A CONJECTURE ABOUT CONJUGACY IN FREE GROUPS

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Abstract

Say that an element of a free group is a pure conjugate if it can be expressed from the generators using exclusively the conjugacy operation. We study free reductions in words representing pure conjugates. Using finite state automata, we attribute to the letters in such words levels that live in some free left distributive system. If a certain conjecture about this system is true, then reduction can occur only between letters lying on the same level. Under this conjecture, we establish restrictions on the form of those identities satisfied by group conjugacy, and we construct unique normal forms for large families of pure conjugates. We also show how to use group conjugacy to solve a problem related to the word problem of left self-distributivity.

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1. Introduction

The conjugacy operation of a group

$$x\hat{\ }y = xyx^{-1}$$

is a typical example of a binary operation that is both left self-distributive and idempotent, *i.e.*, that satisfies the identities

$$x\hat{\ }(y\hat{\ }z) = (x\hat{\ }y)\hat{\ }(x\hat{\ }z), \qquad (LD)$$

$$x\hat{\ }x = x. \qquad (I)$$

Except in trivial cases, this operation does not satisfy the entropic law, and, therefore, a group equipped with conjugacy is not a mode. However, it is closely connected with modes: for instance, using the free differential calculus of [1], we project the conjugacy operation of a free group onto the entropic operation

$$x^\wedge y = (1-t)x + ty$$

of an affine space.

Let us consider the question of axiomatizing group conjugacy. If we consider simultaneously the conjugacy operation $^\wedge$ and the dual operation $^\vee$ defined by $x^\vee y = x^{-1}yx$, then the answer is easy: the identities satisfied by $^\wedge$ and $^\vee$ are the consequences of the four identities expressing that $^\wedge$ and $^\vee$ are self- and mutually left distributive, and of the two additional identities $x^\wedge(x^\vee y) = x^\vee(x^\wedge y) = y$. In other words, the systems $(F, ^\wedge, ^\vee)$ where F is a free group are, up to a symmetry, the free quandles of [10] and [15].

In the case of the operation $^\wedge$ alone, the question remains widely open. It was not even known until recently whether $^\wedge$ is completely axiomatized by (LD) and (I). Actually, it is not: D. Larue has constructed in [11] an infinite series of independent identities that are satisfied by conjugacy and are not consequences of (LD) and (I). An example of such an identity is

$$((x^\wedge y)^\wedge y)^\wedge(x^\wedge z) = (x^\wedge y)^\wedge((y^\wedge x)^\wedge z). \tag{1.1}$$

This example appears also in [7].

The only general positive result about axiomatization of group conjugacy is the result of [11] that those identities satisfied by $^\wedge$ are exactly those identities that hold in every left cancellative binary system satisfying (LD) and (I), *i.e.*, in every idempotent rack in the terminology of [9].

There is no reason to believe that the above mentioned identities give a complete axiomatization of group conjugacy. In this paper, we study the question by proving that certain forms of identities are *a priori* impossible—at least if a certain conjecture about the left self-distributive law is true.

In the sequel, V denotes a fixed countable sequence of variables, and T denotes the set of all formal terms constructed using variables in V and a single binary operator denoted $^\wedge$. We say that two terms t, t' in T are *LD-equivalent*, denoted $t =_{LD} t'$, if the identity $t = t'$ is a consequence of Identity (LD).

Definition. Assume that t_0 and t are terms in T. We say that $t_0 \sqsubset t$ holds, or that t_0 is a *proper prefix* of t, if there exist finitely many terms $t_1, \ldots, t_k$ such that t is $(\ldots((t_0{}^\wedge t_1)^\wedge t_2)\ldots)^\wedge t_k$. We say that $t_0 \sqsubset_{LD} t$ (*resp.* $t_0 \sqsubseteq_{LD} t$) holds if there exist terms t'_0 and t' satisfying $t'_0 =_{LD} t_0$, $t' =_{LD} t$ and $t'_0 \sqsubset t'$ (*resp.* $t'_0 \sqsubset t'$ or $t'_0 = t'$).

Definition. The term t is *LD-monotone* if there exist distinct variables $x_1, \ldots, x_m$ such that $t \sqsubseteq_{LD} x_1{}^\wedge\ldots{}^\wedge x_m$ holds. The identity $t_1 = t_2$ is *LD-monotone* if both t_1 and t_2 are LD-monotone terms.

In the above definition, as everywhere subsequently, missing brackets are to be added on the right: $x^\wedge y^\wedge z$ stands for $(x^\wedge y)^\wedge z$. One can verify that neither (I) nor any of the identities of [11] like (1.1) above is an LD-monotone identity. The main statement we shall discuss is:

Conjecture A. *Group conjugacy satisfies no LD-monotone identity except those that are consequences of left self-distributivity.*

A binary system made of a set equipped with a binary operation that satisfies Identity (LD) is called an *LD-system*. By construction, the binary system $(T/=_{LD}, {}^\wedge)$ is a free LD-system based on V. We denote by Λ this free LD-system, and by π_{LD} the canonical projection of T onto Λ. For t a term, we shall usually write $\bar{t}$ for $\pi_{LD}(t)$. If t is a term and $\bar{t}$ is a, we say that t *represents* a.

A non-trivial property of the left self-distributive law is that the prefix relation of terms induces a (strict) partial ordering on the free LD-system Λ [11] [4]. This partial ordering is still denoted $\sqsubset$.

Definition. An element a of Λ is *decomposable* if there exist distinct variables $x_1, \ldots, x_m$ in V such that a can be written $a = a_1{}^\wedge\ldots{}^\wedge a_n$ with $x_1{}^\wedge\ldots{}^\wedge x_m \sqsupset a_1 \sqsupset \ldots \sqsupset a_n$.

Conjecture B. *The square function of Λ is injective on decomposable elements: if a_1, a_2 are decomposable in Λ and $a_1{}^\wedge a_1 = a_2{}^\wedge a_2$ holds, then a_1 and a_2 are equal.*

Our main result here is:

Proposition 1.1. *Conjecture B implies Conjecture A.*

The previous general study can be translated into a study of the conjugacy operation of a free group. In the sequel, we denote by F a free group based on V.

Definition. An element a of the free group F is a *pure conjugate* if it can be expressed using exclusively variables in V and the conjugacy operation ^. The set of all pure conjugates in F is denoted C. Thus, C is the closure of V under ^ in the LD-system $(F, \hat{})$.

For instance, $xyxy^{-1}x^{-1}$ is a pure conjugate, as we have in F the equality $xyxy^{-1}x^{-1} = ((x\hat{}y)\hat{}x$, as well as $xyxy^{-1}x^{-1} = x\hat{}y\hat{}x$.

Saying that every pure conjugate can be expressed using exclusively ^ and variables means that such an element is the *evaluation* in $(F, \hat{})$ of some term of T. We denote by eval this evaluation mapping.

The very fact that non-trivial algebraic identities are satisfied by the conjugacy operation implies that the mapping eval is not injective. For instance, we have $\mathrm{eval}(x\hat{}x) = \mathrm{eval}(x) = x$.

A natural problem is to find a section for eval, *i.e.*, to construct, for every pure conjugate in F, a distinguished, 'normal' term that represents it. We have no general solution, but, if Conjecture A is true, we can obtain a partial solution.

Definition. A pure conjugate is *monotone* if it can be represented by an LD-monotone term.

We define in Section 8 the notion of a *special* term. Then we prove

Proposition 1.2. *Assume that Conjecture A is true. Then every monotone pure conjugate in F is represented by a unique special term, in an effective way, i.e., there exists an algorithm that, starting with a monotone pure conjugate a, returns the unique special term t such that a is* $\mathrm{eval}(t)$.

One of the possible interests of this result is that, assuming that Conjecture A is true, it gives a large family of terms with pairwise distinct evaluations in $(F, \hat{})$. Because group conjugacy satisfies both (LD) and (I), these terms must be pairwise LDI-inequivalent, with the obvious definition that t and t' are LDI-equivalent if the identity $t = t'$ is a consequence of the conjunction of (LD) and (I). It is not known whether LDI-equivalence is a decidable relation, and very few techniques are available for constructing LDI-inequivalent terms.

In the other direction, we can use group conjugacy to solve problems about LD-systems. Let z be a fixed variable, and T_1 be the set of all terms constructed using z and the operator ^. For t in T, we define the *skeleton* of t to be the term in T_1 obtained from t by replacing each variable with z. It is known that two LD-equivalent terms having the same skeleton must be equal. In other words, if we are given the skeleton of a term t and its class in Λ, there is exactly one way to choose the variables of t. It is natural to ask for an algorithmic solution.

Definition. An element a of Λ is *monotone* if it can be represented by an LD-monotone term.

Proposition 1.3. *There exists an algorithm that, assuming that Conjecture A is true and starting with a monotone element a of Λ and a skeleton s in T_1, returns the (unique) term t with skeleton s that represents a, if such a term exists.*

The organization of the paper is as follows. In Section 2, we fix the framework and consider the case when product and conjugacy are considered simultaneously. In Section 3, we introduce the conjugacy words precisely, and we prove some basic relations involving the so-called **Z**-level of the letters. In Section 4, we describe the connection between the prefixes of the conjugate words and the geometry of the terms that represent them. In Section 5, we introduce automata and show how to use a deterministic automaton to control free reductions in a word. In Section 6, we consider the specific automaton associated with a term. In Section 7, we prove Proposition 1.1 and various related results about conjugate words. In Section 8, we define the notion of a special term and prove the normal form result stated as Proposition 1.2. Finally, we develop in Section 9 the algorithm mentioned in Proposition 1.3.

2. LD-MONOIDS

Besides the pure conjugates, which have been defined in Section 1 as those elements of F that can be obtained from the generators using exclusively the conjugacy operation, it will be useful to consider those elements that can be obtained from the generators using both conjugacy and product.

Definition. An element a of F is a *conjugate* if it can be expressed using exclusively variables in V, the product and the operation $^\wedge$. The set of all conjugates is denoted by $\widetilde{C}$. Thus, $\widetilde{C}$ is the closure of V in $(F, \cdot, ^\wedge)$.

For instance (still assuming that x and y belong to V), the word xy is a conjugate—while Proposition 4.5 below will show that it is not a pure conjugate.

Lemma 2.1. *$(\widetilde{C}, \cdot)$ is the submonoid of $(F, \dot{)}$ generated by C.*

Proof. Let C^* denote the submonoid of F generated by C. The only point we have to prove is that C^* is closed under operation ^. This follows from the formula $(a_1 \ldots a_n){}^\wedge(b_1 \ldots b_m) = c_1 \ldots c_m$, where c_i is $a_1{}^\wedge \ldots {}^\wedge a_n{}^\wedge b_i$. ∎

Definition. The algebraic system $(M, \cdot, 1, {}^\wedge)$ is an *LD-monoid* if $(M, \cdot, 1)$ is a monoid, and the following mixed identities holds in M:

$$x \cdot y = (x{}^\wedge y) \cdot x, \qquad (LDM_1)$$
$$(x \cdot y) {}^\wedge z = x{}^\wedge(y{}^\wedge z), \qquad (LDM_2)$$
$$x{}^\wedge(y \cdot z) = (x{}^\wedge y) \cdot (x{}^\wedge z), \qquad (LDM_3)$$
$$x{}^\wedge 1 = 1. \qquad (LDM_4)$$

Note that, in every LD-monoid, the second operation is left self-distributive and satisfies $1{}^\wedge x = x$ for every x. The following fact is obvious:

Proposition 2.2. *Assume that $(G, \cdot, 1)$ is a group. Then $(G, \cdot, 1, {}^\wedge)$ is an LD-monoid.*

Like pure conjugates, conjugates are the evaluation in $(F, \cdot, {}^\wedge)$ of some terms. Here, we have to consider terms involving two binary operation symbols, say $\cdot$ and ^. We write $\widetilde{T}$ for the set of all such terms, extended with an additional term 1 that represents the unit. We still use eval for the surjective evaluation mapping of $\widetilde{T}$ onto $\widetilde{C}$, as it extends the evaluation mapping of T onto C.

We say that two terms t, t' in $\widetilde{T}$ are *LDM-equivalent*, written $t =_{LDM} t'$, if the identity $t = t'$ is a consequence of the axioms of LD-monoids. The latter are denoted (LDM) in the sequel. As for the case of one operation, the system $(\widetilde{T}/=_{LDM}, \cdot, {}^\wedge)$ is a free LD-monoid based on V. We denote it by $\widetilde{\Lambda}$, and we write π_{LDM} for the canonical projection of $\widetilde{T}$ onto $\widetilde{\Lambda}$. Again, for t in $\widetilde{T}$, we shall usually write $\bar{t}$ for $\pi_{LDM}(t)$.

Definition. The set T^* is the subset of $\widetilde{T}$ consisting of those terms of the form $t_1 \cdot \ldots \cdot t_m$ with t_1, …, t_m in T, *i.e.*, involving the operator ^ only.

Lemma 2.3. [13, 3] *(i) There exists a function* red *of* $\widetilde{T}$ *onto* T^* *that maps every term to an LDM-equivalent term. For* $t = t_1 \cdot \ldots \cdot t_m$ *in* T^* *and* x *in* V, *the term* $\mathrm{red}(t\hat{}x)$ *is* $t_1\hat{}\ldots\hat{}t_m\hat{}x$, *thus it belongs to* T.

(ii) Two terms t, t' *in* T^* *are LDM-equivalent if and only if for each variable* x *the terms* $\mathrm{red}(t\hat{}x)$ *and* $\mathrm{red}(t'\hat{}x)$ *are LD-equivalent.*

(iii) For t, t' *in* T *and* x *a variable, the terms* t *and* t' *are LD-equivalent if and only if the terms* $t\hat{}x$ *and* $t'\hat{}x$ *are LD-equivalent.*

By (i) above, all elements of $\widetilde{\Lambda}$ can be represented by terms in T^*—and, therefore, by the existence of eval, so do all elements of $\widetilde{C}$ (this implies Lemma 2.1). By (ii) and (iii), two terms in T are LDM-equivalent if and only if they are LD-equivalent. So the inclusion of T into $\widetilde{T}$ induces an embedding of Λ into $\widetilde{\Lambda}$, and we shall from now on consider Λ as a sub-LD-system of $\widetilde{\Lambda}$. Then, by (i), every element in $\widetilde{\Lambda}$ is a finite product of elements of Λ. Finally, the mapping $a \mapsto a\hat{}x$ is an injection of $\widetilde{\Lambda}$ into Λ for every variable x in V.

Definition. Let t be a term in T. Then t admits a unique decomposition of the form $t = t_1\hat{}\ldots\hat{}t_m\hat{}x$ with x a variable in V and $t_1, \ldots, t_m$ terms in T. We define $\mathrm{var}_R(t)$ to be x, and t^- and t^+ to be respectively the terms $t_1 \cdot \ldots \cdot t_m$ and $t_1 \cdot \ldots \cdot t_m \cdot x$ in T^* (t^- is 1 if t is x).

Lemma 2.4. *The mappings* var_R, $^-$ *and* $^+$ *induce well-defined mappings of* Λ *respectively into* V, $\widetilde{\Lambda}$ *and* $\widetilde{\Lambda}$.

Proof. When we apply (LD) once to the term $t = t_1\hat{}\ldots\hat{}t_m\hat{}x$, either we apply it inside some subterm t_i, or we replace t with $t_1\hat{}\ldots\hat{}(t_i\hat{}t_{i+1})\hat{}t_i\hat{}\ldots\hat{}t_m\hat{}x$, or we replace t with $t_1\hat{}\ldots\hat{}t_{i+1}\hat{}t'_i\hat{}\ldots\hat{}t_m\hat{}x$, where t_i is $t_{i+1}\hat{}t'_i$. In every case, the value of x, as well as the value of m, is unchanged. So $\mathrm{var}_R(t)$ depends only on the LD-class of t.

Assume then that the terms $t = t_1\hat{}\ldots\hat{}t_m\hat{}x$ and $t' = t'_1\hat{}\ldots\hat{}t'_{m'}\hat{}x'$ are LD-equivalent. By the argument above, we have $m = m'$ and $x = x'$. By Lemma 2.3(ii), the terms t^- and $(t')^-$ are LDM-equivalent if and only if the terms $\mathrm{red}(t^-\hat{}x)$ and $\mathrm{red}((t')^-\hat{}x)$ are LD-equivalent: by construction, this means that the terms t and t' are LD-equivalent. Similarly, the terms t^+ and $(t')^+$ are LDM-equivalent if and only if the terms $\mathrm{red}(t^+\hat{}x)$ and $\mathrm{red}((t')^+\hat{}x)$ are LD-equivalent. Now the latter terms are respectively LD-equivalent to $t\hat{}t$ and $t'\hat{}t'$, and $t =_{LD} t'$ implies $t\hat{}t =_{LD} t'\hat{}t'$. ■

We shall naturally still denote by var_R the mapping of Λ to V induced by var_R, and by $^-$ and $^+$ the mappings of Λ into $\widetilde{\Lambda}$ induced by $^-$ and $^+$. For instance, we have in $\widetilde{\Lambda}$ the equalities

$$\mathrm{var}_R((x\char`^y)\char`^x) = x, \quad ((x\char`^y)\char`^x)^- = x\char`^y, \quad ((x\char`^y)\char`^x)^+ = (x\char`^y)x = xy.$$

Lemma 2.5. *For every element a in Λ, the equalities*

$$a = a^-\char`^\mathrm{var}_R(a) \tag{2.1}$$

$$a^+ = a^- \cdot \mathrm{var}_R(a) = a \cdot a^- \tag{2.2}$$

$$(a\char`^a)^- = a^+ \tag{2.3}$$

hold in $\widetilde{\Lambda}$.

Proof. Assume that $\mathrm{var}_R(a)$ is x. The equalities $a = a^-\char`^x$ and $a^+ = a^- \cdot x$ are obvious from the definition. Then, by $(LDM_1)$, $a^- \cdot x$ is $(a^-\char`^x) \cdot a^-$, *i.e.*, by (2.1), $a \cdot a^-$. Finally, we have $a\char`^a = a^-\char`^x\char`^x = (a^- \cdot x)\char`^x = a^+\char`^x$, hence $(a\char`^a)^-$ is a^+. ∎

3. Conjugate words

As usual, the elements of the free group F are represented by reduced words over the alphabet $V \cup V^{-1}$, where V^{-1} denotes a disjoint copy of V. We denote by W the free monoid of all words over $V \cup V^{-1}$. For w in W, we denote by $\overline{w}$ the *free reduct* of w, *i.e.*, the word obtained from w by iteratively deleting all patterns xx^{-1} and $x^{-1}x$ with x in V. We write $w =_{FG} w'$ if w and w' represent the same element of F. Finally, we denote by w^{-1} the word obtained from w by reversing the ordering of the letters and replacing every letter $x^{\pm 1}$ by its inverse $x^{\mp 1}$.

Definition. The mapping $\mathrm{conj} : \widetilde{T} \longrightarrow W$ is defined inductively by the rules

$$\mathrm{conj}(t) = \begin{cases} t & \text{if } t \text{ is a variable,} \\ \mathrm{conj}(t_0)\mathrm{conj}(t_1)\mathrm{conj}(t_0)^{-1} & \text{if } t \text{ is } t_0\char`^t_1, \\ \mathrm{conj}(t_0)\mathrm{conj}(t_1) & \text{if } t \text{ is } t_0 \cdot t_1. \end{cases}$$

The words of the form $\mathrm{conj}(t)$ for t in T^* are called *conjugate words.* The words of the form $\mathrm{conj}(t)$ for t in T are called *pure conjugate words.*

By construction, we have:

Lemma 3.1. *For every term t,* eval(t) *is* $\overline{\mathrm{conj}(t)}$.

The difference between conjugate words and conjugates is that the former need not be freely reduced in general. For instance, if t is the term $(x\wedge y)\wedge x$, conj(t) is the word $xyx^{-1}xxy^{-1}x^{-1}$, while eval(t) is the reduced word $xyxy^{-1}x^{-1}$. Most of our work in the sequel consists in trying to control the free reductions that may occur in conjugate words.

We shall always consider a word w of length ℓ in W as a mapping of the integer interval $\{1, \ldots, \ell\}$ into $V \cup V^{-1}$. Thus, for $1 \le p \le \ell$, we write $w(p)$ for the p-th letter of w. Similarly, $w\restriction\{p_1, \ldots, p_2\}$ denotes the subword of w that comprises the letters from the p_1-th to the p_2-th.

If $w(p)$ is $x^{\pm 1}$, we say that p is a *position* of x in w. The *sign* sign(p, w) of the position p in the word w is $+$ or $-$ according to whether the letter $w(p)$ lies in V or in V^{-1}.

Proposition 3.2. *Assume that w is a pure conjugate word, or a pure conjugate. Then the length ℓ of w is odd, the median letter of w is positive, and $w(\ell+1-p) = w(p)^{-1}$ holds for every non-median position p.*

Proof. By induction on the term t, the properties are obvious for the words conj(t). Then they are preserved under free reduction. ∎

Definition. Assume that w is a word in W. We write $\|w\|_+$ and $\|w\|_-$ respectively for the total number of positive and of negative positions in w—so that the length of w is always $\|w\|_+ + \|w\|_-$. The $\mathbf{Z}$-*balance* of w is $\|w\|_+ - \|w\|_-$. For p a position in w, the $\mathbf{Z}$-*level* of p in w is the difference

$$\mathbf{Z}\text{-level}(p, w) = \|w\restriction\{1, \ldots, p-1\}\|_+ - \|w\restriction\{1, \ldots, p\}\|_-.$$

Example 3.3. The pure conjugate word conj$((x\wedge y)\wedge x) = xyx^{-1}xxy^{-1}x^{-1}$ has $\mathbf{Z}$-balance $+1$, and the $\mathbf{Z}$-levels of its 7 positions are

p	1	2	3	4	5	6	7
$w(p)$	x	y	x^{-1}	x	x	y^{-1}	x^{-1}
$\mathbf{Z}$-level(p, w)	0	1	1	1	2	2	1

Lemma 3.4. *Assume that w is a word of length ℓ. Then the equality*

$$\mathbf{Z}\text{-level}(\ell+1-p, w) = \mathbf{Z}\text{-balance}(w) + \mathbf{Z}\text{-level}(p, w^{-1}) \tag{3.1}$$

holds for $1 \le p \le \ell$.

Proof. By definition, $w^{-1}(q)$ is $w(\ell+1-q)^{-1}$ for every position q, so we have

$$\begin{aligned}\mathbf{Z}\text{-level}(p, w^{-1}) &= \|w^{-1}{\restriction}\{1,\ldots,p-1\}\|_+ - \|w^{-1}{\restriction}\{1,\ldots,p\}\|_- \\ &= \|w{\restriction}\{\ell+2-p,\ldots,\ell\}\|_- - \|w{\restriction}\{\ell+1-p,\ldots,\ell\}\|_+ \\ &= \|w\|_- - \|w{\restriction}\{1,\ldots,\ell+1-p\}\|_- \\ &\qquad\qquad - \|w\|_+ + \|w{\restriction}\{1,\ldots,\ell-p\}\|_+ \\ &= -\mathbf{Z}\text{-balance}(w) + \mathbf{Z}\text{-level}(\ell+1-p, w),\end{aligned}$$

which gives (3.1). ∎

Proposition 3.5. *Assume that w is a pure conjugate word. Then the* $\mathbf{Z}$*-balance of w is $+1$; position 1 has* $\mathbf{Z}$*-level 0 in w, while all subsequent positions have* $\mathbf{Z}$*-level ≥ 1. If the length of w is at least 2, the last position has* $\mathbf{Z}$*-level 1.*

Proof. The result is proved for $\mathrm{conj}(t)$ inductively on t. Everything is obvious when t is a variable. Assume $t = t_0{}^\wedge t_1$. Write w for $\mathrm{conj}(t)$, w_i for $\mathrm{conj}(t_i)$, $i = 0, 1$, and ℓ, ℓ_0 and ℓ_1 respectively for the lengths of w, w_0 and w_1. By definition, w is $w_0 w_1 w_0^{-1}$, so, by induction hypothesis, $\mathbf{Z}$-balance(w) is $1+1-1 = 1$. For $1 \leq p \leq \ell_0$, we have

$$\mathbf{Z}\text{-level}(p, w) = \mathbf{Z}\text{-level}(p, w_0),$$

so, by induction hypothesis, this number is 0 for $p = 1$ and ≥ 1 for $p \geq 2$. Then, for $1 \leq p \leq \ell_1$, we have

$$\mathbf{Z}\text{-level}(\ell_0 + p, w) = \mathbf{Z}\text{-balance}(w_0) + \mathbf{Z}\text{-level}(p, w_1).$$

By induction hypothesis, $\mathbf{Z}$-balance(w_0) is 1, and $\mathbf{Z}$-level(p, w_1) is ≥ 0, so $\mathbf{Z}$-level$(\ell_0 + p, w) \geq 1$ holds. Finally, for $1 \leq p \leq \ell_0$, we have by Formula (3.1)

$$\begin{aligned}\mathbf{Z}\text{-level}(\ell+1-p, w) &= \mathbf{Z}\text{-balance}(w) + \mathbf{Z}\text{-level}(p, w^{-1}) \\ &= 1 + \mathbf{Z}\text{-level}(p, w_0)\end{aligned}$$

and we conclude as above. ∎

Proposition 3.6. *The mapping* conj *is injective on T^*.*

Proof. Assume that w is $\mathrm{conj}(t_1 \cdot \ldots \cdot t_m)$. We can retrieve the words $\mathrm{conj}(t_1)$, …, $\mathrm{conj}(t_m)$ from w. Indeed, by Proposition 3.5, the $\mathbf{Z}$-balance of w is m. The first position corresponding to $\mathrm{conj}(t_m)$ in w has $\mathbf{Z}$-level $m-1$, while all subsequent positions have level $\geq m$. So $\mathrm{conj}(t_m)$ can be isolated, and, by repeating the process, so can $\mathrm{conj}(t_{m-1})$, …, $\mathrm{conj}(t_2)$.

It suffices now to prove injectivity on T. Assume $w = \mathrm{conj}(t)$. Then w has length 1 if and only if t is a variable, and, in this case, w is that variable. Otherwise, assume $t = t_0 {}^\wedge t_1$. Then w is $\mathrm{conj}(t_0)\mathrm{conj}(t_1)\mathrm{conj}(t_0)^{-1}$, and it suffices to recognize where the subword $\mathrm{conj}(t_1)$ begins in w. By Proposition 3.5, the first position coming from $\mathrm{conj}(t_1)$ has $\mathbf{Z}$-level 1, while all subsequent positions have level ≥ 2, except the last position, which also has level 1. ■

Definition. The term t is *injective* if every variable occurs at most once in t.

Corollary 3.7. *The mapping* eval *is injective on injective terms of* T^*.

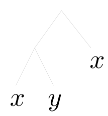
Proof. An easy induction shows that, if t is an injective term in T^*, then the word $\mathrm{conj}(t)$ is freely reduced. Indeed, assume $t = t_0 {}^\wedge t_1$ or $t = t_0 \cdot t_1$. By induction hypothesis, the words $\mathrm{conj}(t_0)$ and $\mathrm{conj}(t_1)$ are reduced. Now no reduction can occur between the words $\mathrm{conj}(t_0)$ or $\mathrm{conj}(t_0)^{-1}$ and $\mathrm{conj}(t_1)$, as these words involve disjoint variables. So, for t an injective term, $\mathrm{eval}(t)$ is merely equal to $\mathrm{conj}(t)$ and injectivity follows from Proposition 3.5. ■

4. Addresses and cuts

Here we describe a correspondence between the geometry of the terms and the positions in the associated conjugate words.

In order to be able to refer precisely to the variables in a term, we introduce *addresses*. To this end, we consider a term of T as a binary tree. The tree associated with a variable has a single node labelled with this variable. The tree associated with $t_0 {}^\wedge t_1$ admits the tree associated with t_0 as its left subtree and the tree associated with t_1 as its right subtree.

For instance, the tree associated with the term $(x^\wedge y)^\wedge x$ is



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We introduce now addresses for the nodes of binary trees: the address of a node consists of a finite sequence of 0's and 1's that describes the path from the root of the tree to the considered node, 0 standing for a left forking and 1 for a right forking. We use Λ (empty word) to represent the address of the root of the tree. If t is a term in T, we say that u is an *address in* t if u is the address of a terminal node of (the tree associated with) t, and we write $t(u)$ for the variable that occurs at this node.

For instance, there are 3 addresses in the term $t = (x\hat{}y)\hat{}x$, namely 00, 01 and 1, and we have $t(00) = t(1) = x$ and $t(01) = y$.

There exists a natural left–right ordering of the addresses: we say that u_1 lies *on the right* of u_2 if there exists an address u such that u_1 begins with $u1$ and u_2 begins with $u0$.

Definition. Assume that u_1, u_2 are addresses. We say that u_1 *covers* u_2 if there exists an address u and a positive integer k such that u_1 is $u1^k$ and u_2 begins with $u0$. A *stack* in the term t is a finite sequence of addresses $(u_1, \ldots, u_m)$ in t such that u_i covers u_{i+1} for each i.

Example 4.1. There are 7 stacks in the term $(x\hat{}y)\hat{}x$, namely (00), (01), (01, 00), (1), (1, 01, 00), (1, 01), and (1, 00).

Lemma 4.2. *Assume that t_0 and t_1 are terms in T and t is $t_0\hat{}t_1$. Assume that 1^k is the rightmost address in t. The stacks in t are: the sequences $(0u_1, \ldots, 0u_m)$ with $(u_1, \ldots, u_m)$ a stack in t_0, the sequences $(1u_1, \ldots, 1u_m)$ with $(u_1, \ldots, u_m)$ a stack in t_1, and the sequences $(1^k, 0u_1, \ldots, 0u_m)$ with $(u_1, \ldots, u_m)$ a stack in t_0.*

Proof. The addresses covered by $0u$ are exactly those addresses of the form $0u'$ where u' is covered by u. If u contains at least one 0, the addresses covered by $1u$ are exactly those addresses of the form $1u'$ where u' is covered by u. Finally, the addresses covered by 1^k are all addresses of the form 1^i0u with $i < k$. ■

Definition. Assume that t is a term in T and p is a position in the word $\mathrm{conj}(t)$. The *origin* of p in t is the sequence of addresses defined inductively by the following rules:

(i) Assume that t is a variable. Then p is necessarily 1, and the origin of p in t is (Λ);

(ii) Assume that t is $t_0\hat{}t_1$, and $1 \le p \le \ell_0$ holds, where ℓ_0 is the length of $\mathrm{conj}(t_0)$. Let $(u_1, \ldots, u_m)$ be the origin of p in t_0, ℓ be the length of $\mathrm{conj}(t)$, and 1^k be the rightmost address in t. Then the origins of p and $\ell + 1 - p$ in t are respectively $(0u_1, \ldots, 0u_m)$ and $(1^k, 0u_1, \ldots, 0u_m)$,

(iii) Assume that t is $t_0{}^\wedge t_1$, and $1 \le p \le \ell_1$ holds, where ℓ_i is the length of $\mathrm{conj}(t_i)$. Let $(u_1, \ldots, u_m)$ be the origin of p in t_1. Then the origin of $\ell_0 + p$ in t is $(1u_1, \ldots, 1u_m)$.

Example 4.3. Let (once more) t be the term $(x^\wedge y)^\wedge x$. Then the origins of the 7 letters in the word $\mathrm{conj}(t)$ are as follows:

p	1	2	3	4	5	6	7
$w(p)$	x	y	x^{-1}	x	x	y^{-1}	x^{-1}
origin of p	(00)	(01)	$(01, 00)$	(1)	$(1, 01, 00)$	$(1, 01)$	$(1, 00)$

Proposition 4.4. *For every term t in T, the origin mapping establishes a bijection between the positions in the word $\mathrm{conj}(t)$ and the stacks in the term t.*

Proof. Induction on t. The result is straightforward if t is a variable. Assume $t = t_0{}^\wedge t_1$, and let ℓ_i be the length of the word $\mathrm{conj}(t_i)$ for $i = 0, 1$, and ℓ be the length of $\mathrm{conj}(t)$. The positions in $\mathrm{conj}(t)$ are the ℓ_0 positions in $\mathrm{conj}(t_0)$, followed by ℓ_1 positions of the form $\ell_0 + p$ with p a position in $\mathrm{conj}(t_1)$, followed by ℓ_0 positions of the form $\ell + 1 - p$ with p a position in $\mathrm{conj}(t_0)$. Then the induction hypothesis and Lemma 4.2 give the result. ∎

Definition. Assume that t is a term in T, u is an address in t and p is a position in $\mathrm{conj}(t)$. We say that p *comes from* u in t if u is the last component in the origin of p in t.

Proposition 4.5. *Assume that t is a term in T and u is an address in t that contains i letters 0 and j letters 1. Let w be the word $\mathrm{conj}(t)$. Then there are 2^i positions in w that come from u. If $p_1 < \ldots < p_{2^i}$ are these positions, we have for every k*

$$w(p_k) = t(u)^{(-1)^{k+1}}, \qquad \mathbf{Z}\text{-level}(p_k, w) = j + F(k),$$

where F is the universal function inductively defined by $F(1) = 0$ and $F(k) = 1 + F(2^{m+1} - k + 1)$ for $2^m < k \le 2^{m+1}$. Moreover, the length of the origin of p_k in t, i.e., the number of addresses it comprises, is $F(k) + 1$.

Proof. Straightforward using induction on t. ∎

For instance, we see that, if t is $(x\hat{}y)\hat{}x$, then 4 positions in the word $\mathrm{conj}(t)$ come from 00, namely those underlined in $\underline{x}y\underline{x}^{-1}x\underline{x}y^{-1}\underline{x}^{-1}$. The corresponding **Z**-levels are 0, 1, 2, and 1, and the associated stacks, namely (00), (01, 00), (1, 01, 00) and (1, 00), have respective length 1, 2, 3, and 2.

Definition. Assume that t is a term in T, and that u is an address in t. The *cut* of t at u is the term $\mathrm{cut}(t,u)$ inductively defined by the following rules:

(i) If t is a variable and u is Λ, then $\mathrm{cut}(t,u)$ is t;

(ii) If t is $t_0\hat{}t_1$ and u is $0u_0$ for some address u_0 in t_0, then $\mathrm{cut}(t,u)$ is $\mathrm{cut}(t_0,u_0)$;

(iii) If t is $t_0\hat{}t_1$ and u is $1u_1$ for some address u_1 in t_1, then $\mathrm{cut}(t,u)$ is $t_0\hat{}\mathrm{cut}(t_1,u_1)$.

The term $\mathrm{cut}(t,u)$ is the term obtained from t by forgetting everything on the right of u. For instance, $t=(x\hat{}y)\hat{}x$ has 3 cuts, namely

$$\mathrm{cut}(t,00)=x, \qquad \mathrm{cut}(t,01)=x\hat{}y, \qquad \mathrm{cut}(t,1)=t.$$

Definition. Assume that t is a term in T, and ℓ is the length of the word $\mathrm{conj}(t)$. The mapping $t^\sharp$ of $\{1,\ldots,\ell\}$ into Λ is defined by

$$t^\sharp(p)=\pi_{LD}(\mathrm{cut}(t,u_1)\hat{}\ldots\hat{}\mathrm{cut}(t,u_m)),$$

where $(u_1,\ldots,u_m)$ is the origin of p in t. The $\widetilde{\Lambda}$-*level* of p in w is $t^\sharp(p)^-$.

Example 4.6. Letting once more t be the term $(x\hat{}y)\hat{}x$, we find:

p	1	2	3	4	5	6	7
$w(p)$	x	y	x^{-1}	x	x	y^{-1}	x^{-1}
$t^\sharp(p)$	x	$x\hat{}y$	$(x\hat{}y)\hat{}x$	$(x\hat{}y)\hat{}x$	$x\hat{}y\hat{}x$	$((x\hat{}y)\hat{}x)\hat{}x\hat{}y$	$((x\hat{}y)\hat{}x)\hat{}x$
$\widetilde{\Lambda}$-level(p,w)	1	x	$x\hat{}y$	$x\hat{}y$	xy	$((x\hat{}y)\hat{}x)\,x$	$(x\hat{}y)\hat{}x$

Lemma 4.7. *Assume that t_0 and t_1 are terms in T, and t is $t_0\hat{}t_1$. Let ℓ, ℓ_0 and ℓ_1 be respectively the length of the words $\mathrm{conj}(t)$, $\mathrm{conj}(t_0)$ and $\mathrm{conj}(t_1)$. Then the equality*

$$t^\sharp(p)=\begin{cases} t_0^\sharp(p) & \text{if } 1\le p\le \ell_0 \text{ holds,} \qquad (4.1)\\ \overline{t_0}\ \hat{}\ t_1^\sharp(p_1) & \text{if } p \text{ is } \ell_0+p_1 \text{ with } 1\le p_1\le \ell_1, \qquad (4.2)\\ \overline{t}\ \hat{}\ t_0^\sharp(p_0) & \text{if } p \text{ is } \ell+1-p_0 \text{ with } 1\le p_0\le \ell_0 \qquad (4.3)\end{cases}$$

holds for every position p in $\mathrm{conj}(t)$.

Proof. Assume that the origin of p_0 in t_0 is $(u_1, \ldots, u_m)$. Then the origin of p_0 in t is $(0u_1, \ldots, 0u_m)$, and we have

$$\begin{aligned} t^\sharp(p_0) &= \pi_{LD}(\mathrm{cut}(t, 0u_1)\hat{}\ldots\hat{}\mathrm{cut}(t, 0u_m)) \\ &= \pi_{LD}(\mathrm{cut}(t_0, u_1)\hat{}\ldots\hat{}\mathrm{cut}(t_0, u_m)) = t_0^\sharp(p_0). \end{aligned}$$

Similarly, assume that the origin of p_1 in t_1 is $(u_1, \ldots, u_m)$. Then the origin of $\ell_0 + p_1$ in t is $(1u_1, \ldots, 1u_m)$, and we have

$$\begin{aligned} t^\sharp(\ell_0 + p_1) &= \pi_{LD}(\mathrm{cut}(t, 1u_1)\hat{}\ldots\hat{}\mathrm{cut}(t, 1u_m)) \\ &= \pi_{LD}((t_0\hat{}\mathrm{cut}(t_1, u_1))\hat{}\ldots\hat{}(t_0\hat{}\mathrm{cut}(t_1, u_m))) \\ &= \overline{t_0} \mathbin{\hat{}} \pi_{LD}(\mathrm{cut}(t_1, u_1)\hat{}\ldots\hat{}\mathrm{cut}(t_1, u_m)) = \overline{t_0} \mathbin{\hat{}} t_1^\sharp(p_1). \end{aligned}$$

Finally, the origin of $\ell + 1 - p_0$ in t is $(1^k, 0u_1, \ldots, 0u_m)$, where 1^k is the rightmost address in t, and we have

$$\begin{aligned} t^\sharp(\ell + 1 - p) &= \pi_{LD}(\mathrm{cut}(t, 1^k)\hat{}\mathrm{cut}(t, 0u_1)\hat{}\ldots\hat{}\mathrm{cut}(t, 0u_m)) \\ &= \pi_{LD}(t\hat{}\mathrm{cut}(t_0, u_1)\hat{}\ldots\hat{}\mathrm{cut}(t_0, u_m)) = \overline{t} \mathbin{\hat{}} t_0^\sharp(p_0). \end{aligned}$$ ∎

Proposition 4.8. *Assume that t is a term in T, that w is the word* $\mathrm{conj}(t)$, *and that ℓ is the length of w. Let $\ell' = (\ell + 1)/2$. Then we have*

$$\begin{gathered} t^\sharp(\ell') = \overline{t}, \qquad t^\sharp(\ell' + p) = \overline{t} \mathbin{\hat{}} t(\ell' - p) \quad \text{for } 1 \le p < \ell', \\ \widetilde{\Lambda}\text{-level}(1, w) = 1, \quad \widetilde{\Lambda}\text{-level}(\ell', w) = \overline{t^-}, \quad \widetilde{\Lambda}\text{-level}(\ell, w) = \overline{t}. \end{gathered}$$

Proof. An obvious induction using Lemma 4.7. Observe that the origin of the median position ℓ' is a stack of the form (1^k). ∎

The fact that $(F, \cdot, \hat{})$ is an LD-monoid implies that the evaluation mapping of $\widetilde{T}$ onto $\widetilde{C}$ factors through $\widetilde{\Lambda}$, and its restriction to T factors through Λ. We shall denote by $\overline{\mathrm{eval}}$ the surjective homomorphism of $\widetilde{\Lambda}$ onto $\widetilde{C}$ such that $\mathrm{eval}(t) = \overline{\mathrm{eval}}(\overline{t})$ holds for every term t in $\widetilde{T}$.

Proposition 4.9. *Assume that t is a term in T, w is* $\mathrm{conj}(t)$ *and p is a position in w. Then the equivalences*

$$w\restriction\{1, \ldots, p\} =_{FG} \overline{\mathrm{eval}}(t^\sharp(p)^{\mathrm{sign}(p,w)}) \tag{4.4}$$

$$w\restriction\{1, \ldots, p-1\} =_{FG} \overline{\mathrm{eval}}(t^\sharp(p)^{-\mathrm{sign}(p,w)}) \tag{4.5}$$

hold in W.

Proof. Induction on t. If t is a variable, the result is obvious. So assume $t = t_0{}^\wedge t_1$. Write w_i for $\mathrm{conj}(t_i)$. Assume first $1 \le p \le \ell_0$. Using the induction hypothesis and (4.1), we find

$$\begin{aligned} w\restriction\{1,\dots,p\} &= w_0\restriction\{1,\dots,p\} \\ &=_{FG} \overline{\mathrm{eval}}(t_0^\sharp(p)^{\mathrm{sign}(p,w_0)}) = \overline{\mathrm{eval}}(t^\sharp(p)^{\mathrm{sign}(p,w)}). \end{aligned}$$

The computation is similar for $p-1$.

Assume now $p = \ell_0 + p_1$ with $1 \le p_1 \le \ell_1$. Then, by (4.2), we have $t^\sharp(p) = \overline{t_0} \,{}^\wedge\, t_1^\sharp(p_1)$, which implies

$$t^\sharp(p)^{\mathrm{sign}(p,w)} = \overline{t_0} \cdot t_1^\sharp(p_1)^{\mathrm{sign}(p_1,w_1)}.$$

Thus, using the induction hypothesis, we find

$$\begin{aligned} w\restriction\{1,\dots,p\} &= w_0 \cdot (w\restriction\{\ell_0+1,\dots,\ell_0+p_1\}) \\ &= w_0 \cdot (w_1\restriction\{1,\dots,p_1\}) \\ &=_{FG} \overline{\mathrm{eval}}(\overline{t_0} \cdot t_1^\sharp(p_1)^{\mathrm{sign}(p_1,w_1)}) = \overline{\mathrm{eval}}(t^\sharp(p)^{\mathrm{sign}(p,w)}). \end{aligned}$$

Again the computation is similar for $p-1$. We just have to notice that, if p_1 is 1, then the sign of p_1 in w_1 is $+$, so that (4.5) claims that w_0 is equivalent to $\mathrm{eval}(t_0)$, which is obvious.

Finally assume $p = \ell + 1 - p_0$, where p_0 is a position in w_0. By (4.3), we have $t^\sharp(p) = \overline{t_0}{}^\wedge t^\sharp(p_0)$ and $\mathrm{sign}(p,w) = -\mathrm{sign}(p_0,w_0)$, which implies

$$t^\sharp(p)^{\mathrm{sign}(p,w)} = \overline{t} \cdot t_0^\sharp(p_0)^{-\mathrm{sign}(p_0,w_0)}.$$

Using the induction hypothesis, we deduce

$$\begin{aligned} w\restriction\{1,\dots,p\} &= w_0 w_1 w_0^{-1} \cdot (w_0\restriction\{1,\dots,p_0-1\}) \\ &=_{FG} w \cdot (w_0\restriction\{1,\dots,p_0-1\}) \\ &=_{FG} \overline{\mathrm{eval}}(\overline{t} \cdot t_0^\sharp(p_0)^{-\mathrm{sign}(p_0,w_0)}) = \overline{\mathrm{eval}}(t^\sharp(p)^{\mathrm{sign}(p,w)}). \end{aligned}$$

Again, the computation is similar for $p-1$. ∎

Corollary 4.10. *Every prefix of a conjugate (viewed as a reduced word) is still a conjugate.*

Definition. Assume that t is a term in T and w is $\mathrm{conj}(t)$. A *main position* in w is a position p whose origin in t is a stack of length 1.

By Proposition 4.5, one main position in the word $\mathrm{conj}(t)$ is associated with each address u in t, namely the least position that comes from u. For instance, if t is $(x^\wedge y)^\wedge x$, there are 3 addresses in t, and the corresponding 3 main positions in the word $\mathrm{conj}(t)$ are underlined in $\underline{x}\underline{y}x^{-1}\underline{x}xy^{-1}x^{-1}$.

Corollary 4.11. *Assume that t is a term in T and p is a main position in the word $\mathrm{conj}(t)$. Then the equivalence*

$$\mathrm{conj}(t)\restriction\{1,\ldots,p-1\} =_{FG} \mathrm{conj}(\mathrm{cut}(t,u)^-)$$

holds, where u denotes the address p comes from in t.

Proof. Obvious from Formula (4.5), as, in the present case, $t^\sharp(p)$ is the LD-class of the term $\mathrm{cut}(t,u)$. ■

5. Automata

Our main task is to *control reductions* in conjugate words. The first, obvious remark is that **Z**-level provides us with a useful tool. In the sequel, we write $\mathrm{lg}(w)$ for the length of the word w.

Definition. Assume that the word w' is obtained from the word w by one step of free reduction, say by deleting two letters x, x^{-1} or x^{-1}, x at positions p_0, p_0+1. We let $H_{(w,w')}$ denote the partial mapping of $\{1,\ldots,\mathrm{lg}(w)\}$ onto $\{1,\ldots,\mathrm{lg}(w')\}$ defined by $H_{(w,w')}(p) = p$ for $p < p_0$ and $H_{(w,w')}(p) = p-2$ for $p > p_0+1$. If $\vec{w} = (w_0,\ldots,w_n)$ is a sequence of one step free reductions, we denote by $H_{\vec{w}}$ the product $H_{(w_n,w_{n-1})}\circ\cdots\circ H_{(w_1,w_2)}\circ H_{(w_1,w_0)}$.

By construction, $H_{\vec{w}}$ is the partial surjection of $\{1,\ldots,\mathrm{lg}(w_0)\}$ onto $\{1,\ldots,\mathrm{lg}(w_n)\}$ that specifies the positions in w_n of those letters in w_0 that have not vanished in the considered sequence of reductions.

Definition. Assume that w freely reduces to w'. We say that a sequence of positions $(p'_1,\ldots,p'_m)$ is an *heir* of a sequence of positions $(p_1,\ldots,p_m)$ *for* (w,w') if there exists at least one sequence $\vec{w} = (w_0,\ldots,w_n)$ such that w_0 is w, w_n is w', and p'_i is $H_{\vec{w}}(p_i)$ for every i.

Lemma 5.1. *Assume that (p') is an heir of (p) for (w, w'). Then the* $\mathbf{Z}$*-level of p' in w' is equal to the* $\mathbf{Z}$*-level of p in w.*

Proof. It suffices to consider the case of a single reduction. Assume that p_0 and p_0+1 have been deleted. For $p < p_0$, it is obvious that the $\mathbf{Z}$-levels of p in w and w' are equal. Assume now $p > p_0 + 1$, and let p' be $p - 2$. Then (p') is an heir of (p), and we have

$$\|w'\lceil\{1, \ldots, p'-1\}\|_+ = \|w\lceil\{1, \ldots, p-1\}\|_+ - 1,$$
$$\|w'\lceil\{1, \ldots, p'\}\|_- = \|w\lceil\{1, \ldots, p\}\|_- - 1,$$

and, therefore, the $\mathbf{Z}$-levels are equal. ∎

Definition. Assume that w is a word. Two positions $p_1 < p_2$ in w are *mutually reducible* if both $w\lceil\{p_1, \ldots, p_2\}$ and $w\lceil\{p_1+1, \ldots, p_2-1\}$ freely reduce to the empty word.

In other words, p_1 and p_2 are mutually reducible if and only if there is a sequence of reductions from w in which all positions between p_1 and p_2 vanish, and p_1 and p_2 then vanish simultaneously. This in particular implies that there exists a letter x such that $w(p_1)$ is $x^{\pm 1}$ and $w(p_2)$ is $x^{\mp 1}$.

Proposition 5.2. *Assume that p_1 and p_2 are mutually reducible positions in w. Then the* $\mathbf{Z}$*-levels of p_1 and p_2 in w must be equal.*

Proof. The hypothesis means that there exists a word w' and a pair of positions $(p', p'+1)$ in w' that is an heir of (p_1, p_2) for (w, w'), and, in addition, $w(p_1)$ and $w(p_2)$ are mutually inverse letters. Now it is clear that, both in the case of xx^{-1} and of $x^{-1}x$, the $\mathbf{Z}$-levels of p' and $p'+1$ in w' are equal. We conclude by Lemma 5.1. ∎

Definition. The position p is *strong* in the word w if it is mutually reducible with no other position.

Proposition 5.3. *Assume that t is a term in T. Then, for each variable x that occurs in t, the first position of x in the word* $\mathrm{conj}(t)$ *is strong.*

This results from the following stronger result:

Lemma 5.4. *Assume that w is* $\mathrm{conj}(t)$ *for some term t, and that p_0 is a position of the letter x in w such that, for all positions $p < p_0$ of $x^{\pm 1}$ in w, the* $\mathbf{Z}$*-level of p in w is strictly higher than the* $\mathbf{Z}$*-level of p_0 in w. Then p is strong in w.*

Proof. Assume that p_0 is mutually reducible with p'_0 in $\mathrm{conj}(t)$. By Proposition 5.2, the $\mathbf{Z}$-level of p'_0 is equal to the $\mathbf{Z}$-level h of p_0. Hence, by hypothesis, we must have $p'_0 > p_0$. By Proposition 4.5, p_0 must be a positive position, so p'_0 is a negative one. Now, by Proposition 4.5 again, a negative position of x on $\mathbf{Z}$-level h in a word $\mathrm{conj}(t)$ is always preceded by a positive position on level $h-1$. So there must exist a position $p_1 < p'_0$ of x on level $h-1$. By hypothesis, we must have $p_0 < p_1$. Now the hypothesis that p_0 and p'_0 are mutually reducible implies that there exists a position p'_1 with $p_0 < p'_1 < p'_0$ that is mutually reducible with p_1. If we have chosen p_1 to be minimal, p_1 satisfies the same hypotheses as p_0. So the argument repeats indefinitely, and we find an infinite series of positions with $\mathbf{Z}$-levels h, $h-1$, $h-2$, *etc.*, which is impossible. ∎

For instance, we deduce that, in the word $\mathrm{conj}((x\hat{}y)\hat{}x) = xyx^{-1}xxy^{-1}x^{-1}$, the first two positions are strong. Of course, we cannot expect all positions to be strong, since free reductions must happen in the word $\mathrm{conj}(t)$ whenever t is not an injective term.

Our idea now is to replace $\mathbf{Z}$-levels by more subtle levels that provide a better control.

An important feature about $\mathbf{Z}$-levels is their *local* character. As one easily verifies, in order to compute the $\mathbf{Z}$-level of p in w, it suffices to know the $\mathbf{Z}$-level of $p-1$ and the (signs of) the letters occurring at $p-1$ and p. Such a local character is reminiscent of the action of an *automaton* that reads a word.

Definition. Let Σ be a (finite) set. An *automaton* A with *state set* Σ and alphabet V consists of a set of triples in $\Sigma \times V \times \Sigma$ that we call the *transitions* of A, and, in addition, of a distinguished state that we call the *start state*. We shall write $\sigma \xrightarrow{x} \sigma'$ for the triple (σ, x, σ'). The automaton A is *deterministic* if, for every σ in Σ and x in V, there exists at most one transition of the form $\sigma \xrightarrow{x} \sigma'$, and, for every σ' in Σ and every x in V, there exists at most one transition of the form $\sigma \xrightarrow{x} \sigma'$.

Example 5.5. (The augmentation automaton) The collection $\{(\sigma, x, \sigma+1); \sigma \in \mathbf{Z}, x \in S\}$ completed with the distinguished state 0 is a deterministic automaton with state set $\mathbf{Z}$ and alphabet V. This automaton will be denoted $A_{\mathbf{Z}}$. We call it the *augmentation* automaton, as it is connected with the standard augmentation mapping of F onto $\mathbf{Z}$. It is usual to associate with an automaton an oriented graph with labelled edges: vertices correspond to states, and there is a x-labelled arrow from state σ to state σ' if and only if $\sigma \xrightarrow{x} \sigma'$ is a transition of the considered automaton. The start state is indicated with an entering arrow. The graph associated with $A_{\mathbf{Z}}$ is displayed in Figure 5.1.

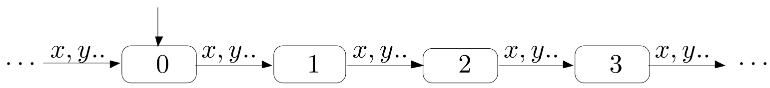

Figure 5.1: The augmentation automaton $A_{\mathbf{Z}}$

Definition. Assume that A is an automaton with state set Σ and alphabet V, and that $w = x_1^{e_1} \ldots x_\ell^{e_\ell}$ is a word in W. A *reading* of w by A is a sequence of states $(\sigma_0, \ldots, \sigma_\ell)$ such that σ_0 is the start state of A and, for every i, if e_i is $+1$, then $\sigma_i \xrightarrow{x_i} \sigma_{i+1}$ is a transition of A, and, if e_i is -1, then $\sigma_{i+1} \xrightarrow{x_i} \sigma_i$ is a transition of A. The word w *can be read by* A if there exists at least one reading of w by A.

For instance, the word $xyx^{-1}xxy^{-1}x^{-1}$ can be read by the automaton $A_{\mathbf{Z}}$ of Example 5.5. The (unique) reading is

$$0 \xrightarrow{x} 1 \xrightarrow{y} 2 \xrightarrow{x^{-1}} 1 \xrightarrow{x} 2 \xrightarrow{x} 3 \xrightarrow{y^{-1}} 2 \xrightarrow{x^{-1}} 1.$$

Lemma 5.6. *Assume that A is a deterministic automaton on V. Then, for every word w in W, there exists at most one reading of w by A.*

Proof. This is obvious as, at each step, there exists, by definition, at most one possible transition. ∎

Definition. Assume that A is a deterministic automaton on V, and that the word w in W can be read by A. Let $(\sigma_0, \ldots, \sigma_\ell)$ the reading of w by A. For p a position in w, the *A-level* of p in w is defined to be σ_{p-1} if p is a positive position, and to be σ_p if p is a negative position. The *A-balance* of w is σ_ℓ.

Proposition 5.7. *Every word can be read by the automaton $A_{\mathbf{Z}}$. The $A_{\mathbf{Z}}$-balance a word is equal to its $\mathbf{Z}$-balance, and the $A_{\mathbf{Z}}$-level of a position coincides with its $\mathbf{Z}$-level.*

Proof. Straightforward from the explicit definitions. For every word w in W, the final state reached after w has been read by the automaton $A_{\mathbf{Z}}$ is $\|w\|_+ - \|w\|_-$. ∎

We have seen in Proposition 5.2 that mutually reducible positions must have the same $\mathbf{Z}$-level, hence the same $A_{\mathbf{Z}}$-level. This property extends to the A-level for every deterministic automaton.

Proposition 5.8. *Assume that A is a deterministic automaton, and that the word w can be read by A. Assume that w reduces to w'. Then w' can be read by A, the A-balance of w' is equal to the A-balance of w. If (p') is an heir of (p) for (w, w'), then the A-level of p' in w' is equal to the A-level of p in w.*

Proof. It suffices to consider the case of a single reduction. Assume that p_0 and p_0+1 have been deleted in the reduction of w to w'. Let $(\sigma_0, \dots, \sigma_\ell)$ be the reading of w by A. We claim that the states σ_{p_0-1} and σ_{p_0+1} coincide. Indeed, either $w(p_0)$ is a letter x of V, and, by definition, we have both $\sigma_{p_0-1} \xrightarrow{x} \sigma_{p_0}$ and $\sigma_{p_0+1} \xrightarrow{x} \sigma_{p_0}$, or $w(p_0)$ is a letter x^{-1}, and, by definition, we have both $\sigma_{p_0} \xrightarrow{x} \sigma_{p_0-1}$ and $\sigma_{p_0} \xrightarrow{x} \sigma_{p_0+1}$. In both cases, $\sigma_{p_0-1} = \sigma_{p_0+1}$ follows from the hypothesis that A is deterministic. We deduce that $(\sigma_0, \dots, \sigma_{p_0-1}, \sigma_{p_0+2} \dots, \sigma_\ell)$ is the reading of w' by A. ∎

Proposition 5.9. *Assume that A is a deterministic automaton, and that the word w can be read by A. Assume that p_1 and p_2 are mutually reducible positions in w. Then the A-levels of p_1 and p_2 in w must be equal.*

Proof. It suffices to repeat the proof of Proposition 5.3 using Proposition 5.8 in place of Proposition 5.2. ∎

Example 5.10. (The Cayley automaton) We let A_F be the automaton with state set F and alphabet V that admits as transitions all triples $a \xrightarrow{x} ax$ for a in F and x in V. The start set is the unit 1. Then A_F is a deterministic automaton. The graph of A_F is the Cayley graph of the group F, so it is natural to call A_F the *Cayley automaton* of F. As for $A_{\mathbf{Z}}$, it is easy to verify that every word in W can be read by A_F. Then the A_F-balance of the word w is simply the projection $\overline{w}$ of w in F. For instance, the reading of the word $xyx^{-1}xxy^{-1}x^{-1}$ by A_F is

$$1 \xrightarrow{x} x \xrightarrow{y} xy \xrightarrow{x^{-1}} xyx^{-1} \xrightarrow{x} xy \xrightarrow{x} xyx \xrightarrow{y^{-1}} xyxy^{-1} \xrightarrow{x^{-1}} xyxy^{-1}x^{-1}.$$

We have displayed in Figure 5.2 the fragment of the graph of A_F involved in the previous reading.

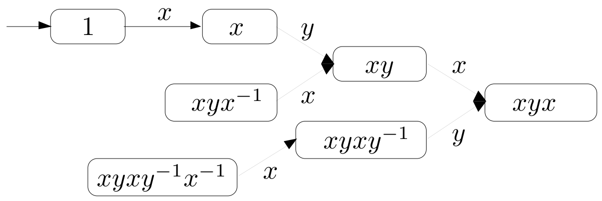

Figure 5.2: The Cayley automaton A_F (fragment)

6. The automaton of a term

The Cayley automaton of Example 5.10 detects all possible reductions in the words: if two positions are mutually reducible, they they must have the same A_F-level, and, conversely, it is easy to verify that, if two positions have the same A_F-level, then they are mutually reducible. Actually, this result is essentially a tautology, and it probably cannot really help us here. We introduce now new, less trivial automata with state sets included in the free LD-monoid $\widetilde{\Lambda}$.

Definition. Assume that t is a term in T. The transitions of the *automaton* A_t are the triples

$$t^\sharp(p)^- \xrightarrow{x} t^\sharp(p)^+$$

where p is a position in the word $\mathrm{conj}(t)$ and x is the letter of V that occurs at position p in $\mathrm{conj}(t)$—*i.e.*, x is $\mathrm{var}_R(t^\sharp(p))$. The start state of A_t is 1.

Example 6.1. Let again t be the term $(x\hat{}y)\hat{}x$. Then the graph of the automaton A_t is displayed in Figure 6.1. We see that this automaton happens to be deterministic. The word $\mathrm{conj}(t) = xyx^{-1}xxy^{-1}x^{-1}$ can be read by A_t, and the corresponding states are

$$1 \xrightarrow{x} x \xrightarrow{y} xy \xrightarrow{x^{-1}} x\hat{}y \xrightarrow{x} xy \xrightarrow{x} xyx \xrightarrow{y^{-1}} ((x\hat{}y)\hat{}x)x \xrightarrow{x^{-1}} (x\hat{}y)\hat{}x.$$

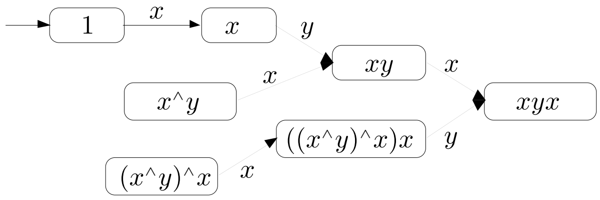

Figure 6.1: The automaton of the term $(x\,\hat{}\,y)\hat{}\,x$

On the other hand, the automaton of the term $x\,\hat{}\,x$ is displayed on Figure 6.2, and we see that it is not deterministic, since two x-labelled arrows arrive at state xx.

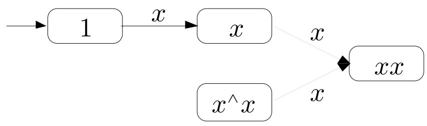

Figure 6.2: The automaton of the term $x\,\hat{}\,x$

Lemma 6.2. *For every state a of the automaton A_t and every x in V, there is at most one transition of the form $a \xrightarrow{x} a'$ in A_t.*

Proof. According to Lemma 2.5 and to the definition of A_t, we see that, if $a \xrightarrow{x} a'$ is a transition of A_t, then the equality $a' = a\,x$ holds in the monoid $\widetilde{\Lambda}$. ■

Lemma 6.3. *Assume that the term t is $t_0\hat{}\,t_1$. Then the transitions of the automaton A_t are:*

(i) all transitions $a \xrightarrow{x} a'$ of A_{t_0};

(ii) all transitions $\overline{t_0}\, a \xrightarrow{x} \overline{t_0}\, a'$ for $a \xrightarrow{x} a'$ a transition of A_{t_1};

(iii) all transitions $\bar{t}\, a \xrightarrow{x} \bar{t}\, a'$ for $a \xrightarrow{x} a'$ a transition of A_{t_0}.

Proof. This is a restatement of Lemma 4.7. ■

Proposition 6.4. *Assume that t is a term in T, let w be the word* conj(t) *and ℓ be the length of w. Then the sequence*

$$(1, t^\sharp(1)^{\mathrm{sign}(1,w)}, t^\sharp(2)^{\mathrm{sign}(2,w)}, \ldots, t^\sharp(\ell)^{\mathrm{sign}(\ell,w)}) \qquad (6.1)$$

is a reading of w by A_t that ends with state $\overline{t}$. If A_t is deterministic, then, for every position p, the A_t-level of p in w is its $\widetilde{A}$-level and the A_t-balance of w is $\overline{t}$.

Proof. Use induction on t. Everything is obvious if t is a variable. So assume $t = t_0{}^\wedge t_1$. We write w_i for $\operatorname{conj}(t_i)$, and ℓ_i for the length of w_i. By induction hypothesis, the sequence

$$(1, t_0^\sharp(1)^{\operatorname{sign}(1,w_0)}, \ldots, t_0^\sharp(\ell_0)^{\operatorname{sign}(\ell_0,w_0)}) \tag{6.2}$$

is a reading of w_0 by A_{t_0} that ends with state $\overline{t_0}$. By (4.1), (6.2) is also the sequence

$$(1, t^\sharp(1)^{\operatorname{sign}(1,w)}, \ldots, t^\sharp(\ell_0)^{\operatorname{sign}(\ell_0,w)}), \tag{6.3}$$

and, therefore, it is a reading of w_0 by A_t.

Now, always by induction hypothesis, the sequence

$$(1, t_1^\sharp(1)^{\operatorname{sign}(1,w_1)}, \ldots, t_1^\sharp(\ell_1)^{\operatorname{sign}(\ell_1,w_1)}) \tag{6.4}$$

is a reading of the word w_1 by the automaton A_{t_1} that ends with state $\overline{t_1}$. By (4.2), multiplying all states in (6.4) by $\overline{t_0}$ on the left gives the sequence

$$(1, t^\sharp(\ell_0+1)^{\operatorname{sign}(\ell_0+1,w)}, \ldots, t^\sharp(\ell_0+\ell_1)^{\operatorname{sign}(\ell_0+\ell_1,w)}), \tag{6.5}$$

which is therefore a reading of the word w_1 by A_t that starts with state $\overline{t_0} \cdot 1 = \overline{t_0}$ and ends with state $\overline{t_0} \cdot \overline{t_1}$.

Similarly, by (4.3), multiplying all states in (6.3) by $\overline{t}$ on the left gives the sequence

$$(\overline{t}, t^\sharp(\ell)^{\operatorname{sign}(\ell,w)}, \ldots, t^\sharp(\ell+1-\ell_0)^{\operatorname{sign}(\ell+1-\ell_0,w)}), \tag{6.6}$$

a reading of w_0 by A_t that starts with state $\overline{t}$ and ends with state $\overline{t} \cdot \overline{t_0}$. Reversing (6.6) gives the sequence

$$(t^\sharp(\ell+1-\ell_0)^{\operatorname{sign}(\ell+1-\ell_0,w)}, \ldots, t^\sharp(\ell)^{\operatorname{sign}(\ell,w)}, \overline{t}), \tag{6.7}$$

hence a reading of the word w_0^{-1} by A_t that starts with state $\overline{t} \cdot \overline{t_0}$ and ends with state $\overline{t}$. Now, by Identity (LDM_1), we have in the LD-monoid $\widetilde{A}$

$$\overline{t} \cdot \overline{t_0} = \overline{t_0{}^\wedge t_1} \cdot \overline{t_0} = \overline{(t_0{}^\wedge t_1) \cdot t_0} = \overline{t_0 \cdot t_1} = \overline{t_0} \cdot \overline{t_1}.$$

Concatenating the three sequences (6.3), (6.5) and $(6,7)$ yields sequence $(6,1)$, which is therefore a reading of the word $w_0 w_1 w_0^{-1} = w$ by A_t which starts with state 1 and ends with state $\overline{t}$.

The formulas for the A_t-level and the A_t-balance are then straightforward consequences of our definitions. ■

Proposition 6.5. *Assume that t is a term in T and w is* $\mathrm{conj}(t)$. *Assume that p_1 and p_2 are positions of mutually inverse letters in w, i.e., there exists x in V such that $w(p_1)$ is $x^{\pm 1}$ and $w(p_2)$ is $x^{\mp 1}$.*

(i) If the $\widetilde{\Lambda}$-levels of p_1 and p_2 in w are equal, then p_1 and p_2 are mutually reducible in w.

(ii) If, in addition, the automaton A_t is deterministic, then the converse of (i) is true.

Proof. (i) Assume that p_1 and p_2 have the same $\widetilde{\Lambda}$-level in w. Assume for instance $p_1 < p_2$. Assume first that p_1 is a positive position in w. By Proposition 4.9, we have

$$\begin{aligned} w{\restriction}\{1,\ldots,p_1-1\} &=_{FG} \overline{\mathrm{eval}}(t^\sharp(p_1)^-) = \overline{\mathrm{eval}}(\widetilde{\Lambda}\text{-level}(p_1,w)) \\ &= \overline{\mathrm{eval}}(\widetilde{\Lambda}\text{-level}(p_2,w)) = \overline{\mathrm{eval}}(t^\sharp(p_2)^-) =_{FG} w{\restriction}\{1,\ldots,p_2\}. \end{aligned}$$

Hence, the word $w{\restriction}\{p_1,\ldots,p_2\}$ reduces to 1, and p_1 and p_2 are mutually reducible. Similarly, if p_1 is negative and p_2 is positive, we find that $w{\restriction}\{1,\ldots,p_1\}$ and $w{\restriction}\{1,\ldots,p_2-1\}$ are equivalent. Hence, the word $w{\restriction}\{p_1+1,\ldots,p_2-1\}$ reduces to 1, and, again, p_1 and p_2 are mutually reducible.

(ii) Assume now that the automaton A_t is deterministic. If p_1 and p_2 are mutually reducible, then, by Proposition 5.10, the A_t-levels of p_1 and p_2 must be equal. ∎

Corollary 6.6. *Assume that t is a term in T such that A_t is deterministic. Let w be the word* $\mathrm{conj}(t)$. *Then two positions p_1, p_2 in w are mutually reducible if and only if $t^\sharp(p_1)$ and $t^\sharp(p_2)$ are equal in Λ and, in addition, $w(p_1)$ and $w(p_2)$ have opposite signs.*

The key technical point that enables the automaton A_t to control the reductions in conjugate words is the following result:

Lemma 6.7. *Assume that p is a main position in the word* $\mathrm{conj}(t)$. *Then $t^\sharp(p') \sqsupset t^\sharp(p)$ holds for every position $p' > p$ in* $\mathrm{conj}(t)$.

Proof. We prove inductively on t the statement:

> *$t^\sharp(p') \sqsupset t^\sharp(p)$ holds for $p^+ \geq p' > p$, where p^+ is the main position that immediately follows p in* $\mathrm{conj}(t)$, *if such a position exists, and for $p' > p$ otherwise.*

As the relation $\sqsupset$ is transitive, this implies the lemma. Now the statement is vacuously true if t is a variable. Assume $t = t_0{}^\wedge t_1$. Let u be the address p comes from in t. Assume first that u begins with 0. Then $t^\sharp(p)$ is $t_0^\sharp(p)$. If p is not the last main position in $\mathrm{conj}(t_0)$, then the next main position p^+ in $\mathrm{conj}(t)$ is the next main position in $\mathrm{conj}(t_0)$, and the induction hypothesis gives the result. Assume now that p is the last main position in the word $\mathrm{conj}(t_0)$. This means that the address p comes from an address of the form 01^i and that p^+ comes from an address of the form 10^j. The induction hypothesis gives $t^\sharp(p') = t_0^\sharp(p') \sqsupset t_0^\sharp(p) = t^\sharp(p)$ for $p^+ > p' > p$, so it remains to consider the case of p^+ itself. Now, by construction, we have

$$t^\sharp(p) = \overline{\mathrm{cut}(t, 01^i)} = \overline{t_0}, \qquad t^\sharp(p^+) = \overline{\mathrm{cut}(t, 10^j)} = \overline{t_0}{}^\wedge t_1(0^j),$$

so $t^\sharp(p^+) \sqsupset t^\sharp(p)$ clearly holds.

Assume now that u begins with 1. Again, if p is not the last main position in $\mathrm{conj}(t)$, then p is $\ell_0 + p_1$ where ℓ_0 is the length of $\mathrm{conj}(t_0)$ and p_1 is a certain main position in $\mathrm{conj}(t_1)$. In this case, p^+ is $\ell_0 + p_1^+$, where p_1^+ is the main position in $\mathrm{conj}(t_1)$ that immediately follows p_1. By induction hypothesis, we have $t_1^\sharp(p_1') \sqsupset t_1^\sharp(p_1)$ for $p_1^+ \geq p_1' \geq p_1$. Multiplying on the left by $\overline{t_0}$ does not change the ordering, and this gives the desired result. It remains to consider the case when p is the last main position in $\mathrm{conj}(t)$. Then p is $\ell_0 + p_1$, where p_1 is the last main position in t_1. The previous argument gives $t^\sharp(p') \sqsupset t^\sharp(p)$ for $\ell_1 + \ell_0 \geq p' > p$, where ℓ_1 is the length of $\mathrm{conj}(t_1)$. It remains to consider the case $p' > \ell_0 + \ell_1$. Now, in this case, u has the form 1^j, and $t^\sharp(p)$ is $\overline{t}$. By construction, $t^\sharp(p')$ is $\overline{t}{}^\wedge t_0^\sharp(p_0)$ for some p_0, and, therefore, $t^\sharp(p') \sqsupset t^\sharp(p)$ is clear. ∎

Definition. The position p is *semistrong* in the word w if it is mutually reducible with no position $p' > p$.

Proposition 6.8. *Assume that the automaton A_t is deterministic. Then every main position in the word* $\mathrm{conj}(t)$ *is semistrong.*

Proof. Assume that p is a main position in the word $\mathrm{conj}(t)$. By Proposition 6.5, p can be mutually reducible only with a position on the same $\widetilde{A}$-level. Now, by Lemma 6.6, no position $p' > p$ can lie on the same $\widetilde{A}$-level as p, since $\sqsupset$ is a strict partial ordering. ∎

Definition. Assume that t is a term in T, and s is a term in T^*. Then s is a *subcut* of t if there exists an address u in t such that s is $\mathrm{cut}(t, u)^-$.

All proper prefixes of a term t are subcuts of t. Indeed, if we have $t = (\ldots((t_0{}^\wedge t_1)^\wedge t_2)\ldots)^\wedge t_k$, then t_0 is equal to $\mathrm{cut}(t, 0^{k-1}10^j)^-$, where j is the unique integer such that $0^{k-1}10^j$ is an address in t.

Proposition 6.9. *Assume that the automaton A_t is deterministic. Then the mapping* eval *is injective on subcuts of t—hence, in particular, on prefixes of t.*

Proof. Assume that u_1, u_2 are distinct addresses in t, and let p_1, p_2 be the associated main positions in $\mathrm{conj}(t)$. We assume $p_1 < p_2$. By Corollary 4.11, we have

$$\mathrm{conj}(t)\lceil\{1, \ldots, p_i - 1\} =_{FG} \mathrm{conj}(\mathrm{cut}(t, u_i)^-)$$

for $i = 1, 2$. Assume that $\mathrm{eval}(\mathrm{cut}(t, u_1)^-)$ and $\mathrm{eval}(\mathrm{cut}(t, u_2)^-)$ are equal. We deduce that the word $\mathrm{conj}(t)\lceil\{p_1, \ldots, p_2 - 1\}$ reduces to 1, so, in particular, the position p_1 must be mutually reducible with some position p'_1 with $p_1 < p'_1 < p_2$: this contradicts Proposition 6.7. ∎

Observe that, if the automaton A_t is deterministic and s is a subcut of t, then the reduced word $\underline{\mathrm{eval}}(s)$ can be read by A_t, and, by Proposition 4.5, its A_t-balance is the class $\overline{s}$. Thus, as far as subcuts of t are concerned, the automaton A_t provides a section for the projection $\overline{\mathrm{eval}}$.

7. Monotone terms

We now come back to the questions considered in Section 1.

Definition. An element a of Λ is *monotone* if there exist distinct variables $x_1, \ldots, x_m$ in V such that $a \sqsubseteq x_1{}^\wedge\ldots{}^\wedge x_m$ holds. An element a of $\widetilde{\Lambda}$ is *monotone* if there exists a monotone element b in Λ such that a is b^-. The sets of all monotone elements in Λ and $\widetilde{\Lambda}$ are denoted respectively Λ_{mono} and $\widetilde{\Lambda}_{\mathrm{mono}}$.

Using the results of [6], one easily shows that, if a is an element in Λ satisfying $a \sqsubset x_1{}^\wedge\ldots{}^\wedge x_m$ with $x_1, \ldots, x_m$ being distinct elements of V, then $a^\wedge x_1 \sqsubseteq x_1{}^\wedge\ldots{}^\wedge x_m$ holds as well. It follows that Λ_{mono} is included in $\widetilde{\Lambda}_{\mathrm{mono}}$, *i.e.*, our two definitions of monotonicity are compatible.

Note that, by definition, an element of Λ is monotone if and only if it can be represented by an LD-monotone term, and, similarly, an element of $\widetilde{\Lambda}$ is monotone if it can be represented by some term t^- where t is an LD-monotone term in T. Such terms t^- will be called *LDM-monotone* terms, and an identity will be called LDM-monotone if its two members are LDM-monotone terms in T^*.

Conjecture $\widetilde{\mathrm{A}}$. *Every LDM-monotone identity that holds in every system $(G, \cdot, {}^\wedge)$, where $(G, \cdot)$ is a group, is a consequence of (LDM).*

An LD-monotone identity is an LDM-identity, and, by Lemma 2.3, every consequence of (LDM) that involves only terms in T is a consequence of (LD). Hence Conjecture $\widetilde{\mathrm{A}}$ implies Conjecture A.

Proposition 7.1. *Conjecture $\widetilde{\mathrm{A}}$ is true if and only if the mapping* eval *is injective on $\widetilde{\Lambda}_{\mathrm{mono}}$.*

Proof. Assume that t_1 and t_2 are LDM-monotone terms in T^* and that the reduced words $\mathrm{eval}(t_1)$ and $\mathrm{eval}(t_2)$ are equal in the free group F. Let G be any group, and f be a mapping of V into G. Because F is a free group, $\mathrm{eval}(f(t_1)) = \mathrm{eval}(f(t_2))$ holds as well, where $f(t)$ denotes the result of replacing every variable x in t with its image under f. This means that the identity $t_1 = t_2$ holds in $(G, \cdot, {}^\wedge)$. So, if Conjecture $\widetilde{\mathrm{A}}$ is true, $t_1 = t_2$ follows from (LDM), *i.e.*, t_1 and t_2 represent the same element of $\widetilde{\Lambda}_{\mathrm{mono}}$.

Conversely, if some LDM-monotone identity $t_1 = t_2$ holds in every LD-monoid $(G, \cdot, {}^\wedge)$, it holds in particular in $(F, \cdot, {}^\wedge)$, and we have $\mathrm{eval}(t_1) = \mathrm{eval}(t_2)$. If Conjecture $\widetilde{\mathrm{A}}$ is false, this applies to at least one identity where t_1 and t_2 are not LDM-equivalent, which means that their images in $\widetilde{\Lambda}_{\mathrm{mono}}$ are not equal. ∎

Definition. If w is a word in W, $\mathrm{Var}(w)$ denotes the enumeration without repetition of those variables that appear in w, ordered according to their leftmost position (ignoring the sign). Similarly, if t is a term in T, $\mathrm{Var}(t)$ is the enumeration without repetition of those variables that appear in t ordered according to their leftmost address. The definition is extended to T^* by concatenating the sequences and removing the possible repetitions.

Lemma 7.2. *For every term t in T^*, the sequences* $\mathrm{Var}(t)$, $\mathrm{Var}(\mathrm{conj}(t))$, *and* $\mathrm{Var}(\mathrm{eval}(t))$ *coincide.*

Proof. An immediate induction gives the first equality. For the second, we may assume t in T, and we have to check that the free reductions in the word $\mathrm{conj}(t)$ cannot the change the order of the leftmost positions of each letter. Now, by Proposition 5.3, each such position is strong in $\mathrm{conj}(t)$. ∎

Lemma 7.3. *Assume that t_1 and t_2 are LD-monotone terms in T and* $\mathrm{eval}(t_1^-) = \mathrm{eval}(t_2^-)$ *holds. Then there exist distinct variables $x_1, \ldots, x_m$ such that both $t_1 \sqsubseteq_{LD} x_1{}^\wedge\ldots{}^\wedge x_m$ and $t_2 \sqsubseteq_{LD} x_1{}^\wedge\ldots{}^\wedge x_m$ hold.*

Proof. By definition, there exist distinct variables x_1, ..., x_m such that $t_1 \sqsubseteq_{LD} x_1{}^\wedge \ldots {}^\wedge x_m$ holds. By definition, there must exist terms t'_1 and t'_0 respectively LD-equivalent to t_1 and to $x_1{}^\wedge \ldots {}^\wedge x_m$ such that t'_1 is a prefix of t'_0. By Lemma 7.2, the sequence $\mathrm{Var}(t_1)$ is equal to $\mathrm{Var}(\mathrm{conj}(t'_1))$, and the sequence $\mathrm{Var}(x_1{}^\wedge \ldots {}^\wedge x_m)$, which is $(x_1, \ldots, x_m)$, is equal to $\mathrm{Var}(\mathrm{conj}(t'_0))$. Now, by construction, the word $\mathrm{conj}(t'_1)$ is a prefix of the word $\mathrm{conj}(t'_0)$, so we conclude that $\mathrm{Var}(t_1)$ is an initial segment of $(x_1, \ldots, x_m)$. If, moreover, m is chosen to be minimal, then $\mathrm{Var}(t_1)$ is exactly $(x_1, \ldots, x_m)$. Now $\mathrm{Var}(t_1^-)$ is equal to $\mathrm{Var}(t_1)$, *i.e.*, to $(x_1, \ldots, x_m)$, unless if t_1 is LD-equivalent to $x_1{}^\wedge \ldots {}^\wedge x_m$, in which case $\mathrm{Var}(t_1^-)$ is $(x_1, \ldots, x_{m-1})$.

Assume now that t_1 and t_2 are LD-monotone terms satisfying $\mathrm{eval}(t_1^-) = \mathrm{eval}(t_2^-)$. There exists variables x_1, ..., x_m and y_1, ..., y_n such that $t_1 \sqsubseteq_{LD} x_1{}^\wedge \ldots {}^\wedge x_m$ and $t_2 \sqsubseteq_{LD} y_1{}^\wedge \ldots {}^\wedge y_n$ hold. Assume that m and n have been chosen to be minimal. Assume first $t_1 \sqsubset_{LD} x_1{}^\wedge \ldots {}^\wedge x_m$ and $t_2 \sqsubset_{LD} y_1{}^\wedge \ldots {}^\wedge y_n$. By the argument above, we find

$$(x_1, \ldots, x_m) = \mathrm{Var}(t_1^-) = \mathrm{Var}(\mathrm{eval}(t_1^-)) = \mathrm{Var}(t_2^-) = (y_1, \ldots, y_n),$$

and, therefore, $t_2 \sqsubseteq_{LD} x_1{}^\wedge \ldots {}^\wedge x_m$ holds. Assume now $t_1 =_{LD} x_1{}^\wedge \ldots {}^\wedge x_m$ and $t_2 \sqsubset_{LD} y_1{}^\wedge \ldots {}^\wedge y_n$. The previous argument gives now $(x_1, \ldots, x_{m-1}) = (y_1, \ldots, y_n)$, which implies $t_2 \sqsubseteq_{LD} x_1{}^\wedge \ldots {}^\wedge x_{m-1}$, and, therefore, $t_2 \sqsubseteq_{LD} x_1{}^\wedge \ldots {}^\wedge x_m$ as $\sqsubseteq_{LD}$ is a transitive relation and $x_1{}^\wedge \ldots {}^\wedge x_{m-1} \sqsubseteq_{LD} x_1{}^\wedge \ldots {}^\wedge x_m$ holds trivially. The argument is similar for $t_1 \sqsubset_{LD} x_1{}^\wedge \ldots {}^\wedge x_m$ and $t_2 =_{LD} y_1{}^\wedge \ldots {}^\wedge y_n$. Finally assume $t_1 =_{LD} x_1{}^\wedge \ldots {}^\wedge x_m$ and $t_2 =_{LD} y_1{}^\wedge \ldots {}^\wedge y_n$. The previous argument gives $(x_1, \ldots, x_{m-1}) = (y_1, \ldots, y_{n-1})$. Let t'_2 be the term obtained from t_2 by replacing y_n with x_m. Because y_m occurs only at the rightmost address in t_2, we have $(t'_2)^- = t_2^-$, and $t'_2 =_{LD} y_1{}^\wedge \ldots {}^\wedge y_{n-1}{}^\wedge x_m = x_1{}^\wedge \ldots {}^\wedge x_{m-1}{}^\wedge x_m$. ∎

Definition. For t in T, the *derived* term ∂t is defined inductively by the rules

$$\partial t = \begin{cases} t & \text{if } t \text{ is a variable,} \\ \partial t_0{}^{\wedge\wedge} \partial t_1 & \text{if } t \text{ is } t_0{}^\wedge t_1, \end{cases}$$

where, for s, t in T, $s^{\wedge\wedge}t$ itself is defined inductively by $s^{\wedge\wedge}t = s^\wedge t$ if t is a variable, and $s^{\wedge\wedge}t = (s^{\wedge\wedge}t_0)^\wedge(s^{\wedge\wedge}t_1)$ if t is $t_0{}^\wedge t_1$.

Definition. The term t' is an *LD-expansion* of the term t if t' is obtained from t by iteratively replacing some subterm of the form $s_0{}^\wedge s_1{}^\wedge s_2$ by the corresponding term $(s_0{}^\wedge s_1)^\wedge(s_0{}^\wedge s_2)$.

It is clear that, if t' is an LD-expansion of t, then t' and t are LD-equivalent. An easy induction shows that ∂t is always an LD-expansion of t, and, that, if t_0 is a prefix of t, then ∂t_0 is a prefix of ∂t. The terms $\partial^k t$ are cofinal in the LD-equivalence class of t in the following sense:

Proposition 7.4. [2] *Assume that t, t' are terms in T. Then t and t' are LD-equivalent if and only if $\partial^k t$ is an LD-expansion of t' for k large enough.*

Corollary 7.5. *Assume that t is a term in T and a is an element of Λ. Then $a \sqsubseteq \overline{t}$ holds if and only if, for every k large enough, a can be represented by a prefix of $\partial^k t$.*

Definition. For $n > 0$ and $k \geq 0$, the automaton $A_{m,k}$ is defined to be the automaton A_t, where t is the term $\partial^k(z_1{}^\wedge\ldots{}^\wedge z_m)$ and $(z_1, z_2, \ldots)$ is some fixed sequence of distinct variables in V.

Proposition 7.6. *Assume that the automaton $A_{m,k}$ is deterministic for every m, k. Then Conjecture $\widetilde{\mathrm{A}}$ is true.*

Proof. By Proposition 7.1, it suffices to prove that the mapping eval is injective on $\widetilde{\Lambda}_{\mathrm{mono}}$. Assume that a_1, a_2 are elements of Λ_{mono} satisfying $\mathrm{eval}(a_1^-) = \mathrm{eval}(a_2^-)$. By Lemma 7.3, up to changing the names of the variables, we may assume that $a_i \sqsubseteq z_1{}^\wedge\ldots{}^\wedge z_m$ holds in $\widetilde{\Lambda}$ for $i = 1, 2$. By Corollary 7.5, there exists $k \geq 0$ such that a_1 and a_2 can be represented by prefixes of $\partial^k(z_1{}^\wedge\ldots z_m)$, say t_1 and t_2. Thus t_1^- and t_2^-, which represent a_1^- and a_2^- by definition, are subcuts of $\partial^k(z_1{}^\wedge\ldots z_m)$. If the automaton $A_{m,k}$ is deterministic, Proposition 6.9 forces a_1^- and a_2^- to be equal. ∎

Now clearly Proposition 1.1 follows from:

Proposition 7.7. *Assume that Conjecture B is true. Then the automaton $A_{m,k}$ is deterministic for every m, k.*

Proof. Let t be the term $\partial^k(x_1{}^\wedge\ldots{}^\wedge x_m)$. We wish to prove that A_t is deterministic. According to Lemma 6.2, it suffices to show that, if $a_1 \xrightarrow{x} a'$ and $a_2 \xrightarrow{x} a'$ are transitions of A_t, then a_1 and a_2 must be equal. Now, by Lemma 2.5, the previous condition implies $a' = a_1 \cdot x = a_2 \cdot x$ in $\widetilde{\Lambda}$. Let b_i be $a_i{}^\wedge x$ for $i = 1, 2$. By construction, b_1 is a value of of the function $t^\sharp$. Hence, by definition, there exist a stack $(u_1, \ldots, u_n)$ in t such that b_1 is $\pi_{LD}(\mathrm{cut}(t, u_1){}^\wedge\ldots{}^\wedge\mathrm{cut}(t, u_n))$. Now, if the address u' lies on the right of the address u, the relation $\mathrm{cut}(t, u) \sqsupset_{LD} \mathrm{cut}(t, u')$ always holds [6]. Hence b_1 is decomposable, and so is b_2. Now, by Formula (2.3), we have, for $i = 1, 2$,

$$(b_i{}^\wedge b_i)^- = b_i^+ = (a_i{}^\wedge x)^+ = a_i \cdot x = a',$$

and $b_1{}^\wedge b_1$ is equal to $b_2{}^\wedge b_2$. If Conjecture B is true, this implies $b_1 = b_2$, hence $a_1 = b_1^- = b_2^- = a_2$, and A_t is deterministic. ■

Let us mention that the converse of Proposition 7.7 is true: using techniques of [6], one can show that, up to a change of variables, every decomposable element of Λ is a value of $\partial^k(z_1{}^\wedge \ldots {}^\wedge z_m)^\sharp$ for k, m large enough, and, therefore, if distinct decomposable elements had the same square in Λ, then the corresponding automaton $A_{k,m}$ could not be deterministic.

Computer experiments involving more than 10^4 cuts of $\partial^k(z_1{}^\wedge z_2{}^\wedge \ldots)$ with $k \leq 5$ have failed to provide any counterexample to Conjectures A or B. We have presently no proof of them, except for the cases associated with $k = 0$, $k = 1$ and, partially, $k = 2$, where specific arguments exist—thus, defining an LD-monotone term of degree k to be one that is LD-equivalent to a prefix of $\partial^k(z_1{}^\wedge z_2{}^\wedge \ldots)$, we may state that group conjugacy satisfies no LD-monotone identity of degree ≤ 2 except those that are consequences of (LD). The missing piece for a general proof seems to be a normal form for the decomposable elements of Λ in the spirit of the special terms constructed in the next section for representing the monotone elements.

8. Special terms

We turn to the construction of normal forms. Under Conjecture A, we know that the mapping eval is injective on the subset Λ_{mono} of Λ. The techniques of [6], or, alternatively, [13] and [14], make it easy to construct a unique normal form for the elements of Λ_{mono}, thus leading to a normal form for those elements of C that come from LD-monotone terms.

The key point is that there exists a simple relation between the set Λ_{mono} and the free LD-system on one generator Λ_1.

As in Section 1, we fix a variable z and we denote by T_1 the set of all terms constructed using only z and the operator ${}^\wedge$. We write ϕ for the forgetful projection of T onto T_1 that replaces every variable with z. Of course, ϕ induces a surjective homomorphism, still denoted ϕ, of Λ onto Λ_1.

Proposition 8.1. *The mapping ϕ induces a surjection of Λ_{mono} onto Λ_1.*

Proof. We construct sections for ϕ whose images are included in $\Lambda_{\rm mono}$ as follows. Let $\vec{x} = (x_1, x_2, \ldots)$ be an (infinite) sequence of distinct variables. Assume that a is an element of Λ_1. By Proposition 7.4, there exist integers m and k such that a is represented by some prefix of the term $\partial^k(z\char`^\ldots\char`^z)$, m times z. By construction, the prefixes of a term are exactly the cuts of this term associated with addresses of the form 0^k. So, we have an equality of the form $a = \pi_{LD}(\mathrm{cut}(\partial^k(z\char`^\ldots\char`^z), 0^k))$. Then we define $\psi_{\vec{x}}(a)$ as $\pi_{LD}(\mathrm{cut}(\partial^k(x_1\char`^\ldots\char`^x_m), u))$. Due to the compatibility between cuts and derivability, $\psi_{\vec{x}}$ is well-defined. Then, by construction, every element a of the image of $\psi_{\vec{x}}$ satisfies $a \sqsubseteq x_1\char`^\ldots\char`^x_m$ for m large enough, and the image of $\psi_{\vec{x}}$ is included in $\Lambda_{\rm mono}$. ∎

The image of $\psi_{\vec{x}}$ is exactly the subset of $\Lambda_{\rm mono}$ consisting of those elements a for which the sequence $\mathrm{Var}(a)$ is an initial segment of $\vec{x}$. So every element in $\Lambda_{\rm mono}$ belongs to the image of some mapping $\psi_{\vec{x}}$. By [6] (or [13]), normal forms are known for Λ_1. Then it suffices to transport them to $\Lambda_{\rm mono}$ using the mappings $\psi_{\vec{x}}$.

Definition. The *left height* $\mathrm{ht}_L(t)$ of a term t is the maximal number of 0's in an address in t.

Definition. Assume that t is a term in T. If $\mathrm{ht}_L(t)$ is at most 1, we let $\Theta(t)$ be t. Otherwise, t has a unique decomposition $t = t_1\char`^\ldots\char`^t_{m+1}$ where $\mathrm{ht}_L(t_m) = \mathrm{ht}_L(t) - 1$ and $\mathrm{ht}_L(t_{m+1}) < \mathrm{ht}_L(t)$. Then we let $\Theta(t)$ be $t_1\char`^\ldots\char`^t_m\char`^x$, where x is the leftmost variable in t.

Definition. Assume that t and t' are terms in T, say $t = t_1\char`^\ldots\char`^t_m\char`^x$, $t' = t'_1\char`^\ldots\char`^t'_{m'}\char`^x'$, where x and x' are variables. We say that $t >_{\rm Lex} t'$ holds if either there exists k such that $t_1 = t'_1$, …, $t_{k-1} = t'_{k-1}$ and $t_k >_{\rm Lex} t'_k$ hold, or $m > m'$ and $t_1 = t'_1$, …, $t_{m'} = t'_{m'}$ hold. Thus, the relation $>_{\rm Lex}$ is a partial lexicographical ordering of the terms (partial, as we fixed no ordering on the variables).

Definition. (i) A *special* term of degree 0 is a term of the form $x_1\char`^\ldots\char`^x_m$ with x_1, …, x_m distinct variables;

(ii) For $k \geq 1$, a *special* term of degree k is a term of the form $t_1\char`^\ldots\char`^t_{m+1}$ where $m \geq 1$, t_1, …, t_m are special terms of degree $k-1$, t_m is a special term of degree $\leq k-1$, and $\Theta(t_i) >_{\rm Lex} t_{i+1}$ holds for all $i \leq m$.

By definition, the normal terms of [6] involve an infinite series of variables denoted a, b, c, … Then a term t is special in the present sense if and only if there exists a sequence of distinct variables $(x_1, x_2, \ldots)$ and a term t' normal in the sense of [6] such that t is obtained from t' by replacing a with x_1, b with $x_1{}^\wedge x_2$, c with $x_1{}^\wedge x_2{}^\wedge x_3$, *etc.* Repeating in the present framework the proof of [6, Theorem 3.5] gives:

Proposition 8.2. *(i) Every monotone element of Λ is represented by a unique special term.*

(ii) There is an effective function that maps every LD-monotone term to the unique special term it is LD-equivalent to. This function lies in the complexity class $\mathrm{DSPACE}(\exp^*(\mathcal{O}(2^n)))$, *where* $\exp^*(x)$ *denotes a tower of base* 2 *exponentials of height* x.

It is clear that Proposition 8.2 implies Proposition 1.2.

9. An algorithm

We recall that ϕ denotes the forgetful projection of T onto T_1 that replaces every variable with z, a unique fixed variable. The value of $\phi(t)$ is called the *skeleton* of t. Let us say that two terms t, t' are *LD-comparable* if $t \sqsubseteq_{LD} t'$ or $t' \sqsubseteq_{LD} t$ holds.

By the results of [2] completed in [4], we have:

Proposition 9.1. *Two terms t, t' with the same skeleton are LD-comparable if and only if they are equal.*

Corollary 9.2. *For every skeleton s and every sequence of distinct variables $(x_1, \ldots, x_m)$, there exists at most one LD-monotone term t with skeleton s such that* $\mathrm{Var}(t)$ *is* $(x_1, \ldots, x_m)$.

We consider here the problem of algorithmically finding the unique term t mentioned in Corollary 9.2, *i.e.*, we start with a skeleton s and we wish to replace the variables z with variables among $x_1, \ldots, x_m$ so as to obtain an LD-monotone term, if this is possible.

The problem is certainly decidable. Indeed we can always use the 'stupid' algorithm consisting of systematically considering all possible choices (there are finitely many of them), and, for each of them, testing whether the term so obtained is LD-equivalent to $x_1{}^\wedge \ldots {}^\wedge x_m$ or one of its prefixes using one of the algorithms of [4].

If Conjecture A is true, we can solve the question in a much better way by using group conjugacy and the algorithm below, which was first considered in [5].

Assume that f is a mapping of V into itself. For w in W, we denote by $f(w)$ the word obtained from w by replacing every letter $x^{\pm 1}$ with the corresponding letter $f(x)^{\pm 1}$. For t in T, we denote by $f(t)$ the term obtained from t by replacing every variable x with the corresponding vaiable $f(x)$.

We fix two disjoint infinite series of variables $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$, supposed to be included in V. For s in T_1, we write $\widetilde{s}$ for the injective term obtained from s by substituting y_1, y_2, $\ldots$ for the variables z of s starting from the left. For instance, if s is $(z\hat{}z)\hat{}z$, then $\widetilde{s}$ is $(y_1\hat{}y_2)\hat{}y_3$.

Definition. Assume that w is a word in V. We say that w is *solvable* if the first letter $y_j^{\pm 1}$ in w is a y_j, and it is immediately preceded by a letter x_i^{-1}. In this case, we define $\varphi(w)$ to be the mapping of V into itself that maps y_j to x_i and keeps all other variables unchanged.

According to Proposition 3.2, every word conj(t) for t in T is 'skew-symmetric'. We use below half-conj(t) to denote the first half of the word conj(t), defined as the prefix of conj(t) with length $(\ell + 1)/2$ if conj(t) has length ℓ.

Algorithm 9.3. *Input*: A skeleton s in T_1.
Process: Let n be the number of addresses in s.
Start with
$\quad f_0 := \mathrm{id}, \quad w_0 := x_n^{-1} \ldots x_2^{-1} x_1^{-1}\text{half-conj}(\widetilde{s})$;
For $k := 1$ to n, if w_{k-1} is solvable, do
$\quad f_k := \varphi(w_{k-1}) \circ f_{k-1}, \quad w_k := \overline{f_k(w_{k-1})}$;
Output: The term $f_n(s)$, if no obstruction has occurred.

Example 9.4. Let us consider the skeleton $s = (z\hat{}z)\hat{}z$. Then $\widetilde{s}$ is $(y_1\hat{}y_2)\hat{}y_3$, and half-conj($\widetilde{s}$) is $y_1 y_2 y_1^{-1} y_3$. Algorithm 9.3 running on s gives:

$f_0 = \mathrm{id}$,	w_0	$:=$	x_3^{-1}	x_2^{-1}	x_1^{-1}	y_1	y_2	y_1^{-1}	y_3,
$f_1 : y_1 \mapsto x_1$,	w_1	$=$	x_3^{-1}	x_2^{-1}			y_2	x_1^{-1}	y_3,
$f_2 : y_1 \mapsto x_1, y_2 \mapsto x_2$,	w_2	$=$	x_3^{-1}					x_1^{-1}	y_3,
$f_2 : y_1 \mapsto x_1, y_2 \mapsto x_2, y_3 \mapsto x_1$,	w_3	$=$	x_3^{-1}.						

Finally, the term $f_3(s)$ is $(x_1\hat{}x_2)\hat{}x_1$, an LD-monotone term with skeleton s.

Proposition 9.5. *Assume that Conjecture A is true. Then Algorithm 9.3 is correct, i.e., it gives the unique solution of the problem when it exists.*

Proof. Assume that t is an LD-monotone term and let s be the skeleton of t. We assume that $t \sqsubseteq_{LD} x_1{}^\wedge\ldots{}^\wedge x_m$ holds. We first consider the case $t =_{LD} x_1{}^\wedge\ldots{}^\wedge x_m$. Let $u_1, \ldots, u_n$ be the enumeration of the addresses in s (*i.e.*, in t) from left to right, and let $p_1, \ldots, p_n$ be the associated main positions in the word $\mathrm{conj}(t)$. We use f_k and w_k for the function and the word occurring at step k in the algorithm running on s, and we use t_k for the term $f_k(\widetilde{s})$—if it exists. We show inductively on k that f_k maps y_j to $t(u_j)$ for $j = 1, \ldots, k$, that w_k exists and that it satisfies

$$w_k =_{FG} \text{half-conj}(t)^{-1} \cdot \text{half-conj}(t_k).$$

If k is 0, everything is obvious. Otherwise, by induction hypothesis, the variables $y_1, \ldots, y_{k-1}$ have been replaced by variables x_i in the word w_{k-1}. So y_k is the leftmost y variable in w_{k-1}. Its first position in w_{k-1} corresponds to the main position p_k coming from address u_k. By induction hypothesis, we have

$$(\text{half})\text{-conj}(t_k)\restriction\{1, \ldots, p_k - 1\} = (\text{half})\text{-conj}(t)\restriction\{1, \ldots, p_k - 1\}, \qquad (9.1)$$

and, therefore

$$\begin{aligned}\text{half-conj}(t)^{-1}\cdot&\text{half-conj}(t_k)\restriction\{1, \ldots, p_k - 1\}\\ &= \text{half-conj}(t)^{-1} \cdot \text{half-conj}(t)\restriction\{1, \ldots, p_k - 1\}\\ &=_{FG} (\text{half-conj}(t)\restriction\{p_k, \ldots\})^{-1}.\end{aligned}$$

By Proposition 6.7, position p_k is semistrong in the word $\mathrm{conj}(t)$. This implies that the first letter of $\text{half-conj}(t)\restriction\{p_k, \ldots\}$, which is $t(u_k)$ by construction, cannot vanish in a subsequent free reduction. It follows that the reduct of the word $\text{half-conj}(t)^{-1}(\text{half-conj}(t_k)\restriction\{1, \ldots, p_k - 1\})$ ends with the letter $t(u_k)^{-1}$. This means that the word w_{k-1} is solvable, and that f_k is defined to map y_k to $t(u_k)$. Then w_k exists, and, because w_k is $f_k(w_{k-1})$ while t_k is $f_k(t_{k-1})$, Relation (9.1) for $k-1$ implies Relation (9.1) for k.

Finally, if we relax the hypothesis from $t =_{LD} x_1{}^\wedge\ldots{}^\wedge x_m$ to $t \sqsubseteq_{LD} x_1{}^\wedge\ldots{}^\wedge x_m$, the argument remains valid. Indeed, there exist in this case terms $t'_1, \ldots, t'_r$ such that the term $t' = (\ldots((t^\wedge t'_1)^\wedge t'_2)^\wedge\ldots)^\wedge t'_r$ satisfies $t' =_{LD} x_1{}^\wedge\ldots{}^\wedge x_m$. Now the word $\mathrm{conj}(t)$ is a prefix of the word $\mathrm{conj}(t')$, and running Algorithm 9.3 on the skeleton of t is merely the beginning of running it in the skeleton of t'. In the latter case, we know that the algorithm gives the desired subtitution, so it does it as well in the former case. ∎

Remark. Replacing the conjugate words by half-conjugate words in Algorithm 9.3 is just a matter of shortening the words, as only the first half is really involved. We could of course use the full conjugate words as well.

A natural question is whether Algorithm 9.3 always succeeds or, independently of Conjecture A, whether every skeleton is the skeleton of an LD-monotone term. The answer is negative. For instance, one can prove that the term $((z\hat{~}z)\hat{~}z)\hat{~}z\hat{~}z\hat{~}z$ cannot be the skeleton of any LD-monotone term—but its LD-expansion $((z\hat{~}z)\hat{~}z)\hat{~}(z\hat{~}z)\hat{~}z\hat{~}z$ is the skeleton of $((x_1\hat{~}x_2)\hat{~}x_1)\hat{~}(x_1\hat{~}x_2)\hat{~}x_3\hat{~}x_4$, an LD-monotone term. This example is an illustration of the following result:

Proposition 9.6. *Every skeleton has an LD-expansion that is the skeleton of an LD-monotone term.*

Proof. As was mentioned above, every term in T_1 admits an expansion that is a prefix of a term of the form $\partial^k(z\hat{~}\ldots\hat{~}z)$. Now any image of the latter prefix under a mapping $\psi_{\vec{x}}$ is an LD-monotone term. ∎

Let us finally mention that all constructions and conjectures developed here can be extended without any change to the case where the terms $z_1\hat{~}\ldots\hat{~}z_m$ are replaced with any other injective term.

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