let G be the group of fractions of M. Show that the intersection of the subgroups of G generated by M_1 and M_2 is not $\{1\}$.*

Solution. (ii) No word in $\{\mathtt{a}, \mathtt{c}\}^*$ is eligible for any of the defining relations of M, so two distinct such words represent distinct elements of M_1. (iii) In G, we have $\mathtt{c}^{-1}\mathtt{a} = \mathtt{db}^{-1}$ but $\mathtt{db}^{-1}$ cannot be expressed as $f^{-1}g$ with f, g in M_2 since, otherwise, M_2 would not be free.

Exercice 11 (**weakly right-cancellative**).— *Say that a category $\mathcal{C}$ is weakly right-cancellative if $gh = h$ implies that g is invertible.* (i) *Observe that a right-cancellative category is weakly right-cancellative;* (ii) *Assume that $\mathcal{C}$ is a left-cancellative category. Show that $\mathcal{C}$ is weakly right-cancellative if and only if, for all f, g in $\mathcal{C}$, the relation $f \mathrel{\widetilde{\prec}} g$ is equivalent to the conjunction of $f \mathrel{\widetilde{\preccurlyeq}} g$ and “$g = g'f$ holds for no g' in $\mathcal{C}^\times$”.*

Solution. (ii) The conjunction of $f \mathrel{\widetilde{\preccurlyeq}} g$ and “$g = g'f$ holds for no g' in $\mathcal{C}^\times$” always implies $f \mathrel{\widetilde{\prec}} g$. Assume that $\mathcal{C}$ is weakly right-cancellative and $g \mathrel{\widetilde{\succ}} f$ holds. Then we have $g = g'f$ for some $g' \notin \mathcal{C}^\times$. Assume $g = g''f$: if g'' is invertible, we deduce $f = g''^{-1}g = g''^{-1}g'g$. The assumption that $\mathcal{C}$ is weakly right-cancellative implies that $g''^{-1}g'$ is invertible, hence that g' is invertible. So $f \mathrel{\widetilde{\prec}} g$ implies “$g = g'f$ holds for no g' in $\mathcal{C}^\times$”. Conversely, assume that $f \mathrel{\widetilde{\prec}} g$ is equivalent to the conjunction of $f \mathrel{\widetilde{\preccurlyeq}} g$ and “$g = g'f$ holds for no g' in $\mathcal{C}^\times$”. Assume $gh = h$. If x is the source of h, we have $1_x h = h$ and 1_x is invertible. Then the assumption implies that $h \mathrel{\widetilde{\prec}} h$ fails, which, as $h \preccurlyeq h$ is true, means that $h = gh$ holds for no non-invertible g.

Exercice 13 (**increasing sequences**).— *Assume that $\mathcal{C}$ is a left-cancellative. For $\mathcal{S}$ included in $\mathcal{C}$, put $\mathrm{Div}_{\mathcal{S}}(h) = \{f \in \mathcal{C} \mid \exists g \in \mathcal{S}(fg = h)\}$. Show that the restriction of $\widetilde{\prec}$ to $\mathcal{S}$ is well-founded (that is, admits no infinite descending sequence) if and only if, for every g in $\mathcal{S}$, every strictly increasing sequence in $\mathrm{Div}_{\mathcal{S}}(g)$ with respect to left-divisibility is finite.*

Solution. Assume that $\mathcal{S}$ is not right-Noetherian. Let $g_0, g_1, ...$ be an infinite descending sequence with respect to proper right-divisibility in $\mathcal{S}$. For each i, choose a (necessarily non-invertible) element f_i satisfying $g_{i-1} = f_i g_i$. Then we have $g_0 = f_1 g_1 = (f_1 f_2) g_2 = ...$, and the sequence 1_x (x the source of g_0), $f_1, f_1 f_2, ...$ is $\prec$-increasing in $\mathrm{Div}_{\mathcal{S}}(g_0)$.

Conversely, assume that g_0 lies in $\mathcal{S}$ and $h_1 \prec h_2 \prec ...$ is a strictly increasing sequence in $\mathrm{Div}_{\mathcal{S}}(g_0)$. Then, for each i, there exists a non-invertible element f_i

satisfying $h_i f_i = h_{i+1}$. On the other hand, as h_i belongs to $\mathrm{Div}_{\mathcal{S}}(g_0)$, there exists g_i in $\mathcal{S}$ satisfying $h_i g_i = g_0$. We find $g_0 = h_i g_i = h_{i+1} g_{i+1} = h_i f_i g_{i+1}$. By left-cancelling h_i, we deduce $g_i = f_i g_{i+1}$, hence g_{i+1} is a proper right-divisor of g_i for each i. So the sequence $g_0, g_1, ...$ witnesses that $\mathcal{S}$ is not right-Noetherian.

Exercice 15 (**left-generating**).— *Assume that $\mathcal{C}$ is a left-cancellative category that is right-Noetherian. Say that a subfamily $\mathcal{S}$ of $\mathcal{C}$* left-generates *(resp.* right-generates*) $\mathcal{C}$ if every non-invertible element of $\mathcal{C}$ admits at least one non-invertible left-divisor (resp. right-divisor) belonging to $\mathcal{S}$.* (i) *Show that $\mathcal{C}$ is right-generated by its atoms.* (ii) *Show that, if $\mathcal{S}$ is a subfamily of $\mathcal{C}$ that left-generates $\mathcal{C}$, then $\mathcal{S} \cup \mathcal{C}^\times$ generates $\mathcal{C}$.* (iii) *Conversely, show that, if $\mathcal{S} \cup \mathcal{C}^\times$ generates $\mathcal{C}$ and $\mathcal{C}^\times \mathcal{S} \subseteq \mathcal{S}^\sharp$ holds, then $\mathcal{S}$ left-generates $\mathcal{C}$.*

Solution. (ii) Assume that $\mathcal{S}$ left-generates $\mathcal{C}$. Let g be an arbitrary element of $\mathcal{C}$. If g is invertible, g belongs to $\mathcal{C}^\times$. Otherwise, by assumption, there exist a non-invertible element g_1 in $\mathcal{S}$ and g' in $\mathcal{C}$ satisfying $g = g_1 g'$. If g' is invertible, g belongs to $\mathcal{S}\,\mathcal{C}^\times$. Otherwise, there exist a non-invertible element g_2 in $\mathcal{S}$ and g'' satisfying $g' = g_2 g''$, and so on. By Proposition II.2.28, the sequence 1_x, g_1, $g_1 g_2$, $\cdots$, which is increasing with respect to proper left-divisibility and lies in $\mathrm{Div}(g)$, must be finite, yielding ℓ and $g = g_1 \cdots g_\ell \epsilon$ with $g_1, ..., g_\ell$ in $\mathcal{S}$ and ϵ in $\mathcal{C}^\times$.

(iii) Assume that $\mathcal{S} \cup \mathcal{C}^\times$ generates $\mathcal{C}$ and $\mathcal{C}^\times \mathcal{S} \subseteq \mathcal{S}^\sharp$ holds. Let g be a non-invertible element of $\mathcal{C}$. Let $(g_1, ..., g_p)$ be a decomposition of g such that g_i lies in $\mathcal{S} \cup \mathcal{C}^\times$ for every i. As g is not invertible, there exists i such that g_i is not invertible. Assume that i is minimal with this property. Then $g_1 \cdots g_{i-1}$ is invertible and, as $\mathcal{C}^\times \mathcal{S} \subseteq \mathcal{S}^\sharp$ holds, there exists g' in $\mathcal{S} \setminus \mathcal{C}^\times$ and ϵ in $\mathcal{C}^\times$ satisfying $g_1 \cdots g_i = g' \epsilon$. Then g' is a non-invertible element of $\mathcal{S}$ left-dividing g. Hence $\mathcal{S}$ left-generates $\mathcal{C}$.

Exercice 20 (**equivalence**).— *Assume that $(\mathcal{S}, \mathcal{R})$ is a category presentation. Say that an element s of $\mathcal{S}$ is $\mathcal{R}$-right-invertible if $sw \equiv^+_{\mathcal{R}} \varepsilon_x$ (x the source of s) holds for some w in $\mathcal{S}^*$. Show that a category presentation $(\mathcal{S}, \mathcal{R})$ is complete with respect to right-reversing if and only if, for all u, v in $\mathcal{S}^*$, the following are equivalent:* (i) *u and v are $\mathcal{R}$-equivalent (that is, $u \equiv^+_{\mathcal{R}} v$ holds),* (ii) *$\overline{u}v \curvearrowright_{\mathcal{R}} v'\overline{u'}$ holds for some $\mathcal{R}$-equivalent paths u', v' in $\mathcal{S}^*$ all of which entries are $\mathcal{R}$-right-invertible.*

Solution. Assume that right-reversing is complete for $(\mathcal{S}, \mathcal{R})$ and $u \equiv^+_{\mathcal{R}} v$ holds. Denoting by y the common target of u and v, we have $u\varepsilon_y \equiv^+_{\mathcal{R}} v\varepsilon_y$. Hence, by definition of completeness, there exist u', v', w satisfying $\overline{u}v \curvearrowright_{\mathcal{R}} v'\overline{u'}$, $\varepsilon_y \equiv^+_{\mathcal{R}} u'w$, and $\varepsilon_y \equiv^+_{\mathcal{R}} u'w$. Hence, in $\mathcal{C}$, we have $[u']^+ [w]^+ = 1_y = [v']^+ [w]^+$, whence $[u']^+ = [v']^+ = ([w]^+)^{-1}$. So u' and v' are $\mathcal{R}$-equivalent and, as their classes are invertible, they must consist of invertible entries.

Conversely, assume that the condition of (ii) is satisfied, and that uv' and vu' are $\mathcal{R}$-equivalent. By (ii), there exist $\mathcal{R}$-equivalent $\mathcal{S}$-paths u_0, v_0 such that $\overline{(uv')}(vu')$ is right-reversible to $v_0\overline{u_0}$ and all entries in u_0 and v_0 are invertible. By Lemma II.4.23, the reversing of $\overline{(uv')}(vu')$ to $v_0\overline{u_0}$ splits into four reversings. By construction, all entries in u_1, u_2, v_1, v_2 are invertible. Put $w = u'''v_2\overline{u_2 u_1}$. By assumption, we have $v_1 v_2 \equiv^+_{\mathcal{R}} u_1 u_2$, whence $\overline{v_2 v_1} \equiv^+_{\mathcal{R}} \overline{u_2 u_1}$. We deduce

$$u' \equiv^+ u''u'''v_2\overline{u_2 u_1} = u''w, \text{ and } v' \equiv^+ v''u'''v_2\overline{v_2 v_1} \equiv^+ u''w,$$

which means that (u, v, u', v') factorizing through right-reversing. Hence right-reversing is complete for $(\mathcal{S}, \mathcal{R})$.

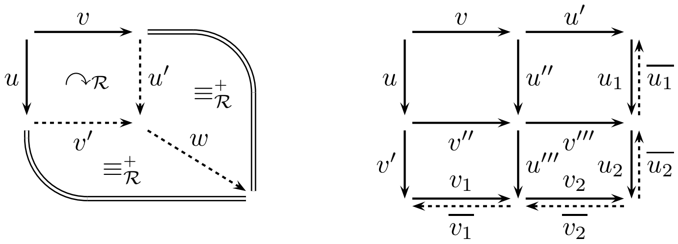


FIGURE 2. Solution to Exercise 20

Exercice 22 (**complete** *vs.* **cube, complemented case**).— *Assume that* $(\mathcal{S},\mathcal{R})$ *is a right-complemented presentation, associated with the syntactic right-complement* θ*. Show the equivalence of the following three properties:* (i) *Right-reversing is complete for* $(\mathcal{S},\mathcal{R})$*;* (ii) *The map* θ^* *is compatible with* $\equiv^+_{\mathcal{R}}$*-equivalence in the sense that, if* $u' \equiv^+_{\mathcal{R}} u$ *and* $v' \equiv^+_{\mathcal{R}} v$ *hold, then* $\theta^*(u',v')$ *is defined if and only if* $\theta(u,v)$ *is and, if so, they are* $\equiv^+_{\mathcal{R}}$*-equivalent;* (iii) *The* θ*-cube condition is true for every triple of* $\mathcal{S}$*-paths. [Hint: For* (i) $\Rightarrow$ (ii)*, note that, if* $\theta^*(u,v)$ *is defined and* $u' \equiv^+_{\mathcal{R}} u$ *and* $v' \equiv^+_{\mathcal{R}} v$ *hold, then* $(u', v,' \theta^*(u,v), \theta^*(v,u))$ *must be* $\curvearrowright$*-factorable; for* (ii) $\Rightarrow$ (iii)*, compute* $\theta^*(w, u\theta^*(u,v))$ *and* $\theta^*(w, v\theta^*(v,u))$*; for* (iii) $\Rightarrow$ (i)*, use Lemma II.4.61.]*

Solution. (i)$\Rightarrow$(ii) Assume that $\theta^*(u,v)$ exists and $u' \equiv^+_{\mathcal{R}} u$ and $v' \equiv^+_{\mathcal{R}} v$ hold. By Corollary II.4.36, we have $u\theta^*(u,v) \equiv^+_{\mathcal{R}} v\theta^*(v,u)$, whence $u'\theta^*(u,v) \equiv^+_{\mathcal{R}} v'\theta^*(v,u)$. By assumption, the quadruple $(u', v', \theta^*(u,v), \theta^*(v,u)$ is $\curvearrowright$-factorable: this means that $\theta^*(u',v')$ and $\theta^*(v',u')$ are defined and there exists w satisfying

$$\theta^*(u,v) \equiv^+_{\mathcal{R}} \theta^*(u',v')w \quad \text{and} \quad \theta^*(v,u) \equiv^+_{\mathcal{R}} \theta^*(v',u')w.$$

By a symmetric argument, we obtain the existence of w' satisfying

$$\theta^*(u',v') \equiv^+_{\mathcal{R}} \theta^*(u,v)w' \quad \text{and} \quad \theta^*(v',u') \equiv^+_{\mathcal{R}} \theta^*(v,u)w'.$$

Merging, we deduce $\theta^*(u,v) \equiv^+_{\mathcal{R}} \theta^*(u,v)w'w$. By Corollary II.4.45, the category $\langle \mathcal{S} \mid \mathcal{R} \rangle^+$ is left-cancellative, so we deduce $\varepsilon_{\text{-}} \equiv^+_{\mathcal{R}} w'w$, whence $w' = w = \varepsilon_{\text{-}}$ and, finally, $\theta^*(v',u') \equiv^+_{\mathcal{R}} \theta^*(u,v)$.

(ii)$\Rightarrow$(iii) Assume that θ^* is compatible with $\equiv^+_{\mathcal{R}}$ and $\theta^*_3(u,v,w)$ is defined. By Lemma II.4.6, we have $\theta^*_3(u,v,w) = \theta^*(u(RC^*(u,v),w)$. As $u(\theta^*(u,v) \equiv^+_{\mathcal{R}} v\theta^*(v,u)$ holds, the compatibility assumption implies that $\theta^*(v(RC^*(v,u),w)$ is defined as well, and it is $\equiv^+_{\mathcal{R}}$-equivalent to the latter. Now, by Lemma II.4.6 again, the path $\theta^*(v(\theta^*(v,u),w)$ is $\theta^*_3(v,u,w)$, so we conclude that $\theta^*_3(v,u,w)$ is defined and it is $\equiv^+_{\mathcal{R}}$-equivalent to $\theta^*_3(u,v,w)$, that is, the θ-cube condition is true for (u,v,w).

(iii)$\Rightarrow$(i) Lemma II.4.61 gives the result directly.

Exercice 23 (**alternative proof**).— *Assume that* $(\mathcal{S},\mathcal{R})$ *is a right-complemented presentation associated with the syntactic right-complement* θ*, that* $(\mathcal{S},\mathcal{R})$ *is right-Noetherian, and that the* θ*-cube condition is true on* $\mathcal{S}$*.* (i) *Show that, for all* r,s,t *in* $\mathcal{S}$*, the path* $\theta^*(r, s\theta(s,t))$ *is defined if and only if* $\theta^*(r, t\theta(t,s))$ *is and, in this case, the relations* $\theta^*(r, s\theta(s,t)) \equiv^+_{\mathcal{R}} \theta^*(r, t\theta(t,s))$ *and* $\theta^*(s\theta(s,t), r) \equiv^+_{\mathcal{R}} \theta^*(t\theta(t,s), r)$ *are satisfied.* (ii) *Show that the map* θ^* *is compatible with* $\equiv^+_{\mathcal{R}}$*, that is, the conjunction of* $u' \equiv^+_{\mathcal{R}} u$ *and* $v' \equiv^+_{\mathcal{R}} v$ *implies that* $\theta^*(u',v')$ *exists if and only if* $\theta^*(u,v)$ *does and, in this case, they are* $\equiv^+_{\mathcal{R}}$*-equivalent. [Hint: Show using on* $\lambda^*(u\theta^*(u,v))$*, where* λ^* *is a right-Noetherianity witness for* $(\mathcal{S},\mathcal{R})$ *that, if* $\theta^*(u,v)$ *is defined and we have* $u' \equiv^+_{\mathcal{R}} u$ *and* $v' \equiv^+_{\mathcal{R}} v$*, then* $\theta^*(u',v')$ *is defined and we have* $\theta^*(u',v') \equiv^+_{\mathcal{R}} \theta^*(u,v)$

and $\theta^*(v', u') \equiv^+_{\mathcal{R}} \theta^*(v, u)$.*]* (iii) *Apply Exercise 22 to deduce a new proof of Proposition II.4.16 in the right-Noetherian case.*

Solution. (i) Lemma II.4.6 gives

$$\theta^*(s\theta(s,t), r) = \theta^*(\theta(s,t), \theta(s,r)) = \theta^*_3(s,t,r),$$
$$\theta^*(r, s\theta(s,t)) = \theta(r,s)\theta^*(\theta(s,r), \theta(s,t)) = \theta(r,s)\theta^*_3(s,r,t).$$

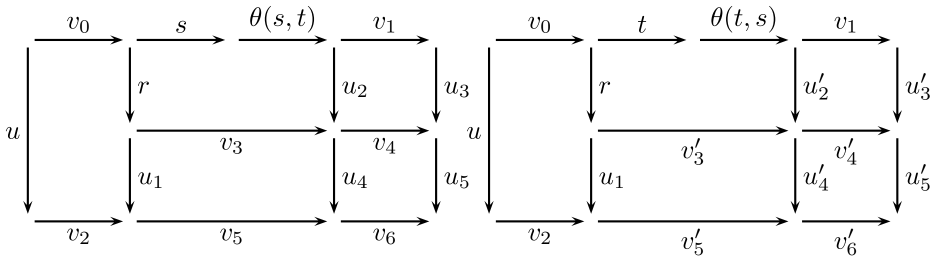
As the θ-cube condition is true on $\{r,s,t\}$, we have $\theta^*_3(s,t,r) \equiv^+_{\mathcal{R}} \theta^*_3(t,r,r)$, so, using the first equality above and its counterpart exchanging s and t, we find

$$\theta^*(s\theta(s,t), r) = \theta^*_3(s,t,r) \equiv^+_{\mathcal{R}} \theta^*_3(t,r,r) = \theta^*(t\theta(t,s), r).$$

Similarly, the θ-cube condition gives the relations $\theta^*_3(s,r,t) \equiv^+_{\mathcal{R}} \theta^*_3(r,s,t)$ and $\theta^*_3(r,t,s) \equiv^+_{\mathcal{R}} \theta^*_3(t,r,s)$, whence, by Corollary II.4.36,

$$\theta^*(r, s\theta(s,t)) = \theta(r,s)\theta^*_3(s,r,t) \equiv^+_{\mathcal{R}} \theta(r,s)\theta^*_3(r,s,t)$$
$$\equiv^+_{\mathcal{R}} \theta(r,t)\theta^*_3(r,t,s) \equiv^+_{\mathcal{R}} \theta(r,t)\theta^*_3(t,r,s) = \theta^*(r, t\theta(t,s)).$$

(ii) The result of (i) is the compatiblity in the case $u' = u = r$ and $v = s\theta(s,t)$, $v' = t\theta(t,s)$, that is, in the basic case of equivalence. We establish the general result using induction on $\lambda^*(u\theta^*(u,v))$ and, for a given value α, on the sum d of the combinatorial distances from u to u' and from v to v'. For an induction, it is sufficient to consider the case $d = 1$, that is, we may assume $u' = u$ and $\mathrm{dist}(v, v') = 1$, that is, there exist s, t in $\mathcal{S}$ and v_0, v_1 in $\mathcal{S}^*$ satisfying $v = v_0 s\theta(s,t)v_1$ and $v' = v_0 t\theta(t,s)v_1$. We assume that $\theta^*(u,v)$ is defined, and our aim is to show that $\theta^*(u',v')$ is defined as well and we have $\theta^*(u',v') \equiv^+_{\mathcal{R}} \theta^*(u,v)$ and $\theta^*(v',u') \equiv^+_{\mathcal{R}} \theta^*(v,u)$. To this end, we compare the reversing grids below:



By assumption, the left grid exists, and we wish to show that the right grid exists as well and that the corresponding paths are pairwise $\equiv^+_{\mathcal{R}}$-equivalent. The rectangles on the left (u and v_0) coincide. Next, we find a rectangle as in (i), namely r and $s\theta(s,t)$ *vs.* r and $t\theta(t,s)$. By (i), u'_2 and v'_3 exist and we have $u'_2 \equiv^+_{\mathcal{R}} u_2$ and $v'_3 \equiv^+_{\mathcal{R}} v_3$. Then consider the median bottom rectangles (u_1 and v'_3): the point is the inequality

$$\lambda^*(\theta^*(u_1 v_5)) < \lambda^*(r u_1 v_5) \leqslant \lambda^*(r u_1 v_5 v_6) \leqslant \lambda^*(v_0 r u_1 v_5 v_6) = \lambda^*(u,v) \leqslant \alpha.$$

As $\theta^*(u_1, v_3)$ exists and we have $v'_3 \equiv^+_{\mathcal{R}} v_3$, the induction hypothesis implies that $\theta^*(u_1, v'_3)$ exists as well and gives $u'_4 \equiv^+_{\mathcal{R}} u_4$ and $v'_5 \equiv^+_{\mathcal{R}} v_5$. The argument is the same for the two right squares.

(iii) So, if the θ-cube condition is true on $\mathcal{S}$, the map θ^* is compatible with $\equiv^+_{\mathcal{R}}$. By Exercise 22, the latter condition implies that right-reversing is complete for $(\mathcal{R}, \mathcal{S})$.

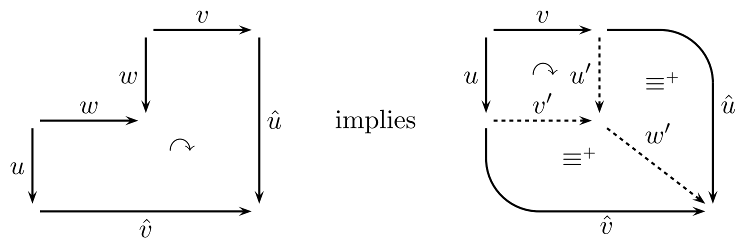


FIGURE 3. Solution to Exercise 24: The cube condition viewed as a factorization property: whenever $\overline{u}w\overline{w}v$ right-reverses to $\hat{v}\overline{\hat{u}}$, the quadruple $(u, v, \hat{u}, \hat{v})$ is $\curvearrowright$-factorable.

Exercice 24 (**cube condition**).— *Assume that* $(\mathcal{S}, \mathcal{R})$ *is a category presentation* (i) *Show that the cube condition is true for* (u, v, w) *if and only if, for all* $\hat{u}, \hat{v}$ *in* $\mathcal{S}^*$ *satisfying* $\overline{u}w\overline{w}v \curvearrowright_{\mathcal{R}} \hat{v}\overline{\hat{u}}$, *the quadruple* $(u, v, \hat{u}, \hat{v})$ *is* $\curvearrowright$*-factorable.* (ii) *Show that, if right-reversing is complete for* $(\mathcal{S}, \mathcal{R})$, *then the cube condition is true for every triple of* $\mathcal{S}$*-paths.*

Solution. (ii) Assume that u, v, w belong to $\mathcal{S}^*$, and we have $\overline{u}w \curvearrowright_{\mathcal{R}} v_1\overline{u_0}$, $\overline{w}v \curvearrowright_{\mathcal{R}} v_0\overline{u_1}$, and $\overline{u_0}v_0 \curvearrowright_{\mathcal{R}} v_2\overline{u_2}$ (so that u, v, w necessarily share the same source). By Proposition II.4.34, we have $uv_1 \equiv^+_{\mathcal{R}} wu_0$, $vu_1 \equiv^+_{\mathcal{R}} wv_0$, and $u_0v_2 \equiv^+_{\mathcal{R}} v_0u_2$, and we deduce $uv_1v_2 \equiv^+_{\mathcal{R}} wu_0v_2 \equiv^+_{\mathcal{R}} wv_0u_2 \equiv^+_{\mathcal{R}} vu_1u_2$. The assumption that right-reversing is complete then implies that (u, v, u_1u_2, v_1v_2) is $\curvearrowright$-factorable, which exactly means that the cube condition is true for (u, v, w).

Chapter III: Normal decompositions

Skipped proofs

Proposition III.1.14 (**power**).— *If* $\mathcal{S}$ *is a subfamily of a left-cancellative category* $\mathcal{C}$ *and* $g_1|\cdots|g_p$ *is* $\mathcal{S}$*-greedy, then* $g_1\cdots g_m|g_{m+1}\cdots g_p$ *is* $\mathcal{S}^m$*-greedy for* $1 \leqslant m \leqslant p$, *that is,*

(III.1.15) *Each relation* $s \preccurlyeq fg_1\cdots g_p$ *with* s *in* $\mathcal{S}^m$ *implies* $s \preccurlyeq fg_1\cdots g_m$.

Proof. (See Figure 4.) We use induction on m. For $m = 1$, the result follows from Proposition III.1.12, and more precisely from (III.1.13). Assume $m \geqslant 2$. Let $s \in \mathcal{S}^m$, say $s = s_1\cdots s_m$ with $s_1, \ldots, s_m$ in $\mathcal{S}$, and $s \preccurlyeq fg_1\cdots g_p$. Then we have $s_1 \preccurlyeq fg_1\cdots g_p$, hence, by (III.1.13), $s_1 \preccurlyeq fg_1$, say $fg_1 = s_1f_1$, and, therefore, $s_2\cdots s_m \preccurlyeq f_1g_2\cdots g_p$. As $s_2\cdots s_m$ belongs to $\mathcal{S}^{m-1}$, the induction hypothesis implies $s_2\cdots s_m \preccurlyeq f_1g_2\cdots g_m$, whence $s_1s_2\cdots s_m \preccurlyeq fg_1g_2\cdots g_m$, as expected. ☐

Lemma III.2.52.— *For* f, g, f', g' *in a cancellative category* $\mathcal{C}$ *satisfying* $f'g = g'f$, *the following conditions are equivalent:*

(i) *The elements* f' *and* g' *are left-disjoint;*

(ii) *The element* $f'g$ *is a weak left-lcm of* f *and* g.

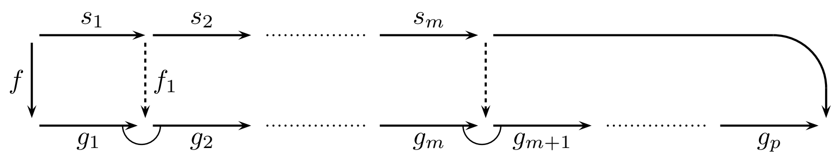


FIGURE 4. Inductive proof of Proposition III.1.14.

Proof. Assume (i), and assume that $f''g = g''f$ is a common left-multiple of f and g such that $f''g$ and $f'g$ admit a common left-multiple, say $h'(f''g) = h''(f'g)$. By assumption we also have $h'(g''f) = h''(g'f)$ and, because $\mathcal{C}$ is assumed to be right-cancellative, we deduce $h'f'' = h''f'$ and $h'g'' = h''g'$. Thus (h', h'') witnesses for $(f'', g'') \bowtie (f', g')$. As f' and g' are left-disjoint, we deduce $h' \preccurlyeq h''$, that is, there exists h satisfying $h'' = h'h$. We deduce $h'(f''g) = h''(f'g) = h'h(f'g)$, whence $f''g = h(f'g)$ by left-cancelling h'. This shows that $f''g$ is a left-multiple of $f'g$, and the latter is a weak left-lcm of f and g. So (i) implies (ii).

Assume now (ii), and assume that (h', h'') witnesses for $(f'', g'') \bowtie (f', g')$, that is, we have $h'f'' = h''f'$ and $h'g'' = h''g'$. We deduce $h'f''g = h''f'g = h''g'f = h'g''f$, whence $f''g = g''f$ by left-cancelling h'. So $f''g$ is a common left-multiple of f and g. Moreover, the above equalities show that $f''g$ and $f'g$ admit a common left-multiple. As $f'g$ is a weak left-lcm of f and g, we deduce that $f''g$ is a left-multiple of $f'g$, that is, there exists h satisfying $f''g = hf'g$, hence $h''f'g = h'f''g = h'hf'g$. Right-cancelling $f'g$, we deduce $h'' = h'h$, that is, $h' \preccurlyeq h''$. Hence f' and g' are left-disjoint, and (ii) implies (i). ☐

Proposition III.2.53 (**symmetric normal exist**).— *If $\mathcal{S}$ is a Garside family in a cancellative category $\mathcal{C}$, the following conditions are equivalent:*

(i) *For all f, g in $\mathcal{C}$ admitting a common right-multiple, there exists a symmetric $\mathcal{S}$-normal path $\overline{u}v$ satisfying $(f, g) \bowtie ([u]^+, [v]^+)$;*

(ii) *The category $\mathcal{C}$ admits conditional weak left-lcms.*

Proof. Assume (i), and let f, g be two elements of $\mathcal{C}$ that admit a common left-multiple, say $\hat{f}g = \hat{g}f$. By (i) applied to $\hat{f}$ and $\hat{g}$, there exists a symmetric $\mathcal{S}$-normal path $\overline{u}v$ such that, putting $f' = [u]^+$ and $g' = [v]^+$, we have $(\hat{f}, \hat{g}) \bowtie (f', g')$, that is, there exist $h', \hat{h}$ satisfying $h'\hat{f} = \hat{h}f'$ and $h'\hat{g} = \hat{h}g'$. By Proposition III.2.11, f' and g' are left-disjoint, so, by Lemma III.2.52, $f'g$, which is also $g'f$, is a weak left-lcm of f and g. Hence $\mathcal{C}$ admits conditional weak left-lcms, and (i) implies (ii).

Conversely, assume (ii), and let f, g be two elements of $\mathcal{C}$ that admit a common right-multiple, say $fg' = gf'$. By (ii), there exists a weak left-lcm of f' and g', say $f''g' = g''f'$, and h satisfying $fg' = hf''g'$, hence $f = hf''$ by right-cancelling g' and, similarly, $g = hg''$. By Lemma III.2.52, the elements f'' and g'' are left-disjoint. Let u be an $\mathcal{S}$-normal decomposition of f'' and v be an $\mathcal{S}$-normal decomposition of g''. By (the trivial part of) Proposition III.2.11, the first entries of u and v are left-disjoint since f'' and g'' are, so $\overline{u}v$ is symmetric $\mathcal{S}$-normal. Finally, the pair $(h, 1_x)$ (with x the source of f and g) witnesses for $(f, g) \bowtie (f'', g'')$. So (ii) implies (i). ☐

Lemma III.2.55.— *A cancellative category $\mathcal{C}$ admits conditional weak left-lcms if and only if $\mathcal{C}$ is a strong Garside family in itself.*

Proof. Assume that $\mathcal{C}$ is strong and $fs = gt$ holds. Then there exist f', g', h such that f' and g' are left-disjoint and we have $f's = h't$, $f = hf'$, and $g = hg'$. By Lemma III.2.52, $f'g$ is a weak left-lcm of f and g, of which fg is a left-multiple. So $\mathcal{C}$ admits conditional weak left-lcms.

Conversely, assume that $\mathcal{C}$ admits conditional weak left-lcms, and $fs = gt$ holds. Then there exists a weak left-lcm $f's$ of s and t of which fs is a left-multiple. By Lemma III.2.52 again, f' and g' are left-disjoint, and the condition of Definition III.2.54 is satisfied. So $\mathcal{C}$ is strong. $\square$

Proposition III.2.56 (**symmetric normal, short case III**).— *If $\mathcal{S}$ is a strong Garside family in a cancellative category $\mathcal{C}$ admitting conditional weak left-lcms, Algorithm III.2.42 running on a positive–negative $\mathcal{S}^\sharp$-path $v\overline{u}$ such that $[u]^+$ and $[v]^+$ admit a common left-multiple, say $\hat{f}[v]^+ = \hat{g}[u]^+$, returns a symmetric $\mathcal{S}$-normal path $\overline{u''}v''$ satisfying $(f, g) \bowtie ([u'']^+, [v'']^+)$ and $[u''v]^+ = [v''u]^+$; moreover there exists h satisfying $f = h[u'']^+$ and $g = h[v'']^+$.*

Once again, Proposition III.2.56 reduces to Proposition III.2.44 in the case of a left-Ore category as it then says that $\overline{u}v$ is a decomposition of $[v\overline{u}]$ in $\mathcal{E}nv(\mathcal{C})$.

Proof. The argument is the same as for Proposition III.2.44, the only difference being that, at each left-reversing step, one has to check the existence of a factorization of the initial equality $f[v] = g[u]$. The principle is explained in Figure 5: the induction hypothesis that is maintained at each step in the construction of the rectangular diagram is that, for every local North-West corner in the current diagram, there exists a factorizing arrow coming from the top-left object x. When one more tile is added, the defining property of a strong Garside family guarantees that one can add a new tile in which the left and top arrows represent left-disjoint elements and there exists a factorizing arrow coming from x. The rest of the proof is unchanged as, in particular, the third domino rule is still valid in the extended context. $\square$

Solution to selected exercises

Exercice 28 (**invertible**).— *Assume that $\mathcal{C}$ is a left-cancellative category and $\mathcal{S}$ is included in $\mathcal{C}$. Show that, if $g_1 \cdots g_p$ belongs to $\mathcal{S}^\sharp$, then $g_1|\cdots|g_p$ being $\mathcal{S}$-greedy implies that $g_2, ..., g_p$ are invertible.*

Solution. The element $g_1 \cdots g_p$ lies in $\mathcal{S}^\sharp$ is equal to, hence left-divides, itself. By Proposition III.1.12, $g_1|g_2 \cdots g_p$ is $\mathcal{S}$-greedy, so we deduce that $g_1 \cdots g_p$ left-divides g_1, say $g_1 = g_1 \cdots g_p g'$. Left-cancelling g_1, we deduce that $g_2 \cdots g_p g'$ is an identity-element, hence $g_2, ..., g_p$, and g' must be invertible.

Exercice 29 (**deformation**).— *Assume that $\mathcal{C}$ is a left-cancellative category. Show that a path $g_1|\cdots|g_q$ is a $\mathcal{C}^\times$-deformation of $f_1|\cdots|f_p$ if and only if $g_1 \cdots g_i =^\times f_1 \cdots f_i$ holds for $1 \leqslant i \leqslant \max(p, q)$, the shorter path being extended by identity-elements if needed.*

Solution. Let $r = \max(p, q)$. Assume that $\epsilon_0, ..., \epsilon_r$ are invertible elements witnessing that $g_1|\cdots|g_r$ is a $\mathcal{C}^\times$-deformation of $f_1|\cdots|f_r$. For every i, we deduce $f_1 \cdots f_i \epsilon_i = \epsilon_0 g_1 \cdots g_r$, whence $g_1 \cdots g_i =^\times f_1 \cdots f_i$ since ϵ_0 is an identity-element.

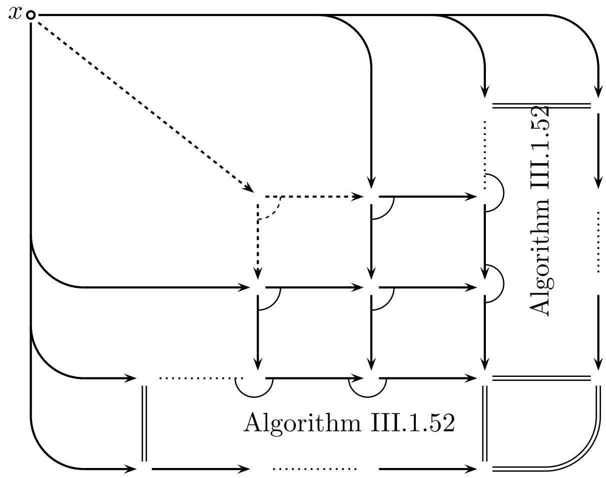


FIGURE 5. Proof of Proposition III.2.56: when the rectangular grid of Figure III.18 is constructed, there exists at each step an arrow connecting the top–left corner to each local North-West corner of the current diagram.

Conversely, assume $g_1 \cdots g_i =^{\times} f_1 \cdots f_i$ for every i, say $g_1 \cdots g_i = f_1 \cdots f_i \epsilon_i$ with ϵ_i in $\mathcal{C}_i$. Set $\epsilon_0 = 1_x$ where x is the source of f_1. Then we have $f_1 \epsilon_1 = g_1$ by construction. Assume $i \geqslant 2$. Then we obtain $f_1 \cdots f_{i-1} f_i \epsilon_i = g_1 \cdots g_{i-1} g_i = f_1 \cdots f_{i-1} \epsilon_{i-1} g_i$, whence $f_i \epsilon_i = \epsilon_{i-1} g_i$ by left-cancelling $f_1 \cdots f_{i-1}$. So $g_1 | \cdots | g_r$ is a $\mathcal{C}^{\times}$-deformation of $f_1 | \cdots | f_r$.

Exercice 33 (**left-disjoint**).— *Assume that $\mathcal{C}$ is a left-cancellative category, f and g are left-disjoint elements of $\mathcal{C}$, and f left-divides g. Show that f is invertible.*

Solution. Let x be the common source of f and g. By assumption we have $f \preccurlyeq 1_x f$ and $f \preccurlyeq 1_x g$. By definition of f and g being left-disjoint, this implies $f \preccurlyeq 1_x$, hence f must be invertible.

Exercice 34 (**normal decomposition**).— *Give a direct argument from deriving Proposition III.2.20 from Corollary III.2.50 in the case when $\mathcal{S}$ is strong.*

Solution. Let gf^{-1} be an element of $\mathcal{C}\mathcal{C}^{-1}$. Let $s_1 | \cdots | s_p$ be an $\mathcal{S}$-normal decomposition of f, and let $t_1 | \cdots | t_q$ be an $\mathcal{S}$-normal decomposition of g. We prove the existence of an $\mathcal{S}$-normal decomposition for gf^{-1} using an induction on p to construct a rectangular diagram consisting of p rows of q tiles as in Lemma III.2.31. As $t_1 | \cdots | t_q$ is $\mathcal{S}$-normal, Corollary III.2.50 inductively implies that the elements of every horizontal line of the diagram make an $\mathcal{S}$-normal path, and so do in particular the elements $t'_1 | \cdots | t'_q$ of the top line. Similarly, as $s_1 | \cdots | s_p$ is $\mathcal{S}$-normal, Corollary III.2.50 again inductively implies that the elements of every vertical line of the diagram make an $\mathcal{S}$-normal path, and so do in particular the elements $s'_1 | \cdots | s'_p$ of the left line. Finally, s'_1 and t'_1 are left-disjoint by construction. Hence $\overline{s'_p} | \cdots | \overline{s'_1} | t'_1 | \cdots | t'_q$ is an $\mathcal{S}$-normal decomposition of gf^{-1}.

Exercice 35 (**Garside base**).— (i) *Let $\mathcal{G}$ be the category whose diagram is displayed on Figure 6 left, and let $S = \{\mathtt{a}, \mathtt{b}\}$. Show that $\mathcal{G}$ is a groupoid with nine elements, $\mathcal{S}$ is a Garside base in $\mathcal{G}$, the subcategory $\mathcal{C}$ of $\mathcal{G}$ generated by $\mathcal{S}$ contains* *no nontrivial invertible element, but $\mathcal{C}$ is not an Ore category. Conclusion?* (ii) *Let $\mathcal{G}$ be the category whose diagram is displayed on Figure 6 right, let $\mathcal{S} = \{\epsilon, \mathtt{a}\}$, and let $\mathcal{C}$ be the subcategory of $\mathcal{G}$ generated by $\mathcal{S}$. Show that $\mathcal{G}$ is a groupoid and every element of $\mathcal{G}$ admits a decomposition that is symmetric $\mathcal{S}$-normal in $\mathcal{C}$. Show that $\epsilon\mathtt{a}$ admits a symmetric $\mathcal{S}$-normal decomposition and no $\mathcal{S}$-normal decomposition. Is $\mathcal{S}$ a Garside family in $\mathcal{C}$? Conclusion?*

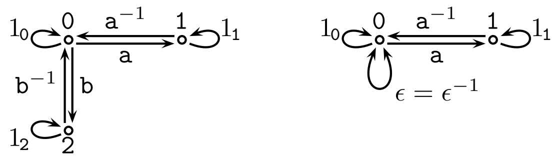


Figure 6. Diagrams of the categories of Exercise 35.

Solution. (i) The nine elements of $\mathcal{G}$ are 1_0, 1_1, 1_2, $\mathtt{a}$, $\mathtt{b}$, $\mathtt{a}^{-1}$, $\mathtt{b}^{-1}$, $\mathtt{a}^{-1}\mathtt{b}$, and $\mathtt{b}^{-1}\mathtt{a}$. That $\mathcal{S}$ is a Garside base in $\mathcal{G}$ follows from a direct inspection. The subcategory of $\mathcal{G}$ generated by $\mathcal{S}$ comprises five elements: 1_0, 1_1, 1_2, $\mathtt{a}$, and $\mathtt{b}$, none of which is invertible. Next, $\mathcal{C}$ is not an Ore category since the elements $\mathtt{a}$ and $\mathtt{b}$ have non common right-multiple although they share the same source. So, in Proposition III.2.25, the conclusion cannot be strengthened to claim that $\mathcal{C}$ necessarily is an Ore category (whereas the assumption of Proposition III.2.24 cannot be weakened to only assume that $\mathcal{C}$ is a left-Ore category).

(ii) The elements of $\mathcal{G}$ are 1_0, 1_1, ϵ, $\mathtt{a}$, $\epsilon\mathtt{a}$, $\mathtt{a}^{-1}$, and $\mathtt{a}^{-1}\epsilon$; those of $\mathcal{C}$ are 1_0, 1_1, ϵ, $\mathtt{a}$, $\epsilon\mathtt{a}$. Then $\epsilon\mathtt{a}$ admits the symmetric $\mathcal{S}$-normal decomposition $\overline{\epsilon}|\mathtt{a}$, but admits no $\mathcal{S}$-normal decomposition. Hence $\mathcal{S}$ is not a Garside family in $\mathcal{C}$. So, in Proposition III.2.25, we cannot simply drop the assumption about nontrivial invertible elements.

Chapter IV: Recognizing Garside families

Skipped proofs

Lemma IV.1.13.— *Assume that $\mathcal{S}$ is a subfamily of a left-cancellative category $\mathcal{C}$.*

(ii) *The family $\mathcal{S}$ is closed under right-comultiple if and only if $\mathcal{S}^\sharp$ is.*

(iii) *If $\mathcal{S}$ is closed under right-complement and $\mathcal{C}^\times\mathcal{S} \subseteq \mathcal{S}$ holds, then $\mathcal{S}^\sharp$ is closed under right-complement too.*

Proof. (ii) Assume that $\mathcal{S}^\sharp$ is closed under right-comultiple, and we have $sg = tf$ with s, t in $\mathcal{S}$. The assumption implies the existence of s', t', h satisfying $st' = ts' \in \mathcal{S}^\sharp$, $f = s'h$, and $g = t'h$. If st' is invertible, then s and t must be invertible too, so s is a common right-multiple of s and t lying in $\mathcal{S}$ of which sg is a right-multiple. Assume now that st' is not invertible. Then there exists ϵ in $\mathcal{C}^\times$ such that $st'\epsilon^{-1}$ lies in $\mathcal{S}$. Then we have $s(t'\epsilon^{-1}) = t(s'\epsilon^{-1}) \in \mathcal{S}$, $f = (s'\epsilon^{-1})(\epsilon h)$, and $g = (t'\epsilon^{-1})(\epsilon h)$. Hence $s'\epsilon^{-1}$, $t'\epsilon^{-1}$, and ϵh witness for $\mathcal{S}$ being closed under right-comultiple.

Conversely, assume that $\mathcal{S}$ is closed under right-comultiple and we have $sg = tf$ with $s, t \in \mathcal{S}^\sharp$. Assume first that s or t is invertible, say s. Put $s' = 1_y$ where y is the target of t, $t' = s^{-1}t$, and $h = f$. Then we have $st' = ts'$, $f = s'h$, $g = t'h$, and s' and t' are invertible, hence lie in $\mathcal{S}^\sharp$, and, by assumption, so does st', which is t. So st' is a common right-multiple of s and t that lies in $\mathcal{S}^\sharp$ and of which sg is a right-multiple. Assume now that neither s nor t is invertible. Then s and t lie in $\mathcal{S}\mathcal{C}^\times$, and there exist s', t' in $\mathcal{S}$ and ϵ, ϵ' in $\mathcal{C}^\times$ satisfying $s = s'\epsilon$ and $t = t'\epsilon'$, see Figure 7. Then we have $s'(\epsilon g) = t'(\epsilon' f)$ with s', t' in $\mathcal{S}$. As $\mathcal{S}$ is closed under right-comultiple, there must exist s'', t'', and h satisfying

$$s't'' = t's'' \in \mathcal{S}, \quad \epsilon' f = s''h, \quad \text{and} \quad \epsilon g = t''h.$$

As ϵ and ϵ' are invertible, we can put $s''_1 = \epsilon'^{-1}s''$ and $t''_1 = \epsilon^{-1}t''$. Then we have $f = s''_1 h$ and $g = t''_1 h$, and $st''_1 = s't'' = t's'' = ts''_1 \in \mathcal{S} \subseteq \mathcal{S}^\sharp$. So, again, st''_1 is a common right-multiple of s and t that lies in $\mathcal{S}^\sharp$ and of which sg is a right-multiple. Hence $\mathcal{S}^\sharp$ is closed under right-comultiple.

(iii) Assume now that $\mathcal{S}$ is closed under right-complement, $\mathcal{C}^\times\mathcal{S} \subseteq \mathcal{S}$ holds, and we have $s, t \in \mathcal{S}^\sharp$ and $sg = tf$. We follow the same scheme as for (ii), and keep the same notation. Assume first that s or t is invertible, say s. Put $s'' = 1_y$ (y the target of t, $t' = s^{-1}t$, and $h = f$. Then we have $st' = ts''$, $f = s''h$, and $g = t'h$. Moreover, s'' belongs to $\mathcal{S}^\sharp$ by definition and t', which belongs to $\mathcal{C}^\times\mathcal{S}^\sharp$, hence to $\mathcal{C}^\times \cup \mathcal{C}^\times\mathcal{S}\mathcal{C}^\times$, belongs to $\mathcal{S}^\sharp$ as $\mathcal{C}^\times\mathcal{S}$ is included in $\mathcal{S}$. Assume now that neither s nor t is invertible. Then we write $s = s'\epsilon$ and $t = t'\epsilon'$ with s', t' in $\mathcal{S}$ and ϵ, ϵ' in $\mathcal{C}^\times$, see Figure 7 again. We have $s'(\epsilon g) = t'(\epsilon' f)$ so, as $\mathcal{S}$ is closed under right-complement, there exist s'', t'' in $\mathcal{S}$ and h in $\mathcal{C}$ satisfying $s't'' = t's''$, $\epsilon g = t''h$, and $\epsilon' f = s''h$. Put $s''_1 = \epsilon'^{-1}s''$ and $t''_1 = \epsilon^{-1}t''$. Then we have $st''_1 = ts''_1$, $f = s''_1 h$, and $g = t''_1 h$. Moreover, by construction, s''_1 and t''_1 belong to $\mathcal{C}^\times\mathcal{S}$, hence, by assumption, to $\mathcal{S}^\sharp$. Hence $\mathcal{S}^\sharp$ is closed under right-complement. □

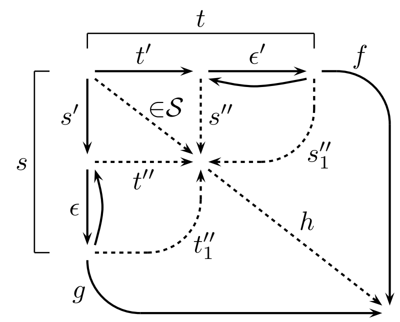


FIGURE 7. Proof of Lemma IV.1.13.

Lemma IV.2.39.— *For every subfamily $\mathcal{S}$ of a left-cancellative category $\mathcal{C}$ that is right-Noetherian and admits unique conditional right-lcms, the following conditions are equivalent:*

(i) *The family $\mathcal{S}^\sharp$ is closed under right-complement (in the sense of Definition IV.1.3);*

(ii) *The family $\mathcal{S}^\sharp$ is closed under $\backslash$, that is, if s and t belong to $\mathcal{S}^\sharp$, then so does $s\backslash t$ when defined, that is, when s and t admit a common right-multiple.*

Proof. Assume that $\mathcal{S}^\sharp$ is closed under right-complement. Let s, t be elements of $\mathcal{S}^\sharp$ that admit a common right-multiple, and let h be the right-lcm of s and t. By assumption, we have $h = s(s\backslash t) = t(t\backslash s)$. As $\mathcal{S}^\sharp$ is closed under right-complement, there exist s', t' in $\mathcal{S}^\sharp$ and h' satisfying $st' = ts'$, $t\backslash s = s'h'$, and $s\backslash t = t'h'$, whence $h = (st')h'$. By definition of the right-lcm, h' must be an identity-element and, therefore, $s\backslash t$ and $t\backslash s$ belong to $\mathcal{S}^\sharp$. So $\mathcal{S}^\sharp$ is closed under the right-complement operation, and (i) implies (ii).

Conversely, assume that $\mathcal{S}^\sharp$ is closed under the right-complement operation. Assume that s, t belong to $\mathcal{S}^\sharp$ and $sg = tf$ holds. Then sg is a common right-multiple of s and t, hence it is a right-multiple of their right-lcm, which is $s(s\backslash t)$ and $t(t\backslash s)$. So there exists h satisfying $sg = s(s\backslash t)h$. Left-cancelling s, we deduce $g = (s\backslash t)h$ and, symmetrically, $f = (t\backslash s)h$. Then $t\backslash s$, $s\backslash t$, and h witness that the expected instance of closure under right-complement is satisfied. So (ii) implies (i). □

Solution to selected exercises

Exercice 38 (**multiplication by invertible**).— *Assume that $\mathcal{C}$ is a cancellative category, and $\mathcal{S}$ is a subfamily of $\mathcal{C}$ that is closed under left-divisor and contains at least one element with source x for each object x. Prove $\mathcal{S}^\sharp = \mathcal{S}$.*

Solution. Assume $g \in \mathcal{S}$ and $\epsilon \in \mathcal{C}^\times$. Then we have $g = g\epsilon\epsilon^{-1}$, whence $g\epsilon \preccurlyeq g$, and $g\epsilon \in \mathcal{S}$. So $\mathcal{S}^\sharp$ is included in $\mathcal{S}$.

Exercice 40 (**head** *vs.* **lcm**).— *Assume that $\mathcal{C}$ is a left-cancellative category, $\mathcal{S}$ is included in $\mathcal{C}$, and g belongs to $\mathcal{C} \setminus \mathcal{C}^\times$. Show that s is an $\mathcal{S}$-head of g if and only if it is a right-lcm of* $\mathrm{Div}(g) \cap \mathcal{S}$.

Solution. If s is an $\mathcal{S}$-head of g, then there exists g' so that $s|g'$ is an $\mathcal{S}$-greedy decomposition of g. By Lemma IV.1.21, we deduce that, for t in $\mathcal{S}$, the relation $t \preccurlyeq g$ implies $t \preccurlyeq s$: this means that every element of $\mathrm{Div}(g) \cap \mathcal{S}$ divides s and, therefore, that s is a left-lcm of $\mathrm{Div}(g) \cap \mathcal{S}$.

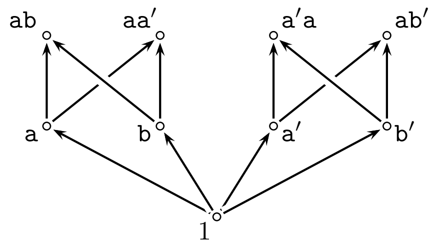
Exercice 41 (**closed under right-comultiple**).— *Assume that $\mathcal{C}$ is a left-cancellative category, $\mathcal{S}$ is a subfamily of $\mathcal{C}$, and there exists $H : \mathcal{C} \setminus \mathcal{C}^\times \to \mathcal{S}$ satisfying (IV.1.46). Show that $\mathcal{S}$ is closed under right-comultiple.*

Solution. Assume $f, g \in \mathcal{S}$ and $f\hat{g} = g\hat{f}$. If $f\hat{g}$ is invertible, then everything is obvious. Otherwise, $H(f\hat{g})$ is defined, and, by (IV.1.46)(i), we have $H(f\hat{g}) \preccurlyeq f\hat{g}$. On the other hand, we have $f \in \mathcal{S}$ and $f \preccurlyeq f\hat{g}$, whence $f \preccurlyeq H(f\hat{g})$ by (IV.1.46)(iii). A symmetric argument gives $g \preccurlyeq H(f\hat{g})$ as we have $f\hat{g} = g\hat{f}$. So $H(f\hat{g})$, which belongs to $\mathcal{S}$ by construction, is a common right-multiple of f and g of the expected type.

Exercice 42 (**power**).— *Assume that $\mathcal{S}$ is a Garside family a left-cancellative category $\mathcal{C}$. Show that, if $g_1|\cdots|g_p$ is an $\mathcal{S}^m$-normal decomposition of g and, for every i, the path $s_{i,1}|\cdots|s_{i,m}$ is an $\mathcal{S}$-normal decomposition of g_i, then the path $s_{1,1}|\cdots|s_{1,m}|s_{2,1}|\cdots|s_{2,m}|\cdots|s_{p,1}|\cdots|s_{p,m}$ is an $\mathcal{S}$-normal decomposition of g.*

Solution. By assumption, every $s_{i,j}$ lies in $\mathcal{S}^\sharp$ and $s_{i,j}|s_{i,j+1}$ is $\mathcal{S}$-greedy for all i, j. So the point is to show that $s_{i,m}|s_{i+1,1}$ is $\mathcal{S}$-greedy. Now assume $t \preccurlyeq s_{i,m}s_{i+1,1}$ with t in $\mathcal{S}$. Then we deduce $s_{i,1}\cdots s_{i,m-1}t \preccurlyeq s_{i,1}\cdots s_{i,m-1}s_{i,m}s_{i+1,1}$, whence $s_{i,1}\cdots s_{i,m-1}t \preccurlyeq g_ig_{i+1}$. As $g_i|g_{i+1}$ is $\mathcal{S}^m$-greedy, we deduce $s_{i,1}\cdots s_{i,m-1}t \preccurlyeq g_i$, that is, $s_{i,1}\cdots s_{i,m-1}t \preccurlyeq s_{i,1}\cdots s_{i,m-1}s_{i,m}$, which implies $t \preccurlyeq s_{i,m}$ since $\mathcal{C}$ is left-cancellative. As $\mathcal{S}$ is a Garside family, Corollary IV.1.31 implies that this is enough to conclude that $s_{i,m}|s_{i+1,1}$ is $\mathcal{S}$-greedy.

Exercice 45 (**no conditional right-lcm**).— *Let M be the monoid generated by* $\mathtt{a}, \mathtt{b}, \mathtt{a}', \mathtt{b}'$ *subject to the relations* $\mathtt{ab}{=}\mathtt{ba}, \mathtt{a}'\mathtt{b}'{=}\mathtt{b}'\mathtt{a}', \mathtt{aa}'{=}\mathtt{bb}', \mathtt{a}'\mathtt{a}{=}\mathtt{b}'\mathtt{b}$. (i) *Show that the cube condition is satisfied on* $\{\mathtt{a}, \mathtt{b}, \mathtt{a}', \mathtt{b}'\}$*, and that right- and left-reversing are complete for the above presentation.* (ii) *Show that M is cancellative and admits right-mcms.* (iii) *Show that* $\mathtt{a}$ *and* $\mathtt{b}$ *admit two right-mcms in M, but they admit no right-lcm.* (iv) *Let* $S = \{\mathtt{a}, \mathtt{b}, \mathtt{a}', \mathtt{b}', \mathtt{ab}, \mathtt{a}'\mathtt{b}', \mathtt{aa}', \mathtt{a}'\mathtt{a}\}$*, see diagram on the side. Show that S is closed under right-mcm, and deduce that $\mathcal{S}$ is a Garside family in M.* (v) *Show that the (unique) strict S-normal decomposition of the element* $\mathtt{a}^2\mathtt{b}'\mathtt{a}'^2$ *is* $\mathtt{ab}|\mathtt{a}'\mathtt{b}'|\mathtt{b}'$.



Solution. (i) There are several relations of the form $\mathtt{a}\cdots = \mathtt{b}\cdots$, but, as there is no relation $\mathtt{a}\cdots = \mathtt{a}'\cdots$, there is no mixed cube to consider: the only triples to check are $(\mathtt{a}, \mathtt{a}, \mathtt{b})$ and the like, and this is easy.

(ii) The presentation is homogeneous, hence (strongly) Noetherian. By Lemma II.4.62, right-reversing is complete. Hence M is right-cancellative and admits right-mcms. By symmetry of the relations, left-reversing is complete as well, and M is right-cancellative.

(iii) By assumption, $\mathtt{ab}$ and $\mathtt{aa}'$ are common right-multiples of $\mathtt{a}$ and $\mathtt{b}$. Owing to their length, they must be minimal. As right-reversing is complete, every common right-multiple of $\mathtt{a}$ and $\mathtt{b}$ is a right-multiple or $\mathtt{ab}$ and $\mathtt{aa}'$. Hence the latter are the only mcms of $\mathtt{a}$ and $\mathtt{b}$. Finally neither is a right-multiple of the other, so $\mathtt{a}$ and $\mathtt{b}$ admit no right-lcm.

(iv) The set S generates M, and it is closed under right-divisor and right-mcm: $\mathtt{ab}$ and $\mathtt{aa}'$ are common right-multiples of $\mathtt{a}$ and $\mathtt{b}$, $\mathtt{a}'\mathtt{b}'$ and $\mathtt{a}'\mathtt{a}$ are the two right-mcms of $\mathtt{a}'$ and $\mathtt{b}'$, whereas none of the pairs $\{\mathtt{a}, \mathtt{a}'\}$, $\{\mathtt{a}, \mathtt{b}'\}$, $\{\mathtt{b}, \mathtt{b}'\}$, and $\{\mathtt{b}, \mathtt{a}'\}$ admits a common right-multiple. Hence, by Corollary IV.2.26, S is a Garside family in M.

(v) Push the letters to the left as much as possible.

Exercice 47 (**solid**).— *Assume that $\mathcal{C}$ is a left-cancellative category and $\mathcal{S}$ is a generating subfamily of $\mathcal{C}$.* (i) *Show that $\mathcal{S}$ is solid in $\mathcal{C}$ if and only if $\mathcal{S}$ includes $\mathbf{1}_\mathcal{C}$ and it is closed under right-quotient.* (ii) *Assume moreover that $\mathcal{S}$ is closed under right-divisor. Show that $\mathcal{S}$ includes $\mathcal{C}^\times \setminus \mathbf{1}_\mathcal{C}$, that $\epsilon \in \mathcal{S} \cap \mathcal{C}^\times$ implies $\epsilon^{-1} \in \mathcal{S}$, and that and $\mathcal{C}^\times\mathcal{S} = \mathcal{S}$ holds, but that that $\mathcal{S}$ need not be solid.*

Solution. (ii) Assume that ϵ is a nontrivial invertible element, with target y. As $\mathcal{S}$ generates $\mathcal{C}$, there exists at least one element ϵ' of $\mathcal{S}$ that right-divides ϵ. Then 1_y right-divides ϵ', which lies in $\mathcal{S}$, hence 1_y must lie in $\mathcal{S}$. Next, ϵ right-divides 1_y, which lies in $\mathcal{S}$, hence ϵ must lie in $\mathcal{S}$. Now assume $\epsilon \in \mathcal{S} \cap \mathcal{C}^\times$. Then either ϵ is an identity-element, in which case it coincides with ϵ^{-1} and the latter lies in $\mathcal{S}$, or ϵ^{-1} is invertible and is not an identity-element, in which case it belongs to $\mathcal{S}$ by the above argument. For $\mathcal{C}^\times\mathcal{S} = \mathcal{S}$, the proof is as for Lemma IV.2.2. If y is an object that is the target of no non-identity-element, then 1_y: need not belong to $\mathcal{S}$. For instance, in the monoid $\{1\}$, the empty set satisfies the condition.

Exercice 48 (**solid**).— *Let M be the monoid $\langle \mathtt{a}, \mathtt{e} \mid \mathtt{ea} = \mathtt{a}, \mathtt{e}^2 = 1\rangle^+$.* (i) *Show that every element of M has a unique expression of the form $\mathtt{a}^p\mathtt{e}^q$ with $p \geqslant 0$ and $q \in \{0, 1\}$, and that M is left-cancellative.* (ii) *Let $S = \{1, \mathtt{a}, \mathtt{e}\}$. Show that S is a solid Garside family in M, but that $S = S^\sharp$ does not hold.*

Solution. (i) Every word in $\{\mathtt{a}, \mathtt{e}\}^*$ is equivalent to a word of the form $\mathtt{a}^p$ or $\mathtt{a}^p\mathtt{e}$. Conversely, it is impossible that two distinct words of this form are equivalent, as the number of $\mathtt{a}$ is an invariant, and so is the parity of the number of $\mathtt{e}$ that follows the last $\mathtt{a}$. The formulas $\mathtt{a} \cdot \mathtt{a}^p\mathtt{e}^q = \mathtt{a}^{p+1}\mathtt{e}^q$ and $\mathtt{e} \cdot \mathtt{a}^p\mathtt{e}^q = \mathtt{a}^p\mathtt{e}^q$ for $p \geqslant 1$, plus $\mathtt{e} \cdot \mathtt{e}^q = \mathtt{e}^{q+1 \mod 2}$ show that, for every s in $\{\mathtt{a}, \mathtt{e}\}$, the value of g can be recovered from s and sg. (ii) S generates M. The explicit formula for the multiplication shows that $\mathtt{a}$ is right-divisible only by 1 and itself, and so does $\mathtt{e}$. Hence S is closed under right-divisor, and it is solid. For $p \geqslant 1$, define $H(\mathtt{a}^p\mathtt{e}^q) = a$. Then H is a S-head function. Hence, by Proposition IV.2.7(i), S is a Garside family in M. On the other hand, $\mathtt{ae}$ is an element of $S^\sharp \setminus S$.

Exercice 49 (**not solid**).— *Let $M = \langle \mathtt{a}, \mathtt{e} \mid \mathtt{ea} = \mathtt{ae}, \mathtt{e}^2 = 1\rangle^+$, and $S = \{\mathtt{a}, \mathtt{e}\}$.* (i) *Show that M is left-cancellative. [Hint: M is $\mathbb{N} \times \mathbb{Z}/2\mathbb{Z}$]* (ii) *Show that S is a Garside family of M, but S is not solid in M. [Hint: $\mathtt{ea}$ right-divides $\mathtt{a}$, but does not belong to S.]*

Solution. (i) Every word of $\{\mathtt{a}, \mathtt{e}\}^*$ is equivalent modulo the relations to a word of the form $\mathtt{a}^p\mathtt{e}^q$ with $q \leqslant 1$. As $(1, \dot{0})$ and $(0, \dot{1})$ satisfy in $\mathbb{N}{\times}\mathbb{Z}/2\mathbb{Z}$ the defining relations of M, there exists a well defined homomorphism F of M to $\mathbb{N}{\times}\mathbb{Z}/2\mathbb{Z}$ that satisfies $F(\mathtt{a}) = (1, \dot{0})$ and $F(\mathtt{e}) = (0, \dot{1})$. As $(1, \dot{0})$ and $(0, \dot{1})$ generate $\mathbb{N}{\times}\mathbb{Z}/2\mathbb{Z}$, the homomorphism F is surjective. As the images under F of pairwise distinct words $\mathtt{a}^p\mathtt{e}^q$ are distinct, F is injective, hence it is a isomorphism. Hence $\mathbb{N}{\times}\mathbb{Z}/2\mathbb{Z}$ is left-cancellative, so is M. (ii) By definition, S generates M. Next, the invertible elements of M are 1 and $\mathtt{e}$. The equality $\mathtt{ea} = \mathtt{ae}$ implies $M^\times S \subseteq S^\sharp$. Finally, the three elements of S^2, namely $\mathtt{a}^2, \mathtt{ae}$, and 1, admit S-normal decompositions, namely for instance $\mathtt{a}|\mathtt{a}$, $\mathtt{a}|\mathtt{e}$, and ε. So S is a Garside family in M. Now $\mathtt{ae}$ does not lie in S, but we have $\mathtt{a} = \mathtt{e}(\mathtt{ae})$, so S is not closed under right-divisor, hence is not solid.

Exercice 50 (**recognizing Garside, right-lcm solid case**).— *Assume that $\mathcal{S}$ is a solid subfamily in a left-cancellative category $\mathcal{C}$ that is right-Noetherian and admits conditional right-lcms. Show that $\mathcal{S}$ is a Garside family in $\mathcal{C}$ if and only if $\mathcal{S}$ generates $\mathcal{C}$ and it is weakly closed under right-lcm.*

Solution. If $\mathcal{S}$ is a (solid or not solid) Garside family in $\mathcal{C}$, then, by Corollary IV.2.29, $\mathcal{S}$ is weakly closed under right-lcm. Conversely, assume that $\mathcal{S}$ is solid, generates $\mathcal{C}$, and is weakly closed under right-lcm. By definition, $\mathcal{S}$ is closed under right-divisor, hence, by Lemma IV.1.13(i), so is $\mathcal{S}^\sharp$. Then Corollary IV.2.29(iii) implies that $\mathcal{S}$ is a Garside family in $\mathcal{C}$.

Exercice 52 (**local right-divisibility**).— *Assume that $\mathcal{C}$ is a left-cancellative category and $\mathcal{S}$ is a generating subfamily of $\mathcal{C}$ that is closed under right-divisor.* (i) *Show that the transitive closure of $\succcurlyeq_{\mathcal{S}}$ is the restriction of $\succcurlyeq$ to $\mathcal{S}$.* (ii) *Show that* *the transitive closure of* $\widetilde{\prec}_{\mathcal{S}}$ *is almost the restriction of* $\widetilde{\prec}$ *to* $\mathcal{S}$*, in the following sense: if* $s \mathrel{\widetilde{\prec}} t$ *holds, there exists* $s' \mathrel{{}^{\times}\!\!=} s$ *satisfying* $s' \mathrel{\widetilde{\prec}^*_{\mathcal{S}}} t$*.*

Solution. Let $\widetilde{\preccurlyeq}^*_{\mathcal{S}}$ be the transitive closure of $\widetilde{\preccurlyeq}_{\mathcal{S}}$. As $s \mathrel{\widetilde{\preccurlyeq}_{\mathcal{S}}} t$ implies $s \mathrel{\widetilde{\preccurlyeq}} t$ trivially and $\widetilde{\preccurlyeq}$ is transitive, $s \mathrel{\widetilde{\preccurlyeq}^*_{\mathcal{S}}} t$ implies $s \mathrel{\widetilde{\preccurlyeq}} t$. Conversely, assume $s \mathrel{\widetilde{\preccurlyeq}} t$, say $t = gs$. As $\mathcal{S}$ generates $\mathcal{C}$, there exist $s_1, ..., s_q$ in $\mathcal{S}$ satisfying $g = s_1 \cdots s_p$. So we have $t = s_1 \cdots s_p s$. Put $t_i = s_i \cdots s_p s$ for $1 \leqslant i \leqslant p$. Each element t_i right-divides t, an element of $\mathcal{S}$, so it belongs to $\mathcal{S}$. Now, by construction, we have $s = \widetilde{\preccurlyeq}_{\mathcal{S}} t_1 \mathrel{\widetilde{\preccurlyeq}_{\mathcal{S}}} \cdots \mathrel{\widetilde{\preccurlyeq}_{\mathcal{S}}} t_p = t$, whence $s \mathrel{\widetilde{\preccurlyeq}^*_{\mathcal{S}}} t$.

(ii) Let $\widetilde{\prec}^*_{\mathcal{S}}$ be the transitive closure of $\widetilde{\prec}_{\mathcal{S}}$. By Lemma IV.2.15, $s \mathrel{\widetilde{\prec}_{\mathcal{S}}} t$ implies $s \mathrel{\widetilde{\prec}} t$, hence $s \mathrel{\widetilde{\prec}^*_{\mathcal{S}}} t$ implies $s \mathrel{\widetilde{\prec}} t$ since $\widetilde{\prec}$ is transitive. Conversely, assume $s \mathrel{\widetilde{\prec}} t$, say $t = gs$ with $g \notin \mathcal{C}^{\times}$. As $\mathcal{S}$ generates $\mathcal{C}$, there exist $s_1, ..., s_q$ in $\mathcal{S}$ satisfying $g = s_1 \cdots s_p$. Assume that p has been chosen minimal. Then $s_1, ..., s_{p-1}$ are not invertible: if s_i is invertible, then $s_i s_{i+1}$ right-divides s_{i+1}, hence belongs to $\mathcal{S}$, and therefore grouping s_i and s_{i+1} would provide a shorter decomposition. We have $t = s_1 \cdots s_p s$. Put $t_i = s_i \cdots s_p s$ for $1 \leqslant i \leqslant p$. Each element t_i right-divides t, an element of $\mathcal{S}$, so it belongs to $\mathcal{S}$. As $\mathcal{S}$ is closed under right-divisor, we have $\mathcal{C}^{\times}\mathcal{S} \subseteq \mathcal{S}$. So we can assume that $s_1, ..., s_{p-1}$ are non-invertible. Hence we have $s_1 s \mathrel{\widetilde{\prec}^*_{\mathcal{S}}} t$.

Exercice 53 (**local left-divisibility**).— *Assume that* $\mathcal{S}$ *is a subfamily of a left-cancellative category* $\mathcal{C}$*.* (i) *Show that* $s \preccurlyeq_{\mathcal{S}} t$ *implies* $s \preccurlyeq t$*, and that* $s \preccurlyeq_{\mathcal{S}} t$ *is equivalent to* $s \preccurlyeq t$ *whenever* $\mathcal{S}$ *is closed under right-quotient in* $\mathcal{C}$*.* (ii) *Show that, if* $\mathcal{S}^{\times} = \mathcal{C}^{\times} \cap \mathcal{S}$ *holds, then* $s \prec_{\mathcal{S}} t$ *implies* $s \prec t$*.* (iii) *Show that, if* $\mathcal{S}$ *is closed under right-divisor, then* $\preccurlyeq_{\mathcal{S}}$ *is the restriction of* $\preccurlyeq$ *to* $\mathcal{S}$ *and, if* $\mathcal{S}^{\times} = \mathcal{C}^{\times} \cap \mathcal{S}$ *holds,* $\prec_{\mathcal{S}}$ *is the restriction of* $\prec$ *to* $\mathcal{S}$*.*

Solution. (iii) First $s \preccurlyeq_{\mathcal{S}} t$ implies $s \preccurlyeq t$. Conversely, assume $s, t \in \mathcal{S}$ and $sg' = t$. As t belongs to $\mathcal{S}$, the assumption that $\mathcal{S}$ is closed under right-divisor implies that g' belongs to $\mathcal{S}$, hence witnesses for $s \preccurlyeq_{\mathcal{S}} t$. Next, by Lemma IV.2.15, $s \prec_{\mathcal{S}} t$ implies $s \prec t$. Conversely, assume $st' = t$ with $t' \notin \mathcal{C}^{\times}$. As above, t' must belong to $\mathcal{S}$, and it cannot belong to $\mathcal{S}^{\times}$. So $s \prec_{\mathcal{S}} t$ holds.

Exercice 55 (**locally right-Noetherian**).— *Assume that* $\mathcal{C}$ *is a left-cancellative category and* $\mathcal{S}$ *is a subfamily of* $\mathcal{C}$*.* (i) *Prove that* $\mathcal{S}$ *is locally right-Noetherian if and only if, for every* s *in* $\mathcal{S}^{\sharp}$*, every* $\prec_{\mathcal{S}}$*-increasing sequence in* $\mathrm{Div}_{\mathcal{S}}(s)$ *is finite.* (ii) *Assume that* $\mathcal{S}$ *is locally right-Noetherian and closed under right-divisor. Show that* $\mathcal{S}^{\sharp}$ *is locally right-Noetherian. [Hint: For* $\mathcal{X} \subseteq \mathcal{S}^{\sharp}$ *introduce* $\mathcal{X}' = \{s \in \mathcal{S} \mid \exists \epsilon, \epsilon' \in \mathcal{C}^{\times} \, (\epsilon s \epsilon' \in \mathcal{X})\}$*, and construct a* $\widetilde{\prec}$*-minimal element in* $\mathcal{X}$ *from a* $\widetilde{\prec}$*-minimal element in* $\mathcal{X}'$*.]*

Solution. (ii) Let $\mathcal{X}$ be a nonempty subfamily of $\mathcal{S}^{\sharp}$. Put

$$\mathcal{X}' = \{s \in \mathcal{S} \mid \exists \epsilon, \epsilon' \in \mathcal{C}^{\times} \, (\epsilon s \epsilon' \in \mathcal{X})\}.$$

We have $\mathcal{X} \subseteq \mathcal{X}'$, whence $\mathcal{X}' \neq \emptyset$. As $\mathcal{X}'$ is included in $\mathcal{S}$, it contains a $\widetilde{\prec}_{\mathcal{S}}$-minimal element, say s_0. By definition, there exists ϵ_0, ϵ'_0 invertible such that $\epsilon_0 s_0 \epsilon'_0$ lies in $\mathcal{X}$. Put $t_0 = \epsilon_0 s_0 \epsilon'_0$. We claim that t_0 is $\widetilde{\prec}_{\mathcal{S}^{\sharp}}$-minimal in $\mathcal{X}$. Indeed, assume $t_0 = rt$ with r in $\mathcal{S}^{\sharp}$ and t in $\mathcal{X}$. We have to prove that r lies in $(\mathcal{S}^{\sharp})^{\times}$. As $\mathcal{S}^{\sharp}$ is solid, it suffices to show that r lies in $\mathcal{C}^{\times}$. By definition, r belongs to $\mathcal{S}\mathcal{C}^{\times} \cup \mathcal{C}^{\times}$. If r belongs to $\mathcal{C}^{\times}$, we are done. Otherwise, write $r = r'\epsilon$ with r' in $\mathcal{S}$ and ϵ in $\mathcal{C}^{\times}$. Then we have $s_0 = (\epsilon_0^{-1} r')(\epsilon t {\epsilon'_0}^{-1})$. As s_0 lies in $\mathcal{S}$ and $\mathcal{S}$ is closed under right-divisor,

$\epsilon t\epsilon_0'^{-1}$ lies in $\mathcal{S}$. As t belongs to $\mathcal{X}$, we deduce that $\epsilon t\epsilon_0'^{-1}$ lies in $\mathcal{X}'$. Then, by the choice of s_0, the elements $\epsilon_0^{-1}r'$, and therefore r' and $r'\epsilon$, that is, r, must be invertible in $\mathcal{C}$. Hence t_0 is $\widetilde{\prec}_{\mathcal{S}^\sharp}$-minimal in $\mathcal{X}$, and $\widetilde{\prec}_{\mathcal{S}^\sharp}$ is a well-founded relation, that is, $\mathcal{S}^\sharp$ is locally right-Noetherian.

Chapter V: Bounded Garside families

Skipped proofs

Proposition V.1.59 (**right-cancellative II**).— *If $\mathcal{S}$ is a Garside family in a left-cancellative category $\mathcal{C}$ that is right-bounded by a target-injective map Δ, and ϕ_Δ preserves $\mathcal{S}$-normality and is surjective on $\mathcal{C}^\times$, then the following conditions are equivalent:*

(i) *The category $\mathcal{C}$ is right-cancellative;*
(ii) *The functor ϕ_Δ is injective on $\mathcal{C}$;*
(iii) *The functor ϕ_Δ is injective on $\mathcal{S}^\sharp$.*

Proof. By Proposition V.1.36, (i) and (ii) are equivalent, and (ii) obviously implies (iii).

Now assume that ϕ_Δ is injective on $\mathcal{S}^\sharp$. We will prove that it is injective on $\mathcal{C}$. First, we claim that $\phi_\Delta(s') =^\times \phi_\Delta(s)$ implies $s =^\times s'$ for s, s' in $\mathcal{S}^\sharp$. Indeed, assume $\phi_\Delta(s') = \phi_\Delta(s)\,\epsilon$ with $\epsilon \in \mathcal{C}^\times$. If ϵ is trivial, the injectivity of ϕ_Δ on $\mathcal{S}^\sharp$ implies $s' = s$. Otherwise, as ϕ_Δ is surjective on $\mathcal{C}^\times$, there exists ϵ' in $\mathcal{C}^\times$ satisfying $\phi_\Delta(\epsilon') = \epsilon$. As Δ is target-injective, the assumption that $\phi_\Delta(s)\phi_\Delta(\epsilon')$ is defined implies that $s\epsilon'$ is defined too: if y is the target of s and x' is the source of ϵ', we obtain $\phi_\Delta(y) = \phi_\Delta(x')$, whence $y = x'$ as ϕ_Δ is injective on objects. We deduce $\phi_\Delta(s') = \phi_\Delta(s\epsilon')$. As s belongs to $\mathcal{S}^\sharp$, so does $s\epsilon'$, and the assumption that ϕ_Δ is injective on $\mathcal{S}^\sharp$ then implies $s' = s\epsilon'$, hence $s' =^\times s$.

Now we prove using induction on $p \geqslant 1$ the statement: $\phi_\Delta(g) = \phi_\Delta(g')$ implies $g = g'$ for all g, g' satisfying $\max(\|g\|_\mathcal{S}, \|g'\|_\mathcal{S}) \leqslant p$. For $p = 1$, as $\|g\|_\mathcal{S} \leqslant 1$ implies $g \in \mathcal{S}^\sharp$, the result follows from the assumption that ϕ_Δ is injective on $\mathcal{S}^\sharp$. Assume $p \geqslant 2$, and $\phi_\Delta(g) = \phi_\Delta(g')$ with $\max(\|g\|_\mathcal{S}, \|g'\|_\mathcal{S}) \leqslant p$. Let $s_1|\cdots|s_p$ and $s_1'|\cdots|s_p'$ be $\mathcal{S}$-normal decompositions of g and g'. As ϕ_Δ is a functor and it preserves normality, $\phi_\Delta(s_1)|\cdots|\phi_\Delta(s_p)$ and $\phi_\Delta(s_1')|\cdots|\phi_\Delta(s_p')$ are $\mathcal{S}$-normal decompositions of $\phi_\Delta(g)$ and $\phi_\Delta(g')$. By Proposition III.1.25 (normal unique), $\phi_\Delta(g) = \phi_\Delta(g')$ implies $\phi_\Delta(s_1) =^\times \phi_\Delta(s_1')$, whence, by the claim above, $s_1' =^\times s_1$. Hence s_1 is an $\mathcal{S}$-head for g and g', and we can write $g = s_1 g_1$, $g' = s_1 g_1'$, with $\max(\|g_1\|_\mathcal{S}, \|g_1'\|_\mathcal{S}) \leqslant p-1$. Then $\phi_\Delta(g') = \phi_\Delta(g)$ implies $\phi_\Delta(s_1)\phi_\Delta(g_1) = \phi_\Delta(s_1)\phi_\Delta(g_1')$, whence $\phi_\Delta(g_1) = \phi_\Delta(g_1')$. The induction hypothesis implies $g_1 = g_1'$, whence $g' = g$. So ϕ_Δ is injective on $\mathcal{C}$, and (iii) implies (ii). □

Proposition V.2.34 (**right-cancellativity III**).— *If $\mathcal{C}$ is a left-cancellative category and Δ is a Garside map of $\mathcal{C}$ that preserves Δ-normality, then $\mathcal{C}$ is right-cancellative if and only if ϕ_Δ is injective on $(\mathcal{D}iv(\Delta))^2$.*

Proof. If $\mathcal{C}$ is right-cancellative, Proposition V.1.36 implies that ϕ_Δ is injective on $\mathcal{C}$, hence a fortiori on $(\mathcal{D}iv(\Delta))^2$, so the condition is necessary.

Conversely, assume that ϕ_Δ is injective on $(\mathcal{D}iv(\Delta))^2$. First, ϕ_Δ must be injective on $\mathbf{1}_\mathcal{C}$, which is included in $(\mathcal{D}iv(\Delta))^2$, and, therefore, on $\mathcal{O}bj(\mathcal{C})$, that is, Δ must be target-injective. Next, as $\mathbf{1}_\mathcal{C}$ is included in $\mathcal{D}iv(\Delta)$, the assumption implies that ϕ_Δ is injective on $\mathcal{D}iv(\Delta)$.

We claim that ϕ_Δ induces a permutation of $\mathcal{C}^\times$. Indeed, as ϕ_Δ is a functor, it maps $\mathcal{C}^\times$ into itself. Now, the assumption that Δ is a Garside map implies that $\mathcal{D}iv(\Delta)$ is bounded by Δ, hence, by Lemma V.2.7, ϕ_Δ induces a surjective map of $\mathcal{D}iv(\Delta)$ into itself, hence a permutation of $\mathcal{D}iv(\Delta)$ as it is injective on $\mathcal{D}iv(\Delta)$. Assume $\epsilon \in \mathcal{C}^\times$. Then ϵ and ϵ^{-1} belong to $\mathcal{D}iv(\Delta)$, so there exist s, t in $\mathcal{D}iv(\Delta)$ satisfying $\phi_\Delta(s) = \epsilon$ and $\phi_\Delta(t) = \epsilon^{-1}$. As Δ is target-injective, st is defined. Indeed, let y be the target of s and x' be the source of t. As ϕ_Δ is a functor, $\phi_\Delta(y)$ is the target of $\phi_\Delta(s)$, that is, of ϵ, whereas $\phi_\Delta(x')$ is the source of $\phi_\Delta(t)$, that is, of ϵ^{-1}. Hence we have $\phi_\Delta(y) = \phi_\Delta(x')$ and, therefore, $y = x'$, that is, st is defined. The argument for ts is symmetric. So st and ts belong to $(\mathcal{D}iv(\Delta))^2$. The assumption that ϕ_Δ is injective on $(\mathcal{D}iv(\Delta))^2$ implies $st = 1_x$ and $ts = 1_y$, where x (*resp.* y) is the source (*resp.* target) of s. So s belongs to $\mathcal{C}^\times$, and ϕ_Δ induces a permutation of $\mathcal{C}^\times$. Then, Proposition V.1.59 implies that $\mathcal{C}$ is right-cancellative.

Note that the assumptions of Proposition V.2.34 can be slightly weakened: the only assumptions used in the proof is that ϕ_Δ is injective on $\mathcal{D}iv(\Delta)$ and that, for g in $(\mathcal{D}iv(\Delta))^2$, the relation $\phi_\Delta(g) \in \mathbf{1}_\mathcal{C}$ implies $g \in \mathbf{1}_\mathcal{C}$. We do not know whether the latter condition can be skipped. □

Lemma V.2.38.— (i) *A left-cancellative category that is left-Noetherian and admits left-gcds admits conditional right-lcms.*

(ii) *In a cancellative category that admits conditional right-lcms, any two elements of $\mathcal{C}$ that admit a common left-multiple admit a right-gcd.*

Proof. (i) We first show that every nonempty family of elements of $\mathcal{C}$ sharing the same source admits a left-gcd. Let $\mathcal{S}$ be a nonempty family of elements of $\mathcal{C}$ that share the same source. An obvious induction shows that, in $\mathcal{C}$, every finite nonempty family of elements of $\mathcal{C}$ sharing the same source has a left-gcd. For Y a finite nonempty subset of $\mathcal{S}$, choose a left-gcd g_Y for Y, and let $\mathcal{S}'$ be the family of all elements g_Y. As $\mathcal{C}$ is left-Noetherian, there exists an element g_X of $\mathcal{S}'$ that is $\prec$-minimal in $\mathcal{S}'$. We claim that g_X is a left-gcd for $\mathcal{S}$. Indeed, let g be an arbitrary element of $\mathcal{S}$. The point is to prove $g_X \preccurlyeq g$. Now, by construction, we have $g_{X\cup\{g\}} \preccurlyeq g$ and $g_{X\cup\{g\}} \preccurlyeq g_X$. As g_X is $\prec$-minimal in $\mathcal{S}'$, we must have $g_{X\cup\{g\}} =^\times g_X$, whence $g_X \preccurlyeq g_{X\cup\{g\}} \preccurlyeq g$, as expected. So $\mathcal{S}$ has a left-gcd. Then Lemma II.2.21 guarantees that $\mathcal{C}$ admits conditional right-lcms.

(ii) (See Figure 8.) Let f, g be two elements of $\mathcal{C}$ that admit a common left-multiple, say $f'g = g'f$, hence share the same target. The elements f' and g' admit a common right-multiple, namely $f'g$, hence they admit a right-lcm, say $f'g'' = g'f''$. By definition of a right-lcm, there exists h satisfying $f = f''h$ and $g = g''h$. By construction, h is a common right-divisor of f and g.

Let $\hat{h}$ be a common right-divisor of f and g. So there exist $\hat{f}, \hat{g}$ satisfying $f = \hat{f}\hat{h}$ and $g = \hat{g}\hat{h}$. Then we have $f'\hat{g}\hat{h} = f'g = g'f = g'\hat{f}\hat{h}$, whence $f'\hat{g} = g'\hat{f}$ by right-cancelling $\hat{h}$. So $f'\hat{g}$ is a common right-multiple of f' and g', hence it is a right-multiple of their right-lcm, that is, there exists h' satisfying $f'\hat{g} = (f'g'')h'$, whence $\hat{g} = g''h'$ by left-cancelling f'. We deduce $g''h = g = \hat{g}\hat{h} = g''h'\hat{h}$, whence $h = h'\hat{h}$ by right-cancelling $\hat{h}$. So $\hat{h}$ right-divides h, which shows that h is a right-gcd of f and g. $\square$

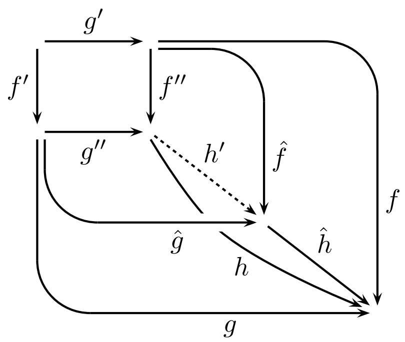


FIGURE 8. Construction of a right-gcd using a right-lcm.

Lemma V.3.40.— *If Δ is a Garside map in a cancellative category $\mathcal{C}$, then, for all s, t in $\mathcal{D}iv(\Delta)$, we have $s \preccurlyeq t \ \Leftrightarrow\ \partial s \mathrel{\widetilde{\succcurlyeq}} \partial t$ and $s \mathrel{\widetilde{\succcurlyeq}} t \ \Leftrightarrow\ \widetilde{\partial} s \preccurlyeq \widetilde{\partial} t$.*

Proof. As ∂ and $\widetilde{\partial}$ are bijective, everything is clear. For instance, assume $s \preccurlyeq t$. Then we have $st' = t$ for some t', which necessarily belongs to $\mathcal{D}iv(\Delta)$. Calling x the common source of s and t, we deduce

$$\Delta(x) = s\,\partial s = t\,\partial t = s\,t'\,\partial t,$$

whence $\partial s = t'\,\partial t$, and $\partial s \mathrel{\widetilde{\succcurlyeq}} \partial t$. The other implications are similar. $\square$

Lemma V.3.46.— *If Δ is a Garside map in a cancellative category $\mathcal{C}$ and f, g are elements of $\mathcal{C}(x, \text{-})$, then, for every m satisfying $m \geqslant \max(\sup_\Delta(f), \sup_\Delta(g))$ and for every h in $\mathrm{Div}(\Delta^{[m]}(x))$, the following conditions are equivalent:*
(i) *The element h is a right-lcm of f and g;*
(ii) *The element $\partial^{[m]}h$ is a right-gcd of $\partial^{[m]}f$ and $\partial^{[m]}g$.*

Proof. Assume first that $m = 1$. As $\mathcal{D}iv(\Delta)$ is closed under right-comultiple, h is a right-lcm of f and g if and only if the three relations

$$f \preccurlyeq h,\ g \preccurlyeq h, \text{ and } \forall h' \in \mathcal{D}iv(\Delta)\ ((f \preccurlyeq h' \text{ and } g \preccurlyeq h') \text{ implies } h \preccurlyeq h')$$

are satisfied. By the duality formulas of Lemma V.3.40, this is equivalent to

$$\partial f \mathrel{\widetilde{\succcurlyeq}} \partial h,\ \partial g \mathrel{\widetilde{\succcurlyeq}} \partial h, \text{ and } \forall h' \in \widetilde{\mathcal{D}iv}(\Delta)\ ((\partial f \mathrel{\widetilde{\succcurlyeq}} h' \text{ and } \partial g \mathrel{\widetilde{\succcurlyeq}} h') \text{ implies } \partial h \mathrel{\widetilde{\succcurlyeq}} h'),$$

hence, as $\widetilde{\mathcal{D}iv}(\Delta)$ coincides with $\mathcal{D}iv(\Delta)$, to ∂h being a right-gcd of ∂f and ∂g.

Assume now that m is arbitrary. By Proposition V.2.32(iv), $\Delta^{[m]}$ is a Garside map in $\mathcal{C}$, and $\partial^{[m]}$ is the associated duality map. Applying the above result to $\Delta^{[m]}$ and $\partial^{[m]}$ gives the expected equivalence.

(Observe that, in the situation of Lemma V.3.46, $\Delta^{[m]}(x)$ is a common right-multiple of f and g, so every possible right-lcm of f and g must lie in $\mathrm{Div}(\Delta^{[m]}(x))$.)
$\square$

Solution to selected exercises

Exercice 59 (**preserving** $\mathcal{D}iv(\Delta)$).— *Assume that $\mathcal{C}$ is a category, Δ is a map from $\mathcal{O}bj(\mathcal{C})$ to $\mathcal{C}$ and ϕ is a functor from $\mathcal{C}$ into itself that commutes with Δ. Show that ϕ maps $\mathcal{D}iv(\Delta)$ and $\widetilde{\mathcal{D}iv}(\Delta)$ to themselves.*

Solution. Assume $s \in \mathcal{D}iv(\Delta)$, say $s \preccurlyeq \Delta(x)$. By (the easy direction of) Lemma II.2.8, this implies $\phi(s) \preccurlyeq \phi(\Delta(x))$. By assumption, the latter is $\Delta(\phi(x))$, so $\phi(s)$ belongs to $\mathcal{D}iv(\Delta)$. The argument is similar for $\widetilde{\mathcal{D}iv}(\Delta)$.

Exercice 60 (**preserving normality I**).— *Assume that $\mathcal{C}$ is a cancellative category, $\mathcal{S}$ is a Garside family of $\mathcal{C}$, and ϕ is a functor from $\mathcal{C}$ to itself.* (i) *Show that, if ϕ induces a permutation of $\mathcal{S}^\sharp$, then ϕ preserves $\mathcal{S}$-normality.* (ii) *Show that ϕ preserves non-invertibility, that is, $\phi(g)$ is invertible if and only if g is.*

Solution. (i) Assume that $s_1|s_2$ is $\mathcal{S}$-normal. First, by assumption, ϕ maps $\mathcal{S}^\sharp$ to itself, hence $\phi(s_1)$ and $\phi(s_2)$ lie in $\mathcal{S}^\sharp$. Assume that s is an element of $\mathcal{S}$ that satisfies $s \preccurlyeq \phi(s_1)\,\phi(s_2)$. By Lemma II.2.8, we deduce $\phi^{-1}(s) \preccurlyeq s_1 s_2$, whence $\phi^{-1}(s) \preccurlyeq s_1$ as ϕ^{-1} maps $\mathcal{S}^\sharp$ into itself and $s_1|s_2$, which is $\mathcal{S}$-normal by assumption, is also $\mathcal{S}^\sharp$-normal by Lemma III.1.10. Reapplying ϕ, we deduce $s \preccurlyeq \phi(s_1)$, and we conclude that $\phi(s_1)|\phi(s_2)$ is $\mathcal{S}$-normal.

(ii) First, $\phi(g)$ is invertible whenever g is invertible since ϕ is a functor. Conversely, assume that $\phi(g)$ is invertible. Let $g_1|\cdots|g_p$ be an $\mathcal{S}^\sharp$-normal decomposition of g. As ϕ is a functor and preserves normality, $\phi(g_1)|\cdots|\phi(g_p)$ is an $\mathcal{S}^\sharp$-normal decomposition of $\phi(g)$. The assumption that $\phi(g)$ is invertible implies that each of $\phi(g_1), ..., \phi(g_p)$ is invertible, and then the assumption that ϕ is injective on $\mathcal{S}^\sharp$ implies that $g_1, \cdots, g_p$ are invertible. Hence g is invertible.

Exercice 61 (**preserving normality II**).— *Assume that $\mathcal{C}$ is a left-cancellative category and $\mathcal{S}$ is a Garside family of $\mathcal{C}$ that is right-bounded by a map Δ.* (i) *Show that ϕ_Δ preserves normality if and only if there exists an $\mathcal{S}$-head map H satisfying $H(\phi_\Delta(g)) =^\times \phi_\Delta(H(g))$ for every g in $(\mathcal{S}^\sharp)^2$, if and only if, for each $\mathcal{S}$-head map H, the above relation is satisfied.* (ii) *Show that a sufficient condition for ϕ_Δ to preserve normality is that ϕ_Δ preserves left-gcds on $\mathcal{S}^\sharp$, that is, if r, s, t belong to $\mathcal{S}^\sharp$ and r is a left-gcd of s and t, then $\phi_\Delta(r)$ is a left-gcd of $\phi_\Delta(s)$ and $\phi_\Delta(t)$.*

Solution. (i) Assume that ϕ_Δ preserves normality, H is a $\mathcal{S}$-head map, and $g_1|g_2$ lies in $(\mathcal{S}^\sharp)^{[2]}$. Put $g'_1 = H(g_1g_2)$, and let g'_2 satisfy $g'_1g'_2 = g_1g_2$. Then $g'_1|g'_2$ is $\mathcal{S}$-normal, hence, as ϕ_Δ preserves normality, so is $\phi_\Delta(g'_1)|\phi_\Delta(g'_2)$. Moreover, we have $\phi_\Delta(g'_1)\phi_\Delta(g'_2) = \phi_\Delta(g_1g_2)$, hence $\phi_\Delta(g'_1)|\phi_\Delta(g'_2)$ is an $\mathcal{S}$-normal decomposition of $\phi_\Delta(g_1g_2)$. By uniqueness of the head, we must have $H(\phi_\Delta(g_1g_2)) =^\times \phi_\Delta(g'_1)$. Conversely, assume that H is a $\mathcal{S}^\sharp$-head map and $H(\phi_\Delta(g)) =^\times \phi_\Delta(H(g)))$ holds for every g in $(\mathcal{S}^\sharp)^2$. Let $g_1|g_2$ be an $\mathcal{S}$-normal path. By construction, we have $g_1 =^\times H(g_1g_2)$. As ϕ_Δ is a functor, this implies $\phi_\Delta(g_1) \preccurlyeq \phi_\Delta(H(g_1g_2))$ and $\phi_\Delta(H(g_1g_2)) \preccurlyeq \phi_\Delta(g_1)$, hence $\phi_\Delta(g_1) =^\times \phi_\Delta(H(g_1g_2))$. As we have $\phi_\Delta(H(g_1g_2)) =^\times H(\phi_\Delta(g_1g_2))$ by assumption, we deduce $\phi_\Delta(g_1) =^\times H(\phi_\Delta(g_1g_2))$. Hence $\phi_\Delta(g_1)$ is an $\mathcal{S}$-head for $\phi_\Delta(g_1g_2)$. As we have $\phi_\Delta(g_1g_2) = \phi_\Delta(g_1)\,\phi_\Delta(g_2)$, we deduce that $\phi_\Delta(g_1)|\phi_\Delta(g_2)$ is an $\mathcal{S}$-normal decomposition of $\phi_\Delta(g_1g_2)$. Hence ϕ_Δ preserves normality.

(ii) Assume that ϕ_Δ preserves left-gcds on $\mathcal{S}^\sharp$. Let $g_1|g_2$ be an $\mathcal{S}$-normal path. Let x be the source of g_2. By Proposition V.1.53, $\partial_\Delta(g_1)$ and g_2 are left-coprime, that is, 1_x is a left-gcd of $\partial_\Delta(g_1)$ and g_2. If the condition of the statement is satisfied, it follows that $\phi_\Delta(1_x)$, which is $1_{\phi_\Delta(x)}$, is a left-gcd of $\phi_\Delta(\partial_\Delta(g_1))$ and $\phi_\Delta(g_2)$. By (V.1.30), we have $\phi_\Delta(\partial_\Delta(g_1)) = \partial_\Delta(\phi_\Delta(g_1))$. So $\partial_\Delta(\phi_\Delta(g_1))$ and $\phi_\Delta(g_2)$ are left-coprime, hence, by Proposition V.1.53, $\phi_\Delta(g_1)|\phi_\Delta(g_2)$ is $\mathcal{S}$-normal.

Exercice 62 (**normal decomposition**).— *Assume that $\mathcal{C}$ is a left-cancellative category, Δ is a right-Garside map in $\mathcal{C}$ such that ϕ_Δ preserves normality, and f, g are elements of $\mathcal{C}$ such that fg is defined and $f \preccurlyeq \Delta^{[m]}(\text{-})$ holds, say $ff' = \Delta^{[m]}(\text{-})$ with $m \geqslant 1$. Show that f' and g admit a left-gcd and that, if h is such a left-gcd, then concatenating a $\mathcal{D}iv(\Delta)$-normal decomposition of fh and a $\mathcal{D}iv(\Delta)$-normal decomposition of $h^{-1}g$ yields a $\mathcal{D}iv(\Delta)$-normal decomposition of fg. [Hint: First show that concatenating fh and a $\mathcal{D}iv(\Delta^{[m]})$-normal decomposition of $h^{-1}g$ yields a $\mathcal{D}iv(\Delta^{[m]})$-normal decomposition of fg and apply Exercise 42 in Chapter IV.]*

Solution. By Proposition V.1.58, $\Delta^{[m]}$ is a right-Garside map in $\mathcal{C}$ and, by definition, f' right-divides an element $\Delta^{[m]}(\text{-})$ so, by Lemma V.2.36, f' and g admit a left-gcd. Assume that h is a left-gcd of f' and g. Since $\Delta^{[m]}$ is a right-Garside map, gh and $\Delta^{[m]}(\text{-})$, that is, ff', admit a left-gcd, which (by Exercise 8(i) in Chapter II) is of the form fh' with h' a left-gcd of f' and g. By uniqueness, we have $h =^{\times} h'$, whence $fh =^{\times} fh'$. So fh is a left-gcd of fg and $\Delta^{[m]}(\text{-})$, hence it is a $\mathcal{D}iv(\Delta^{[m]})$-head of fg. Hence concatenating fh and a $\mathcal{D}iv(\Delta^{[m]})$-normal decomposition of $h^{-1}g$ yields a $\mathcal{D}iv(\Delta^{[m]})$-normal decomposition of fg. Then Exercise 42 gives the result.

Exercice 64 (**iterated duality**).— *Assume thatΔ is a Garside map in a cancellative category $\mathcal{C}$.* (i) *Show that $\partial^{[m']}(g) = \partial^{[m]}(g)\,\Delta^{[m'-m]}(\phi_\Delta^m(x))$ holds for $m' \geqslant m$ and g in* $\mathrm{Div}(\Delta^{[m]}(x))$. (ii) *Show that $\widetilde{\partial}_{[m']}(g) = \widetilde{\Delta}_{[m'-m]}(\phi_\Delta^{-m}(y))\,\widetilde{\partial}_{[m]}(g)$ holds for $m' \geqslant m$ and g in* $\widetilde{\mathrm{Div}}(\widetilde{\Delta}_{[m]}(y))$.

Solution. (i) By definition, we have

$$g\,\partial^{[m]}(g)\,\Delta^{[m'-m]}(\phi_\Delta^m(x)) = \Delta^{[m]}(x)\,\Delta^{[m'-m]}(\phi_\Delta^m(x)) = \Delta^{[m']}(x) = g\,\partial^{[m']}(g),$$

whence the result by left-cancelling g. The computation is symmetric for (ii).

Chapter VI: Germs

Skipped proofs

(none)

Solutions to selected exercises

Exercice 65 (**not embedding**).— *Let $\mathcal{S}$ consist of fourteen elements* $1, \mathtt{a}, ..., \mathtt{n}$, *all with the same source and target, and $\bullet$ be defined by $1 \bullet x = x \bullet 1 = x$ for each x, plus* $\mathtt{a}\bullet\mathtt{b} = \mathtt{f}$, $\mathtt{f}\bullet\mathtt{c} = \mathtt{g}$, $\mathtt{d}\bullet\mathtt{e} = \mathtt{h}$, $\mathtt{g}\bullet\mathtt{h} = \mathtt{i}$, $\mathtt{c}\bullet\mathtt{d} = \mathtt{j}$, $\mathtt{b}\bullet\mathtt{j} = \mathtt{k}$, $\mathtt{k}\bullet\mathtt{e} = \mathtt{m}$, *and* $\mathtt{a}\bullet\mathtt{m} = \mathtt{n}$. (i) *Shows that $\underline{\mathcal{S}}$ is a germ.* (ii) *Show that, in $\mathcal{S}$, we have* $((\mathtt{a}\bullet\mathtt{b})\bullet\mathtt{c})\bullet(\mathtt{d}\bullet\mathtt{e}) = \mathtt{i} \neq \mathtt{n} = \mathtt{a}\bullet((\mathtt{b}\bullet(\mathtt{c}\bullet\mathtt{d}))\bullet\mathtt{e})$, *whereas, in $\mathcal{M}on(\underline{\mathcal{S}})$, we have $\iota\mathtt{i} = \iota\mathtt{n}$. Conclude.*

Solution. (i) The only nontrivial triples eligible for (VI.1.6) are $(\mathtt{a}, \mathtt{b}, \mathtt{j})$, $(\mathtt{f}, \mathtt{c}, \mathtt{d})$, and $(\mathtt{c}, \mathtt{d}, \mathtt{e})$, for which (VI.1.6) is actually true. (ii) The equalities in $\underline{\mathcal{S}}$ follow from the multiplication table of $\underline{\mathcal{S}}$. On the other hand, in $\mathcal{M}on(\underline{\mathcal{S}})$, we find $\iota\mathtt{i} = ((\iota\mathtt{a}\,\iota\mathtt{b})\,\iota\mathtt{c})\,(\iota\mathtt{d}\,\iota\mathtt{e}) = \iota\mathtt{a}\,((\iota\mathtt{b}\,(\iota\mathtt{c}\,\iota\mathtt{d}))\,\iota\mathtt{e}) = \iota\mathtt{n}$ applying associativity. Thus ι is not injective, and $\mathcal{S}^\sharp$ does not embed in $\mathcal{M}on(\underline{\mathcal{S}})$.

Exercice 66 (multiplying by invertible elements).— (i) *Show that, if $\underline{\mathcal{S}}$ is a left-associative germ, then $\mathcal{S}$ is closed under left-multiplication by invertible elements in $\mathcal{C}at(\underline{\mathcal{S}})$.* (ii) *Show that, if $\underline{\mathcal{S}}$ is an associative germ, $s \bullet t$ is defined, and $t' =^{\times}_{\mathcal{S}} t$ holds, then $s \bullet t'$ is defined as well.*

Solution. (i) Assume that ϵ admits a left-inverse ϵ'—as nothing a priori forces the category $\mathcal{C}at(\underline{\mathcal{S}})$ to be left-cancellative, we have to distinguish between left- and right-inverse—and that ϵg is defined for some g lying in $\mathcal{S}$. Then we have $g = \epsilon'\epsilon g$, so ϵg is a right-divisor of an element of $\mathcal{S}$, hence is an element of $\mathcal{S}$ as the latter is closed under right-divisor. Applying this to the case when g is an identity-element shows that the family of all left-invertible elements is included in $\mathcal{S}$.

Exercice 67 (atoms).— (i) *Show that, if $\underline{\mathcal{S}}$ is a left-associative germ, the atoms of $\mathcal{C}at(\underline{\mathcal{S}})$ are the elements of the form $t\epsilon$ with t an atom of $\underline{\mathcal{S}}$ and ϵ an invertible element of $\underline{\mathcal{S}}$.*

(ii) *Let $\underline{\mathcal{S}}$ be the germ whose table is shown on the right. Show that the monoid $\mathcal{M}on(\underline{\mathcal{S}})$ admits the presentation $\langle \mathtt{a}, \mathtt{e} \mid \mathtt{ea} = \mathtt{a}, \mathtt{e}^2 = 1\rangle^+$ (see Exercise 48) and that $\mathtt{a}$ is the only atom of $\underline{\mathcal{S}}$, whereas the atoms of $\mathcal{M}on(\underline{\mathcal{S}})$ are $\mathtt{a}$ and $\mathtt{ae}$.* (iii) *Show that $\underline{\mathcal{S}}$ is a Garside germ.*

$\bullet$	1	a	e
1	1	a	e
a	a		
e	e	a	1

Solution. (i) Assume $s = t\epsilon$ with t an atom of $\underline{\mathcal{S}}$ and ϵ an invertible element in $\mathcal{S}$. Let $s_1|\cdots|s_p$ be a decomposition of s in $\mathcal{C}at(\underline{\mathcal{S}})$. As $\mathcal{S}$ generates $\mathcal{C}at(\underline{\mathcal{S}})$, each element s_i can be expressed as a product of elements of $\mathcal{S}$, leading to a new decomposition $t_1|\cdots|t_q$ of s in terms of elements of $\mathcal{S}$, whence $t = t_1\cdots t_q\epsilon^{-1}$ in $\mathcal{C}at(\underline{\mathcal{S}})$ and, as $\underline{\mathcal{S}}$ is left-associative, $t = \Pi(t_1|\cdots|t_q|\epsilon^{-1})$ in $\underline{\mathcal{S}}$, where Π is the partial map of (VI.1.15). As t is an atom of $\underline{\mathcal{S}}$, at most one of the entries t_j is non-invertible in $\underline{\mathcal{S}}$. Hence at most one of the entries s_i is non-invertible in $\mathcal{C}at(\underline{\mathcal{S}})$, and s is an atom in $\mathcal{C}at(\underline{\mathcal{S}})$.

Conversely, assume that s is an atom in $\mathcal{C}at(\underline{\mathcal{S}})$. As $\mathcal{S}$ generates $\mathcal{C}at(\underline{\mathcal{S}})$, there exists a decomposition $s_1|\cdots|s_p$ of s into elements of $\mathcal{S}$. As s is an atom in $\mathcal{C}at(\underline{\mathcal{S}})$, at most one entry is not invertible. If every entry is invertible, then s is invertible, contradicting the assumption. So exactly one entry, say s_i, is not invertible. Let $t = s_1\cdots s_{i-1}s_i$ and $\epsilon = s_{i+1}\cdots s_p$. As $s_1, ..., s_{i-1}$ are invertible, t is a right-divisor of s, hence it belongs to $\mathcal{S}$. Moreover, t is an atom of $\mathcal{S}$, since a decomposition of t with more than one non-invertible entry in $\mathcal{S}$ would provide a similar decomposition in $\mathcal{C}at(\underline{\mathcal{S}})$, contradicting the assumption. On the other hand, ϵ is invertible. So s, which is $t\epsilon$, has the expected form.

Exercice 68 (families $\mathfrak{I}_{\underline{\mathcal{S}}}$ and $\mathfrak{J}_{\underline{\mathcal{S}}}$).— *Assume that $\underline{\mathcal{S}}$ is a left-associative germ.* (i) *Show that a path $s_1|s_2$ of $\mathcal{S}^{[2]}$ is $\mathcal{S}$-normal if and only if all elements of $\mathfrak{J}_{\underline{\mathcal{S}}}(s_1, s_2)$ are invertible.* (ii) *Assuming in addition that $\underline{\mathcal{S}}$ is left-cancellative, show that, for $s_1|s_2$ in $\mathcal{S}^{[2]}$, the family $\mathfrak{I}_{\underline{\mathcal{S}}}(s_1, s_2)$ admits common right-multiples if and only if $\mathfrak{J}_{\underline{\mathcal{S}}}(s_1, s_2)$ does.*

Solution. (ii) Assume that $\mathcal{I}_{\underline{\mathcal{S}}}(g_1, g_2)$ admits common right-multiples, and let h, h' belong to $\mathcal{J}_{\underline{\mathcal{S}}}(g_1, g_2)$. Then $g_1 \bullet h$ and $g_2 \bullet h'$ are defined and belong to $\mathcal{I}_{\underline{\mathcal{S}}}(g_1, g_2)$, hence, by assumption, $g_1 \bullet h$ and $g_1 \bullet h'$ admit a common right-multiple, say g''. Then we have $g_1 \preccurlyeq_{\mathcal{S}} g_1 \bullet h \preccurlyeq_{\mathcal{S}} g$, whence $g_1 \preccurlyeq_{\mathcal{S}} g''$. So there exists h'' in $\mathcal{S}$ satisfying $g'' = g_1 \bullet h''$. By Lemma VI.1.19, $g_1 \bullet h \preccurlyeq_{\mathcal{S}} g_1 \bullet h''$ implies $h \preccurlyeq_{\mathcal{S}} h''$, and, similarly, we find $h' \preccurlyeq_{\mathcal{S}} h''$. So h'' is a common right-multiple of h and h' in $\mathcal{S}$. Moreover the assumption that $g_1 \bullet h''$ belongs to $\mathcal{I}_{\underline{\mathcal{S}}}(g_1, g_2)$ implies that h'' belongs to $\mathcal{J}_{\underline{\mathcal{S}}}(g_1, g_2)$. So $\mathcal{J}_{\underline{\mathcal{S}}}(g_1, g_2)$ admits common right-multiples. The converse implication is similar, actually simpler as no cancellation is needed.

Exercice 69 (**positive generators**).— *Assume that* Σ *is a family of positive generators in a group* $\mathcal{G}$ *and* Σ *is closed under inverse, that is,* $g \in \Sigma$ *implies* $g^{-1} \in \Sigma$. (i) *Show that* $\|g\|_\Sigma = \|g^{-1}\|_\Sigma$ *holds for every* g *in* $\mathcal{G}$. (ii) *Show that* $f^{-1} \leqslant_\Sigma g^{-1}$ *is equivalent to* $f \,\widetilde{\geqslant}_\Sigma\, g$.

Solution. (i) An S-word w is a minimal length expression for an element g if and only if w^{-1}, which is also an S-word, is a minimal length expression of g^{-1}. (ii) By definition, $f^{-1} \leqslant_\Sigma g^{-1}$ is equivalent to $\|f^{-1}\|_\Sigma + \|(f^{-1})^{-1}g^{-1}\|_\Sigma = \|g^{-1}\|_\Sigma$, hence, by (i), to $\|f\|_\Sigma + \|fg^{-1}\|_\Sigma = \|g\|_\Sigma$ and to $\|f\|_\Sigma + \|gf^{-1}\|_\Sigma = \|g\|_\Sigma$. The latter is $f \,\widetilde{\leqslant}_\Sigma\, g$.

Exercice 70 (**minimal upper bound**).— *For* $\leqslant$ *a partial ordering on a family* $\mathcal{S}'$ *and* f, g, h *in* $\mathcal{S}'$, *say that* h *is a* minimal upper bound, *or* mub, *for* f *and* g, *if* $f \leqslant h$ *and* $g \leqslant h$ *holds, but there exists no* h' *with* $h' < h$ *satisfying* $f \leqslant h'$ *and* $g \leqslant h'$. *Assume that* $\mathcal{G}$ *is a groupoid,* Σ *positively generates* $\mathcal{G}$, *and* $\mathcal{H}$ *is a subfamily of* $\mathcal{G}$ *that is closed under* Σ*-suffix. Show that* $\mathcal{H}^\Sigma$ *is a Garside germ if and only if, for all* f, g, g', g'' *in* $\mathcal{H}$ *such that* $f \bullet g$ *and* $f \bullet g'$ *are defined and* g'' *is a* $\leqslant_\Sigma$*-mub of* g *and* g', *the product* $f \bullet g''$ *is defined.*

Solution. By Lemmas VI.2.60 and VI.2.62, the germ $\mathcal{H}^\Sigma$ is left-associative, cancellative, and Noetherian. Hence, by Proposition VI.2.44, $\mathcal{H}^\Sigma$ is a Garside germ if and only if it satisfies (VI.2.43). Now, by Lemma VI.2.62, for g, h in $\mathcal{H}$, the relation $g \preccurlyeq_{\mathcal{H}^\Sigma} h$ is equivalent to $g \leqslant_\Sigma h$ and, therefore, g'' is an mcm of g and g' in $\mathcal{H}^\Sigma$ if and only it it is a $\leqslant_\Sigma$-mub of g and g'. So the condition is a direct reformulation of(VI.2.43).

Chapter VII: Subcategories

Skipped proofs

Lemma VII.1.3.— *If* $\mathcal{C}_1$ *is a subcategory of a left-cancellative category* $\mathcal{C}$, *we have*

(VII.1.4) $$\mathcal{C}_1^\times \subseteq \mathcal{C}^\times \cap \mathcal{C}_1,$$

with equality if and only if $\mathcal{C}_1$ *is closed under inverse in* $\mathcal{C}$. *For every subfamily* $\mathcal{S}$ *of* $\mathcal{C}$, *putting* $\mathcal{S}_1 = \mathcal{S} \cap \mathcal{C}_1$ *and* $\mathcal{S}_1^\sharp = \mathcal{S}_1\mathcal{C}_1^\times \cup \mathcal{C}_1^\times$, *we have*

(VII.1.5) $$\mathcal{S}_1^\sharp \subseteq \mathcal{S}^\sharp \cap \mathcal{C}_1.$$

If $\mathcal{C}$ *has no nontrivial invertible element, then so does* $\mathcal{C}_1$, *and (VII.1.4)–(VII.1.5) are equalities.*

Proof. If $\epsilon\epsilon' = 1_x$ holds in $\mathcal{C}_1$, it holds in $\mathcal{C}$ as well, so (VII.1.4) is clear. For (VII.1.4) to be an equality means that every invertible element lying in $\mathcal{C}_1$ belongs to $\mathcal{C}_1^\times$, that is, has an inverse that lies in $\mathcal{C}_1$: this means that $\mathcal{C}_1$ is closed under inverse in $\mathcal{C}$.

Next, assume that $\mathcal{S}$ is included in $\mathcal{C}$, and put $\mathcal{S}_1 = \mathcal{S} \cap \mathcal{C}_1$ and $\mathcal{S}_1^\sharp = \mathcal{S}_1\mathcal{C}_1^\times \cap \mathcal{C}_1^\times$. Then $\mathcal{C}_1^\times$ is included in $\mathcal{C}^\times$ and in $\mathcal{C}_1$, so we deduce $\mathcal{S}_1^\sharp \subseteq (\mathcal{S}\mathcal{C}^\times \cup \mathcal{C}^\times) \cap \mathcal{C}_1 = \mathcal{S}^\sharp \cap \mathcal{C}_1$.

On the other hand, assume $\mathcal{C}^\times = \mathbf{1}_\mathcal{C}$. As an identity-element is its own inverse, it is invertible in every subcategory that contains it, and we obtain $\mathcal{C}_1^\times = \mathbf{1}_\mathcal{C} \cap \mathcal{C}_1 = \mathcal{C}^\times \cap \mathcal{C}_1$ and, for every $\mathcal{S} \subseteq \mathcal{C}$, as $\mathcal{S}^\sharp$ is then $\mathcal{S} \cup \mathbf{1}_\mathcal{C}$, we obtain

$$\mathcal{S}^\sharp \cap \mathcal{C}_1 = (\mathcal{S} \cap \mathcal{C}_1) \cup (\mathbf{1}_\mathcal{C} \cap \mathcal{C}_1) = \mathcal{S}_1 \cup \mathcal{C}_1^\times = \mathcal{S}_1\mathcal{C}_1^\times \cup \mathcal{C}_1^\times = \mathcal{S}_1^\sharp. \qquad \square$$

Lemma VII.1.16.— *Every subcategory that is closed under left- or under right-divisor in a left-cancellative category $\mathcal{C}$ is closed under $=^\times$.*

Proof. Assume that $\mathcal{C}_1$ is a subcategory of $\mathcal{C}$, and we have $g \in \mathcal{C}_1$ and $g' =^\times g$, say $g' = g\epsilon$ with ϵ invertible. If $\mathcal{C}_1$ is closed under left-divisor, we can write $g = g'\epsilon^{-1}$, and g' is a left-divisor of g, hence it belongs to $\mathcal{C}_1$.

On the other hand, assume that $\mathcal{C}_1$ is closed under right-divisor. Let y be the target of g, and y' be that of g'. The assumption that g lies in $\mathcal{C}_1$ implies that y lies in $\mathcal{O}bj(\mathcal{C}_1)$, hence 1_y lies in $\mathcal{C}_1$. Now ϵ^{-1} is a right-divisor of g, hence it lies in $\mathcal{C}_1$, its source y' lies in $\mathcal{O}bj(\mathcal{C}_1)$ and $1_{y'}$ lies in $\mathcal{C}_1$. Now, we have $1_{y'} = \epsilon^{-1}\epsilon$, so ϵ, a right-divisor of 1_z, must lie in $\mathcal{C}_1$. Hence g', that is, $g\epsilon$, lies in $\mathcal{C}_1$. $\square$

Lemma VII.1.17.— *If $\mathcal{C}, \mathcal{C}'$ are left-cancellative categories, $\phi : \mathcal{C} \to \mathcal{C}'$ is a functor, and $\mathcal{C}_1'$ is a subcategory of $\mathcal{C}'$ that is closed under left-divisor (resp. right-divisor), then so is $\phi^{-1}(\mathcal{C}_1')$.*

Proof. Assume that $\mathcal{C}_1'$ is closed under left-divisor. Put $\mathcal{C}_1 = \phi^{-1}(\mathcal{C}_1')$, and assume $f \preccurlyeq g \in \mathcal{C}_1$. By definition, there exists g' satisfying $fg' = g$, which implies $\phi(f)\phi(g') = \phi(g)$, whence $\phi(f) \preccurlyeq \phi(g)$. By assumption, $\phi(g)$ lies in $\mathcal{C}_1'$, hence so does $\phi(f)$ as $\mathcal{C}_1'$ is closed under left-divisor. Hence $\phi(f)$ belongs to $\mathcal{C}_1'$, and f lies in $\mathcal{C}_1$. So $\mathcal{C}_1$ is closed under left-divisor. The argument when $\mathcal{C}_1'$ is closed under right-divisor is symmetric. $\square$

Proposition VII.2.16 (**recognizing compatible**).— *If $\mathcal{S}$ is a Garside family in a left-cancellative category $\mathcal{C}$ and $\mathcal{C}_1$ is a subcategory of $\mathcal{C}$ that is closed under right-quotient in $\mathcal{C}$, then $\mathcal{C}_1$ is compatible with $\mathcal{S}$ if and only if*

(VII.2.17) *The family $\mathcal{S}_1^\sharp$ generates $\mathcal{C}_1$, where we put $\mathcal{S}_1 = \mathcal{S} \cap \mathcal{C}_1$ and $\mathcal{S}_1^\sharp = \mathcal{S}_1\mathcal{C}_1^\times \cup \mathcal{C}_1^\times$,*

(VII.2.18) *Every element of $(\mathcal{S}_1^\sharp)^2$ admits an $\mathcal{S}$-normal decomposition with entries in $\mathcal{S}_1^\sharp$.*

Proof. If $\mathcal{C}_1$ is compatible with $\mathcal{S}$, then, by Proposition VII.2.14, (VII.2.15) holds. This implies in particular that every element of $\mathcal{C}_1$ admits a decomposition with entries in $\mathcal{S}_1^\sharp$, hence that $\mathcal{S}_1^\sharp$ generates $\mathcal{C}_1$, which is (VII.2.17). On the other hand, applying (VII.2.15) to an element of $(\mathcal{S}_1^\sharp)^2$ gives (VII.2.18).

Conversely, assume that (VII.2.17) and (VII.2.18) are satisfied. As $\mathcal{C}_1$ is closed under right-quotient, it is closed under inverse, $\mathcal{S}_1^\sharp \cap \mathcal{C}^\times = \mathcal{C}_1^\times$ holds, hence so does

$\mathcal{S}_1^\sharp(\mathcal{S}_1^\sharp \cap \mathcal{C}^\times) \subseteq \mathcal{S}_1^\sharp$. Then Lemma VII.2.19 is valid for $\mathcal{S}_1^\sharp$, so every element of the subcategory of $\mathcal{C}$ generated by $\mathcal{S}_1^\sharp$, hence of $\mathcal{C}_1$, admits an $\mathcal{S}$-normal decomposition whose entries lie in $\mathcal{S}_1^\sharp$. In this case, (VII.2.15) is satisfied, and, by Proposition VII.2.14, $\mathcal{C}_1$ is compatible with $\mathcal{S}$. □

Lemma VII.2.19.— *Assume that $\mathcal{S}$ is a Garside family in a left-cancellative category $\mathcal{C}$ and $\mathcal{S}'$ is a subfamily of $\mathcal{S}^\sharp$ such that $\mathcal{S}'(\mathcal{S}' \cap \mathcal{C}^\times) \subseteq \mathcal{S}'$ holds and every element of $\mathcal{S}'^2$ admits a $\mathcal{S}$-normal decomposition with entries in $\mathcal{S}'$. Then every element in the subcategory of $\mathcal{C}$ generated by $\mathcal{S}'$ admits a $\mathcal{S}$-normal decomposition with entries in $\mathcal{S}'$.*

Proof. First we claim that every element g of $\mathcal{S}'^2$ admits a length two $\mathcal{S}$-normal decomposition with entries in $\mathcal{S}'$: indeed, as $\mathcal{S}'$ is included in $\mathcal{S}^\sharp$, the $\mathcal{S}$-length of g is at most two, so the possible entries at position 3 and beyond in an $\mathcal{S}$-normal decomposition of g must be invertible, and the assumption $\mathcal{S}'(\mathcal{S}' \cap \mathcal{C}^\times) \subseteq \mathcal{S}'$ implies that the latter can be incorporated in the second entry, yielding an $\mathcal{S}$-normal decomposition of length two.

Then the argument is exactly the same as for Proposition III.1.49 (left multiplication), except that, here, we use two different reference families, namely $\mathcal{S}'$ for the entries of the decomposition and $\mathcal{S}$ for the greedy property. The point is to prove that, for every p, every element of $\mathcal{S}'^p$ admits an $\mathcal{S}$-normal decomposition of length p with entries in $\mathcal{S}'$. We use induction on $p \geqslant 1$. For $p = 1$, the result is obvious as $\mathcal{S}'$ is included in $\mathcal{S}^\sharp$, and, for $p = 2$, the result is the assumption. Let g belong to $\mathcal{S}'^p$ with $p \geqslant 3$. Write $g = sg'$ with $s \in \mathcal{S}'$ and $g' \in \mathcal{S}'^{p-1}$. By induction hypothesis, g' admits an $\mathcal{S}$-normal decomposition $s'_1|\cdots|s'_{p-1}$ with entries in $\mathcal{S}'$. Starting with $s_0 = s$, and applying the assumption $p-1$ times, we find an $\mathcal{S}$-normal decomposition $g_i|s_i$ of $s_{i-1}g'_i$ with entries in $\mathcal{S}'$. By the first domino rule (Proposition III.1.45), $s_1|\cdots|s_{p-1}|s_p$ is an $\mathcal{S}$-normal decomposition of g with entries in $\mathcal{S}'$. □

Lemma VII.4.7.— *If $\mathcal{S}$ is any subfamily of a left-cancellative category $\mathcal{C}$, the identity-functor on $\mathcal{C}$ is correct for inverses (resp. right-comultiples, resp. right-complements, resp. right-diamonds) on $\mathcal{S}$ if and only if $\mathcal{S}$ is closed under inverse (resp. right-comultiple, resp. right-complement, resp. right-diamond) in $\mathcal{C}$.*

Proof. The identity-functor on $\mathcal{C}$ is correct for inverses on $\mathcal{S}_1$ if and only if, for every s in $\mathcal{S}_1$ that is invertible, s^{-1} belongs to $\mathcal{S}_1$. By definition, this means that $\mathcal{S}_1$ is closed under inverse in $\mathcal{C}$.

Next, the identity-functor on $\mathcal{C}$ is correct for right-comultiples on $\mathcal{S}_1$ if and only if, when s, t lie in $\mathcal{S}_1$ and $sg = tf$ holds in $\mathcal{C}$, there exist s', t', and h satisfying $st' = ts'$, $f = s'h$, and $g = t'h$, plus $st' \in \mathcal{S}_1$. By definition, this means that $\mathcal{S}_1$ is closed under right-comultiple in $\mathcal{C}$. The result is similar with right-complements and right-diamonds. □

Solutions to selected exercises

Exercice 72 (**$=^\times$-closed subcategory**).— (i) *Show that a subcategory $\mathcal{C}_1$ of a left-cancellative category $\mathcal{C}$ is $=^\times$-closed if and only if, for each x in $\mathcal{O}bj(\mathcal{C}_1)$, the* *families* $\mathcal{C}^\times(x,\text{-})$ *and* $\mathcal{C}^\times(\text{-},x)$ *are included in* $\mathcal{C}_1$*.* (ii) *Deduce that* $\mathcal{C}_1$ *is* $=^\times$*-closed if and only if* $\mathcal{C}_1^\times$ *is a union of connected components of* $\mathcal{C}^\times$*.*

Solution. Assume that $\mathcal{C}_1$ is $=^\times$-closed and x lies in $\mathcal{O}bj(\mathcal{C}_1)$. Then 1_x belongs to $\mathcal{C}_1$. If ϵ belongs to $\mathcal{C}^\times(x,\text{-})$, we have $\epsilon =^\times 1_x$, whence $\epsilon \in \mathcal{C}_1$. On the other hand, if ϵ belongs to $\mathcal{C}^\times(y,x)$, then ϵ^{-1} belongs to $\mathcal{C}^\times(x,y)$. By the above result, ϵ^{-1} belongs to $\mathcal{C}_1$. It follows that y, the target of ϵ^{-1}, lies in $\mathcal{O}bj(\mathcal{C}_1)$, and, therefore, ϵ belongs to $\mathcal{C}_1$. Conversely, assume that $\mathcal{C}^\times(x,\text{-})$ is included in $\mathcal{C}_1$ for every x in $\mathcal{O}bj(\mathcal{C}_1)$, and let g belong to $\mathcal{C}_1$ and $g' =^\times g$ hold, say $g' = g\epsilon$ with ϵ in $\mathcal{C}^\times$. Let x be the target of g. Then x belongs to $\mathcal{O}bj(\mathcal{C}_1)$, hence ϵ belongs to $\mathcal{C}_1$, so $g \in \mathcal{C}_1$ implies $g'g\epsilon \in \mathcal{C}_1$.

Exercice 73 (**greedy paths**).— *Assume that* $\mathcal{C}$ *is a cancellative category,* $\mathcal{S}$ *is included in* $\mathcal{C}$*, and* $\mathcal{C}_1$ *is a subcategory of* $\mathcal{C}$ *that is closed under left-quotient. Put* $\mathcal{S}_1 = \mathcal{S} \cap \mathcal{C}_1$*. Show that every* $\mathcal{C}_1$*-path that is* $\mathcal{S}$*-greedy in* $\mathcal{C}$ *is* $\mathcal{S}_1$*-greedy in* $\mathcal{C}_1$*.*

Solution. With the notation of the proof of Lemma VII.2.1, we have also $f' = f''g_2$ with f' and g_2 in $\mathcal{C}_1$. This implies $f'' \in \mathcal{S}_1$ as $\mathcal{C}$ is right-cancellative and $\mathcal{C}_1$ is closed under left-quotient in $\mathcal{C}$.

Exercice 74 (**compatibility with** $\mathcal{C}$).— *Assume that* $\mathcal{C}$ *is a left-cancellative category. Show that every subcategory of* $\mathcal{C}$ *that is closed under inverse is compatible with* $\mathcal{C}$ *viewed as a Garside family in itself.*

Solution. Let $\mathcal{C}_1$ be a subcategory of $\mathcal{C}$ that is closed under inverse. Then $\mathcal{C}\cap\mathcal{C}_1 = \mathcal{C}_1$ is a Garside family in $\mathcal{C}_1$. A $\mathcal{C}_1$-path $g_1|\cdots|g_q$ is $\mathcal{C}$-normal if and only if $g_2, ..., g_p$ belong to $\mathcal{C}^\times$, hence if and only if $g_2, ..., g_p$ belong to $\mathcal{C}_1^\times$, hence if and only if $g_1|\cdots|g_q$ is $\mathcal{C}_1$-normal.

Exercice 75 (**not compatible**).— *Let* M *be the free Abelian monoid generated by* $\mathtt{a}$ *and* $\mathtt{b}$*, and let* N *be the submonoid generated by* $\mathtt{a}$ *and* $\mathtt{ab}$*.* (i) *Show that* N *is not closed under right-quotient in* M*.* (ii) *Let* $S = \{1, \mathtt{a}, \mathtt{b}, \mathtt{ab}\}$*. Show that* N *is not compatible with* S*.* (iii) *Let* $S' = \{\mathtt{a}^p\mathtt{b}^\epsilon \mid p \geqslant 0, i \in \{0,1\}\}$*. Show that* N *not compatible with* S'*.*

Solution. (i) The elements $\mathtt{ab}$ and $\mathtt{a}$ lie in N, but the right-quotient $\mathtt{b}$ does not.

(ii) The family $S \cap N$ is not a Garside family in N, because $\mathtt{a}$ and $\mathtt{ab}$, which belong to $S \cap N$, have no common right-multiple belonging to $S \cap N$.

(iii) The family $S'\cap N$ is a Garside family in M, but the S'-normal decomposition of $\mathtt{a}^2\mathtt{b}^2$ is $\mathtt{a}^2\mathtt{b}|\mathtt{b}$, whereas the $(S'\cap N)$-normal decomposition of $\mathtt{a}^2\mathtt{b}^2$ in N is $\mathtt{ab}|\mathtt{ab}$.

Exercice 76 (**not closed under right-quotient**).— (i) *Show that every submonoid* $m\mathbb{N}$ *of the additive monoid* $\mathbb{N}$ *is closed under right-quotient, but that* $2\mathbb{N}+3\mathbb{N}$ *of* $\mathbb{N}$ *is not.* (ii) *Let* M *be the monoid* $\mathbb{N} \rtimes (\mathbb{Z}/2\mathbb{Z})^2$*, where the generator* $\mathtt{a}$ *of* $\mathbb{N}$ *acts on the generators* $\mathtt{e}, \mathtt{f}$ *of* $(\mathbb{Z}/2\mathbb{Z})^2$ *by* $\mathtt{ae} = \mathtt{fa}$ *and* $\mathtt{af} = \mathtt{ea}$*, and let* N *be the submonoid of* M *generated by* $\mathtt{a}$ *and* $\mathtt{e}$*. Show that* M *is left-cancellative, and its elements admit a unique expression of the form* $\mathtt{a}^p\mathtt{e}^i\mathtt{f}^j$ *with* $p \geqslant 0$ *and* $i, j \in \{0,1\}$*, and that* N *is* $M \setminus \{\mathtt{f}, \mathtt{ef}\}$*.* (iii) *Show that* N *is not closed under right-quotient in* M*.* (iv) *Let* $S = \{\mathtt{a}\}$*. Show that* S *is a Garside family in* M *and determine* $S^\sharp$*. Show that* N *is compatible with* S*. [Hint: Show that* $S \cap N$*, which is* S*, is not a Garside family in* N*.]* (v) *Shows that* $S^\sharp \cap N$ *is a Garside family in* N *and* N *is compatible with* $S^\sharp$*.*

Solution. (i) If f and $f+g'$ are multiples of m, then g' is a multiple of m as well. On the other hand, $2\mathbb{N}+3\mathbb{N}$ contains 2 and 3, but does not contain 1 although $3=2+1$ holds.

(iii) The set $M^\times$ consists of $1, \mathtt{e}, \mathtt{f}, \mathtt{ef}$, whereas $N^\times$ consists of 1 and $\mathtt{e}$, so N is closed under inverse in M. On the other hand, as $\mathtt{a}$ and $\mathtt{af}$ belong to N but $\mathtt{f}$ does not, N is not closed under right-quotient in M.

(iv) We have $S^\sharp = \{\mathtt{a}^p\mathtt{e}^i\mathtt{f}^j \mid p, i, j \leqslant 1\}$ (eight elements). Then $\mathtt{af}$, which is $\mathtt{ea}$, belongs to $N^\times S$ but not to $SN^\times \cup N^\times$. So $S\cap N$, which is S, is not a Garside family in N, and the submonoid N is not compatible with S.

(v) Consider $S^\sharp$, which is also a Garside family in M. Then $S^\sharp \cap N$ is $S^\sharp \setminus \{\mathtt{f}, \mathtt{ef}\}$ (six elements), and it is equal to $(S^\sharp \cap N)N^\times \cup N^\times$. Hence $S^\sharp \cap N$ generates N. Next, M and N are Noetherian, and both admit right-lcms. Now a direct inspection shows that $S^\sharp \cap N$ is (weakly) closed under right-lcm and right-divisor in N (here we consider only right-divisors that belong to N, so $\mathtt{f}$, which is a right-divisor of $\mathtt{af}$ in M, is excluded). Hence, by Corollary IV.2.29 (recognizing Garside, right-lcm case), $S^\sharp \cap N$ is a Garside family in N. Finally, a pair $\mathtt{a}^{p_1}\mathtt{e}^{i_1}\mathtt{f}^{j_1}|\mathtt{a}^{p_2}\mathtt{e}^{i_2}\mathtt{f}^{j_2}$ is S-normal in M if and only if we do not have $p_1=0$ and $p_2=1$ and the same condition characterizes $(S^\sharp \cap N)$-normal pairs in N. Hence N is compatible with $S^\sharp$.

Exercice 77 (**not closed under divisor**).— *Let* $M = \langle \mathtt{a}, \mathtt{b} \mid \mathtt{ab} = \mathtt{ba}, \mathtt{a}^2 = \mathtt{b}^2\rangle^+$, *and let* N *be the submonoid of* M *generated by* $\mathtt{a}^2$ *and* $\mathtt{ab}$. *Show that* N *is compatible with every Garside family* S *of* M, *but that* M *is not closed under left- and right-divisor.*

Solution. As seen in Example IV.2.34 (no proper Garside), the only Garside family in M is M itself.

Exercice 78 (**head implies closed**).— *Assume that* $\mathcal{C}$ *is a left-cancellative category,* $\mathcal{S}$ *is a subfamily of* $\mathcal{C}$ *that is closed under right-comultiple in* $\mathcal{C}$, *and* $\mathcal{C}_1$ *is a subcategory of* $\mathcal{C}$. *Put* $\mathcal{S}_1 = \mathcal{S} \cap \mathcal{C}_1$. (i) *Show that, if every element of* $\mathcal{S}$ *admits a* $\mathcal{C}_1$*-head that lies in* $\mathcal{S}_1$, *then* $\mathcal{S}_1$ *is closed under right-comultiple in* $\mathcal{C}$. (ii) *Show that, if, moreover,* $\mathcal{S}_1$ *is closed under right-quotient in* $\mathcal{C}$, *then* $\mathcal{S}_1$ *is closed under right-diamond in* $\mathcal{C}$.

Solution. (See Figure 9.) (i) Assume $sg = tf$ with $s, t \in \mathcal{S}_1$. As $\mathcal{S}$ is closed under right-comultiple, there exist s', t' satisfying $st' = ts' \preccurlyeq sg$ with $st' \in \mathcal{S}$. Let r be a $\mathcal{C}_1$-head of st' that lies in $\mathcal{S}_1$. As s lies in $\mathcal{C}_1$ and st' lies in $\mathcal{S}$, the relation $s \preccurlyeq st'$ implies $s \preccurlyeq r$, so we have $r = st''$ for some t'' in $\mathcal{C}_1$. Similarly, we have $r = ts_1$ for some s_1 in $\mathcal{C}_1$, and, therefore, r witnesses that $\mathcal{S}_1$ is closed under right-comultiple in $\mathcal{C}$. (ii) If, in addition, $\mathcal{S}_1$ is closed under right-quotient in $\mathcal{C}_1$, then s_1 and t_1 must lie in $\mathcal{S}_1$ since s, t, and r do, and $\mathcal{S}_1$ is closed under right-diamond in $\mathcal{C}$.

Exercice 79 (**head on generating family**).— *Assume that* $\mathcal{C}$ *is a left-cancellative category that is right-Noetherian,* $\mathcal{C}_1$ *is a subcategory of* $\mathcal{C}$ *that is closed under inverse, and* $\mathcal{S}$ *is a subfamily of* $\mathcal{C}$ *such that every element of* $\mathcal{S}$ *admits a* $\mathcal{C}_1$*-head that lies in* $\mathcal{S}$. *Assume moreover that* $\mathcal{S}$ *is closed under right-comultiple and that* $\mathcal{S} \cap \mathcal{C}_1$ *generates* $\mathcal{C}_1$ *and is closed under right-quotient in* $\mathcal{C}$. *Show that* $\mathcal{C}_1$ *is a head-subcategory of* $\mathcal{C}$. *[Hint: Apply Exercise 78.]*

Solution. Exercise 78 implies that $\mathcal{S} \cap \mathcal{C}_1$ is closed under right-diamond in $\mathcal{C}$. Then Proposition IV.1.15 (factorization grid) implies that the subcategory of $\mathcal{C}$

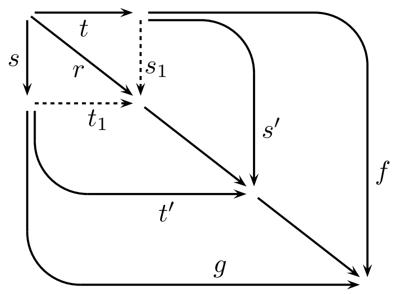


FIGURE 9. Solution to Exercise 79

generated by $\mathcal{S} \cap \mathcal{C}_1$, which is $\mathcal{C}_1$ by assumption, is closed under right-diamond in $\mathcal{C}$. By Lemma VII.1.8, it follows that $\mathcal{C}_1$ is closed under right-quotient in $\mathcal{C}$. Then, as $\mathcal{C}$ is right-Noetherian, Proposition VII.1.21 implies that $\mathcal{C}_1$ is a head-subcategory of $\mathcal{C}$.

Exercice 80 (**transitivity of compatibility**).— *Assume that $\mathcal{S}$ is a Garside family in a left-cancellative category $\mathcal{C}$, $\mathcal{C}_1$ is a subcategory of $\mathcal{C}$ that is compatible with $\mathcal{S}$, and $\mathcal{C}_2$ is a subcategory of $\mathcal{C}_1$ that is compatible with $\mathcal{S}_1 = \mathcal{S} \cap \mathcal{C}_1$. Show that $\mathcal{C}_2$ is compatible with $\mathcal{S}$.*

Solution. Put $\mathcal{S}_2 = \mathcal{S}_1 \cap \mathcal{C}_2$. By assumption, we have $\mathcal{C}_2^\times = \mathcal{C}_1^\times \cap \mathcal{C}_2 = (\mathcal{C}^\times \cap \mathcal{C}_1) \cap \mathcal{C}_2 = \mathcal{C}^\times \cap \mathcal{C}_2$. Therefore, $\mathcal{C}_2$ in closed under inverse in $\mathcal{C}$. Next, as $\mathcal{C}_1$ is compatible with $\mathcal{S}$, the family $\mathcal{S}_1$ is a Garside family in $\mathcal{C}_1$. Now, as $\mathcal{C}_2$ is compatible with $\mathcal{S}_1$ in $\mathcal{C}_1$, the family $\mathcal{S}_2$, which is $\mathcal{S}_1 \cap \mathcal{C}_2$, is a Garside family in $\mathcal{C}_2$. Finally, as $\mathcal{S}_2 = \mathcal{S}_1 \cap \mathcal{C}_2$ holds, an $\mathcal{C}_2$ path is $\mathcal{S}_2$-normal in $\mathcal{C}_2$ if and only if it is $\mathcal{S}_1$-normal in $\mathcal{C}_1$, hence if and only if it is $\mathcal{S}$-normal in $\mathcal{C}$. Hence $\mathcal{C}_2$ is compatible with $\mathcal{S}$.

Exercice 81 (**transitivity of head-subcategory**).— *Assume that $\mathcal{C}$ is a left-cancellative category, $\mathcal{C}_1$ is a head-subcategory of $\mathcal{C}$, and $\mathcal{C}_2$ is a subcategory of $\mathcal{C}_1$. Show that $\mathcal{C}_2$ is a head-subcategory of $\mathcal{C}$ if and only if it is a head-subcategory of $\mathcal{C}_1$.*

Solution. Assume that $\mathcal{C}_2$ is a head-subcategory of $\mathcal{C}$. If ϵ belongs to $\mathcal{C}_1^\times \cap \mathcal{C}_2$, then it belongs to $\mathcal{C}^\times \cap \mathcal{C}_2$, hence ϵ^{-1} belongs to $\mathcal{C}_2$, so $\mathcal{C}_2$ is closed under inverse in $\mathcal{C}_1$. Moreover, every element of $\mathcal{C}_1$ admits a $\mathcal{C}_2$-head since every element of $\mathcal{C}$ does. Conversely assume that $\mathcal{C}_2$ is a head-subcategory of $\mathcal{C}_1$. If ϵ belongs to $\mathcal{C}^\times \cap \mathcal{C}_2$, then it belongs to $\mathcal{C}^\times \cap \mathcal{C}_1$, hence to $\mathcal{C}_1^\times$ since $\mathcal{C}_1$ is a head-subcategory of $\mathcal{C}$. So ϵ belongs to $\mathcal{C}_1^\times \cap \mathcal{C}_2$, hence to $\mathcal{C}_2 i$ since $\mathcal{C}_2$ is a head-subcategory of $\mathcal{C}_1$. So $\mathcal{C}_2$ is closed under inverse in $\mathcal{C}$. Next assume $g \in \mathcal{C}$. Let g' be a $\mathcal{C}_1$-head of g in $\mathcal{C}$, and g'' be a $\mathcal{C}_2$-head of g' in $\mathcal{C}_1$. Assume $h \in \mathcal{C}_2$ and $h \preccurlyeq g$. As h belongs to $\mathcal{C}_1$, we must have $h \preccurlyeq g'$. Then, as h belongs to $\mathcal{C}_2$, we must have $h \preccurlyeq g''$. So g'' is a $\mathcal{C}_2$-head of g, and $\mathcal{C}_2$ is a head-subcategory of $\mathcal{C}$.

Exercice 82 (**recognizing compatible IV**).— *Assume that $\mathcal{S}$ is a Garside family in a left-cancellative category $\mathcal{C}$ and $\mathcal{C}_1$ is a subcategory of $\mathcal{C}$ that is closed under right-quotient in $\mathcal{C}$. Show that $\mathcal{C}_1$ is compatible with $\mathcal{S}$ if and only if, putting $\mathcal{S}_1 = \mathcal{S} \cap \mathcal{C}_1$,* (i) *the family $\mathcal{S}_1$ is a Garside family in $\mathcal{C}_1$, and* (ii) *a $\mathcal{C}_1$-path is strictly $\mathcal{S}_1$-normal in $\mathcal{C}_1$ if and only if it is strictly $\mathcal{S}$-normal in $\mathcal{C}$.*

Solution. Assume that $\mathcal{C}_1$ is compatible with $\mathcal{S}$. By definition, $\mathcal{S}_1$ is a Garside family in $\mathcal{C}_1$, so (i) is satisfied. Next, assume that $g_1|\cdots|g_q$ is a strict $\mathcal{S}$-normal $\mathcal{C}_1$-path. By assumption, $g_1|\cdots|g_q$ is an $\mathcal{S}_1$-normal path in $\mathcal{C}_1$. Moreover, by definition, $g_1, ..., g_q$ are not invertible in $\mathcal{C}$, hence they are not invertible either in $\mathcal{C}_1$, and $g_1, ..., g_{q-1}$ belong to $\mathcal{S}$ and to $\mathcal{C}_1$, hence to $\mathcal{S}_1$. So $g_1|\cdots|g_q$ is strictly $\mathcal{S}_1$-normal in $\mathcal{C}_1$. Conversely, assume that $g_1|\cdots|g_q$ is strictly $\mathcal{S}_1$-normal in $\mathcal{C}_1$. As $\mathcal{C}_1$ is compatible with $\mathcal{S}$, the path $g_1|\cdots|g_q$ is $\mathcal{S}$-normal in $\mathcal{C}$. Moreover, the assumption that $g_1, ..., g_{q-1}$ belong to $\mathcal{S}_1$ implies that they belong to $\mathcal{S}$. Finally, the assumption that no g_i is invertible in $\mathcal{C}_1$ implies that they are not invertible in $\mathcal{C}$ either. So (ii) is satisfied.

Conversely, assume that $\mathcal{C}_1$ satisfies (i) and (ii). Assume that $g_1|g_2$ is an $\mathcal{S}_1$-path. Then g_1g_2 is invertible in $\mathcal{C}_1$ if and only if it is invertible in $\mathcal{C}$. In this case, $g_1|g_2$ is both $\mathcal{S}_1$-normal in $\mathcal{C}_1$ and $\mathcal{S}$-normal in $\mathcal{C}$. Otherwise, if g_1g_2 belongs to $\mathcal{S}_1\mathcal{C}_1^\times$, then $g_1|g_2$ is $\mathcal{S}_1$-normal in $\mathcal{C}_1$ if and only if g_2 is invertible in $\mathcal{C}_1$ and, similarly, it is $\mathcal{S}$-normal in $\mathcal{C}$ if and only if g_2 is invertible in $\mathcal{C}$, so both conditions are equivalent. Finally, if g_1g_2 has $\mathcal{S}_1$-length 2, it admits a strict $\mathcal{S}_1$-normal decomposition in $\mathcal{C}_1$, say $h_1|h_2$. Then $g_1|g_2$ is $\mathcal{S}_1$-normal in $\mathcal{C}_1$ if and only if there exist ϵ invertible in $\mathcal{C}_1$ satisfying $g_1 = h_1\epsilon$ and $h_2 = \epsilon g_2$; similarly, $g_1|g_2$ is $\mathcal{S}$-normal in $\mathcal{C}$ if and only if there exist ϵ invertible in $\mathcal{C}$ satisfying $g_1 = h_1\epsilon$ and $h_2 = \epsilon g_2$: both conditions are equivalent again. So an $\mathcal{S}_1$-path is $\mathcal{S}_1$-normal in $\mathcal{C}_1$ if and only if it is $\mathcal{S}$-normal in $\mathcal{C}$, and, by definition, $\mathcal{C}_1$ is compatible with $\mathcal{S}$.

Exercice 83 (inverse image).— *Assume that $\mathcal{C}, \mathcal{C}'$ are left-cancellative categories, ϕ is a functor from $\mathcal{C}$ to $\mathcal{C}'$, and $\mathcal{C}'_1$ is a subcategory of $\mathcal{C}'$ that is closed under left- and right-divisor. Show that the subcategory $\phi^{-1}(\mathcal{C}'_1)$ is compatible with every Garside family of $\mathcal{C}$.* (ii) *Let B^+ be the Artin–Tits monoid of type B as defined in Example VII.4.21. Show that the map ϕ defined by $\phi(\sigma_0) = 1$ and $\phi(\sigma_i) = 0$ for $i \geqslant 1$ extends into a homomorphism of B^+ to $\mathbb{N}$, and that the submonoid $N = \{g \in M \mid \phi(g) = 0\}$ of B^+ is compatible with every Garside family of B^+.*

Solution. (i) By Proposition VII.1.18, the subcategory $\phi^{-1}(\mathcal{C}'_1)$ is closed under right-quotient and under $=^\times$. Then apply Proposition VII.2.21. (ii) Use (i)

Exercice 84 (intersection).— *Assume that $\mathcal{S}$ is a Garside family in a left-cancellative category $\mathcal{C}$.* (i) *Let $\mathcal{F}$ be the family of all subcategories of $\mathcal{C}$ that are closed under right-quotient, compatible with $\mathcal{S}$, and $=^\times$-closed. Show that every intersection of elements of $\mathcal{F}$ belongs to $\mathcal{F}$.* (ii) *Same question when “$=^\times$-closed” is replaced with “including $\mathcal{C}^\times$”.* (iii) *Same question when $\mathcal{C}$ contains no nontrivial invertible element and “$=^\times$-closed” is skipped.*

Solution. (i) Let $(\mathcal{C}_i)_{i \in I}$ be a family of $=^\times$-closed subcategories of $\mathcal{C}$ that are compatible with $\mathcal{S}$. First, an intersection of subcategories is a subcategory. Next, an intersection of $=^\times$-closed families is $=^\times$-closed and, similarly, an intersection of families that are closed under right-quotient is closed under right-quotient. Finally, let g belong to $\bigcap \mathcal{C}_i$. Then g admits an $\mathcal{S}$-normal decomposition $s_1|\cdots|s_p$. By Lemma VII.2.20, the latter is $(\mathcal{S} \cap \mathcal{C}_i)$-normal for every i, hence it its entries lie in every subfamily $\mathcal{C}_i$, hence in their intersection. By Proposition VII.2.21 we deduce that $\bigcap \mathcal{C}_i$ is compatible with $\mathcal{S}$.

(ii) An intersection of families that include $\mathcal{C}^\times$ includes $\mathcal{C}^\times$. On the other hand, a subcategory that includes $\mathcal{C}^\times$ is $=^\times$-closed and we apply (i).

(iii) In this case, every subcategory is $=^{\times}$-closed, and we apply (i).

Exercice 85 (**fixed points**).— *Assume that* $\mathcal{C}$ *is a left-cancellative category and* $\phi : \mathcal{C} \to \mathcal{C}$ *is a functor. Show that the fixed point subcategory* $\mathcal{C}^{\phi}$ *is compatible with* $\mathcal{C}$ *viewed as a Garside family in itself.*

Solution. First, assume that ϵ belongs to $\mathcal{C}(x,y) \cap \mathcal{C}^{\phi}$. Necessarily we have $\phi(x) = x$. As ϕ is a functor, we find $\epsilon\epsilon^{-1} = 1_x = \phi(1_x) = \phi(\epsilon\epsilon^{-1}) = \phi(\epsilon)\phi(\epsilon^{-1}) = \epsilon\phi(\epsilon^{-1})$, whence $\epsilon^{-1} = \phi(\epsilon^{-1})$, so $\mathcal{C}^{\times} \cap \mathcal{C}^{\phi} \subseteq (\mathcal{C}^{\phi})^{\times}$ holds. Next, we have $\mathcal{C}^{\sharp} = \mathcal{C}$, so $\mathcal{C}^{\sharp} \cap \mathcal{C}^{\phi} \subseteq \mathcal{S}^{\phi}$ trivially holds, and $\mathcal{C}^{\phi}$ and $\mathcal{C}$ satisfy (the counterpart of) (VII.2.11). On the other hand, every element g of $\mathcal{C}$ admits the $\mathcal{C}$-normal decomposition g, so, in particular, every element of $\mathcal{C}^{\phi}$ has a $\mathcal{C}$-normal decomposition whose entries lie in $\mathcal{C}^{\phi}$. So (the counterpart of) (VII.2.12) is satisfied and, by Proposition VII.2.10, $\mathcal{C}^{\phi}$ is compatible with $\mathcal{C}$.

Exercice 86 (**connection between closure properties**).— *Assume that* $\mathcal{S}$ *is a subfamily in a left-cancellative category* $\mathcal{C}$ *and* $\mathcal{S}_1$ *is a subfamily of* $\mathcal{S}$. (i) *Show that, if* $\mathcal{S}_1$ *is closed under product, inverse, and right-complement in* $\mathcal{S}$*, then* $\mathcal{S}_1$ *is closed under right-quotient in* $\mathcal{S}$. (ii) *Assume that* $\mathcal{S}_1$ *is closed under product and right-complement in* $\mathcal{S}$*. Show that, if* $\mathcal{S}$ *is closed under left-divisor in* $\mathcal{C}$*, then* $\mathcal{S}_1$ *is closed under right-comultiple in* $\mathcal{S}$*. Show that, if* $\mathcal{S}$ *is closed under right-diamond in* $\mathcal{C}$*, then* $\mathcal{S}ub(\mathcal{S}_1)$ *is closed under right-comultiple in* $\mathcal{S}$. (iii) *Show that, if* $\mathcal{S}_1$ *is closed under identity and product in* $\mathcal{S}$*, then* $\mathcal{S}_1$ *is closed under inverse and right-diamond in* $\mathcal{S}$ *if and only if* $\mathcal{S}_1$ *is closed under right-quotient and right-comultiple in* $\mathcal{S}$.

Solution. (i) The argument is the same as for Lemma VII.1.8. Assume $sg = t$ with s, t in $\mathcal{S}_1$ and g in $\mathcal{S}$. As $\mathcal{S}_1$ is closed under right-complement in $\mathcal{S}$, there exist s', t' in $\mathcal{S}_1$ and r in $\mathcal{S}$ satisfying $st' = ts'$, $1_y = s'r$, and $g = t'r$, where y is the target of t. The second equality implies that s' is invertible, and the assumption that $\mathcal{S}_1$ is closed under inverse in $\mathcal{S}$ then implies that s'^{-1}, that is, r, lies in $\mathcal{S}_1$. Hence $t'r$, that is g, belongs to $\mathcal{S}_1^{[2]} \cap \mathcal{S}$, hence to $\mathcal{S}_1$ as the latter is closed under product in $\mathcal{S}$. So $\mathcal{S}_1$ is closed under right-quotient in $\mathcal{S}$.

(iii) Assume that $\mathcal{S}_1$ is closed under inverse and right-diamond in $\mathcal{S}$. Then, by definition, $\mathcal{S}_1$ is closed under right-comultiple and right-complement, and, by Exercice 89 (transfer of closure), it is closed under right-quotient.

Conversely, assume that $\mathcal{S}_1$ is closed under right-quotient and right-comultiple in $\mathcal{S}$. First, the assumption that $\mathcal{S}_1$ is closed under right-quotient trivially implies that $\mathcal{S}_1$ is closed under inverse. Next, by the same argument as in Lemma IV.1.8, the assumption that $\mathcal{S}_1$ is closed under right-comultiple and right-quotient implies that it is closed under right-diamond.

Exercice 87 (**subgerm**).— *Assume that* $\underline{\mathcal{S}}$ *is a left-cancellative germ and* $\underline{\mathcal{S}}_1$ *is a subgerm of* $\mathcal{S}$ *such that the relation* $\preccurlyeq_{\mathcal{S}_1}$ *is the restriction to* $\mathcal{S}_1$ *of the relation* $\preccurlyeq_{\mathcal{S}}$*. Show that* $\mathcal{S}_1$ *is closed under right-quotient in* $\mathcal{S}$.

Solution. Assume $f \bullet h = g$ with $f, g \in \mathcal{S}_1$. By definition, $f \preccurlyeq_{\mathcal{S}} g$ holds, hence so does $f \preccurlyeq_{\mathcal{S}_1} g$. This means that there exists h_1 in $\mathcal{S}_1$ satisfying $f \bullet h_1 = g$. The assumption that $\mathcal{S}$ is left-cancellative implies $h_1 = h$, whence $h \in \mathcal{S}$. So $\mathcal{S}_1$ is closed under right-quotient in $\mathcal{S}$.

Exercice 88 (**transitivity of closure**).— *Assume that $\underline{\mathcal{S}}_1$ is a subgerm of a Garside germ $\underline{\mathcal{S}}$, the subcategory $\mathcal{S}ub(\mathcal{S}_1)$ is closed under right-quotient and right-diamond in $\mathcal{C}at(\underline{\mathcal{S}})$, and $\mathcal{S}_1$ is closed under inverse and right-complement (resp. right-diamond) in $\mathcal{S}ub(\mathcal{S}_1)$. Show that $\mathcal{S}_1$ is closed under inverse and right-complement (resp. right-diamond) in $\mathcal{S}$.*

Solution. By Lemma VII.1.7, the assumption that $\mathcal{S}ub(\mathcal{S}_1)$ is closed under right-quotient implies that it is closed under inverse, and Lemma VII.3.7, which is valid since the assumption that $\underline{\mathcal{S}}$ is a Garside germ implies that $\mathcal{S}$ is closed under right-quotient in $\mathcal{C}at(\underline{\mathcal{S}})$, then implies that $\mathcal{S}_1$ is closed under inverse in $\mathcal{S}$. Next, by Lemma VII.3.8 applied with $\mathcal{S}ub(\mathcal{S}_1)$ in place of $\mathcal{S}$, the assumption that $\mathcal{S}_1$ is closed under right-complement (*resp.* right-diamond) in $\mathcal{S}ub(\mathcal{S}_1)$ implies that $\mathcal{S}_1$ is closed under right-complement (*resp.* right-diamond) in $\mathcal{C}at(\underline{\mathcal{S}})$. Now, as $\mathcal{S}$ is a solid Garside family in $\mathcal{C}at(\underline{\mathcal{S}})$, it is closed under right-divisor in $\mathcal{C}at(\underline{\mathcal{S}})$, hence a fortiori under right-quotient. Applying once more Lemma VII.3.8, we deduce from the fact that $\mathcal{S}_1$ is closed under right-complement (*resp.* right-diamond) in $\mathcal{C}at(\underline{\mathcal{S}})$ that $\mathcal{S}_1$ is closed under right-complement (*resp.* right-diamond) in $\mathcal{S}$.

Exercice 89 (**transfer of closure**).— *Assume that $\mathcal{C}$ is a left-cancellative category, $\mathcal{S}$ is a subfamily of $\mathcal{C}$, and $\mathcal{S}_1$ is a subfamily of $\mathcal{S}$ that is closed under identity and product.* (i) *Show that, if $\mathcal{S}ub(\mathcal{S}_1)$ is closed under right-quotient in $\mathcal{C}$, then $\mathcal{S}_1$ is closed under right-quotient in $\mathcal{S}$.* (ii) *Show that, if, moreover, $\mathcal{S}$ is closed under right-quotient in $\mathcal{C}$, then $\mathcal{S}_1$ is closed under right-quotient in $\mathcal{S}ub(\mathcal{S}_1)$.*

Solution. (i) Assume that t belongs to $\mathcal{S}$ and s and st belong to $\mathcal{S}_1$. Then s and st belong to $\mathcal{S}ub(\mathcal{S}_1)$, so, as the latter is assumed to be closed under right-quotient, t must belong to $\mathcal{S}ub(\mathcal{S}_1)$, hence to $\mathcal{S}ub(\mathcal{S}_1) \cap \mathcal{S}$. By Proposition VII.3.3, the latter is $\mathcal{S}_1$. So $\mathcal{S}_1$ is closed under right-quotient in $\mathcal{S}$.

(ii) Assume that $\mathcal{S}$ is closed under right-quotient in $\mathcal{C}$. Assume that t belongs to $\mathcal{S}ub(\mathcal{S}_1)$ and s and st belong to $\mathcal{S}_1$. Then s and st belong to $\mathcal{S}$, so the assumption implies that t belong to $\mathcal{S}$, hence to $\mathcal{S}ub(\mathcal{S}_1) \cap \mathcal{S}$, which is $\mathcal{S}_1$ by Proposition VII.3.3. So $\mathcal{S}_1$ is closed under right-quotient in $\mathcal{S}ub(\mathcal{S}_1)$.

Exercice 90 (**braid subgerm**).— *Let $\underline{\mathcal{S}}$ be the six-element Garside germ associated with the divisors of Δ_3 in the braid monoid B_3^+.* (i) *Describe the subgerm $\underline{\mathcal{S}}_1$ of $\underline{\mathcal{S}}$ generated by σ_1 and σ_2. Compare $\mathcal{M}on(\underline{\mathcal{S}}_1)$ and $\mathcal{S}ub(\mathcal{S}_1)$ (describe them explicitly).* (ii) *Same questions with σ_1 and $\sigma_2\sigma_1$. Is $\mathcal{S}_1$ closed under right-quotient in $\underline{\mathcal{S}}$ in this case?*

Solution. (i) The subgerm of $\underline{\mathcal{S}}$ generated by σ_1 and σ_2 is the closure of $\{\sigma_1, \sigma_2\}$ under identity and product in $\underline{\mathcal{S}}$: so it contains 1, and, as $\sigma_1 \bullet \sigma_2$ is defined in $\underline{\mathcal{S}}$, it contains $\sigma_1 \bullet \sigma_2$, that is, $\sigma_1\sigma_2$. Then it contains $\sigma_1\sigma_2 \bullet \sigma_1$, which is Δ_3, etc. Finally, one finds $\underline{\mathcal{S}}_1 = \underline{\mathcal{S}}$, whence $\mathcal{M}on(\underline{\mathcal{S}}_1) = \mathcal{S}ub(\mathcal{S}_1) = B_3^+$.

(ii) The closure of $\{\sigma_1, \sigma_2\sigma_1\}$ under identity and product is $\{1, \sigma_1, \sigma_2\sigma_1, \Delta_3\}$. In the associated germ $\underline{\mathcal{S}}_1$, the only nontrivial product is $\sigma_1 \bullet_1 \sigma_2\sigma_1 = \Delta_3$, so $\mathcal{M}on(\underline{\mathcal{S}}_1)$ is $\langle \mathtt{a}, \mathtt{b}, \Delta_3 \mid \mathtt{ab} = \Delta_3 \rangle^+$, that is, a free monoid based on $\mathtt{a}$ and $\mathtt{b}$. On the other hand, in $\mathcal{S}ub(\mathcal{S}_1)$, the relation $(\sigma_2\sigma_1)^3 = \Delta_3^2$ holds, corresponding to $\mathtt{b}^3 = (\mathtt{ab})^2$, which fails in $\mathcal{M}on(\underline{\mathcal{S}}_1)$: so $\mathcal{M}on(\underline{\mathcal{S}}_1)$ is not isomorphic to $\mathcal{S}ub(\mathcal{S}_1)$. Here $\mathcal{S}ub(\mathcal{S}_1)$ is not closed under right-quotient in $\underline{\mathcal{S}}$, as Δ_3 and $\sigma_2\sigma_1$ lie in $\mathcal{S}ub(\mathcal{S}_1)$, but we have $\Delta_3 = \sigma_2\sigma_1 \bullet \sigma_2$ in $\underline{\mathcal{S}}$ and $\sigma_2 \notin \mathcal{S}_1$.

Exercice 91 ($=^\times$**-closed**).— *Show that, if* $\underline{\mathcal{S}}_1$ *is a subgerm of an associative germ* $\underline{\mathcal{S}}$, *then* $\mathit{Sub}(\mathcal{S}_1)$ *is* $=^\times$*-closed in* $\mathit{Cat}(\underline{\mathcal{S}})$ *if and only if* $\underline{\mathcal{S}}_1$ *is* $=^\times_{\mathcal{S}}$*-closed in* $\mathcal{S}$.

Solution. Assume that $\underline{\mathcal{S}}_1$ is an $=^\times_{\mathcal{S}}$-closed subgerm of $\underline{\mathcal{S}}$. Assume that g is an element of $\mathit{Sub}(\mathcal{S}_1)$ and $g' =^\times g$ holds in $\mathit{Cat}(\underline{\mathcal{S}})$. By definition, g admits an $\mathcal{S}_1$-decomposition, say $s_1|\cdots|s_p$, and, in $\mathit{Cat}(\underline{\mathcal{S}})$, we then have $g' = s_1 \cdots s_p\epsilon$. As $\mathcal{S}$ is closed under left-divisor in $\mathit{Cat}(\underline{\mathcal{S}})$ since $\underline{\mathcal{S}}$ is right-associative, the assumption that s_p lies in $\mathcal{S}$ implies that its left-divisor $s_p\epsilon$ also lies in $\mathcal{S}$, that is, $s_p \bullet \epsilon$ is defined in $\underline{\mathcal{S}}$. The assumption that $\underline{\mathcal{S}}_1$ is $=^\times_{\mathcal{S}}$-closed in $\underline{\mathcal{S}}$ then implies that $s_p \bullet \epsilon$ belongs to $\mathcal{S}_1$, which implies that g', which is $s_1 \cdots s_{p-1}(s_p\epsilon)$, lies in $\mathit{Sub}(\mathcal{S}_1)$. Conversely, assume that $\mathit{Sub}(\mathcal{S}_1)$ is an $=^\times$-closed subcategory of $\mathit{Cat}(\underline{\mathcal{S}})$. Assume that s belongs to $\mathcal{S}_1$ and $s' =^\times_{\mathcal{S}} s$ holds in $\underline{\mathcal{S}}$. This means that there exists ϵ in $\mathcal{S}^\times$ satisfying $s' = s \bullet \epsilon$. Then, in $\mathit{Cat}(\underline{\mathcal{S}})$, we have $s' = s\epsilon$, whence $s' =^\times s$. The assumption that $\mathit{Sub}(\mathcal{S}_1)$ is $=^\times$-closed implies $s' \in \mathit{Sub}(\mathcal{S}_1)$, whence $s' \in \mathit{Sub}(\mathcal{S}_1) \cap \mathcal{S}$. As $\underline{\mathcal{S}}_1$ is a subgerm of $\underline{\mathcal{S}}$, the latter family is $\mathcal{S}_1$.

Exercice 92 (**correct** *vs.* **mcms**).— *Assume that* $\mathcal{C}$ *and* $\mathcal{C}'$ *are left-cancellative categories and* $\mathcal{S}$ *is included in* $\mathcal{C}$. *Assume moreover that* $\mathcal{C}$ *and* $\mathcal{C}'$ *admit right-mcms and* $\mathcal{S}$ *is closed under right-mcm. Show that a functor* ϕ *from* $\mathcal{C}$ *to* $\mathcal{C}'$ *is correct for right-comultiples on* $\mathcal{S}$ *if and only if, for all* s, t *in* $\mathcal{S}$, *every right-mcm of* $\phi(s)$ *and* $\phi(t)$ *is* $=^\times$*-equivalent to the image under* ϕ *of a right-mcm of* s *and* t.

Solution. Assume that ϕ is correct for right-comultiples on $\mathcal{S}$, that s, t lie in $\mathcal{S}$, and that h is a right-mcm of $\phi(s)$ and $\phi(t)$. By definition, there exists a common right-multiple r of s and t that lies in $\mathcal{S}$ and satisfies $\phi(r) \preccurlyeq h$. As $\mathcal{C}$ admits right-mcms, there exists a right-mcm r' of s and t satisfying $r' \preccurlyeq r$ and, as $\mathcal{S}$ is closed under right-mcm, r' lies in $\mathcal{S}$. Then $\phi(r') \preccurlyeq h$ holds, and $\phi(r')$ is a common right-multiple of $\phi(s)$ and $\phi(t)$. As r is minimal, we deduce $\phi(r') =^\times h$. So the condition is necessary. Conversely, assume that ϕ preserves mcms in the sense of the statement. Assume that s, t lie in $\mathcal{S}$, and h is a common right-multiple of $\phi(s)$ and $\phi(t)$. As $\mathcal{C}'$ admits right-mcms, h is a right-multiple of some right-mcm of $\phi(s)$ and $\phi(t)$, hence, by (ii), of some element $\phi(r)$ where r is a right-mcm of s and t. By assumption, r belongs to $\mathcal{S}$, and there exist s', t' in $\mathcal{C}$ satisfying $st' = ts' = r$. Hence ϕ is correct for right-comultiples on $\mathcal{S}$.

Chapter VIII: Conjugacy

Skipped proofs

(none)

Solution to selected exercises

Exercice 97 (**quasi-distance**).— (i) *In Context VIII.3.7, show that* $\ell_\Delta(g) = \sup_\Delta(g^0)$ *holds for every* g *in* $\mathcal{G}$. (ii) *For* g, g' *in* $\mathcal{G}$, *define* $\mathrm{dist}(g, g')$ *to be* ∞ *if* g, g' *do not share the same source, and to be* $\ell_\Delta(g^{-1}g')$ *otherwise. Show that* dist *is a quasi-distance on* $\mathcal{G}$ *that is compatible with* $=_\Delta$.

Solution. (ii) The canonical length is invariant under left- and right-multiplication by Δ and, therefore, dist takes constant values on $=_\Delta$-classes.

Chapter IX: Braids

Skipped proofs

(none)

Solution to selected exercises

Exercice 102 (smallest Garside, right-angled type).— *Assume that B^+ is a right-angled Artin–Tits monoid, that is, B^+ is associated with a Coxeter system (W, Σ) satisfying $m_{s,t} \in \{2, \infty\}$ for all s, t in Σ.* (i) *For $I \subseteq \Sigma$, denote by Δ_I the right-lcm (here the product) of the elements of I, when it exists (that is, when the elements pairwise commute). Show that the divisors of Δ_I are the elements Δ_J with $J \subseteq I$.* (ii) *Deduce that the smallest Garside family in B^+ is finite and consists of the elements Δ_I for $I \subseteq \Sigma$.*

Solution. (i) Clearly $J \subseteq I$ implies $\Delta_J \preccurlyeq \Delta_I$ and $\Delta_J \widetilde{\preccurlyeq} \Delta_I$, since the elements of I commute. Conversely, the point is to see that, if u is a word in S that involves at least one letter not in I, then (the class of) u cannot right-divide Δ_I: this is so since, when left-reversing $\Delta_I \overline{u}$, a negative letter $\overline{s}$ can only vanish when it is adjacent to the positive letter s, and it, if u is a word in I in which some letter is repeated twice, then the second occurrence cannot vanish as we are arguing in the free abelian monoid based on I.

(ii) Let S be the family of all elements Δ_I with I a set of pairwise commuting atoms in B^+. As the smallest Garside family of B^+ includes I and is closed under right-lcm, it must include S. On the other hand, S includes Σ, it is closed under right-lcm by definition, and it is closed under right-divisor by (i). Corollary IV.2.29 (recognizing Garside, right-lcm case) implies that S is a Garside family in B^+.

Exercice 103 (smallest Garside, large type).— *Assume that B^+ is an Artin–Tits monoid of large type, B^+ is associated with a Coxeter system (W, Σ) satisfying $m_{s,t} \geqslant 3$ for all s, t in Σ. Put*

$\Sigma_1 = \{s{\in}\Sigma \mid \forall r{\in}\Sigma\, (m_{r,s}{=}\infty)\}$,

$\Sigma_2 = \{(s,t){\in}\Sigma^2 \mid m_{s,t} < \infty \text{ and } \forall r{\in}\Sigma\, (m_{r,s}{+}m_{r,t}{=}\infty\}$,

$\Sigma_3 = \{(r,s,t){\in}\Sigma^3 \mid m_{r,s}{+}m_{r,t}{+}m_{s,t}{<}\infty\}$,

and $E = \Sigma_1 \cup \{\Delta_{s,t} \mid (s,t) \in \Sigma_2\} \cup \{r\Delta_{s,t} \mid (r,s,t) \in \Sigma_3\}$ (we write $\Delta_{s,t}$ for the right-lcm of s and t when it exists). (i) *Explicitly describe the elements of the closure S of E under right-divisor, and deduce that S is finite.* (ii) *Show that S is closed under right-lcm and deduce that S is a Garside family in B^+.* (iii) *Show that S is the smallest Garside family in B^+.* (iv) *Show that, if Σ has n elements and $m_{s,t} \neq \infty$ holds for all s, t, then E has $3\binom{n}{3}$ elements.* (v) *Show that, if Σ has n elements and $m_{s,t} = m$ holds for all s, t, then S has $(n + 2m - 5)\binom{n}{2} + n + 1$ elements. Apply to $n = m = 3$.*

Solution. (i) For s in Σ_1, the only right-divisors of s are 1 and s; for (s,t) in Σ_2, the right-divisors of $\Delta_{s,t}$ are the elements $(s|t)^{[k]}$ and $(t|s)^{[k]}$ with $k \leqslant m_{s,t}$: this is so because right-divisors are detected using left-reversing (Section II.4) and it is clear that, if u is a word involving a letter different from s and t or if u involves two letters s or two letters t, or if u is of the form $(s|t)^{[k]}$ and $(t|s)^{[k]}$ with $k > m_{s,t}$, then left-reversing $\Delta_{s,t}u^{-1}$ cannot result in a positive word. Finally, for (r,s,t) in Σ_3, the right-divisors of $r\Delta_{s,t}$ are $r\Delta_{s,t}$ and the elements $(s|t)^{[k]}$ and $(t|s)^{[k]}$ with $k \leqslant m_{s,t}$: again, this is so because left-reversing $\Delta_{s,t}u^{-1}$ can result in a positive word only in the cases described above, and it can result in r^{-1} concatenated with a positive word only if u is $r\Delta_{s,t}$ itself.

(ii) We have to analyze when two elements of S may admit a common right-multiple. As there are three cases for each factor, nine cases are to be considered. Many of them are trivial. First, all cases involving an atom of Σ_1 are trivial. Then, in the case of two elements $(s|t)^{[k]}$ and $(s'|t')^{[k']}$, the case $\{s,t\} = \{s',t'\}$ is obvious. Otherwise, for $s' = s$ and $t' \neq t$, the only cases when a common right-multiple may exist are the trivial cases $k = 1$ or $k' = 1$, plus the case $k = k' = 2$, in which case there exists a common right-multiple for $m_{t,t'} < \infty$, and the right-lcm is then $s\Delta_{t,t'}$, an element of S. Finally, for $\#\{s,t,s',t'\} = 4$, no common right-multiple may exist for $k, k' \geqslant 2$, and the remaining cases are treated as above. The cases of an element $r\Delta_{s,t}$ and an element $(s'|t')^{[k']}$ and of two elements $r\Delta_{s,t}$ and $r'\Delta_{s',t'}$ are treated in the same way: once again, the point is that common right-multiples may exist only in the obvious cases. Finally, S is closed under right-lcm. As, by definition, S includes Σ and is closed under right-divisor, Corollary IV.2.29 (recognizing Garside, right-lcm case) implies that S is a Garside family in B^+.

(iii) Conversely, every Garside family S' of B^+ containing 1 must include S. Indeed, S' must include Σ, hence all elements $\Delta_{s,t}$ since it is closed under right-lcm. Moreover, for (r,s,t) in Σ_3, the family S' must contain rs since it is closed under right-divisor and rs right-divides $\Delta_{r,s}$; it contains rt for a similar reason and, therefore, it contains the right-lcm $r\Delta_{s,t}$ of rs and rt. So S' includes E, hence S, and S is the smallest Garside family containing 1 in B^+.

(iv) When no coefficient $m_{s,t}$ is ∞, Σ_1 and Σ_2 are empty and E consists of all elements $r\Delta_{s,t}$ with r,s,t pairwise distinct atoms in B^+. As $m_{s,t}$ does not matter, there exist $3\binom{n}{3}$ such elements for Σ of size n.

(v) By the description of (i), S comprises 1, plus the n atoms, plus, for each of the $\binom{n}{2}$ pairs (s,t), the $2m-3$ right-divisors of $\Delta_{s,t}$ of length $\geqslant 2$, plus, for each of the $\binom{n}{3}$ triples (r,s,t), the 3 elements $r\Delta_{s,t}, s\Delta_{t,r}, t\Delta_{r,s}$, whence the formula. The case $n = m = 3$ corresponds to the affine type $\widetilde{\mathrm{A}}_2$, and confirms that the smallest Garside family containing 1 has 16 elements, as seen in Reference Structure 9, page 111.

Chapter X: Deligne–Lusztig varieties

Skipped proofs

(none)

Solution to selected exercises

(none)

Chapter XI: Left self-distributivity

Skipped proofs

Lemma XI.1.8.— *Assume that F is a partial action of a monoid M on a set X.*

(i) *If the monoid M is left-cancellative, then so is the category $\mathcal{C}(M, F)$.*

(ii) *Conversely, if F is proper and $\mathcal{C}(M, F)$ is left-cancellative, then so is M.*

Proof. (i) Assume that M is left-cancellative and, in $\mathcal{C}(M, F)$, the equality $x \xrightarrow{g} y \cdot y \xrightarrow{h} z = x \xrightarrow{g} y \cdot y \xrightarrow{h'} z'$ holds. This implies $x \xrightarrow{gh} z = x \xrightarrow{gh'} z'$, whence, by definition, $gh = gh'$ and $z = z'$, and $h = h'$ since M is left-cancellative. We deduce $y \xrightarrow{h} z = y \xrightarrow{h'} z'$, and $\mathcal{C}(M, F)$ is left-cancellative.

(ii) Conversely, assume that F is proper and $\mathcal{C}(M, F)$ is left-cancellative. Assume $gh = gh'$. As F is proper, there exists x in X such that both $x \bullet gh$ and $x \bullet gh'$ are defined. Let $z = x \bullet gh$. By (XI.1.3), $x \bullet g$ must be defined and, putting $y = x \bullet g$, we have $z = y \bullet h = y \bullet h'$. Then, in $\mathcal{C}(M, F)$, we have $x \xrightarrow{g} y \cdot y \xrightarrow{h} z = x \xrightarrow{g} y \cdot y \xrightarrow{h'} z'$. As $\mathcal{C}(M, F)$ is left-cancellative, we deduce $y \xrightarrow{h} z = y \xrightarrow{h'} z'$, whence $h = h'$. Hence M is left-cancellative. □

Lemma XI.1.9.— *Assume that F is a partial action of a monoid M on a set X.*

(i) *Assume that $x \bullet g$ is defined. Then $y \xrightarrow{h} \text{-} \preccurlyeq x \xrightarrow{g} \text{-}$ holds in $\mathcal{C}(M, F)$ if and only if we have $y = x$ and $h \preccurlyeq g$ in M.*

(ii) *Assume that $x \bullet f$ and $x \bullet g$ are defined. Then $x \xrightarrow{h} \text{-}$ is a left-gcd of $x \xrightarrow{f} \text{-}$ and $x \xrightarrow{g} \text{-}$ in $\mathcal{C}(M, F)$ if and only if h is a left-gcd of f and g in M.*

Proof. (i) Assume $y \xrightarrow{h} y \bullet h \cdot x' \xrightarrow{g'} \text{-} = (x, g, \text{-})$. Then we have $y = x$ and $hg' = g$, hence $h \preccurlyeq g$. Conversely, assume $h \preccurlyeq g$, say $hg' = g$. By (XI.1.3), the assumption that $x \bullet g$ is defined implies that $x \bullet h$ is defined, say $x \bullet h = x'$. Then we have $x \xrightarrow{h} x' \cdot x' \xrightarrow{g'} \text{-} = x \xrightarrow{g} \text{-}$, whence $x \xrightarrow{h} \text{-} \preccurlyeq x \xrightarrow{g} \text{-}$ in $\mathcal{C}(M, F)$.

(ii) Assume that $x \xrightarrow{h} \text{-}$ is a left-gcd of $x \xrightarrow{f} \text{-}$ and $x \xrightarrow{g} \text{-}$ in $\mathcal{C}(M, F)$. By (i), h left-divides f and g in M. Let h' be a common left-divisor of f and g in M. By (XI.1.3), the assumption that $x \bullet f$ is defined implies that $x \bullet h'$ is defined. Then $x \xrightarrow{h} \text{-}$ is a common left-divisor of $x \xrightarrow{f} \text{-}$ and $x \xrightarrow{g} \text{-}$ in $\mathcal{C}(M, F)$, hence it is a left-divisor of $x \xrightarrow{h} \text{-}$. By (i), this implies $h' \preccurlyeq h$. So h is a left-gcd of f and g.

Conversely, assume that h is a left-gcd of f and g. By (i), $x \xrightarrow{h} \text{-}$ left-divides $x \xrightarrow{f} \text{-}$ and $x \xrightarrow{g} \text{-}$ in $\mathcal{C}(M, F)$. Now, consider an arbitrary common left-divisor of $\mathtt{NF}_{(}x, f, \text{-})$ and $x \xrightarrow{g} \text{-}$. By (i), it must be of the form $x \xrightarrow{h'} \text{-}$ with h' left-dividing f and g. Then h left-divides h in M, and $x \xrightarrow{h'} \text{-}$ left-divides $x \xrightarrow{h} \text{-}$ in $\mathcal{C}(M, F)$. □

Lemma XI.4.16.— *Assume that M is a left-cancellative monoid, F is a proper partial action of M on some set X, and $(\Delta_x)_{x\in X}$ is a right-Garside map on M with respect to F. Assume moreover that S is a family of atoms that generate M. Now assume that $\pi : M \to \underline{M}$ is a surjective homomorphism, $\pi_\bullet : X \to \underline{X}$ is a surjective map, and, for all x in X and g in M,*

(XI.4.17) *The value of $\pi_\bullet(x \bullet g)$ only depends on $\pi_\bullet(x)$ and $\pi(g)$;*

(XI.4.18) *The value of $\pi(\Delta_x)$ only depends on $\pi_\bullet(x)$.*

Assume finally that $\widetilde{\pi} : \underline{S} \to S$ is a section of π such that, for x in X, $\underline{s}$ in $\underline{S}$, and $\underline{w}$ in $\underline{S}^*$,

(XI.4.19) *If $\pi_\bullet(x) \bullet \underline{s}$ is defined, then so is $x \bullet \widetilde{\pi}(\underline{s})$.*

(XI.4.20) *The relation $[\underline{w}] \preccurlyeq \Delta_{\pi_\bullet(x)}$ implies $[\widetilde{\pi}^*(\underline{w})] \preccurlyeq \Delta_x$.*

Define a partial action $\underline{F}$ of $\underline{M}$ on $\underline{X}$ by $\pi_\bullet(x) \bullet \pi(g) = \pi_\bullet(x \bullet g)$, and, for $\underline{x}$ in $\underline{X}$, let $\Delta_{\underline{x}}$ be the common value of $\pi(\Delta_x)$ for x satisfying $\pi_\bullet(x) = \underline{x}$. Then $\underline{F}$ is a proper partial action of $\underline{M}$ on $\underline{X}$ and $(\Delta_{\underline{x}})_{\underline{x}\in\underline{X}}$ is a right-Garside sequence on $\underline{M}$.

Proof. First, (XI.4.17) and (XI.4.18) guarantee that the definitions of $\underline{x} \bullet \underline{g}$ and $\Delta_{\underline{x}}$ are unambiguous. Next $\underline{F}$ is a partial action of $\underline{M}$ on $\underline{X}$. Indeed, if $\underline{g}$ and $\underline{h}$ lie in $\underline{M}$, and if g, h satisfy $\pi(g) = \underline{g}$ and $\pi(h) = \underline{h}$, then we have $\pi(gh) = \underline{g}\underline{h}$ and, for $\underline{x}$ in $\underline{X}$ satisfying $\underline{x} = \pi_\bullet(x)$, we can write

$$(\underline{x} \bullet \underline{g}) \bullet \underline{h} = \pi_\bullet(x \bullet g) \bullet \pi(h) = \pi_\bullet((x \bullet g) \bullet h) = \pi_\bullet(x \bullet gh) = \underline{x} \bullet \underline{g}\underline{h},$$

equality meaning as usual that the involved expressions are simultaneously defined and, in this case, they have the same value.

The partial action $\underline{F}$ is proper. Indeed, assume that $\underline{g}_1, ..., \underline{g}_m$ are elements of $\underline{M}$. As π is surjective, there exists for every i an element g_i of M that satisfies $\pi(g_i) = \underline{g}_i$. As F is proper, there exists x in X such that $x \bullet g_i$ is defined for each i. Then $\pi_\bullet(x) \bullet \underline{g}_i$ is defined as well.

We now check that $(\Delta_{\underline{x}})_{\underline{x}\in\underline{X}}$ is a right-Garside sequence on $\underline{M}$. First, let $\underline{x}$ belong to $\underline{X}$. There exists x in X satisfying $\pi_\bullet(x) = \underline{x}$. By assumption, $x \bullet \Delta_x$ is defined, hence $\underline{x} \bullet \pi(\Delta_x)$ is defined as well, and it is $\pi_\bullet(x \bullet \Delta_x)$. So the sequence $(\Delta_{\underline{x}})_{\underline{x}\in\underline{X}}$ satisfies (XI.1.11).

Next, the assumption that M is generated by $\bigcup_{x\in X} \mathrm{Div}(\Delta_x)$ implies that $\underline{M}$ is generated by $\bigcup_{x\in X} \pi(\mathrm{Div}(\Delta_x))$, which is $\bigcup_{\underline{x}\in\underline{X}} \mathrm{Div}(\Delta_{\underline{x}})$. So, the sequence $(\Delta_{\underline{x}})_{\underline{x}\in\underline{X}}$ satisfies (XI.1.12).

Now, assume $\underline{s} \in \underline{S}$ and $\underline{s} \preccurlyeq \Delta_{\underline{x}}$. Let s satisfy $\pi(g) = \underline{g}$ and x satisfy $\pi_\bullet(x) = \underline{x}$. By assumption, $\pi_\bullet(x) \bullet \underline{s}$ is defined, hence, by (XI.4.19), $x \bullet \widetilde{\pi}(\underline{s})$ is defined. As $\bigcup_x \mathrm{Div}(\Delta_x)$ generates M, the element $\widetilde{\pi}(\underline{s})$ is left-divisible by some non-invertible element s that lies in $\mathrm{Div}(\Delta_y)$ for some y. As S consists of atoms, every element s of S belongs to $\bigcup_y \mathrm{Div}(\Delta_y)$ and, therefore, by (XI.1.15), $s \preccurlyeq \Delta_x$ must hold for every x such that $x \bullet s$ is defined. Applying this to $\widetilde{\pi}(\underline{s})$, we deduce $\widetilde{\pi}(\underline{s}) \preccurlyeq \Delta_x$, whence, by (XI.1.12), $\Delta_x \preccurlyeq \widetilde{\pi}(\underline{s})\Delta_{x\bullet\widetilde{\pi}(\underline{s})}$. Applying π, we deduce $\Delta_{\underline{x}} \preccurlyeq \underline{s}\Delta_{\underline{x}\bullet\underline{s}}$. Then an easy induction on the length of $\underline{g}$ shows that $\underline{g} \preccurlyeq \Delta_{\underline{x}}$ implies $\Delta_{\underline{x}} \preccurlyeq \underline{g}\Delta_{\underline{x}\bullet\underline{g}}$ for every $\underline{g}$ in $\underline{M}$. So, the sequence $(\Delta_{\underline{x}})_{\underline{x}\in\underline{X}}$ satisfies (XI.1.13).

As of (XI.1.14), it is automatically satisfied by the sequence $(\Delta_{\underline{x}})_{\underline{x}\in\underline{X}}$ since any two elements of $\underline{M}$ are supposed to admit a left-gcd.

Finally, assume that $\underline{g}$ is an element of $\underline{M}$ satisfying $\underline{g} \preccurlyeq \Delta_{\underline{x}}$ and $\underline{y} \bullet \underline{g}$ is defined. Let x, y in X satisfy $\pi_\bullet(x) = \underline{x}$, $\pi_\bullet(y) = \underline{y}$. Let $\underline{w}$ be an $\underline{S}$-word representing $\underline{g}$ in $\underline{M}$. By construction, $[\underline{w}]$ left-divides $\Delta_{\pi_\bullet(x)}$ hence, by (XI.4.20), $[\widetilde{\pi}^*(\underline{w})]$ left-divides Δ_x. By (XI.4.19) and an induction on the length of $\underline{w}$, the assumption that $\underline{y} \bullet [\underline{w}]$ is defined implies that $y \bullet [\widetilde{\pi}^*(\underline{w})]$ is defined. As the sequence $(\Delta_x)_{x \in X}$ satisfies (XI.1.15), we deduce that $[\widetilde{\pi}^*(\underline{w})]$ left-divides Δ_y. Applying π, we conclude that $[\underline{w}]$, that is, $\underline{g}$, left-divides $\Delta_{\underline{y}}$. Hence, the sequence $(\Delta_{\underline{x}})_{\underline{x} \in \underline{X}}$ satisfies (XI.1.15) and, therefore, it is a right-Garside sequence in $\underline{M}$. $\square$

Solution to selected exercises

Exercice 104 (skeleton).— *Say that a set of addresses is an* antichain *if it does not contain two addresses, one is a prefix of the other; an antichain is called* maximal *if it is properly included in no antichain.* (i) *Show that a finite maximal antichain is a family $\{\alpha_1, ..., \alpha_n\}$ such that every long enough address admits as a prefix exactly one of the addresses α_i.* (ii) *Show that, for every Σ-word w, there exists a unique finite maximal antichain A_w such that $T \bullet w$ is defined if and only if the skeleton of T includes A_w.*

Solution. (ii) Use induction on the length of w. For w of length one, say $w = \Sigma_\alpha$, the result is true with $T_w^- = T_w^+ = x_1 = \{\alpha 10\}$. Assume now $w = \Sigma_\alpha w'$. It follows from the definition of the action of Σ_α that, for every address γ, there exists a well defined address γ' such that, for every term T such that $T \bullet \Sigma_\alpha$ is defined, the skeleton of $T \bullet \Sigma_\alpha$ contains γ if and only if the skeleton of T contains γ'. Put $\gamma' = \Sigma_\alpha^{-1}(\gamma)$. Then we claim that the result holds with $A_w = \{\alpha 10\} \cup \Sigma_\alpha^{-1}(A_{w'})$. Indeed, $T \bullet w$ is defined if and only if $T \bullet \Sigma_\alpha$ and $(T \bullet \Sigma_\alpha) \bullet w'$ are defined, hence, by induction hypothesis, if and only if the skeleton of T contains $\alpha 10$ and the skeleton of $T \bullet \Sigma_\alpha$ includes $A_{w'}$.

Exercice 105 (preservation).— *Assume that M is a left-cancellative monoid and F is a partial action of M on a set X.* (i) *Show that, if M admits right lcms (resp. conditional right-lcms), then so does the category $\mathcal{C}_F(M,X)$.* (ii) *Show that, if M is right-Noetherian, then so is $\mathcal{C}_F(M,X)$.*

Solution. By definition, $(x, f, y) \preccurlyeq (x', f', y')$ in $\mathcal{C}_F(M,X)$ implies $x' = x$ and $f \preccurlyeq f'$ in M. So the assumption that M is right-Noetherian implies that $\mathcal{C}_F(M,X)$ is right-Noetherian as well. Assume that (x, f, y) and (x, g, z) admit a common right-multiple in $\mathcal{C}_F(M,X)$, say $(x, f, y)(y, g', x') = (x, g, z)(z, f', x')$. Then $fg' = gf'$ holds in M. As M is a right-lcm monoid, f and g admit a right-lcm h, and we have $f \preccurlyeq c$, $g \preccurlyeq h$, and $h \preccurlyeq fg'$. By assumption, $x \bullet fg'$ is defined, hence so is $x \bullet h$, and it is obvious to check that $(x, h, x \bullet h)$ is a right-lcm of (x, f, y) and (x, g, z) in $\mathcal{C}_F(M,X)$.

Exercice 107 (Noetherianity).— *Assume that F is a proper partial action of a monoid M on a set X and there exists a map $\mu : X \to \mathbb{N}$ such that $\mu(x \bullet g) > \mu(x)$ holds whenever g is not invertible. Show that M is Noetherian and every element of M has a finite height.*

Solution. Let g be a non-invertible element of M, and let $g_1 | \cdots | g_p$ be any decomposition of g. At the expense of possibly gathering entries, we may assume that each element g_i is non-invertible. As F is proper, there exists x in X such that

$x \bullet g$ is defined. Then, by (XI.1.3), $x \bullet g_1 \cdots g_i$ is defined for every i. We then obtain $\mu(x) < \mu(x \bullet g_1) < \mu(x \bullet g_1 g_2) < \cdots < \mu(x \bullet g)$, whence $p \leqslant \mu(x \bullet g) - \mu(x)$. So g has a finite height bounded above by $\mu(x \bullet g) - \mu(x)$. In particular, by Proposition II.2.47 (height), M_{LD} is Noetherian.

Exercice 109 (**common multiple**).— *Assume that M is a left-cancellative monoid, F is a partial action of M on X, and $(\Delta_x)_{x \in X}$ is a right-Garside sequence in M with respect to F. Show that, for every x in X, any two elements of* $\mathrm{Def}(x)$ *admit a common right-multiple that lies in* $\mathrm{Def}(x)$.

Solution. Let f, g belong to $\mathrm{Def}(x)$. Then $(x, f, x \bullet f)$ and $(x, g, x \bullet g)$ are two elements of $\mathcal{C}_F(M,X)$ sharing the same source. By Proposition XI.3.25, $\mathcal{C}_F(M,X)$ possesses a right-Garside map, hence any two elements of $\mathcal{C}_F(M,X)$ with the same source admit a common right-multiple. So do in particular $(x, f, x \bullet f)$ and $(x, g, x \bullet g)$. A common right-multiple must be of the form $(x, h, x \bullet h)$ where h is a common right-multiple of f and g in M. Then h lies in $\mathrm{Def}(x)$.

Chapter XII: Ordered groups

Skipped proofs

(none)

Solution to selected exercises

Exercice 111 (**braid ordering**).— *Show that* $1 <_{\mathrm{D}} \sigma_1\sigma_2 \leqslant_{\mathrm{D}} (\sigma_1\sigma_2\sigma_1)^{2p}\sigma_2^q$ *holds in B_3 for $p > 0$.*

Solution. Put $g = (\sigma_1\sigma_2\sigma_1)^{2p}\sigma_2^q$ and $g' = \sigma_1\sigma_2$. We find $g'^{-1}g = \sigma_1(\sigma_1\sigma_2\sigma_1)^{2p-1}\sigma_2^q$. Hence $g' <_{\mathrm{D}} g$ is true.

Exercice 112 (**limit of conjugates**).— *Assuming that $<_{\mathrm{D}}$ is a limit of its conjugates in B_3, show the same result in B_n. [Hint: Use the subgroup of B_n generated by σ_{n-2} and σ_{n-1}, which is isomorphic to B_3.]*

Solution. Let P_n be the positive cone of $<_{\mathrm{D}}$ on B_n, and let H be the subgroup of B_n generated by σ_{n-2} and σ_{n-1}. Then H is isomorphic to B_3 and the $<_{\mathrm{D}}$-ordering of B_n restricted to H corresponds with the $<_{\mathrm{D}}$-ordering of B_3. Moreover, the positive cone for the $<_{\mathrm{D}}$-ordering of H is $P_n \cap H$. Let S be a finite subset of P_n. By assumption, there exists f in H such that $f^{-1}hf$ is lies in P_n for every h in $S \cap H$, and $f(P_n \cap H)f^{-1}$ is distinct from $P_n \cap H$. Assume now $h \in S \setminus H$. By assumption, h is σ_i-positive for some $i < n-2$. Its conjugate $f^{-1}hf$ is σ_i-positive as well, hence it lies in P_n. We deduce $f^{-1}Sf \subseteq P_n$, hence $S \subseteq fP_nf^{-1}$. Finally, fP_nf^{-1} and P_n are distinct, because their intersections with H are distinct.

Exercice 113 (**closure of conjugates**).— *Let P_n be the positive cone of the ordering $<_{\mathrm{D}}$ on B_n considered in Example XII.1.23. Show that the closure of the conjugates of P_n in* $\mathrm{LO}(B_n)$ *is a Cantor set.*

Solution. Let Z_n be the family of all conjugates of P_n. We saw in Example XII.1.23 that P_n is a limit of its conjugates, hence it is a limit point in Z_n. Hence so is every conjugate of P_n, that is, every point of Z_n is a limit point in Z_n. By continuity, every point in the closure of Z_n is a limit point. Next, the closure of Z_n is a closed subspace in a totally disconnected nonempty compact metric space, hence it is itself totally a disconnected, compact, and metric space. As it is nonempty and every point is a limit point, the closure of Z_n is homeomorphic to the Cantor set.

Exercice 114 (space $\mathrm{LO}(B_\infty)$).— *Show that every point in the space $\mathrm{LO}(B_\infty)$ is a limit of its conjugates and that $\mathrm{LO}(B_\infty)$ is homeomorphic to the Cantor set (contrary to the spaces $\mathrm{LO}(B_n)$ for finite n).*

Solution. Consider an arbitrary positive cone P for a left-invariant ordering of B_∞ and suppose S is a finite subset of P. We will show there is a positive cone $\sigma_i P \sigma_i^{-1}$ in B_∞ that also includes S and is distinct from P. Choose n such that S is included in Bn. Then, for each $i > n$, every braid in S commutes with σ_i, so we have $S = \sigma_i S \sigma_i^{-1} \subseteq \sigma_i P \sigma_i^{-1}$. On the other hand, there exists $i > n$ such that the sets P and $\sigma_i P \sigma_i^{-1}$ are different. For otherwise, using sh for the shift endomorphism that maps σ_i to σ_{i+1} for every i, consider the subgroup $\mathrm{sh}^n(B_\infty)$, which is isomorphic to B_∞. The sets $P \cap \mathrm{sh}^n(B_\infty)$ and $(\sigma_i P \sigma_i^{-1}) \cap \mathrm{sh}^n(B_\infty)$ are positive cones for orderings of $\mathrm{sh}^n(B_\infty)$. If $P = \sigma_i P \sigma_i^{-1}$ is true for each $i > n$, then the cone $P \cap \mathrm{sh}^n(B_\infty)$ of $\mathrm{sh}^n(B_\infty)$ is invariant under conjugation by all elements of $\mathrm{sh}^n(B_\infty)$. This would imply that $\mathrm{sh}^n(B_\infty)$ and, therefore, $B\infty$ are bi-orderable, which is not true.

Exercice 116 (non-terminating reversing).— *Assume that (S, R) is a triangular presentation.* (i) *Show that, if a relation of $\widehat{R}$ has the form $s = w$ with $\lg(w) > 1$ and w finishing with s, then the monoid $\langle S \mid R\rangle^+$ is not of right-O-type.* (ii) *Let $(S, \widehat{R})$ be the maximal right-triangular deduced from (S, R). Show that, if a relation of $\widehat{R}$ has the form $s = w$ with w beginning with $(uv)^r us$ with $r \geqslant 1$, u nonempty, and v such that $v^{-1}s$ reverses to a word beginning with s, hence in particular if v is empty or it can be decomposed as $u_1, ..., u_m$ where $u_k s$ is a prefix of w for every k, then $s^{-1}us$ cannot be terminating, and deduce that$\langle S \mid R\rangle^+$ is not of right-O-type.* (iii) *Show that a relation* $\mathtt{a} = \mathtt{babab}^3\mathtt{a}^2...$ *is impossible in a right-triangular presentation for a monoid of right-O-type.*

Solution. (i) If R contains a relation $s = us$ with u nonempty, $s = [u]^+ s$ holds in $\langle S \mid R\rangle^+$, whereas $1 = [u]^+$ fails. So $\langle S \mid R\rangle^+$ is not right-cancellative. (ii) Write the involved relation $s = (uv)^r usw_1$ with $v^{-1}s \curvearrowright_{\widehat{R}} sw_2$. We find

$$\begin{aligned} s^{-1}us \curvearrowright_{\widehat{R}}\ & w_1^{-1}s^{-1}(vu)^{-(r-1)}u^{-1}v^{-1}s \\ \curvearrowright_{\widehat{R}}\ & w_1^{-1}s^{-1}(vu)^{-(r-1)}u^{-1}sw_2 \curvearrowright_{\widehat{R}} \\ & w_1^{-1}s^{-1}(vu)^{-(r-1)}(vu)^{r-1}vusw_1w_2 \\ \curvearrowright_{\widehat{R}}\ & w_1^{-1}s^{-1}vusw_1w_2 \curvearrowright_{\widehat{R}} w_1^{-1}w_2^{-1} \cdot s^{-1}us \cdot w_1w_2. \end{aligned}$$

We deduce that $s^{-1}us \curvearrowright_{\widehat{R}} (w_1^{-1}w_2^{-1})^n \cdot s^{-1}us \cdot (w_1w_2)^n$ holds for every n and, therefore, it is impossible that $s^{-1}us$ leads in finitely many steps to a positive–negative word. Proposition II.4.51 (completeness), right-reversing is complete for

$(S, \widehat{R})$, the elements s and $[u]^+s$ admit no common right-multiple in $\langle S \mid R\rangle^+$, and the latter cannot be of right-O-type.

(iii) The right-hand side of the relation can be written as $(\mathtt{ba})\mathtt{bab}^2(\mathtt{ba})\mathtt{a}...$, eligible for(ii) with $u = \mathtt{ba}$ and $v = \mathtt{bab}\cdot\mathtt{b}$, a product of two words u_1, u_2 such that $u_i\mathtt{a}$ is a prefix of the right-hand term of the relation.

Exercice 117 (**roots of Garside element**).— *Assume that M is a left-cancellative monoid generated by a set S* (i) *Show that, for δ, g in a left-cancellative monoid M, a necessary and sufficient condition for δ to right-dominate g is that there exist $m \geqslant 1$ satisfying $(*)\forall k{\geqslant}0\ (g\delta^{km+m-1} \preccurlyeq \delta^{km+1})$.* (ii) *Assume that δ^m is a right-Garside element in M. Show that δ right-dominates every element g that satisfies $g\delta^{m-1} \preccurlyeq \delta$.*

Solution. (i) If δ right-dominates g, then, by definition, $(*)$ holds with $m = 1$. Conversely, assume $(*)$. Let n be a nonnegative integer. Let k be maximal with $km \leqslant n$. Then we have $n \leqslant km+m-1$, and $(*)$ implies $g\delta^n \preccurlyeq g\delta^{km+m-1} \preccurlyeq \delta^{km+1} \preccurlyeq \delta^{n+1}$, so δ right-dominates s.

(ii) Put $\Delta = \delta^m$, and let ϕ be the (necessarily unique) endomorphism of M witnessing that Δ is right-quasi-central. First, we have $\delta\Delta = \delta^{m+1} = \Delta\phi(\delta)$, whence $\phi(\delta) = \delta$ since M is left-cancellative. Next, we claim that $g \preccurlyeq h$ implies $\phi(g) \preccurlyeq \phi(h)$. Indeed, by definition, $g \preccurlyeq h$ implies the existence of h' satisfying $gh' = h$, whence $\phi(g)\phi(h') = \phi(h)$ since ϕ is an endomorphism. This shows that $\phi(g) \preccurlyeq \phi(h)$ is satisfied. So, in particular, and owing to the above equality, $g \preccurlyeq \delta$ implies $\phi(g) \preccurlyeq \delta$. Now assume $g\delta^{m-1} \preccurlyeq \delta$. Then, for every k, we find

$$g\delta^{km+m-1} = g\delta^{m-1}\Delta^k = \Delta^k\phi^k(g\delta^{m-1}) \preccurlyeq \Delta^k\phi^k(\delta) = \Delta^k\delta = \delta^{km+1},$$

and we conclude that δ right-dominates g by (i).

Exercice 119 (**right-ceiling**).— *Assume that M is a cancellative monoid of right-O-type, and that $s_\ell \cdots s_1$ is a right-top S-word in M such that $[s_\ell \cdots s_1]^+$ is central in M. Show that $s_i = s_1$ must hold for every i, and deduce that ${}^\infty s_1$ is the right-S-ceiling in M.*

Solution. Let $\Delta = s_\ell \cdots s_1$. First, we have $g \preccurlyeq \Delta$ for every g in S^ℓ, so that Δ right-dominates S^ℓ. Hence, by Lemma XII.3.9, the right-S-ceiling is periodic with period $s_\ell \cdots s_1$. Now consider its length $\ell + 1$ final fragment $s_1 s_\ell \cdots s_1$. Then, in M, we have $s_1 s_\ell \cdots s_1 = s_\ell \cdots s_1 s_1$, so $s_1 s_\ell \cdots s_1$ and $s_\ell \cdots s_1 s_1$ are two right-top S-words of length $\ell + 1$. By uniqueness of the right-S-ceiling, these words must coincide, which is possible only for $s_1 = ... = s_\ell$.

Exercice 123 (**no triangular presentation**]).— *Assume that M is a monoid of right-O-type that is generated by $\mathtt{a}, \mathtt{b}, \mathtt{c}$ with $\mathtt{a} \succ \mathtt{b} \succ \mathtt{c}$ and $\mathtt{b}, \mathtt{c}$ satisfying some relation $\mathtt{b} = \mathtt{c}v$ with no $\mathtt{a}$ in v.* (i) *Prove that, unless M is generated by $\mathtt{b}$ and $\mathtt{c}$, there is no way to complete $\mathtt{b} = \mathtt{c}v$ with a relation $\mathtt{a} = \mathtt{b}u$ so as to obtain a presentation of M.* (ii) *Deduce that no right-triangular presentation made of $\mathtt{b} = \mathtt{cbc}$ (Klein bottle relation) or $\mathtt{b} = \mathtt{cb}^2\mathtt{c}$ (Dubrovina–Dubrovin braid relation) plus a relation of the form $\mathtt{a} = \mathtt{b}...$ may define a monoid of right-O-type.*

Solution. For a contradiction, assume that $(\mathtt{a}, \mathtt{b}, \mathtt{c}; \mathtt{a} = \mathtt{b}u, \mathtt{b} = \mathtt{c}v)$ is a presentation of M. If there is no $\mathtt{a}$ in u, the assumption that $\mathtt{a} = \mathtt{b}u$ is valid in M implies that $\mathtt{a}$ belongs to the submonoid generated by $\mathtt{b}$ and $\mathtt{c}$, so M must be generated by $\mathtt{b}$ and $\mathtt{c}$.Assume now that there is at least one $\mathtt{a}$ in u. As $\mathtt{a}$ does not occur in $\mathtt{b} = \mathtt{c}v$, a word containing $\mathtt{a}$ cannot be equivalent to a word not containing $\mathtt{a}$. This implies that $\mathtt{a}$ is preponderant in $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$. Indeed, assume that g, h belong to the submonoid of M generated by $\mathtt{b}$ and $\mathtt{c}$. By the above remark, $h\mathtt{a}g' = g$ is impossible, hence so is $h\mathtt{a} \preccurlyeq g$. As, by assumption, M is of right-O-type, we deduce $g \preccurlyeq h\mathtt{a}$. Then Proposition XII.3.14 gives the result.

Exercice 124 (**Birman–Ko–Lee generators**).— *Put $b_{i,j} = a_{i,j}^{(-1)^{i+1}}$ in the braid group B_n. Show that, for every n, the monoid $B_n^{\oplus}$ is generated by the elements $b_{i,j}$.*

Solution. By Lemma XII.3.13, an element of B_n belongs to $B_n^{\oplus}$ if and only if it is either σ_i-positive for some odd i or σ_i-negative for some even i. The braid relations imply $a_{i,j} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$ for $i < j$, hence $a_{i,j}$ is σ_i-positive, and $b_{i,j}$ is σ_i-positive for odd i and σ_i-negative for even i. Therefore, $b_{i,j}$ belongs to $B_n^{\oplus}$ for all i, j. Conversely, in B_n, we have

$$\sigma_i \cdots \sigma_{n-1} = (\sigma_i \cdots \sigma_{n-2} \sigma_{n-1} \sigma_{n-2}^{-1} \cdots \sigma_i^{-1}) \\ (\sigma_i \cdots \sigma_{n-3} \sigma_{n-2} \sigma_{n-3}^{-1} \cdots \sigma_i^{-1}) \cdots (\sigma_i \sigma_{i+1} \sigma_i^{-1})(\sigma_i),$$

whence $s_i = (\sigma_i \cdots \sigma_{n-1})^{(-1)^{i+1}} = b_{i,n} b_{i,n-1} \cdots b_{i,i+1}$ for odd i, and $= b_{i,i+1} \cdots b_{i,n-1} b_{i,n}$ for even i. Hence s_i belongs to the submonoid of B_n generated by the $b_{i,j}$'s and, finally, $B_n^{\oplus}$ coincides with the latter.

Chapter XIII: Set-theoretic solutions of YBE

Skipped proofs

(none)

Solution to selected exercises

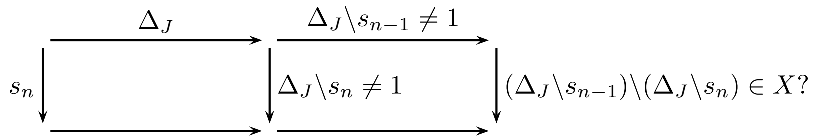
Exercice 125 (**bijective RC-quasigroup**).— *Assume that $(X, \star)$ is an RC-quasigroup. Let $\psi : X \to X$ and $\Psi : X \times X \to X \times X$ be defined by $\psi(a) = a \star a$ and $\Psi(a, b) = (a \star b, b \star a)$. Show that Ψ is injective (resp. bijective) if and only if ψ is.*

Solution. If Ψ is injective, then so is ψ since $\psi(a) = \psi(a')$ implies $\Psi(a, a) = \Psi(a', a')$. On the other hand, if Ψ is surjective, then so is ψ: for every c, there exists (a, b) satisfying $\Psi(a, b) = (c, c)$. As seen in the proof of Lemma XII.2.23(iii), this implies $a = b$, whence $\psi(a) = c$. For the converse, we first compute $(*)$ $\psi(b \star a) = (b \star a) \star (b \star a) = (a \star b) \star (a \star a) = (a \star b) \star \psi(a)$. Assume that ψ is injective and $\Psi(a, b) = \Psi(a', b') = (c, d)$ holds. By $(*)$, we have $c \star \psi(a) = \psi(d) = c \star \psi(a')$. As the left-translation by c and ψ are injective, we deduce $\psi(a) = \psi(a')$, whence $a = a'$, and a symmetric argument gives $b = b'$, so Ψ is injective. Finally, assume that ψ is bijective and c, d belong to X. As the left-translation by c and ψ are surjective, we can find a satisfying $c \star \psi(a) = \psi(d)$, whence, by $(*)$, $\psi(b \star a) = \psi(d)$ and, similarly, we can find b satisfying $\psi(a \star b) = \psi(c)$. As ψ is injective, we deduce $a \star b = c$ and $b \star a = d$, whence $\Psi(a, b) = (c, d)$. So Ψ is surjective, hence bijective.

Exercice 126 (right-complement).— *Assume that $(X, \star)$ is an RC-quasigroup and M is the associated structure monoid.* (i) *Show that, for every element f in $M \cap X^p$, the function from X to $X \cup \{1\}$ that maps t to $f\backslash t$ takes pairwise distinct values in X plus at most p times the value 1.* (ii) *Deduce that, for I a finite subset of X with cardinal n, the right-lcm Δ_I of I lies in X^n.*

Solution. (i) We use induction on p. For $p = 0$, that is, for $f = 1$, we have $f\backslash t = t$, which takes pairwise distinct values in X. For $p = 1$, that is, for f in X, we have $f\backslash f = 1$ and $f\backslash t = f \star t$ for $t \neq f$. As $t \mapsto f \star f$ injective, the expected result is true. Assume $p \geqslant 2$ and write $f = gs$ with g in X^{p-1}. Then, by the formula for an iterated right-complement, we have $f\backslash t = s\backslash(g\backslash t)$, whence $f\backslash t = 1$ if $g\backslash t$ lies in $\{1, s\}$ and $f\backslash t = s \star (g\backslash t)$ otherwise. Then the result follows from the induction hypothesis.

(ii) Use induction on $n \geqslant 0$. For $n = 0$, we have $\Delta_I = 1$. For $n = 1$, say $I = \{s\}$ with s in X, we have $\Delta_I = s$, an element of X. For $n = 2$, say $I = \{s, t\}$ with $s \neq t$, we have $\Delta_I = s(s \star t)$, so the induction starts. Assume $n \geqslant 3$ and $I = \{s_1, ..., s_n\}$. Let $J = \{s_1, ..., s_{n-2}\}$. By induction hypothesis, Δ_J belongs to X^{n-2}, and $\Delta_{J\cup\{s_{n-1}\}}$, which is $\Delta_J(\Delta_J\backslash s_{n-1})$, belongs to X^{n-1}. This implies $\Delta_J\backslash s_{n-1} \neq 1$, so $\Delta_J\backslash s_{n-1}$ must lie in X. For the same reason, $\Delta_J\backslash s_n$ lies in X. Moreover, as s_{n-1} and s_n are distinct, we have $\Delta_J\backslash s_{n-1} \neq \Delta_J\backslash s_n$ by (i). Now, as I is $J \cup \{s_{n-1}\} \cup \{s_n\}$, Proposition II.2.12 (iterated lcm) gives $\Delta_I = \Delta_{J\cup\{s_{n-1}\}} \cdot (\Delta_J\backslash s_{n-1})\backslash(\Delta_J\backslash s_n)$ with a commutative diagram as below, and the question is to know whether the last term may be 1.



Now we saw above that $\Delta_J\backslash s_{n-1}$ and $\Delta_J\backslash s_n$ are distinct elements of X, so the element $(\Delta_J\backslash s_{n-1})\backslash(\Delta_J\backslash s_n)$ is equal to $(\Delta_J\backslash s_{n-1}) \star (\Delta_J\backslash s_n)$, an element of X. Hence Δ_I belongs to X^n, as expected.

Exercice 128 (*I*-structure).— *Assume that $(X, \star)$ is a bijective RC-quasigroup.* (i) *Show (by a direct argument) that the map $\underline{\star}$ from $X^* \times X$ to X defined by $1\underline{\star}t = t$, $s\underline{\star}t = s\star t$ for s in X, and $(u|v)\underline{\star}t = v\underline{\star}(u\underline{\star}t)$ induces a well-defined map from $M \times X$ to X.* (ii) *Show that the map ν from X^* to M defined by $\nu(1) = 1$, $\nu(s) = s$, and $\nu(ws) = \nu(w) \cdot (\nu(w)\underline{\star}s)$ for s in X induces a well-defined map from $\mathbb{N}^{(X)}$ to M.*

Solution. (i) Owing to the presentation of M, it suffices to check that the translations associated with $r|(r \star s)$ and $s|(s \star r)$ coincide. Now, using the RC-law, we find $(r|(r \star s))\underline{\star}t = (r \star s) \star (r \star t) = (s \star r) \star (s \star t) = (s|(s \star t))\underline{\star}t$.

(ii) As $\mathbb{N}^{(X)}$ is the quotient of X^* by the equivalence relation generated by all pairs $(u|s|t|v, u|t|s|v)$ with s, t in X, it suffices to check that the images of such words coincide. Now we find

$$\begin{aligned}\nu(u|s|t) &= \nu(u|s) \cdot (\nu(u|s)\underline{\star}t) = \nu(u) \cdot (\nu(u)\underline{\star}s) \cdot (\nu(u)|(\nu(u)\underline{\star}s)\underline{\star}t)\\ &= \nu(u) \cdot (\nu(u)\underline{\star}s) \cdot ((\nu(u)\underline{\star}s)\underline{\star}(\nu(u)\underline{\star}t))\\ &= \nu(u) \cdot (\nu(u)\underline{\star}s) \cdot ((\nu(u)\underline{\star}s) \star (\nu(u)\underline{\star}t))\end{aligned}$$

and $\nu(u|t|s) = \nu(u) \cdot (\nu(u)\underline{\star}t) \cdot ((\nu(u)\underline{\star}t) \star (\nu(u)\underline{\star}s))$, whence the equality. Then multiplying by v on the right gives the same result.

Exercice 129 (**parabolic submonoid**).— (i) *Assume that* (X, ρ) *is a finite involutive nondegenerate set-theoretic solution of YBE and* M *is the associated structure monoid. Show that a submonoid* M_1 *of* M *is parabolic if and only if there exists a (unique) subset* I *of* X *satisfying* $\rho(I \times I) = I \times I$ *such that* M_1 *is the submonoid of* M *generated by* I. (ii) *Assume that* $(X, \star)$ *is a finite RC-system and* M *is the associated structure monoid. Then a submonoid* M_1 *of* M *is parabolic if and only if there exists a (unique) subset* I *of* X *such that* I *is closed under* $\star$ *and* M_1 *is the submonoid generated by* I. (iii) *Show that, if* $(X, \star)$ *is an infinite RC-quasigroup, there may exist subsets* I *of* X *that are closed under* $\star$ *but the induced RC-system* $(I, \star)$ *is not an RC-quasigroup.*

Solution. (i) Assume that M_1 is a parabolic submonoid of M. Set $I = M_1 \cap X$. Since X generates M and M_1 is closed under factors, M_1 is generated by I. Now, assume that a, b belong to I and $(c, d) = \rho(a, b)$ holds. Then ab belongs to M_1, and $ab = cd$ holds in M. As M_1 is closed under factor, we deduce that c and d lie in I, whence $\rho(I \times I) \subseteq I \times I$. As ρ is involutive, the latter inclusion must be an equality. Conversely assume I is a subset of X satisfying $\rho(I \times I) = I \times I$ and M_1 is the submonoid of M generated by I. The monoid is (right)-Noetherian, so, in order to prove that M_1 is a parabolic submonoid, and owing to Proposition VII.1.32 (parabolic subcategory), it is sufficient to establish that M_1 is closed under factor and right-comultiple, that is, in a context where right-lcms exist, that M_1 is closed under factor and right-lcm. Now, an obvious induction shows that every X-word that is equivalent to an I-word is itself an I-word, implying that M_1 is closed under factor. On the other hand, let ρ_I be the restriction of ρ to $I \times I$. The assumption that (X, ρ) is a set-theoretic solution of YBE implies that so is (I, ρ_I), and the assumption that (X, ρ) is involutive implies that so is (I, ρ_I). Finally, the assumption that (X, ρ) is nondegenerate implies that so is (I, ρ_I): with the notation of Definition XIII.1.8, the left-translations associated with ρ_1 and right-translations associated with ρ_2 are injective on X, hence so are their restrictions to I, and therefore, the latter are bijective since I is finite. Now, let a, b belong to I. As (I, ρ_I) is an involutive nondegenerate solution of YBE, there exist c, d in I satisfying $ac = bd$ in M, hence in M_1, and M_1 is closed under right-lcm in M. Hence M_1 is a parabolic submonoid of M.

(ii) Translate the result of (i) in terms of the operation $\star$.

(iii) Consider $X = \mathbb{Z}$ and $x \star y = y + 1$. Then the restriction of $\star$ to $\mathbb{N}$ does not give an RC-quasigroup.

Chapter XIV: More examples

Skipped proofs

(none)

Solution to selected exercises

(none)

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