A Normal Form for the Free Left Distributive Law.

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ABSTRACT. We construct a new normal form for one variable terms up to left distributivity. The proof that this normal form exists for every term is considerably simpler than the corresponding proof for the forms previously introduced by Richard Laver. In particular the determination of the present normal form can be made in a primitive recursive way.

Throughout the paper $\mathcal{W}$ denotes the set of all wellformed terms constructed using a single variable a and a single binary operator $\bullet$, *i.e.* the free algebra generated by a. Practically we shall use right Polish notation, thus writing $PQ\bullet$ for the product of P and Q. Now we denote by $=_{LD}$ the least congruence on $\mathcal{W}$ which forces the left distributivity identity

$$PQR\bullet\bullet =_{LD} PQ\bullet PR\bullet\bullet \qquad (LD)$$

The quotient $\mathcal{W}/=_{LD}$ is the free left distributive algebra (LD-algebra) generated by a.

The study of free LD-algebras has revealed interesting connections both with the set theory of large cardinals (see [10], [12], [4], [7], [8]) and with the topology of braids (see [5], [9], [6]). In particular the decidability of the relation $=_{LD}$, *i.e.* the word problem for the standard presentation of the free LD-algebra with one generator, proved to be a rather delicate question. It has been solved independently in [3] and [10] assuming some auxiliary assumption which Laver in [10] deduced from a very strong logical assumption. Subsequently this assumption was eliminated in [5], and the proof was completed within elementary arithmetic resulting in an exponential complexity for the relation $=_{LD}$.

The method of [10] for deciding $=_{LD}$-equivalence of terms consists in introducing a unique normal form. A very delicate inductive proof is used to establish that normal forms always exist. In particular the reduction of a given term to the normal form seems to have a high complexity and there is no evidence that it should be even a primitive recursive process. On the other hand the method of [3] and [5] is simpler but it directly compares terms with respect to $=_{LD}$ without refering to any normal form. Therefore it seems weaker in terms of applications.

The aim of this paper is to propose a new normal form using the ideas of [3]. Compared with the approach of [10], the present method uses an additional ingredient, namely a filtration connected with the notion of derivation defined in [2]. An integer degree is associated with this filtration, and it allows simple inductive proofs. The existence and uniqueness of the normal form then appears as nearly immediate, and the reduction of an arbitrary term to its normal form is a primitive recursive process. Moreover this normal form seems to be well fitted for applications. We therefore hope that it could be a useful tool in the quickly developing study of left distributive laws and their applications.

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1. The expanded form

If P, Q are in $\mathcal{W}$, the equivalence $P =_{LD} Q$ holds if and only if one can transform P into Q using a finite sequence of elementary transformations, each of which consists either in replacing some subterm $P_1P_2P_3\bullet\bullet$ by the corresponding subterm $P_1P_2\bullet P_1P_3\bullet\bullet$ or in replacing some subterm $P_1P_2\bullet P_1P_3\bullet\bullet$ by the corresponding subterm $P_1P_2P_3\bullet\bullet$. We shall write $P =_{LD}^{k} Q$ if at most k such elementary transformations are used. Also we say that Q is an *extension* (*resp.* a 1-*extension*) of P if no transformation of the second type is used (*resp.* if no transformation of the second type and at most 1 transformation of the first type is used). A key point is the existence of a canonical common extension ∂P for all 1-extensions of a given term P (the term ∂P thought as a 'smallest' common extension of all 1-extensions of P, allthough this need not be readily true). In order to describe the operation ∂, let us first introduce, for any pair of terms Q, R, the new term $\mathrm{dist}(Q, R)$ obtained from R by replacing each occurrence of a in R by $Qa\bullet$, *i.e.* by distributing Q everywhere in R. Thus the operation dist is defined inductively by the rules

$$\mathrm{dist}(Q, a) = Qa\bullet, \qquad \mathrm{dist}(Q, R'R''\bullet) = \mathrm{dist}(Q, R')\mathrm{dist}(Q, R'')\bullet.$$

Observe that $\operatorname{dist}(Q, R)$ is certainly always $=_{LD}$-equivalent to $QR\bullet$. The idea for constructing ∂P is to use an induction on the size of P, so that, if P is $P'P''\bullet$, the derivative of P is obtained by distributing everywhere (in the sense of dist) the derivative of P' in the derivative of P''.

Definition. For every term P, the *derived* term ∂P is constructed inductively by the rules $\partial a = a$ and $\partial(PP'\bullet) = \operatorname{dist}(\partial P, \partial P')$.

The basic properties of derivation are established in [2]. We shall use here the following ones.

Lemma 1. ([2]) *i) For each term P, the term ∂P is an extension of all 1-extensions of P.*

ii) If P' is an extension of P, then $\partial P'$ is an extension of ∂P.

Corollary 2. *For any terms P, Q, the term $\partial^k P$ is an extension of Q whenever $Q =_{LD}^k P$ holds.*

Proof. The property is obvious for $k = 0$. Assume $R =_{LD}^1 Q =_{LD}^k P$. Then ∂Q is an extension of R both if Q is a 1-extension of R (since ∂Q is an extension of Q) and if R is a 1-extension of Q (by Lemma 1.i). By induction hypothesis $\partial^k P$ is an extension of Q, and by Lemma 1.ii this implies that $\partial^{k+1} P$ is an extension of ∂Q, and therefore of R. ■

Since terms are words, there exists a natural notion of prefix: we say that the term Q is a prefix of P, and write $Q \sqsubseteq P$ if, for some word Z, the word P is equal to QZ. The main result about prefixes is the following

Proposition 3. ([5]) *If Q is a strict prefix of P, then Q is not $=_{LD}$-equivalent to P.*

It is convenient to introduce an following easy generalization of the notion of prefix of a term. By the wellknown properties of the Polish notation, not all word prefixes of a term P need be wellformed terms, but for every such word prefix X there exists a unique integer n such that $X\bullet^n$ is a wellformed term. The terms obtained this way from a term P will be called the *cuts* of P, which corresponds to the following

Definition. The *cuts* of a term are inductively defined as follows. The unique cut of a is a. The cuts of $PP'\bullet$ are the cuts of P and all terms of the form $PQ\bullet$ where Q is a cut of P'.

An equivalent definition is immediate induction shows that every prefix of a term is a cut of that term. Also observe that the relation 'being a cut of' is transitive. A more precise study of the cuts of a term will be made in Section 2. Presently we shall only use some very basic properties.

Lemma 4. *i) If P' is an extension of P, then for every cut Q of P some cut of P' is an extension of Q.*

ii) If Q is a cut of P, then ∂Q is a prefix of ∂P.

iii) The cuts of any term are pairwise $=_{LD}$-unequivalent.

Proof. For (i) we may assume that P' is a 1-extension of P, and then argue inductively on the length of P. Everything is obvious if P is a. Otherwise write P as $P_1P_2\bullet$ and P' as $P_1'P_2'\bullet$. If P_1' is a non trivial 1-extension of P_1, then P_2' must be equal to P_2. If Q is a cut of P_1, then by induction hypothesis some cut of P_1', and therefore of P', is an extension of Q. If Q is $P_1Q_2\bullet$ where Q_2 is a cut of P_2, then $P_1'Q_2\bullet$ is a cut of P' and a 1-extension of Q. So the result holds for P'. The proof is similar if P_1' is equal to P_1 and P_2' is a 1-extension of P_2. The remaining case is when P is $P_1P_2P_3\bullet\bullet$ and P' is $P_1P_2\bullet P_1P_3\bullet\bullet$. In this case the result is clear is Q is either a cut of P_1 or $P_1Q_2\bullet$ where Q_2 is a cut of P_2. Finally if Q is $P_1P_2Q_3\bullet\bullet$ where Q_3 is a cut of P_3, then $P_1P_2\bullet P_1Q_3\bullet\bullet$ is a 1-extension of Q and is a cut of P'.

For (ii) the property is obvious when P is a. If P is $P_1P_2\bullet$, an obvious induction shows that ∂P_1 is a prefix of ∂P. So if Q is a cut of P_1, ∂Q is by induction hypothesis a prefix of ∂P_1, and therefore of ∂P. Now if Q is $P_1Q_2\bullet$ where Q_2 is a cut of P_2, then by induction hypothesis ∂Q_2 is a prefix of ∂P_2, and this easily implies that $\mathrm{dist}(\partial P_1, \partial Q_2)$ is a prefix of $\mathrm{dist}(\partial P_1, \partial P_2)$, which is the desired result.

Finally if Q, Q' are distinct cuts of P, ∂Q and $\partial Q'$ are distinct prefixes of ∂P, and, by Proposition 3, they are $=_{LD}$-unequivalent. This gives the result since Q and Q' are $=_{LD}$-equivalent to ∂Q and $\partial Q'$ respectively. ■

We are ready to introduce the terms which will be used as unique representatives for the $=_{LD}$-classes.

Definition. For P in $\mathcal{W}$ and $k \geq 0$, P-$\overline{\mathbf{EF}}_k$ is the set of all cuts of $\partial^k P$, P-$\mathbf{EF}_0$ is P-$\overline{\mathbf{EF}}_0$ and, for $k \geq 1$, P-$\mathbf{EF}_k$ is the subset of P-$\overline{\mathbf{EF}}_k$ made by the cuts of $\partial^k P$ which are not the image under ∂ of some term in P-$\overline{\mathbf{EF}}_{k-1}$. A term Q is a *P-expanded term* if it belongs to some set P-$\mathbf{EF}_k$ with $k \geq 0$.

The above definition of P-$\mathbf{EF}_k$ makes sense by Lemma 4.ii: if Q belongs to P-$\overline{\mathbf{EF}}_{k-1}$, then ∂Q (which is $=_{LD}$-equivalent to Q) belongs to P-$\overline{\mathbf{EF}}_k$. So P-$\mathbf{EF}_k$ is the set of all 'really new' cuts of $\partial^k P$.

Example. Let P be the term $aaaa\bullet\bullet\bullet$. Then the elements of P-$\mathbf{EF}_0$ are the cuts of P, which are

a, $aa\bullet$, $aaa\bullet\bullet$ and P.

The elements of P-$\mathbf{EF}_1$ are the cuts of ∂P which are not derived from the latter ones. There are 4 such 'new' cuts, namely

$aa\bullet a\bullet$, $aa\bullet aa\bullet\bullet a\bullet$, $aa\bullet aa\bullet\bullet aa\bullet\bullet$ and $aa\bullet aa\bullet\bullet aa\bullet a\bullet\bullet$.

There are 42 cuts in $\partial^2 P$, among which 34 are new. The first (*i.e.* shortest) ones are

$aa\bullet a\bullet a\bullet$, $aa\bullet a\bullet aa\bullet\bullet$, $aa\bullet a\bullet aa\bullet a\bullet\bullet a\bullet a\bullet$, *etc...*

It is very easy to state a first normal form result concerning the P-expanded form. The $=_{LD}$-saturation of the prefix relation $\sqsubseteq$ will be denoted by $\sqsubseteq_{LD}$. So $Q \sqsubseteq_{LD} P$ means that there exist terms P', Q' satisfying $Q' =_{LD} Q$, $P' =_{LD} P$ and $Q' \sqsubseteq P'$. It has been shown in [3] and [10] that the relation $\sqsubseteq_{LD}$ induces a linear ordering on $\mathcal{W}/=_{LD}$. This however will also follow from the results of Section 3. Observe that the relations $\sqsubset$, $\sqsubseteq$ and $<_{Lex}$ coincide on each set P-$\overline{\mathbf{EF}}_k$.

Theorem 5. (existence and uniqueness of the P-expanded form) *Let P be a fixed term. Then any term Q satisfying $Q \sqsubseteq_{LD} P$ is $=_{LD}$-equivalent to a unique P-expanded term.*

Proof. Assume $P =_{LD} P'$, $Q =_{LD} Q'$ and $Q' \sqsubset P'$. By Corollary 2 the term $\partial^j Q'$ is an extension of Q for some j. Let P'' be the term obtained from P' by substituting the prefix $\partial^j Q'$ to the prefix Q'. Then P'' is $=_{LD}$-equivalent to P', and therefore to P. So for some k the term $\partial^k P$ is an extension of P''. Now by Lemma 4.i some cut R of $\partial^k P$ is an extension of the cut $\partial^j Q'$ of P'', and, by transitivity of extension, it also is an extension of Q. If R belongs to P-$\mathbf{EF}_k$, we are done. Otherwise we apply to R the inverse mapping $\overline{\partial}$ of ∂ as many times as possible and eventually obtain an element of P-$\mathbf{EF}_{k'}$ for some k' below k. Since $\overline{\partial}$ preserves the $=_{LD}$-class, we have obtained a P-expanded term which is $=_{LD}$-equivalent to Q.

The uniqueness follows from Lemma 4.iii. Indeed let Q, R be distinct P-expanded terms. Assume that Q belongs to P-$\mathbf{EF}_k$ and R belongs to P-$\mathbf{EF}_\ell$. Assume $\ell \leq k$. Then $\partial^{k-\ell} R$ belongs to P-$\overline{\mathbf{EF}}_k$, is $=_{LD}$-equivalent to R and is certainly different from Q since, in the case $\ell < k$, it does not belong to P-$\mathbf{EF}_k$. By construction either Q is a strict cut of $\partial^{k-\ell} R$, or $\partial^{k-\ell} R$ is a strict cut of Q. In both cases we conclude that these terms cannot be $=_{LD}$-equivalent. ■

Thus we have obtained a unique normal form result for the terms Q satisfying the condition $Q \sqsubseteq_{LD} P$. The unique $=_{LD}$-equivalent P-expanded term which is $=_{LD}$-equivalent to a term Q will be called the *P-expanded form* of Q.

The restriction to an initial segment of $(\mathcal{W}, \sqsubseteq_{LD})$ can be easily dropped as follows. First assume that P is a cut of P'. Then all cuts of P are cuts of P', and, by Lemma 4.ii, $\partial^k P$ is a prefix of $\partial^k P'$ for every $k \geq 1$, so that every P-expanded term is still a P'-expanded term, and, more precisely, the P-expanded terms make an initial segment of the P'-expanded terms with respect to $\sqsubseteq_{LD}$. So for $Q \sqsubseteq_{LD} P$, the P'-expanded form of Q is the P-expanded form of Q.

Now, for a word X built on the alphabet $\{a, \bullet\}$, define the *weight* of X to be the integer $|X|_a - |X|_\bullet$, where $|X|_s$ is the number of s's occurring in X. The terms are exactly the words with weight 1 of which any nonempty prefix has weight ≥ 1. Now a term Q is a cut of a term P if the word $Q^\dagger$ obtained from Q by deleting all final $\bullet$'s is a prefix of P.

Definition. i) An *infinite term* is an infinite word P built on the alphabet $\{a, \bullet\}$ (*i.e.* a sequence of elements of $\{a, \bullet\}$ indexed by the natural numbers) such that the weight of any prefix of P is ≥ 1. The set of all finite or infinite terms is denoted by $\widetilde{\mathcal{W}}$.

ii) For P in $\widetilde{\mathcal{W}}$ and Q in $\mathcal{W}$, we say that Q is a cut of P if the word $Q^\dagger$ is a prefix of the word P, and that $Q \sqsubseteq_{LD} P$ holds if $Q \sqsubseteq_{LD} Q'$ holds for some cut Q' of P. The term P is *cofinal* in $(\mathcal{W}, \sqsubset_{LD})$ if $Q \sqsubseteq_{LD} P$ holds for every (finite) term Q.

The compatibility of these definitions in the case of finite terms is obvious. Clearly the cuts of any (finite or infinite) term P are pairwise comparable (with respect to the relation 'being a cut of'). So by the argument above we can define for an infinite term P the P-expanded terms to be all Q-expanded terms where Q is a cut of P. By Lemma 4.ii, there is no ambiguity in defining the derivation of an infinite term P as the infinite term admitting as prefixes the terms derived from the cuts of P. Then the P-expanded terms still are the cuts of P, augmented with the 'new cuts' of ∂P, augmented with the 'new cuts' of $\partial^2 P$, *etc...*

Definition. i) For every finite term R, R^ω is the infinite term $RRR\ldots$. An infinite term P is *eventually constant* if for some finite term R the term P coincides with R^ω up to a finite prefix.

ii) For every finite term R, $R^{[n]}$ is the term $R^n \bullet^{n-1}$.

Observe that, for any n, the term $R^{[n]}$ is a cut of the infinite term R^ω.

Proposition 6. *Every eventually constant (infinite) term is cofinal in* $(\mathcal{W}, \sqsubseteq_{LD})$.

Proof. We first consider the case of the infinite term a^ω. We have to show that the cuts of a^ω, *i.e.* the terms $a^{[n]}$, are cofinal in $(\mathcal{W}, \sqsubseteq_{LD})$. For Q in $\mathcal{W}$, define the *height* $\mathbf{h}(Q)$ and the *complexity* $\mathbf{c}(Q)$ of Q by $\mathbf{h}(a)=1$, $\mathbf{c}(a)=0$ and

$$\mathbf{h}(QQ'\bullet)=\sup(\mathbf{h}(Q),\mathbf{h}(Q'))+1 \qquad \mathbf{c}(QQ'\bullet)=2\mathbf{c}(Q)+\mathbf{c}(Q')+1.$$

We claim that, for any term Q, the equivalence

$$a^{[n]} =_{LD}^{\mathbf{c}(Q)} Qa^{[n-1]}\bullet \tag{1}$$

holds for every $n \geq \mathbf{h}(Q)$, which implies $Q \sqsubseteq_{LD} a^{[\mathbf{h}(Q)]}$. The equivalence $(^*)$ is proved inductively on Q. If Q is a, $(^*)$ is an equality. Assume that $(^*)$ holds for Q and Q', and that n is $\mathbf{h}(QQ'\bullet)$ at least. Applying the induction hypothesis we have

$$a^{[n]} =_{LD}^{\mathbf{c}(Q)} Qa^{[n-1]}\bullet =_{LD}^{\mathbf{c}(Q')} QQ'a^{[n-2]}\bullet\bullet =_{LD}^{1} QQ'\bullet Qa^{[n-2]}\bullet\bullet =_{LD}^{\mathbf{c}(Q)} QQ'\bullet a^{[n-1]}\bullet\bullet,$$

which establishes (1) for $QQ'\bullet$.

Now let R be any finite term. Then the terms $R^{[n]}$ are cofinal in the sequence of all $a^{[i]}$'s. Actually we show inductively on R that, if $\mathbf{f}(R)$ is the number of final $\bullet$'s in R, the equivalence

$$R^{[n]} =_{LD} a^{[n+\mathbf{f}(R)]} \tag{2}$$

holds for $n \geq \mathbf{h}(R)$. The result is obvious if R is a. Assume that (2) is proved for R and R'. For $n \geq \mathbf{h}(RR'\bullet)$ we have

$$\begin{aligned}(RR'\bullet)^{[n]} =_{LD} RR'^{[n]}\bullet =_{LD} Ra^{[n+\mathbf{f}(R')]}\bullet =_{LD} RR^{[n+\mathbf{f}(R')-\mathbf{f}(R)]}\bullet \\ =_{LD} R^{[n+\mathbf{f}(R')-\mathbf{f}(R)+1]} =_{LD} a^{[n+\mathbf{f}(R')+1]},\end{aligned}$$

which is the desired formula since $\mathbf{f}(RR'\bullet)$ is $\mathbf{f}(R')+1$. This shows that the terms $R^{[n]}$ are cofinal in $(\mathcal{W}, \sqsubseteq_{LD})$. Finally if P is infinite and eventually coincides with R^{ω}, it must have the form $P_1 \ldots P_i RR\ldots$ for some terms P_1, ..., P_i. Then for every n the term $P_1\ldots P_i R^{[n]}\bullet^i$ is a cut of P. Now this term is $=_{LD}$-equivalent to $(P_1\ldots P_i R\bullet^i)^{[n]}$, and by applying the preceding result to the term $P_1\ldots P_i R\bullet^i$ we know that the sequence of all $(P_1\ldots P_i R\bullet^i)^{[n]}$'s is cofinal in $(\mathcal{W}, \sqsubseteq_{LD})$. ■

Not every infinite term is cofinal in $(\mathcal{W}, \sqsubseteq_{LD})$: for instance the cuts of the term $aa\bullet a\bullet a\bullet a\bullet\ldots$ all lie below $aaa\bullet\bullet$. In this counterexample the weights of the prefixes are bounded, but Richard Laver has constructed a non cofinal term of the form $P_1P_2P_3\ldots$. No exact description of the cofinal infinite terms is known.

The previous result establishes not only the existence of the P-expanded form but also yields an upper complexity bound for its computation.

Theorem 7. (existence and complexity of the P-expanded form) *Assume that P is an infinite eventually constant term. Then every (finite) term has a P-expanded form, and the function which maps a term to its P-expanded form lies in the complexity class* DSPACE($\exp^*(O(2^n))$), *where* $\exp^*$ *is the iterated exponential defined by* $\exp^*(0) = 1$, $\exp^*(x+1) = 2^{\exp^*(x)}$.

Proof. The existence is immediate from Proposition 6. For the complexity we begin with the special case of a^ω. Let Q be any term. In the proof of Proposition 6, we have established the equivalence

$$a^{[\mathbf{h}(Q)]} =_{LD}^{\mathbf{c}(Q)} Qa^{[\mathbf{h}(Q)-1]}. \tag{1}$$

It follows that some cut of $\partial^{\mathbf{c}(Q)}a^{[\mathbf{h}(Q)]}$ is an extension of Q. This shows that the $a^{[\mathbf{h}(Q)]}$-expanded form of Q exists, but, more, that this form belongs to $a^{[\mathbf{h}(Q)]}$-$\overline{\mathbf{EF}}_k$ for some $k \le \mathbf{c}(Q)$. Thus the determination of the a^ω-expanded form of Q can be made by exhaustively enumerating all extensions of Q whose size is less than the size of $\partial^{\mathbf{c}(Q)}a^{[\mathbf{h}(Q)]}$ and testing equality of terms. For a term Q with length n (as a word), the height of Q is below $(n+1)/2$, and the complexity of Q is below 2^n. For R with length N, the length of ∂R is bounded by 2^N, so one obtains that the size of $\partial^{\mathbf{c}(Q)}a^{[\mathbf{h}(Q)]}$ is bounded by $\exp^*(O(2^n))$ with $\exp^*$ is as above.

Now let R be any finite term. To extend result from a to R, it suffices to prove that, for some constant α, the equivalence

$$R^{[\mathbf{h}(Q)]} =_{LD}^{\mathbf{c}(Q)+\alpha\mathbf{h}(Q)} QR^{[\mathbf{h}(Q)-1]} \tag{2}$$

holds for every sufficiently large term Q. Now (2) follows from (1) provided that there exists a constant β (depending on R) such that

$$R^{[n]} =_{LD}^{\beta n} a^{[n+(\mathbf{R})]} \tag{3}$$

holds for every n large enough. We prove this 'quantitative' version of formula (2) in the proof of Proposition 6 inductively on R. The result is obvious if R is a. So assume that R is $R'R''\bullet$ and (3) is proved for R' and R'' with respective constants β' and β''. We successively obtain for n large enough (actually for $n \ge \mathbf{h}(R)$)

$$\begin{aligned}
R^{[n]} = R'R''\bullet\ldots R'R''\bullet\bullet^n &=_{LD}^{n} R'R''^{[n]}\bullet \\
&=_{LD}^{\beta''n} R'a^{[n+\mathbf{f}(R'')]} \\
&=_{LD}^{\beta'n} R'R'^{[n+\mathbf{f}(R'')-\mathbf{f}(R')]} = R'^{[n+\mathbf{f}(R'')-\mathbf{f}(R')+1]} \\
&=_{LD}^{\beta'n} a^{[n+\mathbf{f}(R'')+1]} = a^{[n+\mathbf{f}(R)]}.
\end{aligned}$$

This establishes the formula (with $b = 2\beta' + \beta'' + 1$), and completes the proof. ∎

2. The geometry of derivation

Although it theoretically gives a unique representative for every $=_{LD}$-equivalence class, the expanded form is not a very useful tool. The main reason is that no simple intrinsic characterization of the expanded terms is known and that the comparison of P-expanded terms with respect to $\sqsubseteq_{LD}$ is not easy. Also the length of the expanded form of even very simple terms is hopelessly large, which makes any practical use difficult.

But the expanded terms are very good intermediates toward a better normal form. Indeed it now suffices to construct simple normal terms to represent the expanded terms, and the results of Section 1 will guarantee that these normal terms will be representatives for all terms.

The main technical task will be to connect the cuts of a term ∂P with the cuts of the term P. This will heavily rely on the *geometry* of the terms and of left distributivity.

As in [2] and [5], it will be convenient to consider terms as binary trees according to the usual convention that $PQ\bullet$ is the tree admitting P as a left subtree and Q as a right subtree. We use finite sequences of 0's and 1's as addresses for nodes in such trees. The set of such sequences is denoted by $\mathbb{S}$, and the empty sequence (the address of the root of the tree) is denoted by Λ. We shall denote respectively by 0^* and 1^* the subsets of $\mathbb{S}$ made by all 0^i and all 1^i for i a nonnegative integer. For P in $\mathcal{W}$, the set of all addresses of the leaves of P is called the *support* of that term, and is denoted by Supp P. With obvious notations we have the following inductive relations

$$\text{Supp } a = \{\Lambda\}$$
$$\text{Supp}(PP'\bullet) = 0(\text{Supp } P) \cup 1(\text{Supp } P')$$

For instance the support of the term $a^{[4]}$ (*i.e.* $aaaa\bullet\bullet\bullet$) is the set $\{0, 10, 110, 111\}$. There are exactly as many points in the support of P as occurrences of the character a in P viewed as a word. Observe that this address system does not necessarily extend to infinite terms since such terms may have an infinitely long leftmost branch. However it could easily be adapted by considering addresses of the form $\overline{0}^i u$ with u in $\mathbb{S}$ and $\overline{0}$ is an inverse of 0.

Because there is a bijection between the support of P and the occurrences of the character a in the word P, and another bijection between these occurrences and the various cuts of P, we can use the elements of Supp P to index the cuts of P: the idea is that $\text{cut}(P, u)$ will be the cut of P obtained by cutting P at the occurrence of a which has address u and completing with as many $\bullet$'s as is needed to obtain a well-formed term. Formally we start with the following

Definition. i) Let P be any (finite) term. For u in the support of P, the term $\mathrm{cut}(P,u)$ is defined as follows: $\mathrm{cut}(a,\Lambda)$ is a, $\mathrm{cut}(P'P''\bullet,u)$ is $\mathrm{cut}(P',v)$ if u is $0v$ and is $P'\mathrm{cut}(P'',w)\bullet$ if u is $1w$.

ii) The relation $<_{Lex}$ is the lexicographical extension to $\widetilde{\mathcal{W}}$ of the ordering on $\{a,\bullet\}$ defined by $\bullet <_{Lex} a$.

iii) For u, v in $\mathbb{S}$, we write $u < v$, and say that u is *on the left of* v, if some w satisfies both $w0 \sqsubseteq u$ and $w1 \sqsubseteq v$.

We naturally denote by $\sqsubset_{LD}$ the $=_{LD}$-saturation of the strict prefix relation $\sqsubset$. We establish inductively that the mapping cut has the desired properties.

Lemma 1. *For every term P the function $u \mapsto \mathrm{cut}(P,u)$ is a bijection of* Supp P *onto the set of all cuts of P. Moreover for u, v in* Supp P *the following are equivalent*

i) $u < v$;

ii) $\mathrm{cut}(P,u) \sqsubset_{LD} \mathrm{cut}(P,v)$;

iii) $\mathrm{cut}(P,u) <_{Lex} \mathrm{cut}(P,v)$.

Proof. That the cuts of P are exactly the terms $\mathrm{cut}(P,u)$ for u in the support of P is established by an easy induction on P. Now because $<$ is a linear ordering on Supp P and both $\sqsubset_{LD}$ (by Lemma 1.4.iii) and $<_{Lex}$ are linear orderings on the cuts of P, it suffices to show that (i) implies (ii) and (iii). Now $u < v$ implies that $\mathrm{cut}(P,u)$ is a cut of $\mathrm{cut}(P,v)$: because the relation 'is a cut of' is transitive, it suffices to establish the implication when v is the immediate successor of u in Supp P. But in this case u and v have the form $w0^i$ and $w10^j$ for some w, i, j, and the property follows from the definition. Then $u < v$ implies $\mathrm{cut}(P,u) <_{Lex} \mathrm{cut}(P,v)$ since, by the construction of cuts as words, a cut of a term always preceds that term with respect to the lexicographical ordering. By Lemma 1.4.ii the fact that $\mathrm{cut}(P,u)$ is a cut of $\mathrm{cut}(P,v)$ implies that $\partial\mathrm{cut}(P,u)$ is a prefix of $\partial\mathrm{cut}(P,v)$, and therefore that $\mathrm{cut}(P,u) \sqsubset_{LD} \mathrm{cut}(P,v)$ holds. ■

It could be useful in the sequel to have a following geometrical intuition of the cuts (which can be verified using an easy induction). Starting from the term P viewed as a binary tree, the tree $\mathrm{cut}(P,u)$ is obtained from P by deleting the part of that tree which lies on the right of u and lifting the remaining subtrees so that the branching index remains 1 at each inner node. More precisely, if u_1, …, u_p are the prefixes of u such that u_10, …, u_p0 are not prefixes of u, and if P_1, …, P_p are the subtrees of P with root at u_10, …, u_p0 respectively, then $\mathrm{cut}(P,u)$ is exactly $P_1 \ldots P_p a \bullet^p$.

The first step in the study of the cuts of ∂P is naturally the study of the cuts of a term $\mathrm{dist}(Q,R)$ in terms of the cuts of Q and R. This is easy.

Lemma 2. *Let Q, R be arbitrary terms in $\mathcal{W}$.*

i) The support of $\mathrm{dist}(Q,R)$ *is the set*

$$(\mathrm{Supp}\ R)1 \cup (\mathrm{Supp}\ R)0(\mathrm{Supp}\ Q).$$

ii) For v in Supp Q *and w in* Supp R *one has*

$$\mathrm{cut}(\mathrm{dist}(Q,R), w1) = \mathrm{dist}(Q, \mathrm{cut}(R,w)),$$
$$\mathrm{cut}(\mathrm{dist}(Q,R), w0v) = \mathrm{dist}(Q,R_1)\ldots\mathrm{dist}(Q,R_r)\mathrm{cut}(Q,v)\bullet^r$$

where $\mathrm{cut}(R,w)$ *is* $R_1\ldots R_r a\bullet^r$.

Proof. The result is rather clear owing to the geometrical construction of $\mathrm{dist}(Q,R)$ as the tree obtained from R by substituting the tree $Qa\bullet$ to every occurrence of a in R. For a formal proof use induction on R. If R is a, $\mathrm{dist}(Q,R)$ is just $Qa\bullet$, the only cut of R is a itself and everything is obvious. Assume $R = R'R''\bullet$. The formula for the support is easy. Assume that v lies in the support of Q, and w in the support of R'. Then $0w$ belongs to Supp R, and one has

$$\begin{aligned}
\mathrm{cut}(\mathrm{dist}(Q,R), 0w1) &= \mathrm{cut}(\mathrm{dist}(Q,R'), w1) \\
&= \mathrm{dist}(Q, \mathrm{cut}(R', w)) \qquad \text{(ind. hyp. for } R'\text{)} \\
&= \mathrm{dist}(Q, \mathrm{cut}(R, 0w)) \\
\mathrm{cut}(\mathrm{dist}(Q,R), 0w0v) &= \mathrm{cut}(\mathrm{dist}(Q,R'), w0v) \\
&= \mathrm{dist}(Q,R'_1)\ldots\mathrm{dist}(Q,R'_r)\mathrm{cut}(Q,v)\bullet^r
\end{aligned}$$

by induction hypothesis for R' assuming $\mathrm{cut}(R',w) = R'_1\ldots R'_r a\bullet^r$. Now $\mathrm{cut}(R,0w)$ is equal to $\mathrm{cut}(R',w)$, so one has obtained the desired formula.

Assume now that w lies in the support of R'', so that $1w$ belongs to the support of R.

$$\begin{aligned}
\mathrm{cut}(\mathrm{dist}(Q,R), 1w1) &= \mathrm{dist}(Q,R')\mathrm{cut}(\mathrm{dist}(Q,R''), w1)\bullet \\
&= \mathrm{dist}(Q,R')\mathrm{dist}(Q, \mathrm{cut}(R'', w))\bullet \qquad \text{(ind. hyp. for } R''\text{)} \\
&= \mathrm{dist}(Q, R'\mathrm{cut}(R'', w)\bullet) \\
&= \mathrm{dist}(Q, \mathrm{cut}(R, 1w)) \\
\mathrm{cut}(\mathrm{dist}(Q,R), 1w0v) &= \mathrm{dist}(Q,R')\mathrm{cut}(\mathrm{dist}(Q,R''), w0v)\bullet \\
&= \mathrm{dist}(Q,R')\mathrm{dist}(Q,R''_1)\ldots\mathrm{dist}(Q,R''_r)\mathrm{cut}(Q,v)\bullet^{r+1}
\end{aligned}$$

by induction hypothesis for R'' assuming $\mathrm{cut}(R'',w) = R''_1\ldots R''_r a\bullet^r$. Now $\mathrm{cut}(R,1w)$ is $R'\mathrm{cut}(R'',w)\bullet$, that is $R'R''_1\ldots R''_r a\bullet^{r+1}$, and the formula has the desired form. ■

We now introduce the key geometrical notion for the sequel.

Definition. i) For u, v in $\mathbb{S}$, write $u \gg v$ if $w1^j0 \sqsubseteq u$ and $w0 \sqsubseteq v$ hold for some integer j and some w in $\mathbb{S}$.

ii) A *descent* of P is a finite sequence $\langle u_1, \ldots, u_p \rangle$ in Supp P satisfying the condition $u_1 \gg u_2 \gg \ldots \gg u_p$. The set of all descents of P is denotes by Desc$P$.

iii) Let P be a (finite) term, and u be an element of Supp P. Let w be the maximal prefix of u which does not end with 1. The point $\theta_P(u)$ is the $<$-least point in Supp P which admits w as a prefix, *i.e.* the only point in Supp $P \cap w0^*$.

Lemma 3. *For any term P and u, v in the support of P, $u \gg v$ is equivalent to $\theta_P(u) > v$.*

The proof is straightforward. Observe that $\gg$ is an ordering on $\mathbb{S}$, and that $u \gg v$ implies $u > v$ since $\theta_P(u) \leq u$ always holds. We shall denote by $<^*$ the lexicographical extension of the order $<$ to the finite sequences in $\mathbb{S}$. Observe that for any term P, the rightmost point in Supp P has the form 1^i. Then $\langle 1^i \rangle$ is the only descent of P where 1^i occurs, and this descent is the last element of Desc$P$ with respect to $<^*$. The point 1^i will be refered to in the sequel as the final point of Supp P, and similarly the descent $\langle 1^i \rangle$ will be refered to as the *final* descent of P.

Example. The $<^*$-increasing enumeration of Desc($aaaa$•••) is
$\langle 0 \rangle$, $\langle 10 \rangle$, $\langle 10, 0 \rangle$, $\langle 110 \rangle$, $\langle 110, 0 \rangle$, $\langle 110, 10 \rangle$, $\langle 110, 10, 0 \rangle$, $\langle 111 \rangle$.

We turn to the description of the correspondence between the support of ∂P and the descents of P. It could be useful to have the following geometrical intuition of this correspondence. When left distributivity is applied to transform a term $\ldots P'P''P'''$••$\ldots$ into $\ldots P'P''$•$P'P'''$••$\ldots$, we can imagine that the second subterm P' is obtained by letting the original P' *cross* the subterm P'', or, more precisely, the rightmost variable of P''. Let us start with a term P whose variables are pairwise distinct. Consider any point w in the support of ∂P. Then in a transformation from P to ∂P the variable x which occurs at w in ∂P has crossed a certain number of variables of P, say x_1, …, x_{p-1}. The set of these variables does not depend on the way the transformation has been operated: actually $\{x_1, \ldots, x_{p-1}\}$ is the set of the rightmost variables of the subterms of ∂P with roots at w_10, …, $w_{p-1}0$, where w_1, …, w_{p-1} are the prefixes of w such that w_11, …, $w_{p-1}1$ are also prefixes of w but w does not belong to w_i1^* (so that $w \gg w_10$, …, $w \gg w_{p-1}0$ holds). Now if u_1, …, u_p are the addresses of the variables x_1, …, x_{p-1}, x in the support of P enumerated in $<$-decreasing order, $\langle u_1, \ldots, u_p \rangle$ is a descent of P and is the image of w in the above mentioned correspondence.

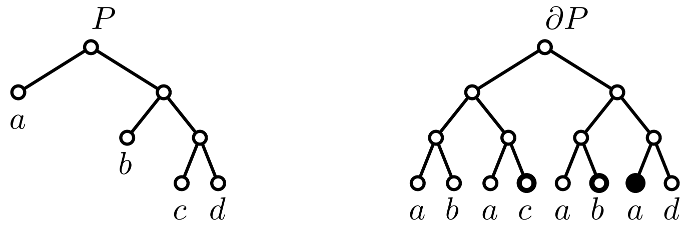

Figure 1

Example. (see Figure 1) Assume that P is $abcd\bullet\bullet\bullet$. Then ∂P is $ab\bullet ac\bullet\bullet ab\bullet ad\bullet\bullet\bullet$. Consider the point 110 in Supp ∂P: the variable at this point is a, which has address 0 in P. There are two prefixes w_i of 110 such that w_i1 is a prefix of 110, namely Λ and 1, and the rightmost variables of the terms below 0 and 10 in ∂P are c and b, with respective addresses 110 and 10 in P. So the point 110 of Supp ∂P corresponds to the descent $\langle 110, 10, 0\rangle$ of P.

In order to obtain sufficiently precise statements for the sequel, we shall have to make the above correspondence completely explicit, which requires some tedious bur easy inductive verifications. For u in $\mathbb{S}$, we denote by $|u|_1$ the number of 1's in u. We add two points $-\infty$ and $+\infty$ to $\mathbb{S}$ with the convention that $+\infty \gg u$ holds for every u in $\mathbb{S} \setminus 1^*$ and $u > -\infty$ holds for every u in $\mathbb{S}$.

Definition. Let P be any term. For $u_1, \ldots, u_p$ in $\mathbb{S} \cup \{\pm\infty\}$ let

$$\varphi_P(u_1, \ldots, u_p) = 0^{\delta_P(u_1,u_2)}10^{\delta_P(u_2,u_3)}1\ldots10^{\delta_P(u_{p-1},u_p)}$$

where

$$\delta_P(u, v) = \mathrm{card}\{x \in \mathrm{Supp}\ P; u \gg x > v\}.$$

For $\langle u_1, \ldots, u_p\rangle$ a non final descent of P, let

$$\Phi_P(\langle u_1, \ldots, u_p\rangle) = \varphi_P(+\infty, u_1, \ldots, u_p)01^{|u_p|_1},$$
$$\Psi_P(\langle u_1, \ldots, u_p\rangle) = \varphi_P(+\infty, u_1, \ldots, u_p, -\infty),$$

and extend Φ_P by $\Phi_P(\langle 1^i\rangle) = 1^i$ where $\langle 1^i\rangle$ is the final descent of P.

The main technical point is the existence of a bijective correspondence between the points of ∂P and the descents of P both in terms of addresses and of the associated cuts.

Lemma 4. *Let P be any term in $\mathcal{W}$.*

i) The mapping Φ_P is an increasing bijection of the set $\mathrm{Desc}P$ *ordered by* $<^*$ *onto* Supp ∂P *ordered by* $<$. *Moreover for every non final descent α of P, the immediate successor of $\Phi_P(\alpha)$ in* Supp ∂P *is $\Psi_P(\alpha)$.*

ii) For every descent $\langle u_1, \ldots, u_p\rangle$ of P one has

$$\mathrm{cut}(\partial P, \Phi_P(\langle u_1, \ldots, u_p\rangle)) = \partial\mathrm{cut}(P, u_1)\ldots\partial\mathrm{cut}(P, u_p)\bullet^{p-1},$$

and, if $\langle u_1, \ldots, u_p\rangle$ is not final,

$$\mathrm{cut}(\partial P, \Psi_P(\langle u_1, \ldots, u_p\rangle)) = \partial\mathrm{cut}(P, u_1)\ldots\partial\mathrm{cut}(P, u_p)a\bullet^{p}.$$

Proof. Use induction on P. If P is a, the only descent of P is $\langle\Lambda\rangle$, $\Phi_P(\langle\Lambda\rangle)$ is by definition Λ, ∂P is a and everything is obvious. Assume from now on that P is $QR\bullet$ and the properties hold for Q and R. We have with obvious notations (using $\frown$ for concatenation of sequences)

$$\mathrm{Desc}P = 1(\mathrm{Desc}^{\mathrm{nf}}R)^\frown 0(\mathrm{Desc}Q) \cup 0(\mathrm{Desc}Q) \cup 1(\mathrm{Desc}R)$$

where $\mathrm{Desc}^{\mathrm{nf}}R$ denotes the set of all non final descents of R. For w non final in Supp R and v non final in Supp Q, one has

$$\delta_P(1w, 0v) = \delta_R(w, -\infty) + 1 + \delta_Q(+\infty, v),$$

and this formula still holds if w is $+\infty$ or v is $-\infty$. So for $\langle w_1, \ldots, w_r\rangle$ a non final descent of R and $\langle v_1, \ldots, v_q\rangle$ a non final descent of Q, one obtains

$$\varphi_P(\langle +\infty, 1w_1, \ldots, 1w_r, 0v_1, \ldots, 0v_q\rangle) = \varphi_R(\langle +\infty, w_1, \ldots, w_r\rangle)0\varphi_Q(\langle v_1, \ldots, v_q\rangle)$$

and therefore

$$\Phi_P(\langle 1w_1, \ldots, 1w_r, 0v_1, \ldots, 0v_q\rangle) = \Psi_R(\langle w_1, \ldots, w_r\rangle)0\Phi_Q(\langle v_1, \ldots, v_q\rangle),$$
$$\Psi_P(\langle 1w_1, \ldots, 1w_r, 0v_1, \ldots, 0v_q\rangle) = \Psi_R(\langle w_1, \ldots, w_r\rangle)0\Psi_Q(\langle v_1, \ldots, v_q\rangle).$$

Similarly one has

$$\Phi_P(\langle 1w_1, \ldots, 1w_r\rangle) = \Phi_R(\langle w_1, \ldots, w_r\rangle)1$$
$$\Psi_P(\langle 1w_1, \ldots, 1w_r\rangle) = \Psi_R(\langle w_1, \ldots, w_r\rangle)0^j,$$
$$\Phi_P(\langle 0v_1, \ldots, 0v_q\rangle) = 0^k\Phi_Q(\langle v_1, \ldots, v_q\rangle)$$
$$\Psi_P(\langle 0v_1, \ldots, 0v_q\rangle) = 0^k\Psi_Q(\langle v_1, \ldots, v_q\rangle),$$

where j and k are the cardinals of the supports of Q and R respectively. Then if v is the final point of Supp Q, one has

$$\delta_P(1w, 0v) = \delta_R(w, -\infty),$$

which gives for $\langle v\rangle$ the final descent of Q

$$\varphi_P(\langle +\infty, 1w_1, \ldots, 1w_r, 0v\rangle) = \varphi_R(\langle +\infty, w_1, \ldots, w_r, -\infty\rangle),$$

whence

$$\Phi_P(\langle 1w_1, \ldots, 1w_r, 0v\rangle) = \Psi_R(\langle w_1, \ldots, w_r\rangle)01^{|v|_1} = \Psi_R(\langle w_1, \ldots, w_r\rangle)0\Phi_Q(\langle v\rangle),$$
$$\Psi_P(\langle 1w_1, \ldots, 1w_r, 0v\rangle) = \Psi_R(\langle w_1, \ldots, w_r\rangle)1.$$

Similarly one obtains (with k as above)

$$\Phi_P(\langle 0v\rangle) = 0^k 1^{|v|_1} = 0^k \Phi_Q(\langle v\rangle),$$
$$\Psi_P(\langle 0v\rangle) = 0^{k-1}1.$$

Finally, if $\langle w\rangle$ is the final descent of R, then $\langle 1w\rangle$ is the final descent of P, and we directly obtain

$$\Phi_P(\langle 1w\rangle) = \Phi_R(\langle w\rangle)1.$$

The explicit formulas above show that the image of Φ_P is made of all $\Psi_R(\gamma)0\Phi_Q(\beta)$ for γ in $\mathrm{Desc}^{\mathrm{nf}}R$ and β in Desc$Q$, together with all $0^k\Phi_Q(\beta)$ for β in Desc$Q$ and all $\Psi_R(\gamma)1$ for γ in Desc$R$. An immediate induction will show that, for any term R, the point 0^{k-1} belongs to the support of ∂R, and is its $<$-least element. Therefore the induction hypothesis implies that the image of Ψ_R augmented with 0^{k-1} is exactly the support of ∂R, and the image of Φ_P is

$$(\mathrm{Supp}\ \partial R)0(\mathrm{Supp}\ \partial Q) \cup (\mathrm{Supp}\ \partial R)1,$$

which, by Lemma 2.i, is the support of dist$(\partial Q, \partial R)$, *i.e.* of ∂P.

By a parallel argument one proves that the image of Ψ_P is all of Supp ∂P except its $<$-least element (*i.e.* $0^{\mathrm{cardSupp}\ P-1}$). By construction $\Phi_P(\alpha) < \Psi_P(\alpha)$ holds for any non final descent α of P. Moreover since $\Phi_P(\alpha)$ and $\Psi_P(\alpha)$ always have the form $w01^i$ and $w10^j$ for some w, i, j, no point u may satisfy $\Phi_P(\alpha) < u < \Psi_P(\alpha)$, so that $\Psi_P(\alpha)$ is certainly the immediate successor of $\Phi_P(\alpha)$ in Supp ∂P. Finally its is easy to verify the following relations

$\Phi_P(0\beta) < \Phi_P(1\gamma) < \Phi_P(1\gamma^\frown 0\beta)$ for β in Desc$Q$ and γ in Desc$R$,

$\Phi_P(1\gamma^\frown 0\beta) < \Phi_P(1\gamma'^\frown 0\beta')$ for β, β' in Desc$Q$ and γ, γ' in Desc$R$ satisfying $\Phi_R(\gamma) < \Phi_R(\gamma')$,

$\Phi_P(1\gamma^\frown 0\beta) < \Phi_P(1\gamma^\frown 0\beta')$ for γ in Desc$R$ and β, β' in Desc$Q$ satisfying $\Phi_Q(\beta) < \Phi_Q(\beta')$,

so that Φ_P must be increasing whenever Φ_Q and Φ_R are. This finishes the proof of (i).

Assume now that $\langle w_1, \ldots, w_r\rangle$ is a non final descent of R, and $\langle v_1, \ldots, v_q\rangle$ is a non final descent of Q. One has as above

$$\begin{aligned}\mathrm{cut}(\partial P, \Phi_P(\langle 1w_1, \ldots, 1w_r, 0v_1, \ldots, 0v_q\rangle)\\ = \mathrm{cut}(\partial P, \Psi_R(\langle w_1, \ldots, w_r\rangle) 0 \Phi_Q(\langle v_1, \ldots, v_q\rangle)).\end{aligned}$$

By the induction hypothesis $\mathrm{cut}(\partial R, \Psi_R\langle w_1, \ldots, w_r\rangle)$ is equal to

$$\partial\mathrm{cut}(R, w_1)\ldots\partial\mathrm{cut}(R, w_r) a \bullet^r$$

and $\mathrm{cut}(\partial Q, F_Q(\langle v_1, \ldots, v_q\rangle)$ is equal to $\partial\mathrm{cut}(Q, v_1)\ldots\partial\mathrm{cut}(Q, v_q)\bullet^{q-1}$, hence by Lemma 2.ii one obtains

$$\begin{aligned}&\mathrm{cut}(\partial P, \Phi_P(\langle 1w_1, \ldots, 1w_r, 0v_1, \ldots, 0v_q\rangle) = \\ &\quad \mathrm{dist}(\partial Q, \partial\mathrm{cut}(R, w_1))\ldots\mathrm{dist}(\partial Q, \partial\mathrm{cut}(R, w_r))\partial\mathrm{cut}(Q, v_1)\ldots\partial\mathrm{cut}(Q, v_q)\bullet^{r+q-1}.\end{aligned}$$

Now $\mathrm{dist}(\partial Q, \partial\mathrm{cut}(R, w))$ is the derived term of $Q\mathrm{cut}(R, w)\bullet$, and the latter term is $\mathrm{cut}(P, 1w)$. On the other hand $\mathrm{cut}(Q, v)$ is equal to $\mathrm{cut}(P, 0v)$. Therefore the last term is equal to

$$\partial\mathrm{cut}(P, 1w_1)\ldots\partial\mathrm{cut}(P, 1w_r)\partial\mathrm{cut}(P, 0v_1), \ldots, \partial\mathrm{cut}(P, 0v_q)\bullet^{r+q-1},$$

which is the desired form. The adaptation to the case of $\mathrm{cut}(\partial P, \Phi_P(\langle 0v_1, \ldots, 0v_q\rangle))$ is straightforward. For the case of a descent $\langle 1w_1, \ldots, 1w_r\rangle$, one obtains

$$\mathrm{cut}(\partial P, \Phi_P(\langle 1w_1, \ldots, 1w_r\rangle)) = \mathrm{cut}(\partial P, \Phi_R(\langle w_1, \ldots, w_r\rangle)1),$$

hence by Lemma 2.ii and the induction hypothesis

$$\begin{aligned}\mathrm{cut}(\partial P, \Phi_P(\langle 1w_1, \ldots, 1w_r\rangle)) &= \mathrm{dist}(\partial Q, \mathrm{cut}(\partial R, \Phi_R(\langle w_1, \ldots, w_r\rangle)1))\\ &= \mathrm{dist}(\partial Q, \mathrm{cut}(R, w_1))\ldots\mathrm{dist}(\partial Q, \mathrm{cut}(R, w_r))\bullet^{r-1},\\ &= \partial\mathrm{cut}(P, 1w_1)\ldots\partial\mathrm{cut}(P, 1w_r)\bullet^{r-1}\end{aligned}$$

which proves the formula for Φ_P. The proof for Ψ_P is similar. The only new case is the case of a descent $\langle 1w_1, \ldots, 1w_r\rangle$. One obtains

$$\mathrm{cut}(\partial P, \Psi_P(\langle 1w_1, \ldots, 1w_r\rangle)) = \mathrm{cut}(\partial P, \Psi_R(\langle w_1, \ldots, w_r\rangle)00^{j-1})).$$

By induction hypothesis $\mathrm{cut}(R, \Psi_R(\langle w_1, \ldots, w_r\rangle))$ must be

$$\partial\mathrm{cut}(R, w_1)\ldots\partial\mathrm{cut}(R, w_r) a \bullet^r,$$

so by Lemma 2.ii. the term above is equal to

$$\mathrm{dist}(\partial Q, \mathrm{cut}(R, w_1))\ldots\mathrm{dist}(\partial Q, \mathrm{cut}(R, w_r))\partial\mathrm{cut}(Q, 0^{j-1}\bullet^r.$$

Now $\mathrm{cut}(\partial Q, 0^{j-1})$ is certainly a, so the final formula is

$$\mathrm{cut}(\partial P, \Psi_P(\langle 1w_1, \ldots, 1w_r\rangle)) = \partial\mathrm{cut}(P, 1w_1)\ldots\partial\mathrm{cut}(P, 1w_r) a \bullet^r,$$

as we wished. ■

Example. (similar to the preceding one) Let P be the term $aaaa\bullet\bullet\bullet$. Then $\langle 110, 10, 0\rangle$ is a descent of P, and $\Phi_P(\langle 110, 10, 0\rangle)$ is 110. Now $\mathrm{cut}(\partial P, 110)$ is $aa\bullet aa\bullet\bullet aa\bullet a\bullet\bullet$, which is equal to $\mathrm{cut}(P, 110)\mathrm{cut}(P, 10)\mathrm{cut}(P, 0)\bullet\bullet$.

It is now easy to characterize the cuts of ∂P which come from some cut of P.

Lemma 5. *Let P be any term. For v in the support of ∂P the following are equivalent*

i) $\mathrm{cut}(\partial P, v)$ *is $=_{LD}$-equivalent to some cut of P;*

ii) v is $\Phi_P(\langle u\rangle)$ for some u in the support of P;

iii) v belongs to 0^*1^*.

Proof. Since by Lemma 4 the term $\mathrm{cut}(\partial P, \Phi_P(\langle u\rangle))$ is always $=_{LD}$-equivalent to $\mathrm{cut}(P, u)$ and two different cuts of ∂P cannot be $=_{LD}$-equivalent, the equivalence of (i) and (ii) is obvious. For (iii) observe that $\Phi_P(\langle u\rangle)$ belongs to 0^*1^* by construction, while for $p \geq 2$ the point $\Phi_P(\langle u_1, \ldots, u_p\rangle)$ cannot belong to this set. ■

It remains to characterize the descents of ∂P in terms of the descents of P.

Lemma 6. *Let P be any term. For every descent $\langle u_1, \ldots, u_p\rangle$ of P one has*

$$\theta_{\partial P}(\Phi_P(\langle u_1, \ldots, u_p\rangle)) = \Phi_P(\langle u_1, \ldots, u_{p-1}, 0^i\rangle),$$

where 0^i is the leftmost point of $\mathrm{Supp}\ P$.

Proof. If u_p is 0^i, then $\Phi_P((\langle u_1, \ldots, u_p\rangle)$ ends with 0, and $\theta_{\partial P}(\Phi_P((\langle u_1, \ldots, u_p\rangle))$ is $\Phi_P((\langle u_1, \ldots, u_p\rangle)$ and the formula is trivial. Now assume that u_p contains at least one 1. If p is at least 2, we have

$$\Phi_P((\langle u_1, \ldots, u_p\rangle) = \varphi_P(+\infty, u_1, \ldots, u_{p-1})10^{\delta_P(u_{p-1}, u_p)}01^{|u_p|_1}$$

with $|u_p|_1 \geq 1$, so that for some k one has

$$\theta_{\partial P}(\Phi_P((\langle u_1, \ldots, u_p\rangle)) = \varphi_P(+\infty, u_1, \ldots, u_{p-1})10^k.$$

On the other hand we have

$$\Phi_P((\langle u_1, \ldots, u_{p-1}, 0^i\rangle) = \varphi_P(+\infty, u_1, \ldots, u_{p-1})10^{\delta_P(u_{p-1}, 0^i)}0,$$

which has precisely the desired form. Finally if p is 1, $\Phi_P(\langle u_1\rangle)$ belongs to 0^*1^*, so that $\theta_{\partial P}(\Phi_P(\langle u_1\rangle))$ is the leftmost point 0^k of $\mathrm{Supp}\ \partial P$, which is also $\Phi_P(\langle 0^i\rangle)$. ■

The description of the correspondence between the points and cuts of ∂P and the points and cuts of P is now complete.

3. The normal form

Our aim is to obtain a canonical description for the expanded terms, *i.e.* for the cuts of the iterated derived terms $\partial^k P$. An inductive scheme is given by Lemma 2.4. Indeed the cuts of $\partial^k P$ are expressed in terms of the cuts of $\partial^{k-1} P$, which in turn are expressed in terms of the cuts of $\partial^{k-2} P$, and so on. Eventually one reaches the cuts of P, which will be considered as atomic objects. The terms obtained by removing all ∂'s in decompositions as above will be called *P-normal* terms. They will turn out be behave very nicely.

In order to work with the cuts of P considered as atomic, it is convenient to introduce for every term P a set of new variables indexed by the cuts of P. A term constructed using these variables and the operator $\bullet$ will be called a *P-term*, and the set of all P-terms will be denoted by $\mathcal{W}_P$. We shall denote by (Q) the variable associated with Q.

Definition. Let P be a fixed term. For every term Q, the *P-factorization* of Q is the P-term $(Q)_P$ defined by

$$\begin{cases} (Q)_P = (Q) & \text{if } Q \text{ is a cut of } P, \\ (Q')_P (Q'')_P \bullet & \text{if } Q \text{ is not a cut of } P \text{ and } Q \text{ is } Q'Q''\bullet. \end{cases}$$

The P-factorization of any term exists since a is always a cut of P. For every P-term Q, the evaluation of Q is the term obtained by replacing each new variable by the corresponding term, *i.e.* simply by removing all parentheses from Q. Observe that any term is equal to the evaluation of its P-factorization. If we order P-terms by the lexicographical extension of the ordering satisying $\bullet <_{\scriptscriptstyle Lex} (Q)$ for each variable (Q) and $(Q) <_{\scriptscriptstyle Lex} (Q')$ if and only if $Q <_{\scriptscriptstyle Lex} Q'$ holds, then P-factorization and evaluation preserve $<_{\scriptscriptstyle Lex}$. Observe that these definitions immediately extend to the case of an infinite term P.

Example. The a^ω-terms are the terms involving the variables (a), $(aa\bullet)$, $(aaa\bullet\bullet)$, *etc.*.. Let Q be the term $aaa\bullet\bullet a\bullet aa\bullet\bullet$: the a^ω-factorization of Q is the term $(aaa\bullet\bullet)(a)\bullet(aa\bullet)\bullet$ whose length is 5. Observe that $(Q)_{a^\omega}$ is not the only a^ω-term whose evaluation is Q: for instance $(a)(aa\bullet)\bullet(a)\bullet(aa\bullet)\bullet$ is another such a^ω-term.

Definition. i) The *degree* of a term Q is the maximal number of 0's occurring in some point in the support of Q.

ii) Let P be a (finite or infinite) term, and Q be a term in $\mathcal{W}$. The *P-degree* $\mathbf{d}_P(Q)$ of Q is the degree of the term $(Q)_P$.

The height of the term Q is the maximal number of 0's and 1's occurring in some point of the support of Q. So the degree of a term is a kind of 'left height'. The a^ω-degree of the term Q considered in the example above is 2 since the support of $(Q)_{a^\omega}$ is $\{00, 01, 1\}$. With the notion of P-degree, we can introduce the crucial notion of P-head of a term. We begin with finite terms.

Definition. Let P be a finite term in $\mathcal{W}$. For any term Q in $\mathcal{W}$, the *P-head* $\Theta_P(Q)$ is defined by

$$\Theta_P(Q) = \begin{cases} \mathrm{cut}(P, \theta_P(u)) & \text{if } \mathbf{d}_P(Q) = 0 \text{ and } Q \text{ is } \mathrm{cut}(P, u), \\ Q_1 \ldots Q_q a \bullet^q & \text{if } \mathbf{d}_P(Q) \geq 1 \text{ and } Q \text{ is } Q_1 \ldots Q_q Q_{q+1} \bullet^q \\ & \quad \text{with } \mathbf{d}_P(Q_q) = \mathbf{d}_P(Q) - 1 \geq \mathbf{d}_P(Q_{q+1}). \end{cases}$$

The definition makes sense since the expression of a term Q as $Q_1 \ldots Q_q Q_{q+1} \bullet^q$ with $\mathbf{d}_P(Q_q) = \mathbf{d}_P(Q) - 1 \geq \mathbf{d}_P(Q_{q+1})$ is unique. The idea is that the P-head of Q is obtained (for terms with P-degree at least 1) by keeping the 'complicated part' of Q (with respect to P-degree) and collapsing the simple final part to a. For instance the $a^{[3]}$-head of the term $aaa\bullet\bullet a\bullet aa\bullet\bullet$ considered above is the term $aaa\bullet\bullet a\bullet a\bullet$ since the index 'q' is 1 in this case. Observe that $\Theta_P(Q)$ is always a cut of Q, which implies $\Theta_P(Q) \leq_{\text{Lex}} Q$ and $\Theta_P(Q) \sqsubseteq_{LD} Q$. The extension to infinite terms is easy thanks to the following compatibility result.

Lemma 1. *Assume that P is a cut of P', and that Q is a proper cut of P. Then $\Theta_P(Q)$ and $\Theta_{P'}(Q)$ coincide.*

Proof. The result is vacuously true if P is a. The point is that, if P is $\mathrm{cut}(P', u)$ and Q is $\mathrm{cut}(P', v')$ (with $v < u$), then Q is $\mathrm{cut}(P, v)$ where v is obtained from v' as follows. Write v' as $u'0w$ where u' is the maximal common prefix to u and v'. Then v is $1^{|u'|_1}0w$. Let R be the subterm of P' whose root has address $u'0$. Then R' is also the subterm of P whose root has address $1^{|u'|_1}0$, and both $\Theta_P(Q)$ and $\Theta_{P'}(Q)$ are $\mathrm{cut}(P', u'0\theta_R(w))$. ■

Definition. Let P be a (finite or infinite) term in $\widetilde{\mathcal{W}}$.

i) P-$\mathbf{NF}_0$ is the set of all cuts of P;

ii) for $k \geq 0$, P-$\mathbf{NF}_{k+1}$ is the set of all terms of the form $Q_1 \ldots Q_{q+1} \bullet^q$ where q is at least 1, $Q_1, \ldots, Q_q$ belong to P-$\mathbf{NF}_k$, Q_{q+1} belongs to the union P-$\overline{\mathbf{NF}}_k$ of all P-$\mathbf{NF}_\ell$ for $\ell \leq k$ and, for $j \leq q$ the condition $\Theta_P(Q_j) >_{\text{Lex}} Q_{j+1}$ holds.

The elements of any set P-$\mathbf{NF}_k$ are called *P-normal* terms.

Example. As above the case of the infinite term a^ω will be fundamental, and we make the convention that reference term when omitted is supposed to be a^ω.

Then the normal terms of degree 0 are exactly the terms $a^{[n]}$. For any n the head of $a^{[n]}$ is $a^{[n]}$ itself: for any $n' > n$, the term $a^{[n]}$ is $\mathrm{cut}(a^{[n']}, 1^{n-1}0)$ and $\theta_{a^{[n']}}(1^{n-1}0)$ is $1^{n-1}0$. So the condition $\Theta(a^{[n]}) >_{\scriptscriptstyle Lex} a^{[m]}$ is equivalent to $n > m$, and the normal terms of degree 1 are the terms $a^{[n_1]}a^{[n_2]}\ldots a^{[n_{q+1}]}\bullet^q$ with $q \geq 1$ and $n_1 > n_2 > \ldots > n_{q+1}$. The table below represents the first elements of $\overline{\mathbf{NF}}_2$ (with respect to $<_{\scriptscriptstyle Lex}$). Actually the terms written below are the factorization of these normal terms using a for (a), b for $(aa\bullet)$, c for $(aaa\bullet\bullet)$. For every $k \geq 0$ the first element of $\mathbf{NF}_{k+1}$ is $ba\bullet(a\bullet)^k$.

$\mathbf{NF}_0$	a	b				c						
$\mathbf{NF}_1$			$ba\bullet$				$ca\bullet$					...
$\mathbf{NF}_2$				$ba\bullet a\bullet$	$ba\bullet b\bullet$			$ca\bullet a\bullet$	$ca\bullet b\bullet$	$ca\bullet ba\bullet\bullet$	$ca\bullet ba\bullet a\bullet\bullet$	

...			$cb\bullet$							$cba\bullet\bullet$
	$ca\bullet ba\bullet b\bullet\bullet$	$ca\bullet c\bullet$		$cb\bullet a\bullet$	$cb\bullet b\bullet$	$cb\bullet ba\bullet$	$cb\bullet ba\bullet a\bullet\bullet$	$cb\bullet ba\bullet b\bullet\bullet$	$cb\bullet c\bullet$	

Lemma 2. *Assume that Q belongs to P-$\mathbf{NF}_k$. Then*

i) Q has P-degree k;

ii) $(Q)_P$ cannot be a strict prefix of (the P-factorization of) any term in P-$\overline{\mathbf{NF}}_k$.

Proof. It is clear that the P-degree of Q is at most k. For proving that this degree cannot be strictly below k, the point is to show that no new 'cut grouping' may happen in the P-factorization of a term in P-$\mathbf{NF}_k$ compared with the P-factorization of its components in P-$\mathbf{NF}_{k-1}$. Clearly it suffices to show the property in the case $k = 1$, *i.e.* to show that no term in P-$\mathbf{NF}_1$ may be a cut of P. A term in P-$\mathbf{NF}_1$ has the form

$$\mathrm{cut}(P, u_1)\ldots\mathrm{cut}(P, u_{q+1})\bullet^q$$

with $\Theta_P(\mathrm{cut}(P, u_j)) >_{\scriptscriptstyle Lex} \mathrm{cut}(P, u_{j+1})$ for $j \leq q$. By Lemma 2.1 the latter condition is equivalent to $\theta_P(u_j) > u_{j+1}$, *i.e.* to $u_j \gg u_{j+1}$. So it is sufficient to prove the following

Claim. *If P is any term and $\langle u_1, \ldots, u_{q+1}\rangle$ is a descent of P, then no cut of P may be $=_{\scriptscriptstyle LD}$-equivalent to the product* $\mathrm{cut}(P, u_1)\ldots\mathrm{cut}(P, u_{q+1})\bullet^q$.

We prove this property by showing that the above product is a 'new' cut in some extension of P, which, by Lemma 1.4, establishes that it cannot be a cut of P, nor even either be $=_{\scriptscriptstyle LD}$-equivalent to a cut of P. We shall first assume $q = 1$. Because $u_1 \gg u_2$ is assumed, there exists points w, v_1, v_2 and an integer j such that u_1 is $w1^j0v_1$ and u_2 is $w0v_2$. For simplicity we assume $w = \Lambda$. We construct successive extensions P', P'', ... of P. In each case there will exist unique points u_i', u_i'', ... satisfying

$$\mathrm{cut}(P, u_i) =_{LD} \mathrm{cut}(P', u_i') =_{LD} \mathrm{cut}(P'', u_i'')\dots$$

for $i = 1, 2$. We choose the extensions so that these points have special geometrical properties. The first step is to obtain P' with $u_1' = 10v_1$, $u_2' = u_2$ (by applying distribution successively at 1^{j-1}, 1^{j-2}, ..., 1). The second step is to obtain P'' with $u_1'' = 101^k$, $u_2'' = u_2$ for some k. Now if P''' is obtained from P'' by applying distributivity at Λ, one obtains

$$\mathrm{cut}(P''', 1u_2) =_{LD} \mathrm{cut}(P'', 101^k)\mathrm{cut}(P'', u_2)\bullet =_{LD} \mathrm{cut}(P, u_1)\mathrm{cut}(P, u_2)\bullet,$$

which gives the result since $\mathrm{cut}(P''', 1u_2)$ is a 'new' cut of P''' (*i.e.* a cut which is not $=_{LD}$-equivalent to any cut in P''). The method can be extended to $q \geq 2$ iteratively.

Finally point (ii) follows, because every term whose Q is a strict prefix has the form $QR_1\bullet R_2\bullet\dots$, and therefore has P-degree at least $k+1$ provided that it is P-normal. ■

Remark. If Q, Q' are distinct cuts of P, the term Q may be a prefix of the term Q', but the atomic term $(Q)_P$ is certainly not a prefix of the atomic term $(Q')_P$: it is therefore essential to use the P-factorization in this matter (as well as at every place in the sequel where prefixes are concerned).

If Q is in P-$\mathbf{NF}_{k+1}$, then $\Theta_P(Q)$ must be $Q_1\dots Q_q a\bullet^q$ where $Q_1\dots Q_q Q_{q+1}\bullet^q$ is the decomposition as in the definition. Thus the integer q is unambiguously defined, and the decomposition is unique. It will be refered to as the *P-normal decomposition* of Q. We shall now easily establish that the P-normal terms are representatives for the P-expanded terms.

Definition. Assume that P is in $\mathcal{W}$. For every P-expanded term Q, the *P-reduction* $\|Q\|_P$ is the term defined as follows. If Q is a cut of P, then $\|Q\|_P$ is Q. If Q is a cut of $\partial^{k+1}P$, then

$$\|Q\|_P = \|Q_1\|_P\dots\|Q_{q+1}\|_P\bullet^q$$

where Q is $\mathrm{cut}(\partial^k P, u)$, u is $\Phi_{\partial^k P}(\langle v_1, \dots, v_{q+1}\rangle)$ and Q_j is $\mathrm{cut}(\partial^k P, v_j)$.

Lemma 3. *Assume that P is a cut of P'. Then the P'-reduction and the P-reduction of any P-expanded term coincide.*

Proof. If Q is a cut of P, Q is a cut of P' too and the result is obvious. If Q is a cut of $\partial^k P$ with $k \geq 1$, then $\partial^k P$ is a prefix of $\partial^k P'$. There exists some integer i verifying

$$\mathrm{cut}(\partial^k P, u) = \mathrm{cut}(\partial^k P', 0^i u)$$

for any u in the support of P. The explicit value shows that there exists a new integer j such that for any descent $\langle u_1, \ldots, u_p \rangle$ of $\partial^k P$ one has

$$\Phi_{P'}(\langle 0^i u_1, \ldots, 0^i u_p \rangle) = 0^j \Phi_P(\langle u_1, \ldots, u_p \rangle).$$

It follows that for any cut Q of $\partial^{k+1} P$, the cuts Q_1, …, Q_{q+1} appearing in the P-reduction of Q coincide with the ones appearing in the P'-reduction of Q. So inductively the P-reduction and the P'-reduction coincide. ■

Therefore there will be no problem to consider the P-reduction even for an infinite term P. The main technical argument is now the following

Lemma 4. *i) For any P-expanded term Q, the term $\|Q\|_P$ is $=_{LD}$-equivalent to Q.*

ii) P-reduction bijectively maps P-$\overline{\mathbf{EF}}_k$ onto P-$\overline{\mathbf{NF}}_k$, and P-$\mathbf{EF}_k$ onto P-$\mathbf{NF}_k$.

iii) P-reduction preserves the ordering $<_{Lex}$ on each set P-$\overline{\mathbf{EF}}_k$.

iv) For every Q in P-$\overline{\mathbf{EF}}_k$ one has

$$\|\Theta_{\partial^k P}(Q)\|_P = \Theta_P(\|Q\|_P).$$

Proof. We may assume that P is a finite term. The results will be proved for the restriction of P-reduction to P-$\overline{\mathbf{EF}}_k$ inductively on $k \geq 0$. For the cuts of P, everything is obvious since P-reduction is the identity mapping. From now on we assume that the results are proved for P-$\overline{\mathbf{EF}}_k$. Let Q be a cut of $\partial^{k+1} P$, say $\mathrm{cut}(\partial^{k+1} P, v)$. Then v is the image under $\Phi_{\partial^k P}$ of some descent $\langle v_1, \ldots, v_{q+1} \rangle$ of $\partial^k P$. Writing Q_j for $\mathrm{cut}(\partial^k P, v_j)$, we have by definition

$$\|Q\|_P = \|Q_1\|_P \cdots \|Q_{q+1}\|_P \bullet^q.$$

By induction hypothesis, $\|Q_j\|_P =_{LD} Q_j$ holds for every j. Now by Lemma 2.4 we have

$$Q =_{LD} Q_1 \ldots Q_{q+1} \bullet^q,$$

and this is enough to deduce $\|Q\|_P =_{LD} Q$. So point (i) is proved for $k+1$. Observe that the injectivity of P-reduction on P-$\overline{\mathbf{EF}}_{k+1}$ follows since we know that the elements of P-$\overline{\mathbf{EF}}_{k+1}$ are pairwise $=_{LD}$-unequivalent.

Next let us assume that the parameter q above is 0. By Lemma 2.5 this happens exactly if the term Q is the image under derivation of some term in P-$\overline{\mathbf{EF}}_k$, *i.e.* if Q is in P-$\overline{\mathbf{EF}}_{k+1} \setminus P$-$\mathbf{EF}_{k+1}$. Then $\|Q\|_P$ is $\|Q_1\|_P$, which by induction hypothesis belongs to $P\text{-}\overline{\mathbf{NF}}_k$. So $\|Q\|_P$ belongs to $P\text{-}\overline{\mathbf{NF}}_{k+1} \setminus P\text{-}\mathbf{NF}_{k+1}$. Now assume that q is at least 1. By induction hypothesis the terms $\|Q_j\|_P$ belong to $P\text{-}\overline{\mathbf{NF}}_k$. If k is 0, $\|Q_1\|_P$, ..., $\|Q_q\|_P$ automatically belong to $P\text{-}\mathbf{NF}_k$ (which is $P\text{-}\overline{\mathbf{NF}}_k$). Assume $k \geq 1$. By definition of a descent the points v_j with $j \leq q$ satisfy $v_j \gg v_{q+1}$, and therefore they cannot belong to 0^*1^* (if w belongs to 0^*1^*, $w \gg w'$ holds for no w'). Then Lemma 2.5 implies that the terms Q_1, ..., Q_q cannot come from $\partial^{k-1}P$, *i.e.* that they belong to $P\text{-}\mathbf{EF}_k$. So by induction hypothesis we can conclude that $\|Q_1\|_P$, ..., $\|Q_q\|_P$ belong to $P\text{-}\mathbf{NF}_k$. In order to conclude that $\|Q\|_P$ belongs to $P\text{-}\mathbf{NF}_{k+1}$, it remains to prove the condition

$$\Theta_P(\|Q_j\|_P) >_{\text{Lex}} \|Q_{j+1}\|_P$$

for $j = 1, \ldots, q$. By hypothesis $v_j \gg v_{j+1}$, and therefore $\theta_P(v_j) > v_{j+1}$, hold. By Lemma 2.1 this implies

$$\Theta_{\partial^k P}(Q_j) = \text{cut}(\partial^k P, \theta_P(v_j)) >_{\text{Lex}} Q_{j+1}.$$

By induction hypothesis we deduce

$$\|\Theta_{\partial^k P}(Q_j)\|_P >_{\text{Lex}} \|Q_{j+1}\|_P,$$

and we know that $\|\Theta_{\partial^k P}(Q_j\|_P)$ is $\Theta_P(\|Q_j\|_P)$, so the desired inequality holds, and $\|Q\|_P$ belongs to $P\text{-}\mathbf{NF}_{k+1}$.

The next (easy) step is to verify that any term Q' in $P\text{-}\mathbf{NF}_{k+1}$ is the image of some term in $P\text{-}\mathbf{EF}_{k+1}$ under reduction. Let $Q'_1 \ldots Q'_{q+1} \bullet^q$ be the normal decomposition of Q. By induction hypothesis, there exist points v_1, ..., v_{q+1} in the support of $\partial^k P$ such that, for $j \leq q+1$, Q'_j is $\|Q_j\|_P$ where Q_j is $\text{cut}(\partial^k P, v_j)$. Moreover the relation $\Theta_P(Q'_j) >_{\text{Lex}} Q'_{j+1}$ implies $\Theta_{\partial^k P}(Q_j) >_{\text{Lex}} Q_{j+1}$ by the same argument as above, and this implies $v_j \gg v_{j+1}$. Hence $\langle v_1, \ldots, v_{q+1}\rangle$ is a descent of $\partial^k P$, and clearly Q' is the reduction of the term $\text{cut}(\partial^{k+1P}, \Phi_{\partial^k P}(\langle v_1, \ldots, v_{q+1}\rangle))$. So the proof of (ii) for $k+1$ is complete.

Let us now assume that Q, Q' belong to $P\text{-}\overline{\mathbf{EF}}_{k+1}$ and $Q <_{\text{Lex}} Q'$ holds. We use the same notations as above for Q and the obviously corresponding ones for Q'. By Lemma 2.4 one has

$$\langle v_1, \ldots, v_{q+1}\rangle <^* \langle v'_1, \ldots, v'_{q'+1}\rangle.$$

The first possibility is that $\langle v_1, \ldots, v_{q+1}\rangle$ is a strict prefix of $\langle v'_1, \ldots, v'_{q'+1}\rangle$. Then the word $Q_1 \ldots Q_{q+1}$ is a strict prefix of the word $Q'_1 \ldots Q'_{q'+1}$, and the word $\|Q_1\|_P \ldots \|Q_{q+1}\|_P$ is a strict prefix of the word $\|Q'_1\|_P \ldots \|Q'_{q'+1}\|_P$. By a simple weight argument this implies

$$\|Q_1\|_P \ldots \|Q_{q+1}\|_P \bullet^q <_{\text{Lex}} \|Q'_1\|_P \ldots \|Q'_{q'+1}\|_P \bullet^{q'},$$

i.e. $\|Q\|_P <_{\text{Lex}} \|Q'\|_P$. The second possibility is that for some $r \leq \inf(q, q')$ one has $v_j = v'_j$ for $j < r$ and $v_r < v'_r$. This implies $Q_j = Q'_j$, and therefore $\|Q_j\|_P = \|Q'_j\|_P$ for $j < r$, and $Q_r <_{\text{Lex}} Q'_r$, whence, by induction hypothesis, $\|Q_r\|_P <_{\text{Lex}} \|Q'_r\|_P$. By Lemma 2.ii this implies $\|Q\|_P <_{\text{Lex}} \|Q'\|_P$ since $\|Q_r\|_P$ cannot be a strict prefix of $\|Q'_r\|_P$. The proof of (iii) for $k+1$ is complete, since the restriction of $<_{\text{Lex}}$ to $P\text{-}\overline{\mathbf{EF}}_{k+1}$ is a linear ordering.

It remains to compute the P-head of the term $\|Q\|_P$. By definition it is (always with the same notations) $\|Q_1\|_P \cdots \|Q_q\|_P a \bullet^q$ since $\|Q_q\|_P$ has P-degree k and $\|Q_{q+1}\|_P$ has P-degree at most k. On the other hand lemma 2.6 tells that $\theta_{\partial^{k+1}P}(v)$ is $\Phi_{\partial^k P}(\theta_P(\langle v_1, \ldots, v_q, 0^i\rangle)$ where 0^i is the leftmost point in the support of $\partial^k P$. So the term $\Theta_{\partial^{k+1}P}(Q)$ is

$$\mathrm{cut}(\partial^{k+1}P, \Phi_{\partial^k P}(\langle v_1, \ldots, v_q, 0^i\rangle).$$

By construction the P-reduction of this term is $\|Q_1\|_P \cdots \|Q_q\|_P a \bullet^q$, for $\mathrm{cut}(\partial^k P, 0^i)$ is a. This establishes point (iv) for $k+1$, and finishes the proof. ■

From the previous result and the results of Section 1 we immediately deduce the main result of this paper.

Theorem 5. (existence, uniqueness and complexity of the normal form) *i) Let P be any finite or infinite term. Every term Q satisfying $Q \sqsubseteq_{LD} P$ is $=_{LD}$-equivalent to exactly one P-normal term, which will be called the P-normal form of Q.*

ii) For P-normal terms, the ordering $\sqsubset_{LD}$ coincide with the lexicographical ordering $<_{\text{Lex}}$.

iii) If P is an eventually constant infinite term, the P-normal form of any term exists, and the function which maps a term to its P-normal form lies in the complexity class $\mathrm{DSPACE}(\exp^*(O(2^n)))$.

The result is clear from Lemma 4. In particular the P-reduction of a term of length N only used terms with length $\leq N$, which justifies the bound of (iii) above.

The inductive construction of normal terms is satisfactory and unambiguous since the normal decomposition is unique. But it does not give a direct geometrical criterion for recognizing normal terms. We shall now establish such a criterion.

Lemma 6. *Assume that R and R' are P-normal terms and $(R')_P$ is a prefix of $(R)_P$. Then, for every P-normal term Q with $\mathbf{d}_P(Q) \leq \mathbf{d}_P(R')$, $Q >_{\text{Lex}} R'$ is equivalent to $Q >_{\text{Lex}} R$.*

Proof. Because $R \geq_{\text{Lex}} R'$ holds by definition, the direct implication is obvious. The converse one is a crucial property of normal terms. Let k be the P-degree of R'. It suffices to prove the result for the case when R has P-degree $k+1$, the general case then follows inductively. Let $R_1 \ldots R_{r+1} \bullet^r$ be the P-normal decomposition of R. By hypothesis R' is R_1. By restricting to a sufficiently large cut of P if necessary we may assume that P is a finite term. Then there exists points $w_1, \ldots,$ w_{r+1} in the support of $\partial^k P$ such that R_j is $\|\text{cut}(\partial^k P, w_j)\|_P$ for $j = 1, \ldots, r+1$. By construction R is

$$\|\text{cut}(\partial^{k+1} P, \Phi_{\partial^k P}(\langle w_1, \ldots, w_{r+1} \rangle))\|_P.$$

Now by hypothesis the P-degree of Q is $\leq k$, which means that for some v in the support of $\partial^k P$ the term Q is is $\|\text{cut}(\partial^k P, v)\|_P$. Then $Q >_{\text{Lex}} R'$ implies $v > w_1$, and therefore $\langle v \rangle <^* \langle w_1, \ldots, w_{r+1} \rangle$. By Lemma 2.3 we deduce

$$\text{cut}(\partial^{k+1} P, \Phi_{\partial^k P}(\langle v \rangle)) >_{\text{Lex}} \text{cut}(\partial^{k+1} P, \Phi_{\partial^k P}(\langle w_1, \ldots, w_{r+1} \rangle)).$$

Applying P-reduction yields $Q >_{\text{Lex}} R$, as was claimed. ■

Remark. The preceding result does not say anything about the cuts of P since by Lemma 2 $(R')_P$ being a strict prefix of $(R)_P$ prevents R from being a cut of P.

Definition. i) A term $QR\bullet$ with positive P-degree is *P-descending* if the conditions $\mathbf{d}_P(Q) \geq \mathbf{d}_P(R) - 1$ and $\Theta_P(Q) >_{\text{Lex}} R$ hold.

ii) For A included in $\mathbb{S}$, the *interior* of A is the set of all strict prefixes of points in A. A subterm of Q is *P-inner* if the address of its root belongs to the interior of the support of $(Q)_P$.

Example. Let Q be the term $aaa\bullet\bullet a\bullet aa\bullet\bullet$. Since the a^ω-factorization of Q is $(aaa\bullet\bullet)(a)\bullet(aa\bullet)\bullet$, the a^ω-inner subterms of Q comprise Q itself and $aaa\bullet\bullet a\bullet$, but not $aa\bullet$. Observe that Q is a^ω-descending: the P-degree of $aaa\bullet\bullet a\bullet$ and $aa\bullet$ are respectively 1 and 0, and $\Theta_{a^\omega}(aaa\bullet\bullet a\bullet)$ is $aaa\bullet\bullet a\bullet$ itself, which is lexicographically after $aa\bullet$.

Theorem 7. (geometric characterization of P-normal terms) *A term Q is P-normal if and only if each P-inner subterm of Q is P-descending.*

Proof. First we prove inductively on k that every P-inner subterm of a term in P-$\mathbf{NF}_k$ is P-descending. If Q is a cut of P, then Q has no P-inner subterm and the property is vacuously true. Assume it proved for k, and let Q belong to P-$\mathbf{NF}_{k+1}$. Let $Q_1 \ldots Q_{q+1} \bullet^q$ be the normal decomposition of Q. By induction hypothesis every P-inner subterm of $Q_1, \ldots, Q_{q+1}$ is P-descending. It remains to show that the terms $Q_j ... Q_{q+1} \bullet^{q-j+1}$ are P-descending, and clearly it suffices to make the verification for $j = 1$, *i.e.* to prove that Q itself is P-descending. Now by construction the P-degree of $Q_2 ... Q_{q+1} \bullet^{q-1}$ is at most $k+1$, while the P-degree of Q_1 is exactly k. Moreover one has $\Theta_P(Q_1) >_{\scriptscriptstyle Lex} Q_2$ by definition of P-normal terms. By Lemma 5 this implies $\Theta_P(Q_1) >_{\scriptscriptstyle Lex} Q_2 ... Q_{q+1} \bullet^{q-1}$, and Q is P-descending. So the property holds for $k+1$.

Conversely assume that Q is a term with P-degree $k+1$ such that every P-inner subterm of Q is P-descending. Write Q as $Q_1 ... Q_{q+1} \bullet^q$ where $Q_1 ... Q_q a \bullet^q$ is $\Theta_P(Q)$. All P-inner subterms of $Q_1, \ldots, Q_{q+1}$ are P-descending and $Q_1, \ldots, Q_{q+1}$ have P-degree at most k, so by the induction hypothesis we conclude that $Q_1, \ldots, Q_{q+1}$ are P-normal. By construction Q_q has P-degree exactly k, so it belongs to P-$\mathbf{NF}_k$, and necessarily the same holds for $Q_1, \ldots, Q_{q-1}$ because of the degree assumptions in Q. Also Q_{q+1} has P-degree at most k and therefore belongs to P-$\overline{\mathbf{NF}}_k$. Now the condition that $Q_j ... Q_{q+1} \bullet^{q-j+1}$ is P-descending implies $\Theta_P(Q_j) >_{\scriptscriptstyle Lex} Q_{j+1} ... Q_{q+1} \bullet^{p-j}$, and, by the trivial direction of Lemma 5, $\Theta_P(Q_j) >_{\scriptscriptstyle Lex} Q_{j+1}$. It follows that Q belongs to P-$\mathbf{NF}_{k+1}$, which finishes the proof. ■

Example. The descent condition solves some delicate questions which appear when one tries to select 'by hand' $=_{\scriptscriptstyle LD}$-representatives. For instance the term $cb \bullet ca \bullet \bullet$ is not normal (its normal form is $cba \bullet \bullet$), while the 'similar' term $cba \bullet \bullet ca \bullet \bullet$ is normal (and its contracted form $cba \bullet a \bullet \bullet$ is not).

We finish this section with alternative descriptions of the P-normal terms which we shall use later. We introduce a variant of the notion of P-head of a term.

Definition. Let P be a finite term. For any term Q in $\mathcal{W}$ distinct of a, the term $\Theta'_P(Q)$ is defined by

$$\Theta'_P(Q) = \begin{cases} \mathrm{cut}(P, v) & \text{if } \mathbf{d}_P(Q) = 0,\ Q \text{ is } \mathrm{cut}(P,u) \text{ and } v \text{ is the immediate} \\ & \text{predecessor of } \theta_P(u) \text{ in Supp } P, \\ Q_1 ... Q_q \bullet^{q-1} & \text{if } \mathbf{d}_P(Q) \geq 1 \text{ and } Q \text{ is } Q_1 ... Q_q Q_{q+1} \bullet^q \\ & \text{with } \mathbf{d}_P(Q_q) = \mathbf{d}_P(Q) - 1 \geq \mathbf{d}_P(Q_{q+1}). \end{cases}$$

We already observed that the term $\Theta_P(Q)$ is always a cut of Q. So is the term $\Theta'_P(Q)$.

Lemma 8. *Assume that Q is P-normal. Then the term $\Theta'_P(Q)$ is the immediate predecessor of $\Theta_P(Q)$ in the family of all cuts of Q ordered by $<_{\scriptscriptstyle Lex}$.*

Proof. If Q is a cut of P, the property is true by definition. Now assume that the P-degree of Q is $k+1$. Then there exists a descent $\langle v_1, \ldots, v_{q+1}\rangle$ of $\partial^k P$ such that Q is the P-reduction of $\mathrm{cut}(\partial^{k+1}P, v)$, where v is the image of $\langle v_1, \ldots, v_{q+1}\rangle$ under $\Phi_{\partial^k P}$. Write Q_j for $\|\mathrm{cut}(\partial^k P, v_j)\|_P$. We have seen in the proof of Lemma 4 that the P-normal decomposition of Q is $Q_1 \ldots Q_{q+1}\bullet^q$, so that $\Theta_P(Q)$ is the P-reduction of

$$\mathrm{cut}(\partial^{k+1}P, \Phi_{\partial^k P}(\langle v_1, \ldots, v_q, 0^i\rangle)),$$

where 0^i is the leftmost point of $\mathrm{Supp}\,\partial^k P$. The same argument shows that $\Theta'_P(Q)$ is

$$\mathrm{cut}(\partial^{k+1}P, \Phi_{\partial^k P}(\langle v_1, \ldots, v_q\rangle)).$$

The result follows, since the descent $\langle v_1, \ldots, v_q\rangle$ is the immediate predecessor of the descent $\langle v_1, \ldots, v_q, 0^i\rangle$ in the set $\mathrm{Desc}\partial^k P$ ordered by $<^*$. ■

As for Θ_P the extension of Θ'_P to the case of an infinite term P is easy.

By Theorem 7 we know that a product $QR\bullet$ can be P-normal only if Q and R are P-normal and $\mathbf{d}_P(Q)$ is at least $\mathbf{d}_P(R)-1$. We shall separates the cases $\mathbf{d}_P(Q) \geq \mathbf{d}_P(R)$ and $\mathbf{d}_P(Q) = \mathbf{d}_P(R)-1$.

Proposition 9. *Assume that Q, R are P-normal with $\mathbf{d}_P(Q) \geq \mathbf{d}_P(R)$. Then the following are equivalent*

i) $QR\bullet$ is P-normal;

ii) $\Theta_P(Q) >_{\text{Lex}} R$ holds;

iii) $\Theta'_P(Q) \geq_{\text{Lex}} R$ holds;

Moreover, if $\mathbf{d}_P(Q)$ is at least 1, the above conditions are equivalent to

iv) $Q >_{\text{Lex}} R$ holds and, if $Q_1 \ldots Q_{q+1}\bullet^q$ is the P-normal decomposition of Q, the word $(Q_1)_P \ldots (Q_q)_P$ is not a strict prefix of $(R)_P$.

Proof. The equivalence of (i) and (ii) is the very definition of P-normal terms. Since $\Theta'_P(Q) <_{\text{Lex}} \Theta_P(Q)$ always holds, clearly (iii) implies (ii). Let k be the P-degree of Q. By construction there exist points v and w in the support of $\partial^k P$ such that Q is the P-reduction of $\mathrm{cut}(\partial^k P, v)$ and R is the P-reduction of $\mathrm{cut}(\partial^k P, w)$. Now $\Theta_P(Q) >_{\text{Lex}} R$ implies $\theta_{\partial^k P}(v) > w$, and therefore w is at most the immediate predecessor of $\theta_{\partial^k P}(v)$ in $\mathrm{Supp}\,\partial^k P$. By Lemma 7 the cut of $\partial^k P$ associated with this immediate predecessor is exactly $\Theta'_P(Q)$. In other words $\Theta_P(Q) >_{\text{Lex}} R$ implies $\Theta'_P(Q) \geq_{\text{Lex}} R$. Finally, assume that the P-degree of Q is at least 1. We use the notations of the proof of Lemma 7 (thus writing $k+1$ for the P-degree of Q). Among all P-normal terms R with P-degree at most $k+1$ which satisfy $Q >_{\text{Lex}} R$, the ones which do not satisfy $\Theta_P(Q) >_{\text{Lex}} R$ are exactly the ones satisfying

$$\Theta'_P(Q) <_{\text{Lex}} R <_{\text{Lex}} Q,$$

which are be the P-reductions of the terms $\mathrm{cut}(\partial^{k+1}P, \Phi_{\partial^k P}(\alpha))$ where α is a descent of $\partial^k P$ satisfying

$$\langle v_1, \ldots, v_q \rangle <^* \alpha <^* \langle v_1, \ldots, v_q, v_{q+1} \rangle.$$

These descents are exactly the descent s with the form $\langle v_1, \ldots, v_q, w_{q+1}, \ldots, w_{r+1} \rangle$, and the corresponding P-normal terms have the form $Q_1 \ldots Q_q R_{q+1} \ldots R_{r+1} \bullet^r$ for some (P-normal) terms $R_{q+1}, \ldots, R_{r+1}$. ■

The case of a product $QR\bullet$ with $\mathbf{d}_P(Q) = \mathbf{d}_P(R) - 1$ is easily deduced from the preceding one.

Proposition 10. *Assume that Q, R are P-normal with $\mathbf{d}_P(Q) = \mathbf{d}_P(R) - 1$. Let R' be the left subterm of R. Then the following are equivalent*

i) $QR\bullet$ is P-normal;

ii) $QR'\bullet$ is P-normal;

iii) $\Theta_P(Q) >_{\text{Lex}} R'$ holds;

iv) $\Theta'_P(Q) \geq_{\text{Lex}} R'$ holds;

Moreover, if $\mathbf{d}_P(Q)$ is at least 1, the above conditions are equivalent to

v) $Q >_{\text{Lex}} R'$ holds and, if $Q_1 \ldots Q_{q+1} \bullet^q$ is the P-normal decomposition of Q, the word $(Q_1)_P \ldots (Q_q)_P$ is not a strict prefix of $(R')_P$.

Proof. The equivalence of (i) and (ii) follows from the definition of P-normal terms. The subsequent equivalences then follow from Proposition 8. ■

It is easy to rephrase the geometrical characterization of P-normal terms given in Theorem 6 using the results above. In particular Proposition 8.iv and 9.v yield criterions which involve only the lexicographical ordering and the prefix relation.

4. Applications

The normal terms defined here do not coincide with the ones constructed by Richard Laver in [10] and [11]. For instance (with the same notations as above), the term *cb•ca•a••* is the one law version of a normal term in the sense of [10], while its a^ω-normal form is *cba••cb•a••*. In both cases an interesting feature is that the restriction of the ordering $\sqsubset_{LD}$ to (P)-normal terms coincide with the natural lexicographical ordering. From the technical point of view, the definition of Laver's normal form entails an inductive decomposition of the terms as $Q_1 Q_2 \bullet Q_3 \bullet Q_4 \bullet \ldots$ while our normal terms are decomposed as $Q_1 Q_2 Q_3 \ldots \bullet\bullet$. It could be therefore natural to call the first one a *left* decomposition, leading to a *left* normal form, while the second decomposition could be called a *right* decomposition thus leading to a *right* normal form.

We shall finish this paper with three properties of the free left distributive law. The first one is already established in [10], and the second one could also certainly be proved using Laver's 'left' normal form. The third one is new. The proofs below mainly appear as an illustration of what can be done using the 'right' P-normal terms.

We begin with a description of the action of mutiplying by P on the left for P-normal terms.

Lemma 1. *Assume that P, Q are finite terms and Q is P^ω-normal. Then the (P^ω-factorization of the) P^ω-normal form of $PQ\bullet$ is the image of (the P^ω-factorization of) Q under the substitution σ defined for terms with P^ω-degree 0 by*

$$\sigma : (P^n\mathrm{cut}(P,u)\bullet^n) \mapsto (P^{n+1}\mathrm{cut}(P,u)\bullet^{n+1}).$$

Proof. For R a P^ω-term write R^σ for the image of R under σ, *i.e.* for the term obtained from R by replacing any variable occurring in R by its image under σ. We shall not distinguish between a term and its P^ω-factorization in the sequel. The definition of σ makes sense since the cuts of P^ω are exactly the terms of the form $P^n\mathrm{cut}(P,u)\bullet^n$ with n any nonnegative integer and u a point in the support of P. It is clear that R^σ is $=_{LD}$-equivalent to $PR\bullet$ for every cut R of P^ω, and therefore Q^σ is $=_{LD}$-equivalent to $PQ\bullet$ for any term Q. So the only point to verify is that, if Q is P^ω-normal, then Q^σ is P^ω-normal as well. We prove this property inductively on Q. If Q is a cut of P^ω the result is clear. Now assume that Q, R and $QR\bullet$ are P^ω-normal. By induction we assume that Q^σ and R^σ are P^ω-normal, and it remains to show that $Q^\sigma R^\sigma\bullet$ is P^ω-normal. We use the criterions established at the end of Section 3. Assume $\mathbf{d}_{P^\omega}(Q) \geq \mathbf{d}_{P^\omega}(R)$. Then $\Theta'_{P^\omega}(Q) \geq_{Lex} R$ holds by Proposition 3.9. Now σ preserves the P^ω-degree, and therefore the term $\Theta'_{P^\omega}(Q^\sigma)$ is exactly $(\Theta'_{P^\omega}(Q))^\sigma$ (the corresponding equality is not true for Θ_{P^ω} because of the additional a). Finally σ preserves the lexicographical ordering $<_{Lex}$, and $\Theta'_{P^\omega}(Q) \geq_{Lex} R$ implies $\Theta'_{P^\omega}(Q^\sigma) \geq_{Lex} R^\sigma$, and $Q^\sigma R^\sigma\bullet$ is P^ω-normal. The argument is similar (using Proposition 3.10) in the case $\mathbf{d}_{P^\omega}(Q) = \mathbf{d}_{P^\omega}(R)-1$. ■

As a first application we have the following property of the ordering $\sqsubset_{LD}$ (stated in [10] and also established by R. Mc Kenzie using a direct proof).

Proposition 2. *For every terms P, Q in $\mathcal{W}$, the relation $Q \sqsubset_{LD} PQ\bullet$ holds, and therefore any strict subterm of a term is a (strict) predecessor of this term with respect to $\sqsubset_{LD}$.*

Proof. Let Q' be the P^ω-normal form of Q. Then $PQ\bullet$ is $=_{LD}$-equivalent to Q'^σ, where σ is as in Lemma 1. Now $Q' <_{Lex} Q'^\sigma$ holds, since

$$P^n \mathrm{cut}(P,u)\bullet^n <_{Lex} P^{n+1}\mathrm{cut}(P,u)\bullet^{n+1}$$

obviously holds for every n and u. So $Q' \sqsubset_{LD} PQ'\bullet$ holds, which is equivalent to $Q \sqsubset_{LD} PQ\bullet$. ■

We now repeat the analysis of Lemma 1 replacing left multiplication by P by left multiplication by $PP\bullet$.

Lemma 3. *Assume that P, Q are finite terms and Q is P^ω-normal. Then the (P^ω-factorization of the) P^ω-normal form of $PP\bullet Q\bullet$ is the image of (the P^ω-factorization of) Q under the substitution τ defined for terms with P^ω-degree at most* 1 *by*

$$\tau : (P^{1+n_1}\mathrm{cut}(P,u_1)\bullet^{1+n_1})\ldots(P^{1+n_r}\mathrm{cut}(P,u_r)\bullet^{1+n_r})(\mathrm{cut}(P,u_{r+1}))\ldots(\mathrm{cut}(P,u_{q+1}))\bullet^q$$

$$\mapsto \begin{cases} (P^{2+n_1}\mathrm{cut}(P,u_1)\bullet^{2+n_1})\ldots(P^{2+n_r}\mathrm{cut}(P,u_r)\bullet^{2+n_r}) \\ \qquad (PP\bullet)(\mathrm{cut}(P,u_{r+1}))\ldots(\mathrm{cut}(P,u_{q+1}))\bullet^{q+1} & \text{if } r < q+1, \\ (P^{2+n_1}\mathrm{cut}(P,u_1)\bullet^{2+n_1})\ldots(P^{2+n_r}\mathrm{cut}(P,u_r)\bullet^{2+n_r})\bullet^q & \text{if } r = q+1. \end{cases}$$

Proof. The formula above is more complicated than the formula of Lemma 1 because the cuts of P^ω may be mapped into terms with P^ω-degree 1. So the least family of terms which is stable under τ is P^ω-$\overline{\mathbf{NF}}_1$. The principle of the proof however remains the same. First is clear that, for any term Q with P^ω-degree at most 1, the term Q^τ is $=_{LD}$-equivalent to $PP\bullet Q\bullet$, and, therefore, R^τ is $=_{LD}$-equivalent to $PP\bullet R\bullet$ for any P^ω-term R. So the only problem is to verify that, if Q is a P^ω-normal term, so is Q^τ. By the explicit value the property is obvious when the P^ω-degree of Q is at most 1. Then we observe that τ preserves the lexicographical ordering. It remains to verify that the descending condition is preserved under τ. For terms with P^ω-degree at least 2, τ preserves Θ'_{P^ω} and we apply the criterions of Propositions 3.9 and 3.10. A direct verification is needed for the terms Q with P^ω-degree 1, say

$$Q = (P^{1+n_1}\mathrm{cut}(P,u_1)\bullet^{1+n_1})\ldots(P^{1+n_r}\mathrm{cut}(P,u_r)\bullet^{1+n_r})(\mathrm{cut}(P,u_{r+1}))\ldots(\mathrm{cut}(P,u_{q+1}))\bullet^q.$$

We assume first $r < q$. One obtains

$$\Theta'_{P^\omega}(Q) = (P^{1+n_1}\mathrm{cut}(P,u_1)\bullet^{1+n_1})\ldots(P^{1+n_r}\mathrm{cut}(P,u_r)\bullet^{1+n_r})(\mathrm{cut}(P,u_{r+1}))\ldots(\mathrm{cut}(P,u_q))\bullet^{q-1}$$

and, in this case $(\Theta'_{P^\omega}(Q))^\tau$ is equal to $\Theta'_{P^\omega}(Q^\tau)$. If r is exactly q, then one has

$$\Theta'_{P^\omega}(Q) = (P^{1+n_1}\mathrm{cut}(P,u_1)\bullet^{1+n_1})\ldots(P^{1+n_r}\mathrm{cut}(P,u_r)\bullet^{1+n_r})\bullet^{q-1},$$

implying

$$(\Theta'_{P^\omega}(Q))^\tau = (P^{2+n_1}\mathrm{cut}(P,u_1)\bullet^{1+n_1})\ldots(P^{2+n_r}\mathrm{cut}(P,u_r)\bullet^{1+n_r})\bullet^{q-1},$$

while

$$\Theta'_{P^\omega}(Q^\tau) = (P^{2+n_1}\mathrm{cut}(P,u_1)\bullet^{1+n_1})\ldots(P^{2+n_r}\mathrm{cut}(P,u_r)\bullet^{1+n_r})(PP\bullet)\bullet^{q}.$$

The explicit value of R^τ (for R a P^ω-normal term with P^ω-degree 1) shows that $(\Theta'_{P^\omega}(Q))^\tau >_{Lex} R^\tau$ (which follows from $\Theta'_{P^\omega}(Q) >_{Lex} R$) implies $\Theta'_{P^\omega}(Q^\tau) >_{Lex} R^\tau$ (although $\Theta'_{P^\omega}(Q^\tau)$ is bigger than $(\Theta'_{P^\omega}(Q))^\tau$). The last case is for $r = q + 1$ (*i.e.* all variables have the form $P^n\mathrm{cut}(P,u)$ with $n \geq 1$) Then τ preserves Θ'_{P^ω} easily. Finally we conclude that the P^ω-normality of $QR\bullet$ implies the P^ω-normality of $Q^\tau R^\tau\bullet$, as was desired. ■

From the above computation we can now deduce (with a rather surprisingly simple proof) an algebraic criterion for left divisibility in the free left distributive algebra. This criterion is reminiscent of similar properties used in [1] in the context of the elementary embeddings in the set theory of measurable cardinals.

Proposition 4. *For every terms P, R in $\mathcal{W}$, the following are equivalent:*
i) there exists a term Q such that R is $=_{LD}$-equivalent to $PQ\bullet$;
ii) the terms $PR\bullet$ and $PP\bullet R\bullet$ are $=_{LD}$-equivalent.

Proof. Condition (ii) is certainly necessary for (i): if $R =_{LD} PQ\bullet$ holds, then one has

$$PR\bullet =_{LD} PPQ\bullet\bullet =_{LD} PP\bullet PQ\bullet\bullet =_{LD} PP\bullet R\bullet.$$

Conversely assume that R does not satisfy condition (i). Let R' be the P^ω-normal form of R. By Lemma 1, some cut of P must occur in R' (in contradistinction with the cuts of P^ω of the form $P^n\mathrm{cut}(P,u)$ with $n \geq 1$). Indeed every cut of P^ω which is not a cut of P lies in the image of the substitution σ, and, therefore, if all variables in R' were such ones, R' would lie in the image of σ and (i) would be true. It follows that some cut of P occurs in the normal form of $PP\bullet R\bullet$ since, by the explicit form of Lemma 3, every cut of P occurring in R' still occurs in the P^ω-normal form of $PP\bullet R\bullet$. Now no cut of P occurs in the P^ω-normal form of $PR\bullet$. Therefore the P^ω-normal forms of $PR\bullet$ and $PP\bullet R\bullet$ certainly do not coincide since they do not involve the same variables. Finally $PR\bullet$ and $PP\bullet R\bullet$ cannot be $=_{LD}$-equivalent since they do not have the same P^ω-normal form. ■

The above result can be seen as a first step toward a quantifier elimination for the theory of free left distributive algebras, a question which seems to be completely open for the moment.

We finally come to a natural geometric question about term extensions which remained open for several years. If the term Q is an extension of the term P, each point in the support of Q can be given a welldefined origin in the support of P as follows. The origin is defined inductively, and for the elementary extension $RR'R''\bullet\bullet \mapsto RR'\bullet RR''\bullet\bullet$ the origin of the points with the form $00x$, $01x$, $10x$ and $11x$ are respectively $0x$, $10x$, $0x$ and $11x$. If we allow terms with several variables, then the origin is immediately readable when the initial term P has pairwise distinct variables. Indeed in this case, if Q is any extension of P and v belongs to the support of Q, then the origin of v in P is the unique u such that the variable of Q at v is the variable of P at u. The problem is to find a similar characterization when P has only one variable. Normal form and cuts give the solution.

Theorem 5. *Assume that P belongs to $\mathcal{W}$ and Q is any extension of P. For v in the support of Q, the origin of v in P is the unique point u in the support of P such that* $(\mathrm{cut}(P,u))$ *is the rightmost variable in the P-normal form of* $\mathrm{cut}(Q,v)$.

The proof is not difficult, but being rather long it will not be given here. The point is to establish the property when Q is some term $\partial^k P$. This case in turn follows from the particular case of ∂P for which the computation can be completed.

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