Transfinite Braids and Left Distributive Operations

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ABSTRACT. We complete Artin's braid group B_∞ with some limit points (with respect to a natural topology), thus obtaining an extended monoid where new left self-distributive operations are defined. This construction provides an effective realization for some free algebraic system involving a left distributive operation and a compatible associative product.

AMS Subject classification: 20F36, 20N02

Here we investigate a rather natural extension of the usual notion of a braid, namely that obtained by considering two infinite series of strands rather than just one as in the case of Artin's braid group B_∞. The corresponding group is very large, but it turns out that a certain submonoid EB_∞ of this group can be described very simply as a completion of B_∞. This completion is obtained by adding upper bounds to some sequences that are increasing for a canonical linear ordering, or, equivalently, that are Cauchy sequences with respect to the associated topology. The basic study of the monoid EB_∞ is the content of the first two sections.

The sequel of the paper is devoted to the study of left self-distributive operations on the monoid EB_∞, *i.e.*, of binary operations $*$ that satisfy the algebraic identity

$$x * (y * z) = (x * y) * (x * z). \qquad (LD)$$

There is nothing gratuitous in this task. Indeed it is well known that deep connections exist between braids (and knots) and self-distributive structures [2], [15], [16], [17], [9] — as well as between the latter and set theory [5], [20], [21], [13] (see also [3] for another relation between braid groups and distributivity). In particular we recall in Section 2 that each new example of an LD-system (defined as a set equipped with a left self-distributive operation) can potentially bring new information about braids using the formalism of braid colourings. So constructing "concrete" LD-systems is a natural aim. It is both easy and difficult. It is easy, since there are very common examples, like lattices, or groups equipped with the conjugacy operation $x * y = xyx^{-1}$. But these examples are rather special since the operations they involve are always idempotent, *i.e.*, the product $x * x$ is always x. This implies severe limitations on the applications such systems can lead to. On the other hand, constructing LD-systems that are not idempotent, and, in particular, constructing realizations of the *free* LD-systems, turns out to be a much more difficult question. In [8] we have constructed a realization of the free LD-system on one generator in terms of braids: Let B_∞ denote Artin's group of (finite) braids on infinitely many strands, and s be the shift endomorphism that maps each generator σ_i to σ_{i+1}. Then the bracket operation defined on B_∞ by

$$\alpha[\beta] = \alpha \; s(\beta) \; \sigma_1 \; s(\alpha^{-1}) \tag{0.1}$$

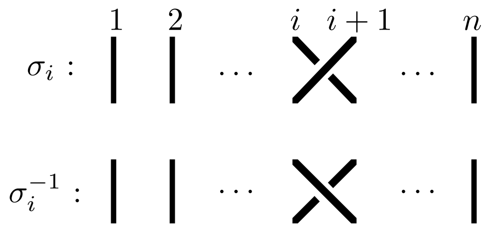
is left distributive, and it satisfies the strong property that every braid β generates under bracket a free LD-system. The existence of such a realization for a free LD-system has led to non-trivial applications, both in algebra and topology. In the first field, it provided the first complete proof that the word problem of the identity (LD) is solvable, a result that R. Laver had previously derived in [20] from an unprovable hypothesis of set theory. In the second one, it led to the existence of a linear ordering of braids ([8], *cf.* Section 2), and, subsequently, to the construction of a very efficient algorithm for comparing braids [11].

In the second part of this paper, we show how to construct left distributive operations in the extended braid monoid EB_∞ —the main result being that EB_∞ appears as the natural framework for such a construction. In particular one obtains a better understanding of the relation between the bracket (0.1) and the classic conjugacy operation. In Section 3 we show how the LD-system made of B_∞ equipped with the bracket (0.1) can be embedded in a richer structure on EB_∞ where the distributive operation and the associative product are connected by strong compatibility identities. Such structures, called here LD-monoids, have already be considered as natural strengthenings of the LD-systems ([4], [20], [14]). They also appear in the context of topology when one wishes to define enhanced braid colourings where not only the strands, but also the regions between the strands, are coloured. The main result here is that the structure of LD-monoid on EB_∞ provides concrete realizations of the free LD-monoid on one generator that constitute an exact counterpart to the realizations of the free LD-system on one generators provided by the braid bracket alone. Finally we show in Section 4 that the braid bracket (0.1), as well as the more general operations of Section 3, can be derived from a certain unique new left distributive operation on EB_∞. The latter appears so as a more basic object, and constitutes perhaps the "atomic core" of the construction.

It is a pleasure for me to thank here C. Kassel for having suggested several improvements in the presentation of this text.

1. Transfinite braids

Let us begin with finite braids. As usual, we define a braid diagram on n strands as a finite concatenation of elementary patterns of the type



Artin's braid group B_n is the quotient of the monoid of n strand braid diagrams equipped with the product induced by concatenation of strands under the congruence generated by the relations $\sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1$ and

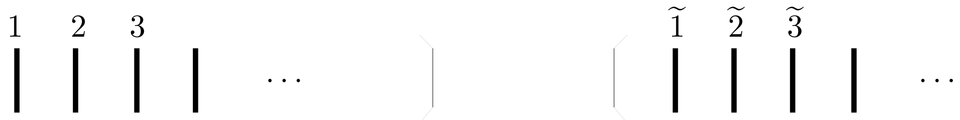
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad (R_1)$$

$$\sigma_i\sigma_j = \sigma_i\sigma_j \qquad \text{for } |i-j| \geq 2 \qquad (R_2)$$

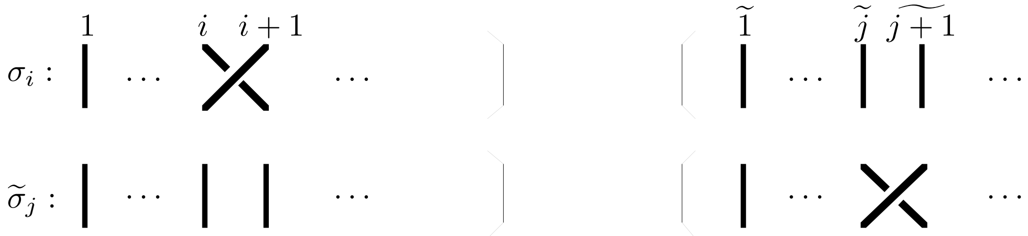
where i, j range over 1, …, $n-1$. As is well-known (see for instance [1]), this congruence corresponds to ambient isotopy when the diagrams are considered as the projection of 3-dimensional figures.

Adding an additional strand on the right of the diagrams gives an injective morphism of B_n into B_{n+1}, so there is no problem in identifying the generators σ_i associated with various values of n. We can also consider braid diagrams built on an infinite sequence of strands indexed by the positive integers: the corresponding group B_∞ is the direct limit of the groups B_n with the previous injective morphisms, and it is the group generated by an infinite sequence of generators σ_i subject to relations (R_1) and (R_2).

Now we can introduce braid diagrams that involve two series of strands indexed by the positive integers and placed as follows:



We consider two sequences of generators σ_i and $\widetilde{\sigma}_j$, such that the effect of $\widetilde{\sigma}_j$ is to cross the strands $\widetilde{j}$ and $\widetilde{j+1}$ of the second series:



Now it is clear that the extension introduced in this way is trivial: the group that appears when the obvious notion of isotopy is considered is merely the direct product of two copies of B_∞, since there is no interaction between the two infinite series of strands. So we introduce new basic diagrams: we shall denote by θ_k the σ-like crossing that sends the k-th strand of the first series to the first position in the second series, as in the figure below

Definition. A *transfinite braid diagram* is a finite concatenation of diagrams of the types σ_i, $\widetilde{\sigma}_j$ and θ_k, and their inverses (horizontal mirror images).

We can see a transfinite braid diagram as the plane projection of a 3-dimensional figure in an obvious way, and introduce the group, denoted $B_{\infty+\infty}$, of all isotopy classes of such transfinite braid diagrams. In order to avoid any ambiguity in the previous notion of isotopy, one should think of the strands as embedded in $\mathbf{R}^3$ in such a way that the basepoints are isolated: we do *not* assume that the point $\widetilde{1}$ is the limit of the points i when i goes to infinity.

Remark. If Γ is any locally finite planar graph, one can naturally associate with Γ a braid group B_Γ by attaching a strand to each vertex of the graph and considering for each edge (v, v') of Γ the operation $\sigma_{(v,v')}$ of crossing the strands at v and v': see [23] for elegant developments on this theme. The present group $B_{\infty+\infty}$ is reminiscent of this approach, except that we do not assume that the underlying graph is locally finite: here the graph is made of two copies of the positive integers, with edges between each integer and its successor, and between each integer in the first family and the first (or, equivalently, any) integer in the second family. Observe that this graph is the graph of some *order* on the set $\mathbf{N} \oplus \mathbf{N}$, namely the canonical well-ordering that turns this set into the ordinal $\omega + \omega$. From this point of view, it would be natural to denote by 0, 1, ... the strands in the first family, by ω, $\omega + 1$, ... the strands in the second family, and, for any two ordinals ξ, η with $\xi < \eta < \omega + \omega$, by $\sigma_{\xi,\eta}$ the generator that corresponds to crossing the strands ξ and η: with these notations σ_i is $\sigma_{i,i+1}$, $\widetilde{\sigma}_j$ is $\sigma_{\omega+j,\omega+j+1}$ and θ_k is $\sigma_{k,\omega}$. Then the present group $B_{\infty+\infty}$ could naturally be named $B_{\omega+\omega}$, and clearly a similar construction is possible for any infinite limit ordinal.

Lemma 1.1. *The group $B_{\infty+\infty}$ is generated by the three sequences of generators σ_i, $\widetilde{\sigma}_j$ and θ_k such that each σ_i commutes with each $\widetilde{\sigma}_j$ and the following* *relations hold:* (R_1), (R_2), *their counterparts* $(\widetilde{R}_1)$, $(\widetilde{R}_2)$ *involving the generators* $\widetilde{\sigma}_j$, *and*

$$\sigma_i\theta_k = \theta_k\sigma_i \quad \text{for } i \le k-2, \tag{R_3}$$

$$\widetilde{\sigma}_j\theta_k = \theta_k\widetilde{\sigma}_{j+1}, \tag{R_4}$$

$$\sigma_{i+1}\theta_k = \theta_k\sigma_i \quad \text{for } i \ge k, \tag{R_5}$$

$$\sigma_k\theta_{k+1}\theta_k = \theta_{k+1}\theta_k\widetilde{\sigma}_1, \tag{R_6}$$

$$\sigma_k\theta_{k+1} = \theta_k. \tag{R_7}$$

Proof. That the above relations are true is easily verified on the diagrams. For instance Figure 1 displays the case of relation (R_6). Now assume conversely that two transfinite diagrams are isotopic: because they involve finitely many elementary diagrams, there exists a finite integer N such that all strands in the first series from the N-th one will behave in a totally parallel and trivial way throughout the whole diagrams. So if we consider the ordinary braid diagrams on one infinite series of strands obtained by collapsing all strands in the first family from the N-th one onto a unique strand, and mapping the strands $\widetilde{1}$, $\widetilde{2}$, ... respectively to $N+1$, $N+2$, ..., then the collapsed diagrams are isotopic, and therefore they are equivalent with respect to relations (R_1) and (R_2). It remains to lift these relations in the original transfinite diagrams, and it is easily seen that each use of (R_1) comes from using (R_1), $(\widetilde{R}_1)$, (R_3) or (R_4), while any use of (R_2) comes from using (R_2), $(\widetilde{R}_2)$, (R_5) or (R_6). Finally (R_7) comes from the fact that θ_k is collapsed to $\sigma_k\sigma_{k+1}\ldots\sigma_{N-1}$. ∎

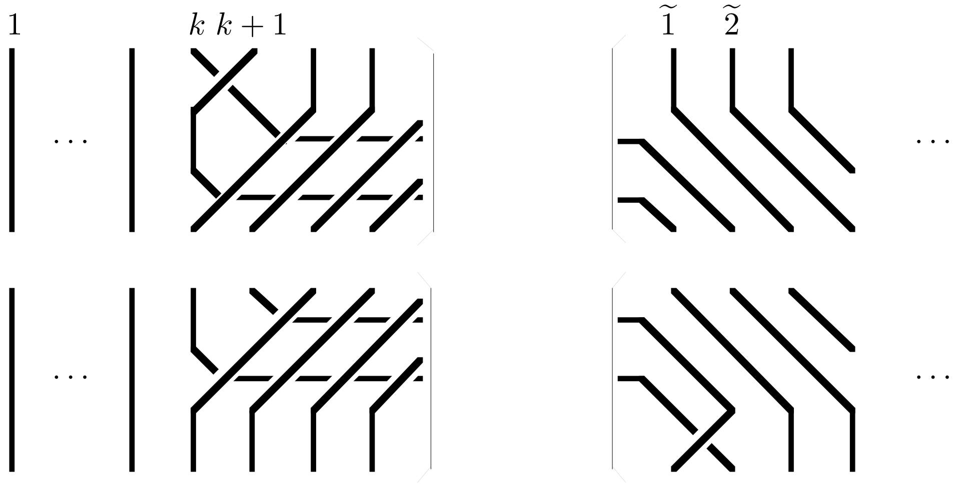


Figure 1: Relation $\sigma_k\theta_{k+1}\theta_k = \theta_{k+1}\theta_k\widetilde{\sigma}_1$

Remark. With the generators θ_k we are far from considering all possible interactions between the strands in the first and the second infinite series:

by definition, and by (R_7), θ_k corresponds to the infinite product $\sigma_k \sigma_{k+1} \ldots$, but nothing corresponds to the infinite product $\sigma_k^{-1} \sigma_{k+1}^{-1} \ldots$ (the inverse of θ_k corresponds to the reversed infinite product $\ldots \sigma_{k+1}^{-1} \sigma_k^{-1}$, *i.e.*, the orientations of the crossings are changed), nor does anything correspond either to the mixed products of the form $\sigma_k^{\varepsilon(k)} \sigma_{k+1}^{\varepsilon(k+1)} \ldots$ where ε is a sequence of ± 1 not eventually equal to 1.

It is clear that the ordinary braid group B_∞ is (isomorphic to) the subgroup of $B_{\infty+\infty}$ generated by all σ_i. So we shall identify B_∞ with this subgroup. Similarly, we write $\widetilde{B}_\infty$ for the subgroup of $B_{\infty+\infty}$ generated by all $\widetilde{\sigma}_j$: $\widetilde{B}_\infty$ is isomorphic to B_∞. For every braid β, we denote by $\widetilde{\beta}$ the translated copy of β obtained by replacing any generator σ_i by its counterpart $\widetilde{\sigma}_i$.

In the sequel of this paper, we shall no longer consider full transfinite braid diagrams, but restrict our attention to their left part, *i.e.*, to the diagrams they induce on the first series of positions. Technically we shall consider the left cosets of $B_{\infty+\infty}$ associated with its subgroup $\widetilde{B}_\infty$, *i.e.*, we consider that two transfinite braids ξ, ξ' are equivalent if ξ' can be written as $\xi\widetilde{\beta}$, where β is an (ordinary) braid. Because $\widetilde{B}_\infty$ is not a normal subgroup of $B_{\infty+\infty}$, there need not exist an induced product on the coset set $B_{\infty+\infty}/\widetilde{B}_\infty$, and the study is not so easy (Section 2 explains in part this difficulty). So, in practice, we shall restrict ourselves to a subset (actually a submonoid) of $B_{\infty+\infty}$.

Definition. A transfinite braid is *θ-positive* if it possesses at least one decomposition where the inverses of the generators θ_k do not occur. The set of all θ-positive transfinite braids is denoted $B^+_{\infty+\infty}$, and the coset set $B^+_{\infty+\infty}/\widetilde{B}_\infty$ is denoted EB_∞.

Lemma 1.2. *Every θ-positive transfinite braid can be written under the form $\alpha\theta_p\theta_{p-1}\ldots\theta_1\widetilde{\beta}$, where $p \geq 0$ and α, β belong to B_∞.*

Proof. We use induction on the length of the decompositions, *i.e.*, we show that multiplying by a generator (of the permitted type) preserves the property. So we start with a transfinite braid $\xi = \alpha\theta_p\ldots\theta_1\widetilde{\beta}$ of the form above, and consider the product $\xi\eta$, where η is either $\sigma_i^{\pm 1}$, or θ_k, or $\widetilde{\sigma}_j^{\pm 1}$. In the first case, we observe that the formula

$$\theta_p\ldots\theta_1\beta = s^p(\beta)\theta_p\ldots\theta_1 \tag{1.1}$$

holds for every β in B_∞ (we recall that s denotes the shift endomorphism of B_∞ that maps σ_i to σ_{i+1} for every i): this follows inductively from (R_5). So we get here

$$\alpha\theta_p\ldots\theta_1\widetilde{\beta}\sigma_i^{\pm 1} = \alpha\theta_p\ldots\theta_1\sigma_i^{\pm 1}\widetilde{\beta} = \alpha\sigma_{i+p}^{\pm 1}\theta_p\ldots\theta_1\widetilde{\beta},$$

which has the desired form. For the second case, we first have $\widetilde{\beta}\theta_k = \theta_k\widetilde{s}(\widetilde{\beta})$ by repeated use of (R_4). Now we have

$$\begin{aligned}\theta_p ...\theta_1\theta_k &= \theta_p ...\theta_1\sigma_{k-1}^{-1} ...\sigma_1^{-1}\theta_1\\ &= \sigma_{p+k-1}^{-1} ...\sigma_{p+1}^{-1}\theta_p ...\theta_1\theta_1\\ &= \sigma_{p+k-1}^{-1} ...\sigma_{p+1}^{-1}\sigma_p\theta_{p+1} ...\sigma_1\theta_2\theta_1\\ &= \sigma_{p+k-1}^{-1} ...\sigma_{p+1}^{-1}\sigma_p ...\sigma_1\theta_{p+1} ...\theta_2\theta_1,\end{aligned}$$

and therefore we have

$$\alpha\theta_1 ...\theta_p\widetilde{\beta}\theta_k = \alpha\sigma_{p+k-1}^{-1} ...\sigma_{p+1}^{-1}\sigma_p ...\sigma_1\theta_{p+1} ...\theta_1\widetilde{s}(\widetilde{\beta}),$$

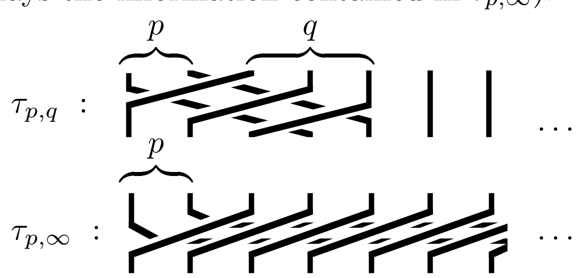
again of the desired form. Finally, for the third case, $\alpha\theta^p\widetilde{\beta}\widetilde{\sigma}_j^{\pm 1}$ has directly the right form. ■

Remark. The previous result does not extend to arbitrary elements of $B_{\infty+\infty}$: for instance the transfinite braid $\theta_1^{-1}\sigma_1^{-1}$ has no decomposition of the form $\alpha\prod_i\theta_{p_i}^{\varepsilon_i}\widetilde{\beta}$. Indeed, an easy induction shows that, in any transfinite braid $\prod_i\theta_{p_i}^{\varepsilon_i}$, the twisting number of the strands 1 and $\widetilde{1}$ (the exponent of σ_1 in the braid obtained by deleting all other strands) is -1, 0 or $+1$, but never -2 as in $\theta_1^{-1}\sigma_1^{-1}$.

Definition. i) For $p, q \geq 1$, the braid $\tau_{p,1}$ is the product $\sigma_p\sigma_{p-1} ...\sigma_1$, and the braid $\tau_{p,q}$ is the product $\tau_{p,1}s(\tau_{p,1}) ...s^{q-1}(\tau_{p,1})$. For $p = 0$ or $q = 0$, the braid $\tau_{p,q}$ is the trivial braid 1.

ii) For $p \geq 1$, the coset $\tau_{p,\infty}$ is $\theta_p ...\theta_1\widetilde{B}_\infty$, and the coset $\tau_{0,\infty}$ is $\widetilde{B}_\infty$ itself.

The braid $\tau_{p,q}$ is the braid that lets the first p strands cross under the next q strands, while $\tau_{p,\infty}$ is the coset associated with the transfinite braid whose effect on the first series of strand is to let the first p ones cross under all other strands in the first series toward the positions $\widetilde{1}$, ..., $\widetilde{p}$ in the second series. So we can take for $\tau_{p,q}$ and $\tau_{p,\infty}$ the following representation (at this point, the figure for $\tau_{p,\infty}$ is still a convention, since we have not yet proved that it adequately displays the information contained in $\tau_{p,\infty}$):



It follows from Lemma 1.2 that every coset in EB_∞ contains at least one element of the form $\alpha\theta_p\ldots\theta_1$, *i.e.*, each element of EB_∞ has at least one decomposition of the form $\alpha\tau_{p,\infty}$ where α belongs to B_∞. Moreover the above computation, and in particular Formula (1.1), show

Proposition 1.3. *The product of $B_{\infty+\infty}$ gives EB_∞ the structure of a monoid, and one has*

$$\alpha\tau_{p,\infty} \cdot \beta\tau_{q,\infty} = \alpha s^p(\beta)\tau_{p+q,\infty}. \tag{1.2}$$

The decomposition of every element of EB_∞ as $\alpha\tau_{p,\infty}$ need not be unique in general. But we can describe exactly the lack of uniqueness. In the sequel, we use the following notation: if H is a subgroup of G, we write $a \equiv a' \pmod{H}$, and say that a and a' are equivalent *modulo* H, when the left cosets aH and $a'H$ are equal.

Lemma 1.4. *For α, α' in EB_∞, the elements $\alpha\tau_{p,\infty}$ and $\alpha'\tau_{p',\infty}$ of EB_∞ are equal if and only if the integers p and p' are equal and the braids α and α' are equivalent* modulo *B_p.*

Proof. Assume $k < p$. By (R_3), (R_6) and (R_4) we have

$$\begin{aligned}\sigma_k\theta_p\ldots\theta_1 &= \theta_p\ldots\theta_{k+2}\sigma_k\theta_{k+1}\theta_k\ldots\theta_1\\ &= \theta_p\ldots\theta_{k+2}\theta_{k+1}\theta_k\widetilde{\sigma}_1\theta_{k-1}\ldots\theta_1\\ &= \theta_p\ldots\theta_{k+2}\theta_{k+1}\theta_k\ldots\theta_1\widetilde{\sigma}_k\end{aligned}$$

So, if $\alpha \equiv \alpha' \pmod{B_p}$ holds, *i.e.*, if α' is $\alpha\gamma$ for some γ in B_p, we have

$$\alpha'\theta_p\ldots\theta_1 = \alpha\theta_p\ldots\theta_1\widetilde{s}(\gamma),$$

and $\alpha\theta_p\ldots\theta_1$ and $\alpha'\theta_p\ldots\theta_1$ are equivalent *modulo* $\widetilde{B}_\infty$.

Conversely assume $\alpha\theta_p\ldots\theta_1 \equiv \alpha'\theta_{p'}\ldots\theta_1 \pmod{\widetilde{B}_\infty}$. As for an ordinary braid, a transfinite braid induces a permutation of the set that indexes the strands, here the disjoint union $\mathbf{N} \oplus \mathbf{N}$. The restriction of this permutation to the first copy of $\mathbf{N}$ is a partial injection of $\mathbf{N}$ into $\mathbf{N}$, and two transfinite braids that are equivalent *modulo* $\widetilde{B}_\infty$ induce the same partial injection. Now the index p in a decomposition $\alpha'\theta_p\ldots\theta_1\widetilde{\beta}$ is the number of integers that have no image in the associated partial injection. Hence p are p' are equal. Now let γ be $\alpha^{-1}\alpha'$, so that the hypothesis is that there exists some braid γ' satisfying

$$\gamma\theta_p\ldots\theta_1 = \theta_p\ldots\theta_1\widetilde{\gamma}'. \tag{1.3}$$

Let us assume that γ belongs to B_{p+q}. By collapsing all strands on the right of the $(p+q)$-th one, (1.3) becomes

$$\gamma\tau_{p,q} = \tau_{p,q}s^q(\gamma').$$

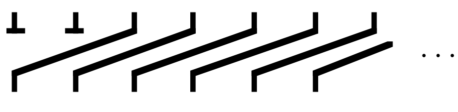
Assume that γ' belongs to $B_{p+q'} \setminus B_{p+q'-1}$ for some positive q'. Then, by the results of [11], the braid γ' admits a decomposition where the generator $\sigma_{p+q'-1}$ occurs but its inverse does not, or conversely. But this implies that $\sigma_{p+q+q'-1}$ occurs in $\tau_{p,q}s^q(\gamma')\tau_{p,q}^{-1}\gamma^{-1}$ and its inverse does not (or conversely), which, by the results of [8], prevents this braid from being trivial. So γ' necessarily belongs to B_p, and this implies that $\tau_{p,q}s^q(\gamma')\tau_{p,q}^{-1}$ is equal to γ'. Hence the hypothesis becomes ${\gamma'}^{-1}\gamma = 1$, which forces γ to belong to B_p as well: in other words $\alpha' \equiv \alpha \pmod{B_p}$ holds. ∎

We can also restate the previous result as

Proposition 1.5. *The mapping $(\alpha, p) \mapsto \alpha\tau_{p,\infty}$ establishes a bijection between the disjoint sum $\coprod_{p\geq 0} B_\infty/B_p$ and EB_∞.*

Observe that the proof of Lemma 1.4 works in particular in the cases $p = 0$ and $p = 1$, according to the convention that B_0 and B_1 both are the trivial group: in other words, the mappings $\alpha \mapsto \alpha\tau_{0,\infty}$ and $\alpha \mapsto \alpha\tau_{1,\infty}$ are both injective on B_∞, *i.e.*, the monoid EB_∞ includes two copies of B_∞. Moreover the first injection is a homomorphism, so there is no danger to identify from now on B_∞ with its image, *i.e.*, to take $\tau_{0,\infty} = 1$. Of course the injection that maps α to $\alpha\tau_{1,\infty}$ is *not* a homomorphism, since the product of any two elements of $B_\infty\tau_{1,\infty}$ belongs to $B_\infty\tau_{2,\infty}$, which is disjoint from $B_\infty\tau_{1,\infty}$. Observe that B_∞ is exactly the set of all invertible elements in the monoid EB_∞: actually the product formula (1.2) shows that EB_∞ is a graded monoid with degree 0 part equal to B_∞.

Remark. In the above graphical representation, $\tau_{p,\infty}$ is a "braid" where the first p strands vanish at infinity. One could think of a more simple representation for such a braid, where the first p strands are simply cut as below

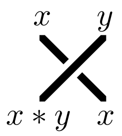


This representation, however, is misleading in the present approach, as it involves a completely different notion of isotopy for which $\tau_{1,\infty}$ and $\sigma_1\tau_{1,\infty}$ are equal, which is not the case in EB_∞.

2. The linear ordering on EB_∞

In this section we show how the monoid EB_∞ can naturally be described as a *completion* of B_∞ with respect to a linear ordering, or, equivalently, to the associated topology (which we shall see has a very simple intrinsic definition in term of the shift endomomorphism).

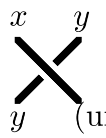
In order to introduce these notions, we first have to briefly present the framework of braid colourings (*cf.* [9]). The idea is to fix a set S equipped with a binary operation $*$ and to attach colours to the strands of the braid diagrams with the convention that colours will be modified at crossings according to the scheme



It is easily verified that the colours at the bottom of the braid will depend only on the braid and on the colours at the top of the braid (but not on the decomposition of the braid that was used) provided that the operation $*$ satisfies the left (self-)distributivity identity

$$x * (y * z) = (x * y) * (x * z), \qquad (LD)$$

i.e., the algebraic system $(S, *)$ is what we called an *LD-system.* Moreover colouring the negative crossings must obey the rule



(unique) z satisfying $x * z = y$

which leads to consider LD-systems where left division is uniquely defined (automorphic sets in [2], or racks of [15]). Actually one can use LD-systems where left division is not necessarily possible, provided it has a unique value when defined (left cancellative LD-systems): then it need not be true that every sequence of initial colours may be propagated through a given braid diagram, but one can show that, for every finite collection of braid diagrams, there always exist sequences of colours that can be propagated through the diagrams. If $\vec{x}$ is a sequence of colours, and u is a braid diagram (*i.e.*, a word written with the letters σ_i and σ_i^{-1}), we denote by $(\vec{x})u$ the final sequence of colours obtained at the bottom of u when $\vec{x}$ is applied at the top of u (if such a sequence exists).

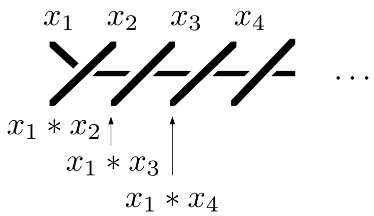
Various braid or knot invariants can be introduced in this way, starting from various known LD-systems (see [9], [15], or [17], [16] for variants and alternative approaches). We consider here the case when colours are taken in a free LD-system $(F, *)$ on one generator e. It is known [8] that this LD-system is left cancellative (so the approach is legitimate), but left division is not always possible, so F-colourings are rather different from those involving the "classic" LD-systems (all related with a group conjugacy). Presently, the main point is that F is equipped with a (unique) linear ordering such that $x < x * y$ always holds ([20], [8]). We denote by $<_{\mathrm{Lex}}$ the lexicographical extension of this order to sequences from F (also we say that a braid word u is a decomposition of the braid α just when α is the equivalence class of u with respect to the braid relations).

Proposition 2.1. ([8], [11]) *For any braids α, β in B_n, the following properties are equivalent, and they define a relation $\alpha < \beta$ that is a linear ordering on B_n:*

- *There exist decompositions u, v of α, β and a sequence $\vec{x}$ in F^n such that $(\vec{x})u$ and $(\vec{x})v$ exist and $(\vec{x})u <_{\mathrm{Lex}} (\vec{x})v$ holds;*
- *For every decompositions u, v of α, β and every sequence $\vec{x}$ in F^n such that $(\vec{x})u$ and $(\vec{x})v$ exist, $(\vec{x})u <_{\mathrm{Lex}} (\vec{x})v$ holds;*
- *The braid $\alpha^{-1}\beta$ has a decomposition where the generator with minimal index occurs only positively.*

The orderings on B_n and B_{n+1} are compatible, so their union, still denoted $<$, is a linear ordering of B_∞. One can show that B_∞ equipped with $<$ is isomorphic to the rationals, *i.e.*, is a dense linear order without endpoint.

Let us come back now to the monoid EB_∞. There is a natural way to extend the notion of a braid colouring to the elements of EB_∞. Indeed, we fix a rule for the diagrams $\tau_{p,\infty}$. Let S be any left cancellative LD-system. Owing to our previous representation, we shall consider for $\tau_{1,\infty}$ the following colouring:



More generally we take

$$(\vec{x})\tau_{p,\infty} = (x_1 * \ldots * x_p * x_{p+1}, x_1 * \ldots * x_p * x_{p+2}, \ldots)$$

Here, and in the sequel, we make the convention that missing brackets, which are significant since we do not assume that the operation $*$ is associative, should be added on the right: $x * y * z$ stands for $x * (y * z)$. Observe that, with the above definition, the sequence of colours $(\vec{x})u\tau_{p,\infty}$ is defined if and only if $(\vec{x})u$ is defined.

We first have to check that our definition makes sense, *i.e.*, that the output colours depend only on the element of EB_∞ represented by the considered diagram.

Lemma 2.2. *The colourings so defined are compatible with the relations of EB_∞.*

Proof. We have to show that, if the braids α, α' are equivalent *modulo* B_p, and if u, u' are any decompositions of α, α' such that $(\vec{x})u$ and $(\vec{x})u'$ are defined, then the sequences $(\vec{x})u\tau_{p,\infty}$ and $(\vec{x})u'\tau_{p,\infty}$ are equal. Now there exists a braid γ in B_p such that α' is $\alpha\gamma$. It is well-known that γ has a decomposition of the form vv'^{-1}, where v, v' are positive braid words, *i.e.*, involve no letter σ_i^{-1}. Then both $(\vec{x})uv$ and $(\vec{x})u'v'$ are defined, which implies that $(\vec{x})uvv'^{-1}$ is defined as well. Write $\vec{y}$ for $(\vec{x})u$, and w for vv'^{-1}. We may assume that w involves only letters $\sigma_i^{\pm 1}$ with $i < p$, and we can use induction on the length of the word w. In other words, it suffices to consider the case when w is a single letter σ_i, *i.e.*, to compare $(\vec{y})\tau_{p,\infty}$ and $(\vec{y})\sigma_i\tau_{p,\infty}$. Now the k-th elements of these sequences are respectively

$$y_1 * \ldots * y_p * y_{k+p}$$
$$y_1 * \ldots * (y_i * y_{i+1}) * y_i * \ldots * y_p * y_{k+p},$$

which are equal since $*$ is assumed to be left distributive. ■

With this extended notion of diagram colouring, we can easily extend the ordering from B_∞ to EB_∞.

Proposition 2.3. *(i) For any elements ξ, η in EB_∞, the following properties are equivalent, and they define a relation $\xi < \eta$ that is a linear ordering on EB_∞:*

- *There exist decompositions $u\tau_{p,\infty}$, $v\tau_{q,\infty}$ of ξ, η and a sequence $\vec{x}$ in $F^{\mathbf{N}}$ such that $(\vec{x})u\tau_{p,\infty}$ and $(\vec{x})v\tau_{q,\infty}$ exist and $(\vec{x})u\tau_{p,\infty} <_{\mathrm{Lex}} (\vec{x})v\tau_{q,\infty}$ holds;*
- *For every decompositions $u\tau_{p,\infty}$, $v\tau_{q,\infty}$ of ξ, η and every sequence $\vec{x}$ in $F^{\mathbf{N}}$ such that $(\vec{x})u\tau_{p,\infty}$ and $(\vec{x})v\tau_{q,\infty}$ exist, $(\vec{x})u\tau_{p,\infty} <_{\mathrm{Lex}} (\vec{x})v\tau_{q,\infty}$ holds.*

(ii) The inequality $\alpha\tau_{p,\infty} < \beta\tau_{q,\infty}$ holds in EB_∞ if and only if $\alpha\tau_{p,n} < \beta\tau_{q,n}$ holds in B_∞ (and EB_∞) for n large enough. So the element $\alpha\tau_{p,\infty}$ is the upper bound of the increasing sequence of braids $(\alpha\tau_{p,n})_{n\geq 0}$.

Proof. By construction the sequence of colours $(\vec{x})u\tau_{p,\infty}$ is the limit of the sequences $(\vec{x})u\tau_{p,n}$ when n goes to infinity: the k-th term of the sequence $(\vec{x})u\tau_{p,n}$ is constant for $n \geq k$. It follows that $(\vec{x})u\tau_{p,\infty} <_{\mathrm{Lex}} (\vec{x})v\tau_{q,\infty}$ holds if and only if $(\vec{x})u\tau_{p,n} <_{\mathrm{Lex}} (\vec{x})v\tau_{q,n}$ holds for n large enough. This proves the equivalence of (i), and shows that $\alpha\tau_{p,\infty} < \beta\tau_{q,\infty}$ holds in EB_∞ if and only if $\alpha\tau_{p,n} < \beta\tau_{q,n}$ holds in B_∞ (and EB_∞) for n large enough. In particular we see that $\alpha\tau_{p,\infty} < \beta$ holds if and only if $\alpha\tau_{p,n} < \beta$ holds for n large enough (if and only if $\alpha\tau_{p,q} < \beta$ holds for any q such that α and β belong to B_{p+q}). This proves (ii), as it is obvious that the sequences $(\tau_{p,n})_{n\geq 0}$ are increasing. ■

So we see that $(EB_\infty, <)$ is the completion of the ordered set $(B_\infty, <)$ obtained by adding upper bounds to all sequences of the form $(\alpha\tau_{p,n})_{n\geq 0}$. That these sequences have no bound in B_∞ for $p \geq 1$ follows from the fact that $<$ is a linear ordering on EB_∞, and B_∞ is a subset of EB_∞ — a direct proof is also easy. Observe that the product of EB_∞ is compatible with this completion: indeed we have

$$\alpha\tau_{p,q+n} \ \cdot \ \beta\tau_{q,p+n} = \alpha s^p(\beta)\tau_{p+q,n}$$

for every n such that β belongs to B_{q+n}. But, on the other hand, the translations associated with $\tau_{p,\infty}$ are not increasing (for $p \geq 1$): for instance $\sigma_2 < \sigma_1$ holds, but $\sigma_2\tau_{1,n} > \sigma_1\sigma_{1,n}$ holds for $n \geq 2$, so we have $\sigma_2\tau_{1,\infty} \geq \sigma_1\tau_{1,\infty}$ (and therefore $\sigma_2\tau_{1,\infty} > \sigma_1\tau_{1,\infty}$ since these terms are not equal according to Lemma 1.3).

The previous notions can be naturally rephrased in terms of topology. First it may be interesting to observe that the topology associated with the braid ordering $<$ has a very simple description.

Proposition 2.4. *The topology of B_∞ induced by the order $<$ is the same as that associated with the ultrametric distance d such that $d(\alpha, \beta)$ is 2^{-n} if $\alpha^{-1}\beta$ belongs to $s^n(B_\infty)$ but not to $s^{n+1}(B_\infty)$. For this topology B_∞ is a topological group.*

Proof. By construction the open ball with radius 2^{-n} centered at α is simply the left coset $\alpha s^n(B_\infty)$, *i.e.*, the set of all braids $\alpha\gamma$ where γ has a decomposition involving no generator $\sigma_i^{\pm 1}$ with $i \leq n$. Let α, β be arbitrary braids, and γ be any element of the open interval (α, β). Assume that α, β and γ belong to B_n. Let us say that a braid is σ_i-positive (*resp.* negative) if it has decomposition where no letter $\sigma_j^{\pm 1}$ occurs, σ_i occurs and σ_i^{-1} does not (*resp.* σ_i^{-1} occurs and σ_i does not). Then the hypothesis $\alpha < \gamma$ implies that the braid $\alpha^{-1}\gamma$ is σ_i-positive for some $i < n$. But then $\alpha^{-1}\gamma s^n(\delta)$ is σ_i-positive for any δ, which means that $\alpha < \gamma s^n(\delta)$ holds as well. Similarly $\beta^{-1}\gamma$ is σ_j-negative for some $j < n$, and again this implies $\gamma < \gamma s^n(\delta)$ for every δ: so the open ball $\gamma s^n(B_\infty)$ is included in the interval (α, β).

Conversely, let us start with an arbitrary open ball $\gamma s^n(B_\infty)$. Let $\gamma s^n(\alpha)$ and $\gamma s^n(\beta)$ be any two points in this ball, and δ be any braid satisfying $\gamma s^n(\alpha) < \delta < \gamma s^n(\beta)$. We claim that δ has to belong to the ball. Indeed we have $s^n(\alpha) < \gamma^{-1}\delta < s^n(\beta)$. If $\gamma^{-1}\delta$ were not in the image of s^n, then for some $i < n$ it would either be σ_i-positive, or be σ_i-negative. In the first case, $s^n(\alpha^{-1})\gamma^{-1}\delta$ is also σ_i-positive, which contradicts $s^n(\alpha) < \gamma^{-1}\delta$. In the second case, $s^n(\beta^{-1})\gamma^{-1}\delta$ is also σ_i-negative, which contradicts $\gamma^{-1}\delta < s^n(\beta)$. So the order topology and that associated with the distance d coincide.

Finally the product and the inverse are continuous operations on B_∞. Indeed, if α, β belong to B_p, and n is at least p, the braid β commutes with every braid in $s^n(B_\infty)$, and we have trivially the equalities

$$\alpha s^n(B_\infty) \cdot \beta s^n(B_\infty) = \alpha\beta\sigma^n(B_\infty)$$
$$(\beta s^n(B_\infty))^{-1} = \beta^{-1}s^n(B_\infty). \qquad \blacksquare$$

From the above remarks it follows that B_∞ equipped with the above topology is homeomorphic to the rationals. Now we see that the sequences $(\alpha\tau_{p,n})_{n\geq 0}$ are Cauchy sequences, and EB_∞ can now be seen as the (partial) topological completion obtained by adding a limit to such sequences. To this end, it suffices to extend the topology of B_∞ to EB_∞ by defining the elementary neighbourhood $s^n(EB_\infty)$ by the formula

$$s^n(\alpha\tau_{p,\infty}) = s^n(\alpha)\tau_{p,n}^{-1}\tau_{p,\infty},$$

deduced from the equality

$$s^n(\tau_{p,q}) = \tau_{p,n}^{-1}\ \tau_{p,n+q}.$$

Details are easy.

Remark. One could also take a more systematic completion of B_∞ obtained by adding limits to *all* Cauchy sequences. With such a construction, one would expect to get rid of the limitations originating in our choice of special Cauchy sequences. In particular we would expect to obtain a group structure on the completion, which would be more satisfactory than the monoid structure of EB_∞. This approach, however, does not seem to work, for the algebraic operations of B_∞ (in contradistinction to those of the rationals) are not regular enough with respect to the ordering. For instance, $\alpha < \beta$ does not imply $\alpha^{-1} > \beta^{-1}$ (consider $\alpha = \sigma_1$, $\beta = \sigma_1^2\sigma_2^{-1}$). For the completion, observe that the sequence $(\tau_{1,n}^{-1})_{n\geq 0}$ is not a Cauchy sequence, and that actually it tends to $-\infty$ in B_∞. Indeed, if β is any braid in B_p, we have $\tau_{1,n}\beta = s(\beta)\tau_{1,n}$ for $n \geq p$, and the latter form is σ_1-positive: so $\tau_{1,n}^{-1} < \beta$ certainly holds. This shows that merely adding inverses for the elements of $EB_\infty \setminus B_\infty$ is impossible in the present framework, and would be possible only in a much bigger framework like $B_{\infty+\infty}$. In this case B_∞ would no longer be dense in any reasonable sense, which would in turn significantly diminish the possible interest of the construction.

As a final general remark, we note that the construction of EB_∞ as a completion of B_∞ enables to extend the linear representations of B_∞ like the classic Burau representation that maps σ_1 to $\begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix}$ at the expense of considering infinite row-finite matrices. For instance, considering $\tau_{1,\infty}$ as the limit of the braids $\tau_{1,n}$ leads to map it to the matrix

$$\begin{pmatrix} 1-t & t & 0 & 0 & \dots \\ 1-t & 0 & t & 0 & \dots \\ 1-t & 0 & 0 & t & \\ \vdots & \vdots & & & \ddots \end{pmatrix}.$$

3. The structure of LD-monoid on EB_∞

We shall now show that the monoid EB_∞ provides a well-fitted framework for constructing new non-trivial algebraic operations that cannot be defined on B_∞ and generalize the bracket operation of (0.1). We begin with the definition of an LD-monoid:

Definition. An *LD-monoid* is a monoid M (with unit denoted 1) equipped with a second binary operation, here denoted as $(x, y) \mapsto {}^x y$, such that ${}^x 1$ is always 1 and the following mixed identities are satisfied:

$$x\ y = {}^x y\ x \tag{LD_1}$$

$$^{xy}z = {}^x({}^y z) \tag{LD_2}$$

$$^x(yz) = {}^x y {}^x z \tag{LD_3}$$

LD-monoids have been introduced (under the name "semi-abelian monoids") in [4] and they have been studied and used in several recent works involving left distributive systems, in particular in connection with set theory (see [20], [14], [13] among other papers). Observe that the exponentiation of an LD-monoid has to be left distributive, since (LD_1) and (LD_2) imply

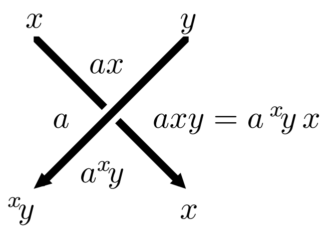
$$^x({}^y z) = {}^{xy}z = {}^{{}^x y x}z = {}^{{}^x y}({}^x z).$$

Similarly, ${}^1 x = x$ always holds, by

$$^1 x = {}^1 x\ 1 = 1\ x = x. \tag{3.1}$$

Observe that (LD_2) together with (3.1) mean that the monoid M acts on the left on itself under exponentiation, and (LD_3) together with the condition ${}^x 1 = 1$ then assert that the action preserves the monoid structure of M.

Computations in LD-monoids are rather easy: a group equipped with its product and the conjugacy operation is an LD-monoid, and, roughly speaking, most of the computations involving conjugacy in a group can be extended to arbitrary LD-monoids. Now it turns out that some LD-monoids are far from the conjugacy of a group, and this is true in particular for *free* LD-monoids. In the context of braids or knots, LD-monoids, or, at least, Identity (LD_1), naturally appear when one wishes to colour not only the strands of a braid diagram, but only the *regions* between the strands, with the convention that the regions separated by a strand coloured x have colours of the form a, ax respectively, as in the figure:



LD-monoids are natural generalizations of abelian monoids: if M is abelian, the trivial exponentiation defined by ${}^xy = y$ gives to M the structure of an LD-monoid. Observe that, if the monoid M can be given such a structure, it certainly satisfies the "skew commutativity" statement

$$(\forall x, y)(\exists z)(zx = xy) \tag{3.2}$$

It is easy to see that, conversely, if (3.2) holds and the element z is unique, then the exponentiation such that ${}^xy$ is the unique z satisfying $zx = xy$ defines on M the structure of an LD-monoid. This is the case when M is a group, and then the unique LD-monoid structure is that associated with the conjugacy ${}^xy = xyx^{-1}$. On the other hand, there exist monoids that satisfy (3.2) but where no structure of LD-monoid can be defined ([4]).

Now we have recalled that a (non-trivial) structure of LD-system can be defined on the braid group B_∞ by using the bracket operation (0.1). A natural question is whether this LD-system can be enriched with a second operation that turns it into an LD-monoid. The previous attempts had failed: in particular we cannot hope to use the product of B_∞, since B_∞ is a group, and therefore the only possible left distributive operation providing a structure of LD-monoid would be the conjugacy, which does not coincide with the bracket. We shall see here that appealing to the extension EB_∞ of B_∞ enables us to answer positively the question: we shall extend the bracket of B_∞ to a new operation on the whole set EB_∞ in such a way that the two operations structure thus obtained is a structure of LD-monoid. Moreover, and this is the main interest in the construction, this LD-monoid includes (many) free LD-monoids, which gives nice "concrete" realizations for the latter structures.

We start with the monoid structure of EB_∞ (and not with the bracket operation of B_∞), and try to define an exponentiation with the desired properties by investigating Condition (3.2) in EB_∞. Let $\alpha\tau_{p,\infty}$ and $\beta\tau_{q,\infty}$ be arbitrary elements of EB_∞: we look for those elements $\gamma\tau_{r,\infty}$ that satisfy in EB_∞

$$\gamma\tau_{r,\infty}\,\alpha\tau_{p,\infty} = \alpha\tau_{p,\infty}\,\beta\tau_{q,\infty} \tag{3.3}$$

(if any). Using the explicit product formula (1.3), this is

$$\gamma s^r(\alpha)\tau_{p+r,\infty} = \alpha s^p(\beta)\tau_{p+q,\infty},$$

which, by Lemma 1.4, is equivalent to

$$r = q \quad \text{and} \quad \gamma s^r(\alpha) \equiv \alpha s^p(\beta) \ (\mathrm{mod}\ B_{p+q}).$$

It follows that the values of $\gamma\tau_{r,\infty}$ such that (3.3) holds are exactly those of the form $\gamma\tau_{q,\infty}$ where γ is any braid in the set $\alpha s^p(\beta)B_{p+q}s^q(\alpha^{-1})$. In other words, the possible candidates for defining an exponentiation on EB_∞ have the form

$$^{\alpha\tau_{p,\infty}}(\beta\tau_{q,\infty}) = \alpha s^p(\beta)\omega s^q(\alpha^{-1})\tau_{q,\infty}, \tag{3.4}$$

where ω is an element of B_{p+q} possibly depending on α and β. By the above computation, Identity (LD_1) will hold for any such operation.

Proposition 3.1. *i) The monoid EB_∞ equipped with the exponentiation*

$$^{\alpha\tau_{p,\infty}}(\beta\tau_{q,\infty}) = \alpha\, s^p(\beta)\,\tau_{p,q}\, s^q(\alpha^{-1})\,\tau_{q,\infty}, \tag{3.5}$$

is an LD-monoid.
ii) The above exponentiation, and that obtained by replacing $\tau_{p,q}$ in (3.5) with its image under the morphism that exchanges σ_i and σ_i^{-1} for every i, are the only possible exponentiations that turn EB_∞ into an LD-monoid and are such that the braids ω in (3.4) depend only on the integers p and q.

Proof. We observed above that any possible exponentiation on EB_∞ satisfying (LD_1) has the form (3.4). In order to effectively obtain a structure of LD-monoid using (3.4) as a definition for exponentiation, we have to verify first that this formula induces a well-defined operation on EB_∞ (since the decomposition as $\alpha\tau_{p,\infty}$ is not unique in general), and to verify the remaining identities in the definition of an LD-monoid. We perform these verifications in the case when the braids ω involved in (3.4) depend on the integers p and q, but not on the braids α, β. In other words, we consider exponentiation operations of the form

$$^{\alpha\tau_{p,\infty}}(\beta\tau_{q,\infty}) = \alpha s^p(\beta)\omega_{p,q} s^q(\alpha^{-1})\tau_{q,\infty}, \tag{3.6}$$

where $\omega_{p,q}$ is some fixed element of B_{p+q}.

Lemma 3.2. *(i) Formula (3.6) induces a well-defined operation on EB_∞ if and only if the following conditions hold for $1 \le i < p$ and $1 \le j < q$:*

$$\sigma_i\ \omega_{p,q}\sigma_{q+i}^{-1} \equiv \omega_{p,q} \pmod{B_q}, \tag{3.7}$$

$$\sigma_{p+j}\ \omega_{p,q} \equiv \omega_{p,q} \pmod{B_q}. \tag{3.8}$$

(ii) Then EB_∞ equipped with its product and the exponentiation so defined is an LD-monoid if and only if the following additional conditions hold for every p, q, r:

$$\omega_{p,0} = 1, \tag{3.9}$$

$$\omega_{p+q,r} \equiv s^p(\omega_{q,r})\omega_{p,r} \pmod{B_q}, \tag{3.10}$$

$$\omega_{p,q+r} \equiv \omega_{p,q}s^q(\omega_{p,r}) \pmod{B_{q+r}}. \tag{3.11}$$

Proof. (i) First we observe that, for all braids α, α', β in B_∞, $\alpha \equiv \alpha' \pmod{B_p}$ holds if and only if $\beta\alpha \equiv \beta\alpha' \pmod{B_p}$ holds, if and only if $\alpha s^p(\beta) \equiv \alpha' s^p(\beta) \pmod{B_p}$ holds: the latter equivalence follows from the fact that any braid in B_p commutes with any braid in the image of s^p.

The first requirement is that $\alpha \equiv \alpha' \pmod{B_p}$ implies ${}^{\alpha\tau_{p,\infty}}(\beta\tau_{q,\infty}) \equiv {}^{\alpha'\tau_{p,\infty}}(\beta\tau_{q,\infty}) \pmod{B_{p+q}}$. So assume $\alpha' = \alpha\delta$, with δ in B_p. By expanding explicitely the exponentiations and using the remarks above, we see that the desired equivalence holds for

$$\delta\ \omega_{p,q}\ s^q(\delta^{-1}) \equiv \delta \pmod{B_{p+q}},$$

which amounts to (3.7) for δ ranging in B_p.

Similarly we see that $\beta' = \beta\delta$ with δ in B_q implies ${}^{\alpha\tau_{p,\infty}}(\beta\tau_{q,\infty}) \equiv {}^{\alpha\tau_{p,\infty}}(\beta'\tau_{q,\infty}) \pmod{B_{p+q}}$ when

$$\delta\ \omega_{p,q} \equiv \omega_{p,q} \pmod{B_{p+q}}$$

holds, which gives (3.8) for δ ranging in B_q.
(ii) Expanding the condition ${}^{\alpha\tau_{p,\infty}}(1) = 1$ gives $\alpha\omega_{p,0}\alpha^{-1} = 1$, hence $\omega_{p,0} = 1$. Then expanding ${}^{\alpha\tau_{p,\infty}\,\beta\tau_{q,\infty}}(\gamma\tau_{r,\infty})$ and ${}^{\alpha\tau_{p,\infty}}({}^{\beta\tau_{q,\infty}}(\gamma\tau_{r,\infty}))$ gives respectively

$$\alpha s^p(\beta)s^{p+q}(\gamma)\omega_{p+q,r}s^{p+r}(\beta^{-1})s^r(\alpha^{-1})\tau_{r,\infty}$$

and

$$\alpha s^p(\beta)s^{p+q}(\gamma)s^p(\omega_{q,r}s^{p+r}(\beta^{-1})\omega_{p,r}s^r(\alpha^{-1})\tau_{r,\infty}.$$

Since $\omega_{p,r}$ belongs to B_{p+r}, it commutes with $s^{p+r}(\beta^{-1})$, and therefore, by cancelling $\alpha s^p(\beta)s^{p+q}(\gamma)$ on the left and $s^{p+r}(\beta^{-1})s^r(\alpha^{-1})$ on the right, which is legal when equivalence *modulo* B_r is considered, we see that the above expressions are equal in EB_∞ if and only if (3.10) holds. Finally, expanding ${}^{\alpha\tau_{p,\infty}}(\beta\tau_{q,\infty}\,\gamma\tau_{r,\infty})$ and ${}^{\alpha\tau_{p,\infty}}(\beta\tau_{q,\infty})\,{}^{\alpha\tau_{p,\infty}}(\gamma\tau_{r,\infty})$ leads to (3.11) by a similar computation. ∎

We continue the proof of Proposition 3.1. By hypothesis the braid $\omega_{1,1}$ belongs to B_2, *i.e.*, it has the form σ_1^m for some exponent m. By (3.10) we must have

$$\omega_{2,1} \equiv s(\omega_{1,1})\omega_{1,1} \pmod{B_1},$$

i.e., $\omega_{2,1}$ is then $\sigma_2^m\sigma_1^m$. Now (3.7) requires

$$\sigma_1\omega_{2,1}\sigma_2^{-1} \equiv \omega_{2,1} \pmod{B_1},$$

that is $\sigma_2^m\sigma_1^m = \sigma_1\sigma_2^m\sigma_1^m\sigma_2^{-1}$, or, equivalently, $\sigma_2^m\sigma_1^m\sigma_2 = \sigma_1\sigma_2^m\sigma_1^m$. This equality holds for $m = \pm 1$, and for no other value. Indeed, denoting by λ_m the quotient $(\sigma_2^m\sigma_1^m\sigma_2)^{-1}(\sigma_1\sigma_2^m\sigma_1^m)$, we can verify inductively for $m \geq 2$ the formulas

$$\lambda_m = \sigma_1\sigma_2^{-(m-2)}\sigma_1\sigma_2^{-m}\sigma_1^{m-2}\sigma_2^{-1}\sigma_1^{m-1}$$
$$\lambda_{-m} = \sigma_2^{-1}\sigma_1^{m-1}\sigma_2^{-1}\sigma_1^{m-2}\sigma_2^{-m}\sigma_1\sigma_2^{-(m-2)}\sigma_1\sigma_2.$$

We observe that the generator σ_1 occurs in these decompositions, while σ_1^{-1} does not: by the results of [8] we can conclude that these quotients are not equal to 1.

Assume that $\omega_{1,1}$ is σ_1. An easy induction gives $\omega_{p,1} = \sigma_p\sigma_{p-1}\ldots\sigma_1 = \tau_{p,1}$ for $p \geq 1$. Then (3.11) implies that $\omega_{p,2}$ has to be equivalent to $\omega_{p,1}\sigma(\omega_{p,1})$ *modulo* B_2. But the value of $\omega_{p,q}$ has to be defined only up to equivalence *modulo* B_q: multiplying $\omega_{p,q}$ by some braid δ in B_q on the right does not change the exponentiation on EB_∞, since the additional factor δ commutes with the term $s^p(\alpha^{-1})$ in (3.4). So we can assume $\omega_{p,2} = \omega_{p,1}\sigma(\omega_{p,1}) = \tau_{p,2}$ without loss of generality, and, similarly, we find $\omega_{p,q} = \tau_{p,q}$ for every $q \geq 1$. Finally, it is clear that taking σ_1^{-1} for $\omega_{1,1}$ amounts to changing the orientation of all crossings. This proves both that the operation (3.5) satisfies all requirements, and that it is, up to changing the orientation of all crossings, the only one of the form (3.6) to do so. ■

So, from now on, EB_∞ is equipped with two operations, namely the product and the exponentiation defined respectively by

$$\alpha\tau_{p,\infty} \cdot \beta\tau_{q,\infty} = \alpha s^p(\beta)\tau_{p+q,\infty},$$
$${}^{\alpha\tau_{p,\infty}}(\beta\tau_{q,\infty}) = \alpha s^p(\beta)\tau_{p,q}s^q(\alpha^{-1})\tau_{q,\infty}.$$

Figure 2 displays the computation of exponentiation. Observe that this exponentiation on EB_∞ extends the two known left distributive operations on B_∞: for α, β in B_∞, we have

$${}^{\alpha}\beta = \alpha\beta\alpha^{-1},$$
$${}^{\alpha\tau_{1,\infty}}(\beta\tau_{1,\infty}) = \alpha s(\beta)\sigma_1 s(\alpha^{-1})\tau_{1,\infty} = \alpha[\beta]\tau_{1,\infty},$$

so the exponentiation on EB_∞ induces the conjugacy operation on the first copy of B_∞, while it induces the bracket of (0.1) on the second copy (that associated with the mapping $\alpha \mapsto \alpha\tau_{1,\infty}$). Thus the present construction is the natural answer to the question of defining an associative product on B_∞ in order to obtain an LD-monoid: this product has to live not in B_∞, but in the extended structure EB_∞.

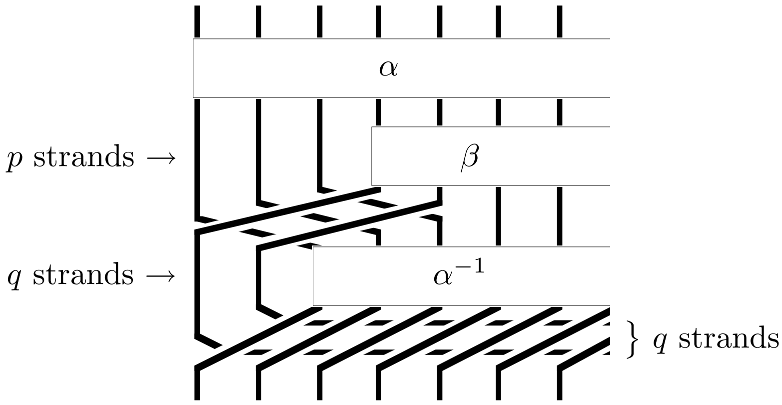


Figure 2: Value of ${}^{\alpha\tau_{p,\infty}}(\beta\tau_{q,\infty})$

Remark. (i) If we remove the hypothesis that the braids $\omega_{p,q}$ do not depend on the braids α and β in (3.6), and write $\omega_{p,q}(\alpha,\beta)$ for the braid involved in the exponentiation of $\alpha\tau_{p,\infty}$ and $\beta\tau_{q,\infty}$, we can see that $\omega_{1,1}(\alpha,\beta)$ still has to be $\sigma_1^{\pm 1}$. More precisely, if $\omega_{1,1}(\alpha,\beta)$ is $\sigma_1^{\varepsilon(\alpha,\beta)}$, the constraint that the function ε must satisfy is

$$\sigma_2^{e(\beta,\gamma)}\sigma_1^{\varepsilon(\alpha,s(\beta)\sigma_1^{\varepsilon(\beta,\gamma)}s(\alpha^{-1}))}\sigma_2^{e(\alpha,\beta)} = \sigma_1^{\varepsilon(\alpha,\beta)}\sigma_2^{\varepsilon(\alpha,\gamma)}\sigma_1^{\varepsilon(\alpha s(\beta)\sigma_1^{\varepsilon(\alpha,\beta)}s(\alpha^{-1}),\alpha s(\gamma)\sigma_1^{\varepsilon(\alpha,\gamma)}s(\alpha^{-1}))}.$$

We could not find any other solution than the constants $\varepsilon(\alpha,\beta) = \pm 1$ (It is easy to show that the only solutions depending on the first variable only are the constants).

(ii) Making the computation in the quotient structure EB_∞ is essential in the above construction. Since $B_{\infty+\infty}$ is a group, the unique way to define an exponentiation on $B_{\infty+\infty}$ to obtain an LD-monoid is to consider the conjugacy operation. We could therefore consider defining an exponentiation on EB_∞ starting from the conjugacy of $B_{\infty+\infty}$. But this simple approach does not work: the conjugacy of $B_{\infty+\infty}$ induces no well-defined operation on EB_∞, since for instance $\theta_1^2\theta_1(\theta_1^2)^{-1}$ is θ_1, while $(\sigma_1\theta_1^2)\theta_1(\sigma_1\theta_1^2)^{-1}$ is $\sigma_1\theta_1\sigma_1^{-1}$, that is $\sigma_1\sigma_2^{-1}\theta_1$, and $\theta_1 \equiv \sigma_1\sigma_2^{-1}\theta_1 \pmod{\widetilde{B}_\infty}$ does *not* hold.

Now the most interesting property of the LD-monoid EB_∞ is

Proposition 3.3. *The sub-LD-monoid of* EB_∞ *generated by any element not in* B_∞ *is a free LD-monoid.*

Proof. By Laver's criterion of [20], a sufficient (and necessary) condition for the sub-LD-monoid generated by an element a in an LD-monoid M to be free is that

$$b \neq {}^{{}^{{}^{b}c_1}\cdot^{\cdot^{\cdot}}}c_\ell$$

holds for every positive ℓ and every b, c_1, ..., c_ℓ in this sub-LD-monoid (in other words the left divisibility relation associated with exponentiation has no cycle). An easy induction shows that any element not equal to 1 in the sub-LD-monoid of EB_∞ generated by $\alpha\tau_{p,\infty}$ has the form $\beta\tau_{q,\infty}$ with q a non-zero multiple of p. So it is enough to show that

$$\beta\tau_{q,\infty} \neq {}^{{}^{{}^{\beta\tau_{q,\infty}}(\gamma_1\tau_{r_1,\infty})}\cdot^{\cdot^{\cdot}}}(\gamma_\ell\tau_{r_\ell,\infty}) \tag{3.12}$$

holds in EB_∞ whenever q is not 0 and r_1, ..., r_ℓ are at least equal to q. Now we observe that, by very definition of the exponentiation, the right hand side of (3.12) has the form

$$\beta\ s^q(\delta_0)\ \tau_{q,r_1}\ s^{r_1}(\delta_1)\ \tau_{r_1,r_2}\ s^{r_2}(\delta_2)\ \ldots\ s^{q_\ell}(\delta_\ell)\ \tau_{q_\ell,\infty}.$$

So, in order to show that (3.12) holds in EB_∞, it suffices to show that, for any braid δ in B_q, a braid of the form

$$s^q(\delta_0)\ \tau_{q,r_1}\ s^{r_1}(\delta_1)\ \tau_{r_1,r_2}\ s^{r_2}(\delta_2)\ \ldots\ s^{r_\ell}(\delta_\ell)\ \delta \tag{3.13}$$

cannot be trivial. Now we observe (*cf.* Figure 3) that the decomposition (3.13) has the property that the generator σ_q occurs in it $\inf(q, r_1) + \ldots + \inf(r_{\ell-1}, r_\ell)$ times, while its inverse σ_q^{-1} does not occur: by the results of [8] (also reproved in [18]), such a braid cannot be trivial. ■

The previous result obviously does not extend to the case of an ordinary braid α: in this case, the mapping $k \mapsto \alpha^k$ gives an isomorphism of the sub-LD-monoid of EB_∞ generated by α onto the LD-monoid made of $(\mathbf{N}, +)$ with the trivial exponentiation ${}^{x}y = y$ (since B_∞ is torsion free due to the existence of the ordering $<$), and the latter is not a free LD-monoid.

To complete this section, we look at the counterparts of the above operations in terms of permutations. Every braid induces a permutation of the integers indexing its strands, which gives a projection of each braid group B_n onto the corresponding symmetric group S_n. This construction has to be modified

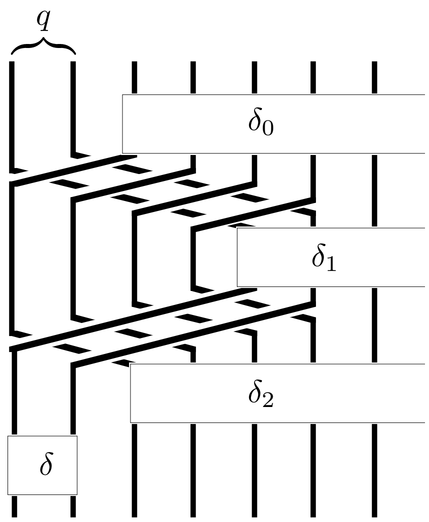


Figure 3: A braid $s^q(\delta_0)\tau_{q,r_1}s^{r_1}(\delta_1)\ldots\delta$ is not trivial: here $q = r_2 = 2$, $r_1 = 4$, and σ_2 occurs 4 times.

in the case of EB_∞ since some strands may vanish. However, we can still associate with every element ξ of EB_∞ the injection $\xi^\vee$ that maps i to j if the strand initially at position j ends at position i. If α is a braid, then $\alpha^\vee$ is the permutation considered above and the mapping $\alpha \mapsto \alpha^\vee$ is a morphism of the group B_∞ into the symmetric group $S_{\mathbf{N}}$. In the general case of EB_∞, the projection still behaves nicely:

Proposition 3.4. *The image of the projection $\xi \mapsto \xi^\vee$ is the set I_∞ of all injections of the positive integers with a finite co-image, and the operations of EB_∞ induce well-defined operations on I_∞: the product of EB_∞ projects onto the composition of I_∞, and the exponentiation of EB_∞ projects onto the (left distributive) exponentiation defined on I_∞ by*

$$^{f}g(n) = \begin{cases} fgf^{-1}(n) & \text{if } n \text{ belongs to the image of } f, \\ n & \text{otherwise.} \end{cases}$$

Proof. First we see that $(\alpha\tau_{p,\infty})^\vee$ is $\alpha^\vee \circ d^p$, where d is the injection that shifts every positive integer by one unit. Let $s^\vee$ denote the morphism of I_∞ such that $s^\vee(f)$ is the injection that maps $n+1$ to $f(n)+1$ and 0 to 0. It is clear that $s(\alpha)^\vee$ is $s^\vee(\alpha^\vee)$, and we obtain

$$\begin{aligned}(\alpha\tau_{p,\infty} \cdot \beta\tau_{q,\infty})^\vee &= (\alpha s^p(\beta)t_{p+q,\infty})^\vee \\ &= \alpha^\vee \circ s^{\vee p}(\beta^\vee) \circ d^{p+q} \\ &= \alpha^\vee \circ d^p \circ \beta^\vee \circ d^q = (\alpha\tau_{p,\infty})^\vee \circ (\beta\tau_{q,\infty})^\vee\end{aligned}$$

since $d^p \circ f$ is always equal to $s^{\vee p}(f) \circ d^p$. For exponentiation, let us assume that ζ is ${}^{\xi}\eta$, where ξ is $\alpha\tau_{p,\infty}$ and η is $\beta\tau_{q,\infty}$. We compute $\zeta^\vee(n)$, and separate two cases. Assume first that n belongs to the image of $\xi^\vee$, *i.e.*, of $\alpha^\vee \circ d^p$: this means that n is $\alpha^\vee(p+m)$ for some $m \geq 1$. Using the fact that $\tau^\vee_{p,q}$ is the identity beyond $p+q$, we obtain

$$\begin{aligned}
\zeta^\vee(n) &= \alpha^\vee \circ s^p(\beta)^\vee \circ \tau^\vee_{p,q} \circ s^q(\alpha^{-1})^\vee \circ d^q(\alpha^\vee(p+m)) \\
&= \alpha^\vee \circ s^p(\beta)^\vee \circ \tau^\vee_{p,q} \circ s^q(\alpha^{-1})^\vee(\alpha^\vee(p+m)+q) \\
&= \alpha^\vee \circ s^p(\beta)^\vee \circ \tau^\vee_{p,q}(p+m+q) \\
&= \alpha^\vee \circ s^p(\beta)^\vee(p+m+q) \\
&= \alpha^\vee(\beta^\vee(m+q)+p) \\
&= \alpha^\vee(\eta^\vee(m)+p) = \xi^\vee(\eta^\vee(m)) = {}^{\xi^\vee}(\eta^\vee)(n)
\end{aligned}$$

Assume now that n does not belong to the image of $\xi^\vee$: this means that n is $\alpha^\vee(m)$ for some $m \leq p$. We obtain

$$\begin{aligned}
\zeta^\vee(n) &= \alpha^\vee \circ s^p(\beta)^\vee \circ \tau^\vee_{p,q} \circ s^q(\alpha^{-1})^\vee \circ d^q(\alpha^\vee(m)) \\
&= \alpha^\vee \circ s^p(\beta)^\vee \circ \tau^\vee_{p,q} \circ s^q(\alpha^{-1})^\vee(\alpha^\vee(m)+q) \\
&= \alpha^\vee \circ s^p(\beta)^\vee \circ \tau^\vee_{p,q}(m+q) \\
&= \alpha^\vee \circ s^p(\beta)^\vee(m) \\
&= \alpha^\vee(m) = n = {}^{\xi^\vee}(\eta^\vee)(n)
\end{aligned}$$

This completes the proof. ■

The set I_∞ equipped with composition and the above exponentiation (as well as any set of injections equipped with these operations) is an LD-monoid, as was noted in [6].

4. A new left distributive operation

Assume that $(S, *)$ is an LD-system and e is an element of S. Under obvious compatibility conditions needed to guarantee the soundness of the construction, two new binary operations can be defined on the left ideal I of S generated by e using the following rules: if x is $x_1 * \ldots * x_p * e$ (we keep the convention that $a * b * c$ stands for $a * (b * c)$), and y is $y_1 * \ldots * y_q * e$, we take

$$x\, y = x_1 * \ldots * x_p * y_1 * \ldots * y_q * e \tag{4.1}$$

$$^{x}y = z_1 * \ldots * z_q * e, \quad \text{where } z_j \text{ is } x_1 * \ldots * x_p * y_j. \tag{4.2}$$

In particular, we have

$$^{x*e}(y * e) = (x * y) * e.$$

Then I equipped with these new operations is an LD-monoid. This approach works in particular when S is a free LD-system generated by a single element e, and, in this case, the LD-monoid derived in the above way is a free LD-monoid generated by $e{*}e$ ([7], [20]). Conversely, if M is any LD-monoid (and specially if M is free or includes free LD-monoids), it is natural to ask if the operations of M (or at least their restrictions to some subset of M) derive from some underlying left distributive operation in the manner of (4.1) and (4.2). Observe that, if such an operation can be found, the compatibility conditions that are needed for (4.1) and (4.2) to make sense (namely various decompositions of the form $x_1{*}\ldots{*}x_p{*}e$ should lead to the same values) are automatically verified since the derived operations are assumed to exist *a priori*. The following construction gives a general method for addressing the question.

Lemma 4.1. [12] *Assume that M is an LD-monoid, and a is any element of M. Then the operation $*$ defined by*

$$x *_a y = {}^{x}a\; y$$

is left distributive, and, on the left ideal of $(M, *_a)$ generated by 1, the product and the exponentiation of M are defined from $*$ using (4.1) and (4.2).

Let us come back to the specific case of EB_∞. We wish to apply the above construction in such a way that the left ideal of $(EB_\infty, *)$ generated by 1, which will be automatically a sub-LD-monoid of M, is not trivial, and, in particular, includes a free LD-monoid. It is easy to see that choosing a braid for the parameter a in the construction of the operation $*$ gives an uninteresting result, as the ideal generated by 1 is included in B_∞. So we consider the case when the parameter is not a braid, and the simplest case is to take $\tau_{1,\infty}$. Then the results are what we can expect:

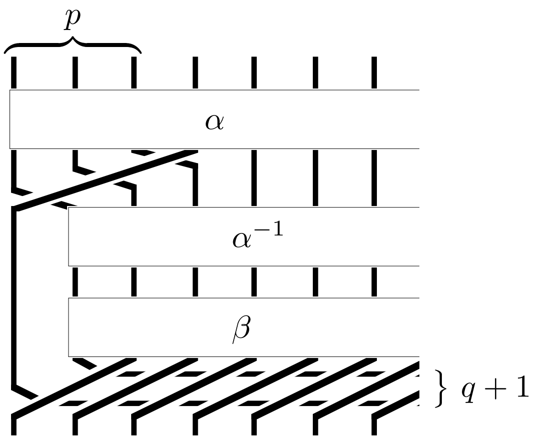


Figure 4: Value of $\alpha\tau_{p,\infty} * \beta\tau_{q,\infty}$

Proposition 4.2. *(i) The formula*

$$(\alpha\tau_{p,\infty}) * (\beta\tau_{q,\infty}) = \alpha\tau_{p,1}s(\alpha^{-1}\beta)\tau_{q+1,\infty}, \tag{4.3}$$

defines a left-distributive operation on EB_∞, and every element of EB_∞ generates under $*$ a free LD-system.
(ii) The operations of the LD-monoid EB_∞ coincide with those defined from the operation $*$ using (4.1) *and* (4.2) *on the left ideal of $(EB_\infty, *)$ generated by* 1.

Proof. Formula (4.3) is the translation of the definition of Lemma 4.1 in the case when a is $\tau_{1,\infty}$. So, it only remains to prove the statement about the monogenerated subsystems of $(EB_\infty, *)$. We shall apply again Laver's criterion and show that the left divisibility relation of $(EB_\infty, *)$ has no cycle. To this end it is sufficient to show that the inequality

$$\alpha\tau_{p,\infty} < \alpha\tau_{p,\infty} * \beta\tau_{q,\infty} \tag{4.4}$$

is always true. By expanding and applying the results of Section 2, this amounts to showing that the braid inequality

$$\alpha\tau_{p,n} < \alpha\tau_{p,1}s(\alpha^{-1}\beta)\tau_{q+1,n}$$

holds for n large enough. Since $\tau_{p,n}$ is equal (for $n \geq 1$) to $\tau_{p,1}s(\tau_{p,n-1})$, this is equivalent to

$$1 < s(\tau_{p,n-1}^{-1})s(\alpha^{-1}\beta)\tau_{q+1,n},$$

which is true since the generator σ_1 occurs once in the right hand side expression, while σ_1^{-1} does not. ■

So we see that the operation $*$ on EB_∞ is some sort of "ancestor" for the operations of Section 3, and, in particular, for the braid bracket defined by (0.1). Of course Formulas (4.1) and (4.2) hold only on the left ideal generated by 1, and not everywhere, but observe that the latter ideal includes in particular the (free) sub-LD-monoid of EB_∞ generated by $\tau_{1,\infty}$. Since every element of EB_∞ generates under $*$ a free LD-system, so does in particular the unit braid 1, and we obtain a simple relization of the free LD-system F. For instance, one can easily compute expressions like

$$1 * 1 = \tau_{1,\infty}, \qquad 1 * 1 * 1 = \tau_{2,\infty}, \quad \textit{etc.}$$
$$(1 * 1) * 1 = \sigma_1 \tau_{1,\infty}, \qquad ((1 * 1) * 1) * 1 = \sigma_1^2 \sigma_2^{-1} \tau_{1,\infty}, \quad \textit{etc.}$$

Variants of the above construction can be obtained by changing the initial parameter $\tau_{1,\infty}$ of (4.6): for instance using $\tau_{m,\infty}$ amounts to considering the operation

$$(\alpha\tau_{p,\infty}) *_m (\beta\tau_{q,\infty}) = \alpha\tau_{p,m} s^m(\alpha^{-1}\beta)\tau_{q+m,\infty}.$$

This variant however adds nothing in terms of the generated LD-systems, as it amounts to grouping the strands of the braids in series of m ones. Note also that the regressing process that leads from the braid bracket to the operation $*$ cannot be repeated (inside EB_∞): the operation $*$ increases the degree of its right argument by exactly one, and therefore there cannot exist any operation on EB_∞ for which the identity (LD_2) could be satisfied.

Instead of using the general method of Lemma 4.1 to guess Formula (4.6), we could alternatively appeal to braid colourings involving a free LD-system F with one generator e, as it has been done to introduce the braid bracket of (0.1) (*cf.* [8]). Indeed Formula (0.1) appears when one tries to construct, for every colour x in F, a canonical braid β_x that produces the colour x in the sense of

$$(e, e, \ldots)\beta_x = (x, e, e, \ldots). \tag{4.5}$$

As shows Figure 5 (left), a possible induction formula is

$$\beta_{x*y} = \beta_x \; s(\beta_y) \; \sigma_1 \; s(\beta_x^{-1}),$$

which directly leads to (0.1) when we require β_{x*y} to be $\beta_x\,[\,\beta_y\,]$.

Now let us consider the question again in the framework of EB_∞ (with the colourings defined in Section 2), and try to construct, for every element x of the free LD-system F a canonical element χ_x of EB_∞ that produces the colour x in the sense of

$$(e, e, \ldots)\chi_x = (x, x, \ldots). \tag{4.6}$$

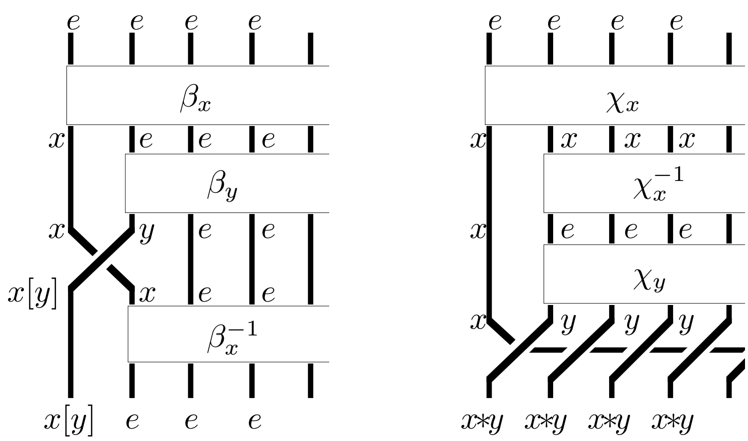


Figure 5: Canonical colourings

(Observe that using braids from the constant sequence $(e, e, \ldots)$ necessary leads to a sequence of colours eventually equal to e, a limitation that disappears when one goes to EB_∞.) Figure 5 (right) shows that the inductive definition

$$\chi_{x*y} = \chi_x \; s(\chi_x^{-1}) \;\; s(\chi_y) \; \tau_{1,\infty}$$

works, and requiring that the mapping χ be a homomorphism leads now to consider on EB_∞ the binary operation

$$\xi * \eta = \xi \; s(\xi^{-1}) \; s(\eta) \; \tau_{1,\infty}. \tag{4.7}$$

However this formula does not directly makes sense since in general there is no inverse in the monoid EB_∞. But, in the group $B_{\infty+\infty}$, we have

$$\theta_p \ldots \theta_1 \; \theta_2^{-1} \ldots \theta_{p+1}^{-1} = \sigma_p \ldots \sigma_1 = \tau_{p,1},$$
$$\theta_{q+1} \ldots \theta_2 = \tau_{q,1}^{-1} \theta_q \ldots \theta_1,$$

which exactly leads (because $\tau_{q,1}^{-1} \tau_{q+1,\infty}$ is $\tau_{q+1,\infty}$) to Formula (4.6).

Remark. Let us consider again the LD-monoid structure of Section 3. We have mentioned that the operations derived using (4.1) and (4.2) on a free LD-system generated by e give rise to a free LD-monoid. Since the bracket on B_∞ has the property that any braid generates a free LD-system, we obtain a way to embed the free LD-monoid (on one generator) into B_∞: this is enough for instance to construct an effective algorithm that decides if two abstract terms (involving one variable and two binary operations) are or not equivalent under the LD-monoid identities. But this does not define an *effective* structure of LD-monoid on (a subset of) B_∞: for instance, assuming that we start with the free LD-system generated by the braid 1, the involved distributive operation, here denoted $\wedge$, is characterized by the equality

$$\alpha[1] \ \wedge \ \beta[1] = \alpha[\beta][1], \tag{4.8}$$

i.e.,

$$\alpha\sigma_1 s(\alpha^{-1}) \ \wedge \ \beta\sigma_1 s(\beta^{-1}) = \alpha s(\beta)\sigma_1 s(\alpha^{-1})\sigma_1 s^2(\alpha)\sigma_2^{-1} s^2(\beta^{-1}) s(\alpha^{-1}).$$

The mapping $\alpha \mapsto \alpha\sigma_1 s(\alpha^{-1})$ is injective on B_∞, but, as long as no explicit inversion formula is known, (4.11) is not an explicit definition. From this point of view, the operations of Section 3 are very different.

A last question is whether the above constructions can be extended to the case of free distributive structures with several generators. For LD-systems and the bracket of (0.1), the extension is possible at the expense of considering extended braids where the strands carry some additional information ("charged braids" [10], see also [19]). It should be easy to extend the present completion process to the case of charged braids since their algebraic properties are quite parallel to those of ordinary braids.

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