ANOTHER USE OF SET THEORY

PATRICK DEHORNOY

Abstract. Here, we analyse some recent applications of set theory to topology and argue that set theory is not only the closed domain where mathematics is usually founded, but also a flexible framework where imperfect intuitions can be precisely formalized and technically elaborated before they possibly migrate toward other branches. This apparently new role is mostly reminiscent of the one played by other external fields like theoretical physics, and we think that it could contribute to revitalize the interest in set theory in the future.

Traditionally, there have been *two* uses of set theory. The first is well-known: after it was created by Georg Cantor at the end of the last century in order to elaborate the notion of transfinite number, set theory acquired a special status in the first decades of this century when it became apparent that every mathematical object could be *represented* as a set, and therefore that set theory could be used as a foundational system. The second comes from the (partial) failure of the former: according to Gödel's second incompleteness theorem, no formal system can exhaustively describe the whole mathematical universe, and this applies in particular to the Zermelo-Fraenkel axiomatization of set theory. This legitimizes the study of all possible extensions of this system, and proves that resorting to such extensions is inevitable to decide some open questions. In this perspective, the point is no longer to actually *prove* the properties, but rather to *calibrate* them in a scale of increasingly strong logical axioms.

Here, we present what could be seen as a *third* and apparently new use of set theory. In the example to be developed, dealing with recent applications of this theory to algebra and topology, set theory has been used to crystallize some intuitions for which other frameworks would have been too rigid, and to elaborate them with the help of its specific tools, including strong axioms. This role is certainly distinct from the previous ones as all strong axioms have been subsequently eliminated from the picture, so that the final results have no link with set theoretical hypotheses, which therefore appear only as technical auxiliaries. However, we shall argue that the significance of set theory in the above case is no less than in the classic applications "of the second type". Actually, the new use described here is very close to that of theoretical physics, when it gives heuristic intuitions that have to be rigorously justified afterwards.

This text is organized as follows: First, we shall give a sketchy description of the technical results that led us to the present analysis (for mathematical details, we refer to [5], or possibly to the original papers [21], [3], [14], [4], [13], [15], [6]). Then, we shall analyze the specific role of set theory in these examples. Finally, we shall briefly discuss the present status of set theory, and the new prospects that are offered by applications like the one we present here

1991 *Mathematics Subject Classification.* 03A05, 00A30, 03A75.

1. Recent applications of set theory to the topology of braids

The notion of an *elementary embedding* has a prominent role in contemporary set theory. Studying its purely algebraic aspects has recently led to several new applications, whose specificity is to involve especially simple and usual mathematical objects, namely Artin's braid groups. Here, we would like to emphasize how naturally these applications cropped up and, to some extent, *had to*.

1.1. **Self-similarity as a strong form of infinity.** The initial intuition in the sequel is the notion of *self-similarity*, that is, the concept of objects that remain similar to themselves when considered on various scales. One knows how fruitful this idea has been in recent mathematical research, in particular in the study of dynamical systems where fractal objects appear, or, to take another area, in the study of coherent spaces in lambda-calculus. Set theory provides another framework to study self-similarity with the help of *large cardinals*, that is, strong axioms of infinity.

Set theory is the study of infinity. When combined with the basic properties of sets, the mere assumption that *one* infinite object exists is sufficient to reveal a deep structure on infinite ordinals and cardinals (Cantor's alephs). It is therefore natural to iterate the process of going from finite to infinite and consider *higher infinities* that behave with respect to usual infinities the way the latter behave with respect to finite objects. This idea of *large cardinals* has met with great success because it has essentially made it possible to fulfill Gödel's program of classifying all extensions of the Zermelo-Fraenkel system: in some technically satisfying sense, these extensions can be classified in terms of the generalized axioms of infinity that they imply (see [10], which is the most comprehensive textbook to date).

Infinite objects can be distinguished from finite ones in many aspects, and a specific generalization can be considered for each. In the present case, we shall take into account the property that infinity entails self-similarity phenomena, namely that an infinite set is so large that it is similar to one of its proper subsets. For instance, the function that maps every natural number to its successor gives a one-to-one correspondence between the set of all nonnegative numbers and the set of all positive numbers: in other words, it is a non-bijective injection of the set into itself.

In the case above, we may observe that the similarity between the set and one of its proper subsets deals not only with the "naked" set, but also with the additional structure provided by the usual ordering. The logical framework of set theory immediately suggests a systematization of this idea: a "large", or, better, a self-similar set will be any set that is similar to one of its proper subsets with respect to *all* notions that are definable by means of a mathematical formula (and not only with respect to order as in the example above). So, technically, the object proving that a set is self-similar will be a proper injection of this set into itself preserving all definable properties (where definable means "definable using a mathematical formula of the set theory language"). As such properties are generally called elementary, an injection like the one above is referred to as *elementary.*

It is not hard to see that all self-similar sets are infinite, but that the converse implication is false: as the example of natural numbers shows, being infinite is not a sufficient condition for being self-similar. Indeed, the successor function is certainly a proper injection that preserves order, but it clearly does not preserve addition (the successor of the sum of two numbers is not the sum of their successors!) although this operation is elementarily definable in the framework of set theory. It is easy to see that no alternative function could work. Actually, a self-similar set can only be huge, and, in particular, its cardinality must be larger than all "usual" cardinals, like the cardinal of the natural numbers, or that of the reals, etc. This shows that one cannot hope to construct in any sense a self-similar set. Moreover, Gödel's incompleteness theorem shows that not only is it impossible to prove that at least one self-similar set exists, but it is also impossible to prove that the existence of such a set is consistent at all. On the one hand, this situation is really poor, and one could think that it might be wiser to renounce taking the notion of self-similarity any further, but, on the other hand, we shall see further down how its study has led to interesting applications.

1.2. **The technical elaboration.** The general study of elementary embeddings in the framework of set theory was developed by Gaifman and Kunen, in particular, in the early 70's (cf. [10]), and became in the 80's one of the most active subjects in the theory, mainly due to its connection with the results by Martin, Steel and Woodin on the axiom of determinacy (see [17]). Research into self-similar sets is one chapter of this study. The main point here is that using the specific tools of set theory has permitted and even, to some extent, forced the transition toward pure algebra.

The first stage in this process has been the introduction of self-similar *ranks* [21]. Ranks are sets of a special kind, which are "broad" enough to force any set that is included in some element of a rank to be itself an element of that rank. One can then consider self-similar ranks. Combining the idea of self-similarity together with the technical potentialities of the ranks gives rise to a new phenomenon connected with the idea of self-reference, an idea known to be rather common in set theory. Indeed, let us assume that I is an elementary embedding proving that the rank R is self-similar. Because of the specific properties of ranks, the mapping I, which acts *on* R and is therefore external with respect to R, can itself be considered as an *element* of R, and therefore, since I applies to every element of R, it can be applied to itself in particular, thus leading to a new object $I(I)$[1]. And, because I preserves *any* definable property, and because being an elementary embedding happens to behave here like a definable property (it is an infinite conjunction of definable properties), the object $I(I)$ is still an elementary embedding of R into itself. More generally, if J and K are any elementary embeddings of R into itself, so is $J(K)$, and, therefore, starting from I, we obtain an (infinite) family $\mathcal{S}_I$ of elementary embeddings I, $I(I)$, $I(I)(I)$, $I(I(I))$, ... that we call here the *closure* of I.

The next stage consisted in observing that every system $\mathcal{S}_I$ associated as above with an elementary embedding I has to satisfy the algebraic identity $x(y(z)) = x(y)(x(z))$, which asserts some self-distributivity of the application operation, and directly follows from the self-similarity notion[2]. Various works have shown that this property is significant insofar as a non-trivial part of the properties of self-similar

[1]To be more precise, I can be approximated by elements of R (its restrictions to earlier ranks), and, applying I to these approximations, one can make sense of "applying I to itself".

[2]Justification is easy: each time the rank R satisfies some relation of the form "y is the image of x under the mapping f", it also has to automatically satisfy the relation "$I(y)$ is the image of $I(x)$ under the mapping $I(f)$" whenever I is an elementary embedding. In other words the

sets and of elementary embeddings comes directly from the fact that the systems $\mathcal{S}_I$ satisfy the self-distributivity identity.

The third stage was to use the ordinals. One knows how important these objects are in set theory. In the present framework, ordinals have enabled us to analyse the systems $\mathcal{S}_I$ and their main algebraic properties. The basic idea is to associate some sort of *norm* with an ordinal value to every elementary embedding — hence in particular to every element of the systems $\mathcal{S}_I$. This approach has been very classic since Scott and Gaifman, and it consists in evaluating the size of a function in terms of the least ordinal that is moved by that function. Such an ordinal is called the *critical ordinal* of the function. In the present case, Richard Laver has observed in [14] that the wellfoundedness property of the ordinals translates into some algebraic property of the systems $\mathcal{S}_I$, namely *acyclicity of divisibility*[3].

This technical statement is crucial for all subsequent applications, because it is "exportable" outside set theory. Indeed general algebra introduces, for the self-distributivity identity as well as for any identity, some sort of maximal systems that satisfy this identity. These so-called *free* systems have the property that any other system satisfying the considered identity can be obtained from them as a quotient algebra. Let use denote by $\mathcal{F}$ the (monogenerated) free system in the case of the self-distributivity identity. The very syntactical form of the acyclicity property implies that the particular system $\mathcal{F}$ has to verify the acyclicity property whenever *at least one* self-distributive system does. So, Laver's result implies that the system $\mathcal{F}$ does verify the acyclicity property, a result of pure algebra. But, as the existence of a self-similar rank, and therefore of a system $\mathcal{S}_I$, is an unprovable axiom of set theory, the only corollary one can extract from this result is the implication (*): "*If there exists at least one self-similar rank, then the system $\mathcal{F}$ is acyclic.*"

1.3. **Elimination of the set-theoretical axiom, and derived applications.**

The preceding result is rather strange, as the proof of a purely algebraic property involving only a "very small" object depends on a hypothesis involving a "very large" object, which has no obvious connection with the previous one. The situation could have come to two different ends: either the set theoretical axiom proved to be necessary, as for some topological properties of the real line (see for instance [10], or [23]), or it could be eliminated by constructing an alternative proof that does not use it. In the present case, and, it seems, for the first time for such a simple algebraic property[4], the second situation turned out to be the right one: three years after Laver's result, the acyclicity property of the system $\mathcal{F}$ has been established without using any reference to the hypothetical systems $\mathcal{S}_I$ or any other logical assumption [4].

An interesting point here is that the very principle of the direct proof of acyclicity has paved the way for new applications, mainly in the field of braid topology. This subject plays a significant role in contemporary mathematics because of its multiple connections with various areas ranging from combinatorics to theoretical physics and knot theory — see for instance [11]. Braids were introduced as mathematical

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equality $I(f(x)) = I(f)(I(x))$ always holds. Now it holds in particular when x and f are also elementary embeddings, and thus one obtains the above identity.

[3]Technically this property asserts that there cannot exist any cycle of elements of $\mathcal{S}_I$ in which each element divides the next one on the left.

[4]The set-theoretical axiom that had been used by Solovay to prove the consistency of all sets of reals having the Baire property was subsequently eliminated by Shelah (cf. [20]).

objects by Emil Artin in the 20's. As a simple formalization of the physical notion of a braid, they are quite natural objects: an n-strand braid is the projection onto a vertical plane of n strands that hang from equidistant points on an horizontal line, and freely cross but are subject to keeping a downward orientation: Figure 1 below shows two 3-strand braids. The construction in [4] relies on the abstract study of the self-distributivity identity and its underlying geometry. The latter is described by a certain characteristic group, and the core of the proof consists in establishing a convenient version of the acyclicity property in this group. The geometry of the braids is also described by a group (Artin's braid group), and the remarkable point is that the groups above are very close to each other from an algebraic point of view (one is a projection of the other). It follows that the geometry of the braids is, to some extent, a projection of the geometry of self-distributivity, which implies that some of the results established for self-distributivity can be projected onto similar results involving braids. This, in particular, happens to be the case for the acyclicity property, whose projection to braids asserts the existence of a certain ordering that had never been discovered before.

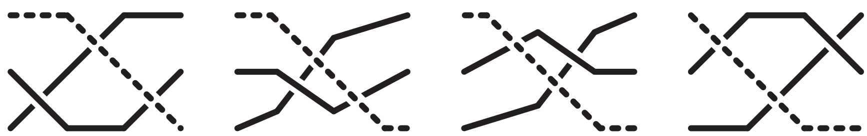

FIGURE 1. Two isotopic 3-strand braids, with two intermediate positions in which the dotted strand moves in front of the other two strands

Several applications of the existence of this ordering have been found. The most "concrete" one is a new solution for the *isotopy problem* of braids. The question is to recognize, by using an algorithmic process, if two braids can be transformed into each other by moving the strands, but without allowing a strand to go through another one. One can for instance verify that the two braids of Figure 1 are isotopic in this sense. The isotopy problem is interesting mainly because of its connection with the famous (incompletely solved) isotopy problem of knots, of which it can be seen as a particular case and an inevitable first stage. It has been often investigated in the past decades, and the solutions successively offered by Artin, Garside, ElRifai and Morton, Thurston have progressed toward better efficiency. A new solution, relying on the braid ordering, and therefore directly originating in the acyclicity property of the system $\mathcal{F}$, is constructed in [6]. Precisely because it uses the new braid ordering that was unknown in the previous solutions, it happens to be much more efficient than the latter, yet it is very simple as it consists in iterating a single reduction operation that possesses a natural geometric definition.

Other applications have been given. Concerning braids, new results about Burau representation, a classic representation of braids by means of matrices, have been established. In algebra, the acyclicity property proved to be crucial in the general study of the self-distributivity identity. It has led in particular to the first solution to the *word problem* of this identity, that is, to an effective process for recognizing which identities are consequences of the considered identity: this is one of the most fundamental and natural questions in the study of any algebraic identity.

On the other hand, still after the same scheme, Laver has established in [15] new properties of the systems $\mathcal{S}_I$, thus giving rise to new implications in the shape of (*) above. This time, the results are combinatorial in nature and involve finite self-distributive systems. Laver has defined, for every number n, a special self-distributive system A_n with 2^n elements, and established various properties of these systems using their construction from the systems $\mathcal{S}_I$. It is easy to give a direct (very simple) alternative construction of the systems A_n that no longer uses the systems $\mathcal{S}_I$, so their existence does not depend on any logical axiom. But the systems $\mathcal{S}_I$ are used in the proof of the combinatorial statements established by Laver, and one obtains such implications as the following one (**): "If there exists a self-similar rank, the number of pairwise distinct elements on the first row of the multiplication table of A_n goes to infinity with n". In contradistinction to implication (*), the final logical status of implication (**) is not known at present: intensive work by Drápal and Dougherty has so far only led to partial results in the direction of the axiom being eliminated, while, in the other direction, it has been shown in [7] that any proof of (**) will require a logical framework that includes a rather strong form of induction. Indeed the function that maps the integer p to the least integer n such that there are at least p distinct elements in the first row of the multiplication table of A_n (it such an n exists) grows faster than Ackermann's function, and, therefore, its existence (more precisely, the fact that it is defined everywhere) cannot be proved inside Primitive Recursive Arithmetic.

2. The role of set theory in these results

Excepting the latter applications whose status still has to be made clear (the most common feeling being that the set-theoretical axiom will finally be eliminated), the most noteworthy point in the above results is the specific role played by set theory, a new role that distinguishes these applications from all previous classic applications of this theory.

2.1. **Are these results applications of set theory?** Because the set theoretical axiom was finally eliminated from the proof of the acyclicity property for the system $\mathcal{F}$, one could deny that this property, and therefore all subsequent corollaries, can legitimately be called an *application* of set theory. Some logicians would have preferred the "exotic" axiom to prove to be necessary. However, we think that the elimination of this axiom does not change anything in the crucial role played by set theory here.

There is nothing casual about the logical dependence that connects the study of self-similar ranks to the recent results about braid topology. We would like the few above mentioned technical details to suggest that not only was the framework of set theory well fitted for the discovery of the results, but also that it was probably the only one to be so well fitted. The formalism of ranks in particular is crucial in order to go from the rather vague idea of self-similarity to the self-distributivity identity, and, therefore, to all subsequent applications. Now, with their strong reflection properties that enable one to nearly consider every function acting on them as one of their elements, ranks are very strange objects, both quite natural in the framework of set theory, and quite artificial in any other framework (except possibly lambda-calculus). Similarly, appealing to ordinals in order to distinguish the elementary embeddings and their iterations is quite natural in set theory, but we have seen that one certainly could not replace the ordinals with the natural numbers for this task: so again the specific tools of set theory are required here.

In the same spirit, the subsequent connection from the self-distributive systems to the braids was similarly to happen: the abstract study of the left self-distributivity identity has been developed precisely because the example of self-distributive systems provided by the systems $\mathcal{S}_I$ of set theory was missing when no set-theoretical hypothesis was assumed, and this process had to lead to braids since the geometry of the self distributivity identity is very close to that of braids, as we mentioned above. Finally, the new solution to the braid isotopy problem comes directly from the braid ordering that had been constructed as a counterpart to the acyclicity property.

2.2. **Set theory as a revealer.** Of course, the fact that a direct proof for the acyclicity property of the system $\mathcal{F}$ has been constructed in an algebraic framework shows that the set theoretical framework was *not* necessary. But we think that it was necessary for discovering a simple and natural proof like Laver's. Indeed there is a large gap of complexity between Laver's proof and that in [4]. Actually, it is more than probable that the latter would never have been considered without the motivation created by Laver's result and the resulting paradoxical implication (*).

So, set theory is not necessary to prove the above mentioned results of algebra and braid topology[5], but one sees that it has played an essential role of crystallization: set theory has given us the technical framework that was suitable for formalizing the initial intuition (here the notion of self-similarity) into precise objects (self-similar sets), and, was is more important, it has offered the specific tools (ranks, ordinals) that have allowed us to discover the properties that were included and somehow hidden there (the acyclicity property of self-distributive systems).

It seems that this sort of relation is new: contrary to the classic examples like the ones in [19], the point is not here to "unveil infinity in finite objects" by isolating properties that are intrinsically connected with the existence of infinite objects, but rather to use infinity, and its specific tools, as a melting pot where previously hidden properties appear without this indicating any link between these properties and the framework that reveals them. We can therefore compare the role played by set theory here with that of a catalyst, or even with that of a photographic film, which reveals a phenomenon but has no connection with it.

2.3. **The "third use" of set theory.** One knows that assuming the existence of (actual) infinity is a logical principle that enables one to establish some properties of finite objects that would otherwise remain inaccessible — see the chapter by J. Paris and L. Harrington in [2]; see also H. Friedman's works, like [8], [9]. The introduction of set theoretical axioms asserting the existence of higher order infinities, like self-similar ranks, can be likened to the introduction of an additional proof principle. It is quite natural that such logical principles should allow one to establish new statements, and, when the use of the additional axiom proves to be necessary, this gives rise to the "second use" of set theory as a gauge for the logical depth of these properties.

[5]However Larue has subsequently given an alternative proof of the acyclicity property that starts from braids and therefore has still less connection with set theory than the one in [4].

But if, on the contrary, the set-theoretical axiom can be eliminated, as in the results we have presented here, one sees a different use, where the additional principle is only a temporary help that lets some properties appear and gives them some sort of plausibility before one discovers their final complete proof, that is, one that makes use only of the basic principles of logic, and not of the short-cuts provided by additional principles.

Incidentally, let us observe that the intuitive or rational *truth* of the considered axioms is rather a secondary question here. Intuition is very uncertain about the existence of self-similar sets, and, even among those who defend a very liberal notion of existence in mathematics, certainly only a few would claim that such sets do exist. This discussion is certainly significant as long as the axiom is used to *prove* statements, since the latter would merely vanish if the axiom turned out to be false. But it becomes irrelevant when the axiom is used only as this revealer that we described here: what is crucial now is no longer the truth of the axiom, but rather its potential richness and its proving power. And, from this viewpoint, the stronger the axiom is, and therefore the closer to contradiction, the more powerful it is likely to be in terms of applications[6].

2.4. **A parallel with theoretical physics.** It is very natural to compare the present situation, where unprovable axioms of set theory are used as revealers and heuristic auxiliaries before a fully rigorous construction is found, with that of theoretical physics. As one knows, physics, especially quantum mechanics, has brought a lot of valuable intuitions to contemporary mathematics. In the most common scheme, physicists offer mathematicians new concepts or statements that rely on heuristic principles, with the task of justifying the soundness of the constructions rigorously, if possible. The Feynman integral is a typical example of a formalism that is not rationally founded, but turns out to be extremely fruitful: one knows that it has led to a lot of new statements, some of which (in particular Witten's invariants of three dimensional manifolds, cf. [1]) have subsequently been established rigorously, that is, without resorting to any heuristic principle. This scheme is the very one we met with the acyclicity property of self-distributive systems: introducing some "heuristic principle", here the hypothesis that a self-similar rank exists viewed as an additional proof principle, has enabled us to discover and justify a property, which was later proved "rigorously", that is, without any exotic principle.

3. Contribution to a defense of set theory

Even without denying the foundational role of set theory and rejecting the well-known dictum by David Hilbert : "*Aus dem Paradies, das Cantor für uns geschaffen hat, soll uns niemand vertreiben können*", one can question the ability of set theory to fit into the recent evolution of mathematics. One may legitimately wonder in particular if alternative or complementary approaches could possibly capture the geometrical aspects of mathematical practice better (cf. [16], [18]), or emphasize the increasing role of effectiveness questions. In any case, one can argue that the contribution of set theory to the foundation of mathematics is now closed, since

[6]It happens that the axiom of a self-similar rank existing is one of the strongest axioms that have been considered to date: in the present hierarchy of large cardinal hypotheses, it lies very close to the top of the scale.

all "big open problems" like the axiom of choice or the continuum hypothesis have received answers that are incomplete but, in some sense, optimal.

On the other hand, even if one admires the tremendous virtuosity of the constructions, one can be a little disappointed by the exotic character often displayed by the classic applications of set theory. No argument forbids very simple statements about the most common objects to depend intrinsically on set theoretical axioms[7], and some recent results suggest that new applications might be discovered in the future [22]. But, up to now, the only properties that have been proved to *necessarily* depend on a set theoretical hypothesis, and therefore constitute "classic" applications of set theory, have always dealt with objects that are very large (uncountable groups, non-metrizable topological spaces...) or "very complicated" (non-Borel subsets of the real line, various monsters used as counterexamples in general topology...). Such objects are clearly rather far from the most common mathematical practice. So, some feeling of doubt and disappointment about set theory and its two main uses is understandable and is actually often perceptible in the mathematical community.

One should not overestimate the importance of the results that have been described here: the study of self-distributive systems remains a rather minor and exotic subject, and, as far as braids, which are definitely central objects in recent mathematics, are concerned, the isotopy problem was solved a long time ago and the recent developments are only quantitative improvements. But one certainly cannot accuse these applications of involving exotic objects, or of having an abstract or ineffective character: one could hardly imagine a more concrete result than the new solution to the isotopy problem of braids, which can be very easily implemented, either by hand or with the help of a computer.

The simplest properties that have been shown to require some form of infinity, like the convergence of Goodstein's sequences [12], involve some constructions that seem less intuitive than the above mentioned braid applications, and in any case they lie far beyond the domain of experimental applicability. On the other hand, it *seems* rather unlikely that "natural" statements about braids or knots (whatever this fuzzy notion means) could really require strong set theoretical hypotheses. If this is true, the only connections one can hope to find between set theory and such fields as low dimensional topology are the ones that belong to the "third type" we have described: here set theory brings its main contribution at the very moment when it disappears as a proof tool.

In spite (or because) of his origin as a set theorist, the author of this text happens to sometimes share some of the pervading doubts about the "real" meaning of the results that involve essentially ineffective and unintuitive objects like large cardinals, or simply uncountable objects, and therefore about the present interest of the traditional uses of set theory. It seems, however, that the existence of the "third use" as described here can respond to such doubts, at least if new similar examples are discovered in the future, and contribute to confirm the interest of continuing the study of set theory. Now, clearly, promoting the "third use" of set theory tends to let one consider it as one theory among other theories rather than as the universal theory that is supposed to found them all. This, in turn, implies

[7] In principle Gödel's arithmetization of proofs and Matijasevič's results imply that a mere diophantine equation can code the provability of any given set-theoretical statement...

that set theory should accept being evaluated in terms of the applications it brings: this is certainly a new challenge for its future development.

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Mathématiques, Université de Caen, 14032 Caen, France
E-mail address: dehornoy@math.unicaen.fr
URL: //www.math.unicaen.fr/∼dehornoy