ON COMPLETENESS OF WORD REVERSING

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Abstract. Word reversing is a combinatorial operation on words that detects pairs of equivalent words in monoids that admit a presentation of a certain form. Here we give conditions for this method to be complete in the sense that *every* pair of equivalent words can be detected by word reversing. In addition, we obtain explicit upper bounds on the complexity of the process. As an application, we show that Artin groups of Coxeter type B embed into Artin groups of type A and are left orderable.

Assume that $\langle S; R\rangle$ is a presentation of monoid, *i.e.*, R is a family of equalities of the form $u = v$ where u, v are words in the free monoid S^*. We say that this presentation is *complemented* (on the right) if R contains no equality $u = v$ where either u or v is empty or u and v begin with the same letter, and, moreover, for every pair (x, y) in S, there exists at most one relation $u = v$ in R such that u begins with x and v begins with y. Thus, for instance, $\langle a, b\ ;\ aba = bb\rangle$ or $\langle a, b, c\ ;\ a = bbc, a = ca, ba = c\rangle$ are typical complemented presentations. So are also all presentations considered in [1] and [17]. In this paper, we investigate in the framework of monoids with a complemented presentation a combinatorial transformation of words that we call *word reversing*.

Saying that the presentation $\langle S; R\rangle$ is right complemented amounts to saying that there exists a partial function

$$f : S \times S \longrightarrow S^*$$

such that the domain of f is a symmetric subset of $S \times S$, $f(x, x)$ is the empty word ε for every x in S and R is the set of all equalities $xf(x, y) = yf(y, x)$ for (x, y) in the domain of f with $x \neq y$. Such a function f will be called a *complement* on S in the sequel, and it will be the object we start from. If f is a complement on the set S, we denote by $\equiv_f$ the congruence relation on S^* generated by all pairs $\{xf(x, y) = yf(y, x)\}$, and by R_f the list of all relations $xf(y, x) = yf(x, y)$ for (x, y) in the domain of f.

In order to define word reversing, we first introduce a disjoint copy S^{-1} of S consisting of a formal inverse x^{-1} for every letter x in S, and, for every word w in $(S \cup S^{-1})^*$, we denote by w^{-1} the word obtained from w by replacing each letter $x^{\pm 1}$ by its inverse $x^{\mp 1}$ and reversing the ordering of letters. Now, we say that the word w is *f-reversible in one step* to the word w' if there exist two letters x, y in S such that w' is obtained from w by replacing some factor $x^{-1}y$ with the corresponding word $f(y, x)f(x, y)^{-1}$. In this case, we write $w \curvearrowright_f w'$. We say naturally that the word w is f-reversible *in p steps* to the word w' if there exists a

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finite sequence of words $(w_0, \ldots, w_p)$ such that w_0 is w, w_p is w' and $w_{i-1} \curvearrowright_f w_i$ holds for $i \leq p$.

Assume that u, v are two words in S^*. It will be easy to show that, if the word $u^{-1}v$ is f-reversible to the empty word, then the words u and v are equivalent with respect to $\equiv_f$, *i.e.*, they represent the same element of the monoid $\langle S\,;\, R_f\rangle$. The question we investigate here is whether the previous sufficient condition is also necessary, in which case we say that word reversing is *complete* for f. In this case, we obtain an simple method for studying the monoid $\langle S\,;\, R_f\rangle$ concretely, as implementing word reversing on a computer is straightforward.

A complement f must satisfy additional conditions for word reversing to be possibly complete for f. We introduce below a property that we call the *local coherence* of f. As the name suggests, this property can be checked in a finite number of steps (provided the set S is finite), and the problem becomes the question of whether local coherence is sufficient for proving completeness. In this paper, we prove several such completeness results, in particular (Corollary 2.5):

Proposition. *Assume that the complement f is locally coherent, and, moreover, it is Noetherian—which means that there is no infinite descending sequence for left division in $\langle S\,;\, R_f\rangle$. Then word reversing is complete for f.*

Besides this result, we associate with every complement an integer parameter called its *degree*, so that, at least when the domain is finite, being locally coherent is equivalent to having a finite degree. We have also (Proposition 4.1):

Proposition. *Assume that the complement f has degree at most 1. Then word reversing is complete for f.*

This result implies in particular that word reversing is *always* complete in the case of two generators.

It seems that monoids with a complemented presentation and word reversing have been considered first in [6], and a number of results have been subsequently obtained in [8], [18], [13], [10]. In particular, [8] includes a weaker version of the above mentioned Corollary 2.5. However, the restrictive hypotheses used in [8] are somehow misleading as they hide the main argument, which is an ordinal induction made possible by the Noetherianity assumption. Also, the argument given in [8] is centered on computing lcm's rather than on proving equivalence. So we think that the present approach is the "right" one. In particular, it allows us to obtain explicit complexity bounds, such as (Corollary 3.3.ii):

Proposition. *Assume that f is a complement on S with the property that the words $f(x,y)$ and $f(y,x)$ have the same length when they exist, f has degree $k \geq 1$, and deviation δ—which means that it is locally coherent with some explicit upper bounds. Assume that u is a word of length ℓ in S^*, and that v is f-equivalent to u, i.e., v represents the same element of $\langle S\,;\, R_f\rangle$. Then the word $u^{-1}v$ is f-reversible to the empty word in a number of steps n that satisfies*

$$d \leq n \leq \ell + \delta \cdot G_k(\ell) \cdot d$$

where d is the combinatorial distance of u and v, i.e., the minimal number of relations in the presentation needed to establish their equivalence, and $G_k(x)$ stands for $(2^x - 1)\, k^{2^x - 1}$.

In contradistinction to the results of earlier papers chiefly dealing with the case where word reversing always halts in a finite number of steps, the current results are relevant even in the case when word reversing may not halt, which make them surprising. Indeed, the existence of small upper bounds for the complexity of word reversing is natural when the latter process always terminates, but it was unexpected in the general case. Roughly speaking, we prove here that word reversing either never halts, or it halts in a relatively small number of steps, namely at most a double exponential in the size of the initial data.

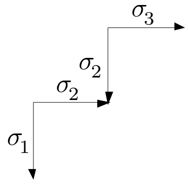
The argument used to prove completeness of word reversing also gives a simple criterion for establishing embedding results for monoids with a complemented presentation. As an example, we consider Artin's monoids. We obtain a short proof of the fact that every Artin monoid of Coxeter type B embeds in an Artin monoid of type A, *i.e.*, in a braid monoid. From here, we can deduce an analogous embedding result for the corresponding groups, which implies in particular that the Artin groups of type B are orderable.

The paper comprises five sections. In Section 1, we introduce word reversing, associate a graph with every reversing process, and show that word reversing is complete if and only if some property called coherence is true. In Section 2, we consider the Noetherian case and show that local coherence then implies full coherence. In Section 3, we study the combinatorial complexity of word reversing and establish upper bounds for the reversing of equivalent words. In Section 4, we consider the special case of complements with degree at most 1, a strengthened coherence hypothesis under which Noetherianity is not needed. Finally, in Section 5, we establish the embedding criterion and mention its application to Artin monoids.

The author wishes to thank Paul-André Melliès: his questions about Example 1.6 below have been the origin of the unexpected Proposition 4.1, and, indirectly, of the equally unexpected results of Section 3.

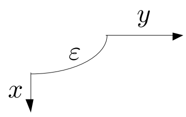
1. Reversing graphs

Assume that f is a complement on S, *i.e.*, a mapping of $S \times S$ to S^* with a symmetric domain and satisfying $f(x, x) = \varepsilon$ for every letter x . We define a f-reversing sequence to be a (finite or infinite) sequence of words $(w_0, w_1, \dots)$ in $(S \cup S^{-1})^*$ such that w_{i-1} is f-reversible in one step to w_i for every i. It will be both natural and useful to associate with every such sequence a labelled graph embedded in the rational plane $\mathbf{Q}^2$. This graph contains three types of edges: *horizontal* S-labelled edges going from a vertex (p, q) to a vertex (p', q) with $p' > p$, *vertical* S-labelled edges going from a vertex (p, q') to a vertex (p, q) with $q < q'$, and unoriented ε-labelled edges. The construction of the graph is inductive. First, we associate a graph $\Gamma(w_0)$ with the initial word w_0 as follows. If w_0 is the empty word, then $\Gamma(w_0)$ consists of solely one vertex, namely $(0, 0)$. Then, for x in S, the graph $\Gamma(wx)$ (*resp.* $\Gamma(wx^{-1})$) is obtained from $\Gamma(w)$ by adding one horizontal (*resp.* vertical) x-labelled edge starting from (*resp.* arriving to) the vertex $(|w|_+, |w|_-)$, where $|w|_+$ and $|w|_-$ denote the number of positive and negative letters in w. Thus, for instance, if w_0 is the word $\sigma_1^{-1}\sigma_2\sigma_2^{-1}\sigma_3$, the graph $\Gamma(w_0)$ will be as follows:



By construction, there exists a unique maximal path in the graph $\Gamma(w_0)$, and w_0 is exactly the label of this path, defined as the sequence consisting of the successive edges in the path, according to the convention that an edge that is labelled x and is crossed contrary to its orientation contributes x^{-1}.

Assume now that the graph $\Gamma(w_0, \dots, w_{n-1})$ has been constructed, and w_n is obtained from w_{n-1} by replacing some factor $x^{-1}y$ with the corresponding factor $f(x,y)f(y,x)^{-1}$. By induction hypothesis, we assume that the word w_{n-1} is traced in $\Gamma(w_0, \dots, w_{n-1})$ from the point $(0,0)$, this meaning that there is a path from $(0,0)$ in the graph such that w_{n-1} is the sequence of the corresponding labels (with the above sign convention). The factor $x^{-1}y$ involved in the reversing process labels some fragment of $\Gamma(w_0, \dots, w_{n-1})$. Then $\Gamma(w_0, \dots, w_n)$ is obtained by adding to $\Gamma(w_0, \dots, w_{n-1})$ horizontal edges labelled $f(x,y)$ and vertical edges labelled $f(y,x)$ in the neighborhood of the above mentionned fragment labelled $x^{-1}y$, according to the generic picture



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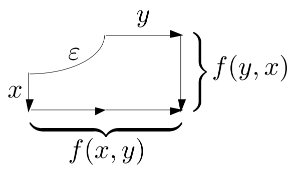


The new horizontal and vertical edges are determined so as to have equal lengths. If $f(x,y)$ and/or $f(y,x)$ is empty, we use instead an ε-labelled edge.

Example 1.1. Every braid monoid, and, more generally, every Artin monoid, admits a right complemented presentation. Let $S = \{\sigma_i \ ; \ i \in I\}$ be a nonempty set. A Coxeter matrix over S is a symmetric matrix $M = (m_{i,j})_{i,j \in I}$ such that $m_{i,i}$ is 1, and $m_{i,j}$ belongs to $\{2, 3, \dots, \infty\}$ for $i \neq j$ (see for instance [2]). The Artin monoid associated with M is the monoid that admits the presentation

$$\langle S \ ; \ \mathrm{prod}(\sigma_i, \sigma_j, m_{i,j}) = \mathrm{prod}(\sigma_j, \sigma_i, m_{j,i}) \text{ for } m_{i,j} < \infty \rangle \tag{1.1}$$

where $\mathrm{prod}(x, y, m)$ stands for the alternating word $xyxyxy\dots$ of length m. This presentation is associated with the complement f defined by

$$f(\sigma_i, \sigma_j) \begin{cases} = \mathrm{prod}(\sigma_i, \sigma_j, m_{i,j} - 1) & \text{if } m_{i,j} \text{ is odd,} \\ = \mathrm{prod}(\sigma_j, \sigma_i, m_{i,j} - 1) & \text{if } m_{i,j} \text{ is even,} \\ \text{undefined} & \text{if } m_{i,j} \text{ is } \infty. \end{cases}$$

Let us consider for instance the type A_3 Coxeter matrix such that $m_{i,i+1}$ is 3 and $m_{i,j}$ is 2 for $|i-j| \geq 2$. The associated monoid is the monoid of positive braids on

4 strands. Then

$$\begin{aligned}\sigma_1^{-1}\sigma_2\sigma_2^{-1}\sigma_3 &\curvearrowright_f \sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3 \curvearrowright_f \sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_3\sigma_2\sigma_3^{-1}\sigma_2^{-1}\\ &\curvearrowright_f \sigma_2\sigma_1\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_2^{-1} \curvearrowright_f \sigma_2\sigma_1\sigma_3\sigma_2\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_2^{-1}\\ &\curvearrowright_f \sigma_2\sigma_1\sigma_3\sigma_2\sigma_3^{-1}\sigma_2^{-1}\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}\\ &\curvearrowright_f \sigma_2\sigma_1\sigma_3\sigma_2\sigma_3^{-1}\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1} \curvearrowright_f \sigma_2\sigma_1\sigma_3\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}\end{aligned}$$

is a typical f-reversing sequence. We cannot extend it further, as, in the last word, every positive letter lies before every negative letter, so there remains no factor $x^{-1}y$ that could be reversed. The graph associated with the previous f-reversing sequence is displayed on Figure 1.1.

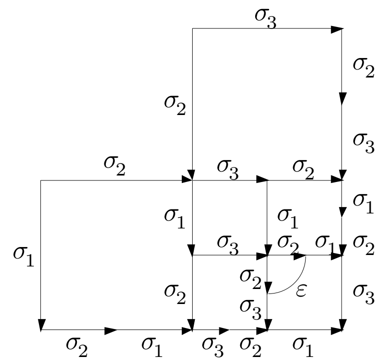


Figure 1.1. A reversing graph

Reversing graphs are connected with Cayley graphs, but, in general, a reversing graph is *not* a fragment of the Cayley graph of the corresponding monoid as we do not identify those vertices that are connected by an ε-labelled edge. Notice also that reversing graphs are planar, as the last word currently added always lies on the right (actually, on the "South-East") of all previous words.

Considering reversing graphs makes it quite intuitive that, for every initial word w, there exists a unique maximal reversing graph that begins from w, and that this graph does not depend on the ordering of the reversing steps used to construct it. Thus, if one f-reversing sequence goes from some word w in $(S \cup S^{-1})^*$ to some terminal word of the form uv^{-1} with u, v in S, then every f-reversing sequence does so. In other words, f-reversing is a confluent transformation. More precisely, we have:

Lemma 1.2. *Assume that f is a complement on S, and that w is a word in $(S \cup S^{-1})^*$ that is f-reversible to w' in p' steps and f-reversible to w'' in p'' steps. Then there exists an integer p at most equal to $p' + p''$ and a word w''' such that w' is f-reversible to w''' in $p - p'$ steps and w'' is f-reversible to w''' in $p - p''$ steps.*

As the (easy) proof appears in [8, Lemma 1.1], we do not repeat it here.

A key point in the sequel will be to extend the initial complement, which is defined only on pairs of letters, into a mapping defined on pairs of words. To this end, we use word reversing again.

Definition. Assume that f is a complement on the set S. For u, v in S^*, we denote by $u\backslash_f v$ the unique word v' in S^* such that $u^{-1}v$ is f-reversible to $v'u'^{-1}$ for some u' in S^*, if such words exist. In this case, we write $u \vee_f v$ for $u(u\backslash_f v)$. The *f-complexity* $c_f(u, v)$ of the pair $\{u, v\}$ is the number of steps needed to reverse the word $u^{-1}v$ to a word of the form $v'u'^{-1}$, if such a number exists, and ∞ otherwise.

By Lemma 1.2, $\backslash_f$ is a well-defined partial mapping of $S^* \times S^*$ into S^*. We shall occasionally use the symbol $\perp$ to mean "undefined", and write $u\backslash_f v = \perp$ when $u\backslash_f v$ does not exist.

By construction, if $u^{-1}v$ is f-reversible to $v'u'^{-1}$, then $v^{-1}u$ is f-reversible to $u'v'^{-1}$, and, therefore, the word u' is $v\backslash_f u$. In other words, $u\backslash_f v$ and $v\backslash_f u$ exist if and only if the f-reversing of the word $u^{-1}v$ comes to an end, and $(u\backslash_f v)(v\backslash_f u)^{-1}$ is the corresponding final word. Observe that the mapping $\backslash_f$ extends f: for $x, y \in S$, $x\backslash_f y$ is defined if and only if $f(x, y)$ is defined, and, in this case, these words are equal.

Lemma 1.3. *Assume that f is a complement on the set S. Then the equalities*

$$(uv)\backslash_f w = v\backslash_f (u\backslash_f w), \tag{1.2}$$

$$w\backslash_f (uv) = (w\backslash_f u)\,((u\backslash_f w)\backslash_f v). \tag{1.3}$$

hold for all u, v, w in S^*—such equalities mean that either both sides are defined and they are equal, or none of them is defined (so they are true equalities when we use the symbol $\perp$).

Proof. It should be clear from Figure 1.2 that, if the words $u\backslash_f w$ and $v\backslash_f (u\backslash_f w)$ exist, then so does the word $(uv)\backslash_f w$ and (1.2) and (1.3) hold. Assume conversely that $(uv)\backslash_f w$ exists. We have to show that $u\backslash_f w$ and $v\backslash_f (u\backslash_f w)$ exist, whence the result is clear. We observe that, for each pair of words (u, w), either $w\backslash_f u$ exists, or there exists an infinite f-reversing sequence starting from $u^{-1}w$. In the latter case, adding v^{-1} at the left of every word in this sequence gives an infinite f-reversing sequence from $v^{-1}u^{-1}w$, and $w\backslash_f (uv)$ cannot exist either. ∎

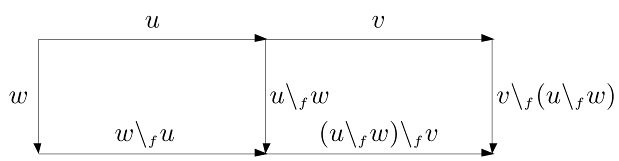


Figure 1.2. Double reversing

Word reversing produces equivalent words. To state a precise result, we first introduce a parameter that counts how many times the basic relations of the presentation are used.

Definition. Assume that f is a complement on S, and u, v belong to S^*. We say that $u \equiv_f^m v$ holds if there exists a sequence of words $w_0 = u$, w_1, $\ldots$, $w_m = v$ such that, for every i, the word w_i is obtained from w_{i-1} by replacing exactly one factor $xf(x, y)$, x, y in S, with the corresponding word $yf(y, x)$. The *f-distance* $d_f(u, v)$ of u and v is the minimal number m such that $u \equiv_f^m v$ holds, if such a number exists, *i.e.*, if u and v are f-equivalent, and ∞ otherwise. We extend the definition with $d_f(u, \perp) = \infty$ and $d_f(\perp, \perp) = 0$.

Proposition 1.4. *Assume that f is a complement on the set S, and that u, v are words in S^* such that $u\backslash_f v$ exists. Then the equivalence*

$$u \vee_f v \equiv_f v \vee_f u \tag{1.4}$$

holds, and, more precisely, we have

$$d_f(u \vee_f v, v \vee_f u) \leq c_f(u,v). \tag{1.5}$$

Proof. We use induction on the integer $m = c_f(u,v)$. For $m = 0$, then u or v must be empty. In this case, we have $u\backslash_f v = v$, $v\backslash_f u = u$, and the result is true. Assume now $m \geq 1$. Then the words u and v are nonempty. Let us write $u = xu_0$, $v = yv_0$, with $x, y \in S$. By repeated uses of Lemma 1.3, we see that there exist words u_1, v_1, …, u_3, v_3 as represented on Figure 1.3. By construction, we have

$$c_f(u,v) = c_f(x,y) + c_f(u_0, f(x,y)) + c_f(f(y,x), v_0) + c_f(u_1, v_2). \tag{1.6}$$

As $c_f(y,x)$ is equal to 1, the three remaining f-complexities are strictly less than m, and the induction hypothesis applies to the corresponding words. In this way, we find

$$\begin{aligned} u(u\backslash_f v) = xu_0v_1v_3 &\equiv_f^{c_f(u_0,f(x,y))} xf(x,y)u_1v_3 \\ &\equiv_f^1 yf(y,x)u_1v_3 \\ &\equiv_f^{c_f(u_1,v_2)} yf(y,x)v_2u_3 \\ &\equiv_f^{c_f(f(y,x),v_0)} yv_0u_2u_3 = v(u\backslash_f v), \end{aligned}$$

which, by (1.6), gives the desired result ■

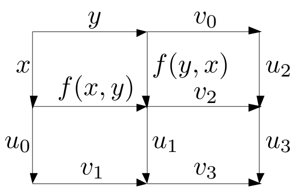


Figure 1.3. Equivalence in reversing

We immediately deduce:

Proposition 1.5. *Assume that f is a complement on the set S, and that u, v are words in S^* such that $u^{-1}v$ is f-reversible to ε. Then u and v are f-equivalent, and $d_f(u,v) \leq c_f(u,v)$ holds.*

As was mentioned in the introduction, we wish to study the possible converse implication.

Definition. Assume that f is a complement on S. We say that word reversing is *complete for f* if the word $u^{-1}v$ is f-reversible to ε whenever u and v are f-equivalent words in S^*.

We can easily see that word reversing need not be complete for every complement.

Example 1.6. Let M be the monoid with presentation

$$\langle a, b, c \,;\; a = b^2a, ac = c, bc = c\rangle.$$

The previous presentation is associated with the complement f defined by

$$f(a,b) = f(a,c) = f(b,c) = \varepsilon, \quad f(b,a) = ba, \quad f(c,a) = f(c,b) = c.$$

Now the words ac and bac both are f-equivalent to c: we have $ac \equiv_f^1 c \equiv_f^1 bc \equiv_f^1 bac$, so the f-distance of ac and bac is (at most) 3. On the other hand, the f-reversing of the word $(ac)^{-1}(bac)$ does not lead to the empty word. Indeed, we find

$$(ac)^{-1}(bac) \curvearrowright_f (bac)^{-1}(ac) \curvearrowright_f (ac)^{-1}(bac) \curvearrowright_f \dots$$

and there exists an infinite f-reversing sequence from w—one that is periodic with period 2. So word reversing is not complete for the current complement f. ■

So, we have to introduce additional hypotheses. The first step in this direction is to consider a new property of complement called *coherence* that we shall prove is equivalent to completeness, and that will turn to be more easy to work with than completeness.

Definition. Assume that f is a complement on the set S. We say that f is *coherent* if the mapping $\backslash_f$ is compatible with the congruence $\equiv_f$, *i.e.*, if $u' \equiv_f u$ and $v' \equiv_f v$ imply $u'\backslash_f v' \equiv_f u\backslash_f v$ for all u, v, u', v' in S^*.

Proposition 1.7. *Assume that f is a complement on S. Then the following are equivalent:*
(i) The complement f is coherent;
(ii) Word reversing is complete for f.

Proof. It is obvious that (i) implies (ii). Indeed, assume that u and v are f-equivalent words. By construction, the word $u^{-1}u$ is f-reversible to the empty word (in exactly n steps if u has n letters). Thus the word $u\backslash_f u$ exists and it is empty. If f is coherent, the hypothesis that v is f-equivalent to u implies that the words $u\backslash_f v$ and $v\backslash_f u$ exist, and that they are f-equivalent to ε. By construction of $\equiv_f$, this is possible only if these words are the empty word, *i.e.*, if $u^{-1}v$ is f-reversible to ε.

Let us now assume that word reversing is complete for f. By symmetry and transitivity, it suffices that we prove the result assuming $u' = u$. So, we assume that u, v, v' belong to S^*, that $u\backslash_f v$ and $v\backslash_f u$ exist, and that v' is f-equivalent to v. By (1.4), the words $u \vee_f v$ and $v \vee_f u$ are f-equivalent. Since v' is f-equivalent to v, we deduce that $u\,(u\backslash_f v)$ is f-equivalent to $v'\,(v\backslash_f u)$. By completeness of word reversing for f, this implies

$$(u\,(u\backslash_f v))\backslash_f(v'\,(v\backslash_f u)) = \varepsilon,$$

which, by Lemma 1.3, implies *a fortiori* $(u\,(u\backslash_f v))\backslash_f v' = \varepsilon$. Using Lemma 1.3 again, this evaluates into

$$(u\backslash_f v)\backslash_f(u\backslash_f v') = \varepsilon.$$

This proves that $u\backslash_f v'$, and, therefore, $v'\backslash_f u$, exist, and that there exists a word w, namely $(u\backslash_f v')\backslash_f(u\backslash_f v)$, such that $u\backslash_f v$ is f-equivalent to $(u\backslash_f v')\, w$. A symmetric argument shows that there must exist a word w' in S^* such that $u\backslash_f v'$ is f-equivalent to $(u\backslash_f v)\, w'$. We deduce that $u\backslash_f v$ is f-equivalent to $(u\backslash_f v)\, w'\, w$, which, by completeness, implies that the word

$$(u\backslash_f v)^{-1}(u\backslash_f v)\, w'\, w$$

is f-reversible to ε. Now, $(u\backslash_f v)^{-1}(u\backslash_f v)$ is f-reversible to ε, so the word $w'w$, which belongs to S^*, is f-reversible to ε: this is possible only if ww' is the empty word ε. So we conclude that $u\backslash_f v'$ and $u\backslash_f v$ are f-equivalent.

Similar computations show that $(v(v'\backslash_f u))\backslash_f u$ exists and is empty, which implies that $v'\backslash_f u$ exists and that $v\backslash_f u \equiv_f v'\backslash_f u\; w$ holds for some positive word w. As above we show that w must be empty, and we conclude that $v\backslash_f u$ and $v'\backslash_f u$ are f-equivalent. ∎

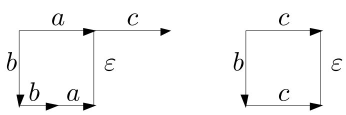
2. Noetherian complements

After the previous proposition, we are left with the question of recognizing whether a given complement is coherent. This property involves arbitrary words, and, therefore, even if the domain S is finite, there is no systematic way of proving coherence. Now, we can restrict to some special occurrences of coherence.

Definition. Assume that f is a complement on the set S. We say that f is *locally coherent* if, for every triple x, y, z in S, we have $z\backslash_f(x \vee_f y) \equiv_f z\backslash_f(y \vee_f x)$ and $(x \vee_f y)\backslash_f z \equiv_f (y \vee_f x)\backslash_f z$.

(Again, the previous equivalences mean either that both words exist and are equivalent, or that none of them exists). Local coherence is a special case of coherence: indeed, the words $x \vee_f y$ and $y \vee_f x$ are f-equivalent whenever $f(x, y)$ is defined. Actually, it involves exactly those minimal cases where a coherence phenomenon may happen. Observe that, if S is finite and f is a locally coherent complement on S, then this local coherence can be checked effectively in a finite number of elementary steps.

Example 2.1. The complement of Example 1.6 is locally coherent. Let us write $C(x, y, z)$ for the local coherence condition that corresponds to the triple of letters (x, y, z). It is easily checked that $C(x, y, z)$ holds whenever at least two letters coincide (*cf.* Lemma 3.1 below). On the other hand, $C(x, y, z)$ and $C(y, x, z)$ are equivalent by construction. So, in order to prove that the current complement f is locally coherent, it suffices that we establish the three conditions $C(a, b, c)$, $C(b, c, a)$ and $C(c, a, b)$. For instance, establishing $C(c, a, b)$ amounts to comparing the f-reversing diagrams



As $bac \equiv_f c$ and $\varepsilon \equiv_f \varepsilon$ hold, the condition is true.

So we are left with the question of whether local coherence implies full coherence. We have seen that the complement of Example 1.6 is not coherent, so the answer cannot be positive in general, and new additional hypotheses are needed. In [8], a variant of the coherence property is proved under the hypothesis that there exists a mapping ν of S^* to the integers that is compatible with $\equiv_f$, takes the value 1 on the elements of S and satisfies $\nu(uv) \geq \nu(u)+\nu(v)$. These hypotheses are misleading in at least two points. On the one hand, the hypothesis $\nu(x) = 1$ for $x \in S$ dismisses all presentations that contain a relation of the form $x = yz$. Such presentations are natural (consider for instance the presentations of braid monoids in terms of permutation braids), and there is no reason to discard them here. On the other hand, considering a function with integer values dismisses lots of monoids that we shall see are eligible.

As the subsequent proof will show, the point is the existence of a convenient induction parameter guaranteeing that f-reversing processes proceed forwards. This is a *well-foundedness* assumption, and, therefore, the natural parameter occurring in the general case is an *ordinal* number rather than a natural number.

Definition. Assume that M is a monoid. We say that M is *right Noetherian* is the right divisibility relation of M has no infinite descending sequence, *i.e.*, there exists no infinite sequence $\ldots a_2 \mid_R a_1 \mid_R a_0$ in M, where $b \mid_R a$ means that a is xb for some $x \neq 1$.

By standard arguments of elementary set theory (see for instance [16]), a monoid M is right Noetherian if and only if there exists a mapping ρ of M into the ordinals such that $b \mid_R a$ implies $\rho(b) < \rho(a)$. In this case, there exists a minimal function ρ with the previous property, namely the function defined by

$$\rho(a) = \begin{cases} 0 & \text{if } a \text{ is } 1, \\ \sup\{\rho(b)+1 \ ; \ b \mid_R a\} & \text{otherwise.} \end{cases}$$

This particular function ρ will be called the *rank* function of M.

Lemma 2.2. *Assume that M is a right Noetherian monoid, and ρ is its rank function. Then the equality*

$$\rho(a) = \sup\{\rho(a_n) + \ldots + \rho(a_1) \ ; \ a = a_1 \ldots a_n\} \tag{2.1}$$

holds for every $a \neq 1$ in M.

Proof. First we prove the inequality $\rho(ac) \geq \rho(c) + \rho(a)$ using induction on $\rho(c)$. The inequality is true when $\rho(c)$ is 0, *i.e.*, when c is 1. Otherwise, we notice that $b \mid_R a$ implies $bc \mid_R ac$. Using the induction hypothesis, we find

$$\begin{aligned} \rho(ac) &\geq \sup\{\rho(bc)+1 \ ; \ b \mid_R a\} \\ &\geq \sup\{\rho(c)+\rho(b)+1 \ ; \ b \mid_R a\} \\ &= \rho(c) + \sup\{\rho(b)+1 \ ; \ b \mid_R a\} = \rho(c)+\rho(a) \end{aligned}$$

From the previous argument, we deduce that the left hand side of (2.1) is always at least equal to its right hand side. On the other hand, $a = a$ is one of the finite decompositions mentioned in the right hand side of (2.1), so $\rho(a)$ is at most equal to the ordinal occurring there. ∎

In the particular case of those monoids that admit a complemented presentation, a mere translation gives the following criterion.

Lemma 2.3. *Assume that f is a complement on S. Then the following are equivalent:*
(i) The monoid $\langle S\,;\, R_f\rangle$ is right Noetherian;
(ii) There exists a mapping ρ of S^* into the ordinals such that
- $\rho(xf(x,y))$ and $\rho(yf(y,x))$ are equal for all x, y in S
- $\rho(u)=\rho(v)$ implies $\rho(xu)=\rho(xv)$ and $\rho(ux)=\rho(vx)$ for every x in S;
- $\rho(xu)>\rho(u)$ holds for every x in S and every u in S^*.

Definition. Assume that f is a complement on S. We say that f is *Noetherian* if it satisfies the previous conditions. In this case, we denote by ρ_f the function of S^* into the ordinals that maps every word u to the rank of the class of u in $\langle S\,;\, R_f\rangle$. The *height* of f is defined to be the supremum of the ordinals $\rho_f(u)$ for u in S^*.

Notice that, if f is a Noetherian complement, then, by construction and by Lemma 2.2, the inequalities

$$\rho_f(uv)\geq\rho_f(u),\quad\text{and}\quad\rho_f(uv)\geq\rho_f(v)$$

hold for all words u, v, and the inequality $\rho_f(uv)>\rho_f(v)$ holds whenever u is not empty. In particular, $\rho_f(u)$ is not 0 if u is not empty.

Simple examples of Noetherian complements appear when the defining relations of the monoid preserve the length of the word—as in the case of Artin monoids. In this case, $\rho_f(u)$ is merely the length of u, and the height of f is ω, the smallest infinite ordinal. However, a typical case where the monoid $\langle S\,;\, R_f\rangle$ is right Noetherian although there is no rank function with integer values is the monoid $\langle a,b\,;\, a=ab\rangle$. The associated rank function is determined by $\rho(uv)=\rho(v)+\rho(u)$, $\rho(a)=\omega$, $\rho(b)=1$, so, for instance, we have

$$\rho(ab)=\rho(b)+\rho(a)=1+\omega=\omega=\rho(a),\quad \rho(ba)=\rho(a)+\rho(b)=\omega+1>\omega=\rho(a).$$

As it stands, the above presentation is not associated with a complement, but $\langle a,b,c,d\,;\, a=cb, c=d, a=d\rangle$ is another presentation of the same monoid that is complemented. The height of the considered complement is the ordinal ω^2.

We prove now the rather natural result that Noetherianity is a sufficient additional condition for deducing full coherence—and, therefore, completeness—from local coherence. In order to give the proof, we introduce a new ordinal parameter that measures the distance between words.

Definition. For u, v in S^*, we define the parameter $e_f(u,v)$ to be $\rho_f(u\vee_f v)$ if $v\backslash_f u$ exists, and ∞ otherwise.

Proposition 2.4. *Assume that f is a locally coherent and Noetherian complement on S. Then f is coherent.*

Proof. We prove inductively on the ordinal $\alpha = e_f(u, v)$ that, if u, v are words in S^* such that $u\backslash_f v$ exists, then, if u', v' are f-equivalent respectively to u, v, then $u'\backslash_f v'$ and $v'\backslash_f u'$ exist and these words are f-equivalent respectively to $u\backslash_f v$ and to $v\backslash_f u$.

For $\alpha = 0$, the only possibility is $u = v = u' = v' = \varepsilon$, and everything is obvious.

Assume now $\alpha > 0$, and use induction on the parameter $m = d_f(u', u) + d_f(v', v)$. For $m = 0$, we have $u' = u$, and $v' = v$, so the result is obvious. Let us now consider the case $m = 1$. Without loss of generality, we may assume $u' = u$ and $v' \equiv_f^1 v$. This means that there exist words v_0, v_1 and letters x, y satisfying $v = v_0 x f(x, y) v_1$ and $v' = v_0 y f(y, x) v_1$. Consider the word $v_0\backslash_f u$, which exists as $v\backslash_f u$ is supposed to exist. If this word is empty, everything is clear, as we have $u'\backslash_f v' = (u\backslash_f v_0) x f(x, y) v_1$, and $u\backslash_f v = (u\backslash_f v_0) y f(y, x) v_1$, an f-equivalent word, and both $v'\backslash_f u'$ and $v\backslash_f u$ are empty. So assume $v_0\backslash_f u = z u_1$, where z belongs to S—see Figure 2.1. Let u_2 and v_2 be respectively the words $(x \vee_f y)\backslash_f z$ and $z\backslash_f (x \vee_f y)$, which exist because $v\backslash_f u$ exists by hypothesis. Similarly, let u_2' and v_2' be respectively the words $(y \vee_f x)\backslash_f z$ and $z\backslash_f (y \vee_f x)$. The hypothesis that f is locally coherent implies that the latter words exist and that $u_2' \equiv_f u_2$ and $v_2' \equiv_f v_2$ hold. Introduce now the words u_3, v_3, u_4, v_4, and u_5, v_5 that appear in the f-reversing of $u^{-1}v$ as shown on Figure 2.1. For instance u_3 is $v_2\backslash_f u_1$, while v_3 is $u_1\backslash_f v_2$. By construction, we have

$$e_f(u_1, v_2) = \rho_f(u_1 v_3) \leq \rho_f(u_1 v_3 v_5) < \rho_f(z u_1 v_3 v_5) \leq \rho_f(v_0 z u_1 v_3 v_5) = \rho_f(u \vee_f v) = \alpha.$$

Hence the induction hypothesis applies to the pairs (u_1, v_2) and (u_1, v_2'), so we deduce that the word $v_2'\backslash_f u_1$, which we shall call u_3', exists and is f-equivalent to $v_2\backslash_f u_1$, *i.e.*, to u_3. A similar argument shows that the word $v_3' = u_1\backslash_f v_2'$ exists and is f-equivalent to v_3.

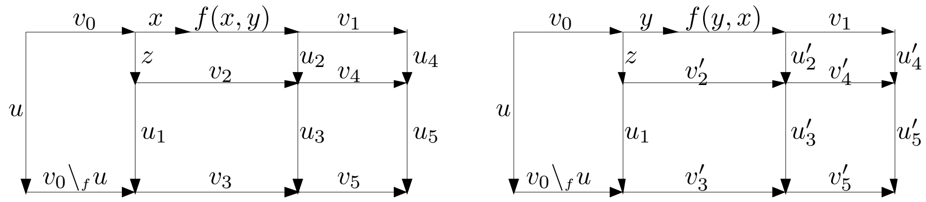


Figure 2.1. Proof of coherence

The same argument again shows that, with obvious notations, the words u_4' and v_4' exist and are respectively f-equivalent to u_4 and v_4. Finally, the argument is the same for u_5' and v_5'. So, we conclude that $u'\backslash_f v'$ exists and it is equal to $(u\backslash_f v_0) v_3' v_5'$, hence f-equivalent to $u\backslash_f v$, which is $(u\backslash_f v_0) v_3 v_5$, while $v'\backslash_f u'$ is equal to $u_4' u_5'$, hence it is f-equivalent to $v\backslash_f u$, which is $u_4 u_5$.

It remains to consider the case $m \geq 2$. In that case, there exists an intermediate pair of words say (u'', v'') satisfying

$$d_f(u'', u) + d_f(v'', v) < m \quad \text{and} \quad d_f(u', u'') + d_f(v', v'') < m.$$

Applying the induction hypothesis to (u, v) and (u'', v''), we deduce that $u''\backslash_f v''$ and $v''\backslash_f u''$ exist, and that they are f-equivalent respectively to $u\backslash_f v$ and $v\backslash_f u$. This implies $\rho_f(u'' \vee_f v'') = \rho_f(u \vee_f v)$, *i.e.*, $e_f(u'', v'') = \alpha$. So, we can in turn apply the induction hypothesis to the pairs (u'', v'') and (u', v'), and conclude that $u'\backslash_f v'$ and $v'\backslash_f u'$ exist and that they are f-equivalent respectively to $u\backslash_f v$ and $v\backslash_f u$. ■

Corollary 2.5. *Assume that f is a locally coherent and Noetherian complement on S. Then word reversing is complete for f.*

3. Effective bounds

We come back now to the previous results from a combinatorial point of view, and investigate the complexity of word reversing. In good cases, we obtain effective upper bounds on the number of reversing steps needed to compare equivalent words.

Our aim is to give an effective version of the coherence property where we measure the distance of $u'\backslash_f v'$ and $u\backslash_f v$ in terms of the distances $d_f(u', u)$ and $d_f(v', v)$. Our result will not apply to every locally coherent Noetherian complement f, but only to those that admit a rank function with finite (= integer) values, *i.e.*, in the case where the height of f is ω. By the results of [13], this happens if and only if, for every word u, the lengths of those words that are f-equivalent to u have a finite upper bound, and, in this case, $\rho_f(u)$ is equal to the latter upper bound.

Definition. Assume that f is a complement on S. The *degree* of f is the maximum of the quantities

$$d_f(z\backslash_f(x\vee_f y), z\backslash_f(y\vee_f x)) + d_f((x\vee_f y)\backslash_f z, (y\vee_f x)\backslash_f z) \tag{3.1}$$

for x, y, z in S. Similarly, the *deviation* of f is the maximum of the differences

$$|c_f(z, y\vee_f x) - c_f(z, x\vee_f y)|$$

for x, y, z in S, with the conventions $\infty - \infty = 0$, and $\infty - n = |n - \infty| = \infty$ for every integer n.

The complement f is locally coherent if and only if the quantity (3.1) is finite for every triple (x, y, z). Hence every complement with a finite degree is locally coherent, and, conversely, if S is a finite set and f is locally coherent, then f has a finite degree. In the latter case, it has also a finite deviation.

As an example, and for future use, we begin with an easy particular case.

Lemma 3.1. *Assume that f is a complement on the set $\{a, b\}$ such that $f(a, b)$ is defined. Then f has degree 0, and its deviation is the maximum of $\lg(f(a, b))$ and $\lg(f(b, a))$.*

Proof. Assume $S = \{a, b\}$. Due to the symmetries, it suffices that we consider the triple (a, b, a). As is clear on Figure 3.1, we have

$$a\backslash_f(a\vee_f b) = f(a, b) = a\backslash_f(b\vee_f a), \quad (a\vee_f b)\backslash_f a = \varepsilon = (b\vee_f a)\backslash_f a,$$

and

$$c_f(a, a\vee_f b) = 1, \quad c_f(a, b\vee_f a) = 1 + \lg(f(b, a)).$$ ∎

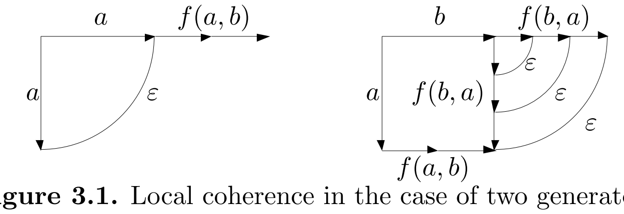


Figure 3.1. Local coherence in the case of two generators

As an additional exemple, let us consider again the complement of Example 1.6. By Lemma 3.1, those triples (x, y, z) where at least two letters coincide contribute 0 to the degree, and 2 to the deviation. Then considering the three remaining cases as displayed in Figure 3.2 shows that the degree and the deviation of f both are equal to 2.

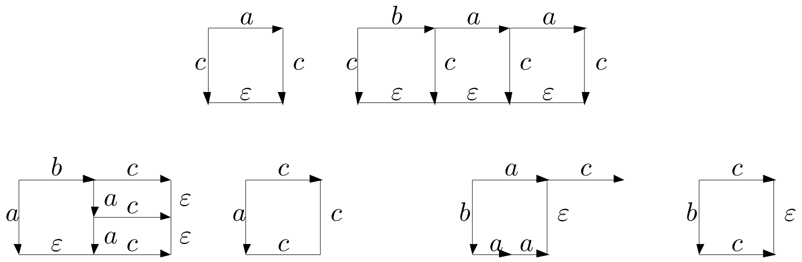


Figure 3.2. Local coherence of Example 1.6

Proposition 3.2. *Assume that f is a complement on S with degree k, height ω, and deviation δ. Let u, v, u', v' be words in S^* such that $u\backslash_f v$ exists, u' is f-equivalent to u and v' is f-equivalent to v. Then $u'\backslash_f v'$ exists as well, and we have*

$$d_f(u'\backslash_f v', u\backslash_f v) + d_f(v'\backslash_f u', v\backslash_f u) \leq F_k(e_f(u, v)) \cdot (d_f(u', u) + d_f(v', v)), \tag{3.2}$$

and

$$c_f(u', v') \leq c_f(u, v) + \delta \cdot G_k(e_f(u, v)) \cdot (d_f(u', u) + d_f(v', v)), \tag{3.3}$$

where $F_k(x)$ denotes $k^{2^{x-1}-1}$ and $G_k(x)$ denotes $(2^x - 1)k^{2^{x-1}}$.

(Formulas (3.2) and (3.3) hold for all words u, v, u', v' provided we use $\perp$ and ∞ when the words are not equivalent or do not exist.)

Proof. We first prove Formula (3.2) by following step by step the proof of Proposition 2.4. So we argue inductively on $r = e_f(u, v)$—which is here an integer—and, for a given value of r, we argue inductively on $m = d_f(u', u) + d_f(v', v)$.

The first case to consider is $r > 0$, $m = 1$. We follow the notations of Figure 2.1. By definition, we have

$$d_f(u'_2, u_2) + d_f(v'_2, v_2) \leq k. \tag{3.4}$$

Now, $e_f(u_1, v_2)$, which is the f-rank of u_1v_3, *i.e.*, of $u_1 \vee_f v_2$, is at most $r-1$. Thus the induction hypothesis implies

$$d_f(u'_3, u_3) + d_f(v'_3, v_3) \leq F_k(r-1) \cdot d_f(v'_2, v_2). \tag{3.5}$$

Similarly, $e_f(u_2, v_1)$, which is the f-rank of u_2v_4, is at most $r-1$. The induction hypothesis implies

$$d_f(u'_4, u_4) + d_f(v'_4, v_4) \leq F_k(r-1) \cdot d_f(u'_2, u_2). \tag{3.6}$$

Finally, $e_f(u_3, v_4)$, which is the f-rank of u_3v_5, is also at most $r-1$. The induction hypothesis implies

$$d_f(u'_5, u_5) + d_f(v'_5, v_5) \leq F_k(r-1) \cdot (d_f(u'_3, u_3) + d_f(u'_4, u_4)).$$

By summing up the above inequalities, we obtain

$$d_f(u'\backslash_f v', u\backslash_f v) + d_f(v'\backslash_f u', v\backslash_f u) \leq k \cdot F_k(r-1)^2 = F_k(r).$$

For $m \geq 2$, we introduce an intermediate pair (u'', v'') between (u, v) and (u', v'), and apply the induction hypothesis for u, v, u'', v'' and u'', v'', u', v', which makes sense as, always by induction hypothesis, $u'' \vee_f v''$ is f-equivalent to $u \vee_f v$, so it has the same f-rank.

Formula (3.3) is then proved similarly. Again, it suffices to consider the case $u' = u$, $v' = v_0 y f(y, x) v_1$ where v is $v_0 x f(x, y) v_1$. With the notations of Figure 2.1, we have

$$c_f(u, v) = c_f(u, v_0) + c_f(z, x \vee_f y) + c_f(u_1, v_2) + c_f(u_2, v_1) + c_f(u_3, v_4)$$
$$c_f(u', v') = c_f(u, v_0) + c_f(z, y \vee_f x) + c_f(u_1, v_2') + c_f(u_2', v_1) + c_f(u_3', v_4')$$

By definition of deviation, we have

$$c_f(z, y \vee_f x) \leq c_f(z, x \vee_f y) + \delta.$$

By induction hypothesis, using the fact that the values of $e_f(u_1, v_2)$, $e_f(u_2, v_1)$ and $e_f(u_3, v_4)$ are at most $r - 1$, we have

$$c_f(u_1, v_2') \leq c_f(u_1, v_2) + \delta\, G_k(r-1)\, d_f(v_2', v_2),$$
$$c_f(u_2', v_1) \leq c_f(u_2, v_1) + \delta\, G_k(r-1)\, d_f(u_2', u_2),$$
$$c_f(u_3', v_4') \leq c_f(u_3, v_4) + \delta\, G_k(r-1)\, (d_f(u_3', u_3) + d_f(v_4', v_4)).$$

By definition, $d_f(u_2', u_2) + d_f(v_2', v_2)$ is bounded above by k. Using Formulas (3.4), (3.5) and (3.6), we see that the number $d_f(u_3', u_3) + d_f(v_4', v_4)$ is bounded above by $kF_k(r-1)$. By summing up, we obtain

$$c_f(u', v') \leq c_f(u, v) + \delta\, (1 + kG_k(r-1) + kF_k(r-1)G_k(r-1)),$$

and the last factor is bounded above by $G_k(r)$. Finally, the induction on m is straightforward using as above an intermediate pair (u'', v'') for $m \geq 2$. ∎

Corollary 3.3. *(i) Assume that f is a complement on S with degree k at least equal to* 1*, height ω, and deviation δ. Assume that u is a word in S^* with length ℓ and f-rank r, and that v is f-equivalent to u. Then the word $u^{-1}v$ is f-reversible to the empty word, and we have*

$$d_f(u, v) \leq c_f(u, v) \leq \ell + \delta \cdot G_k(r) \cdot d_f(u, v). \tag{3.7}$$

(ii) Assume that f is a complement on S with degree k at least equal to 1*, and deviation δ. Assume in addition that f has the property that the words $f(y, x)$ and $f(x, y)$ have the same length when they exist. Assume that u is a word in S^* with length ℓ, and that v is f-equivalent to u. Then the word $u^{-1}v$ is f-reversible to the empty word, and we have*

$$d_f(u, v) \leq c_f(u, v) \leq \ell + \delta \cdot G_k(\ell) \cdot d_f(u, v). \tag{3.8}$$

Proof. For (i), by construction, the complexity $c_f(u, u)$ is equal to ℓ. We then apply Formula (3.3). for the upper bound, and Formula (1.5) for the lower bound. For (ii), the additional hypothesis implies that f is Noetherian and that the f-rank of a word is its length. We then apply (i). ∎

(The initial factor ℓ could disappear provided we modify the definition of complexity so as to take into account only those reversing steps that involve a factor $x^{-1}y$ with $x \neq y$.) If the set S is finite with n elements and f is a Noetherian complement on S, then every word that is f-equivalent to a word u of f-rank r has length at most r, so there are at most n^r such words, and the f-distance between u and such a word is bounded above by n^r. Hence, in this case, Formula (3.7) gives $\ell + \delta\, G_k(r)\, n^r$ as a uniform upper bound for $c_f(u, v)$ when v is f-equivalent to u. Observe that the existence of this bound gives a solution to the word problem of $\langle S\,;\, R_f\rangle$ only if the f-rank function is itself recursive—however, in this case, the word problem can be solved by using a systematic enumeration.

Remark. Even in particular cases, the formulas we have established above give no upper bound for the parameter $c_f(u, v)$ in terms of the lengths of u and v when v is not supposed to be f-equivalent to u. Such a bound cannot exist in general, since we do not assume that the complement is *convergent*, this meaning that every word reversing always terminates within a finite number of steps. However, even if we restrict to convergent complements, no upper bound is known. The Baumslag–Solitar monoid $\langle a, b\,;\, ab = ba^2\rangle$ of [15] gives an easy example where $c_f(u, v)$ can be exponential in the lengths if u and v: $c_f(a, b^n)$ is $2^n - 1$. Let us mention that the complement involved in [7] is Noetherian, coherent and convergent, and the only upper bound on $c_f(u, v)$ we know is a tower of exponentials with exponential height in lengths of u and v. This bound is presumably not optimal, but no fixed iterated exponential seems likely to be an upper bound. Such high bounds about values of $c_f(u, v)$ for arbitrary pairs of words (u, v) are compatible with the current rather low bounds that we have established here for those pairs (u, v) where v is equivalent to u: the words $v\backslash_f u$ and $u\backslash_f v$ may be very long if u and v are not equivalent,.

Let us briefly mention an important special case for which a whole theory can be developed, namely when the length of all words $f(y, x)$ with x, y in S is bounded above by 1, *i.e.*, the complement is either empty, or it is a single letter. In this case, drawing the reversing diagram makes it clear that, for *every* pair of words (u, v), the f-complexity $c_f(u, v)$ is bounded above by $\lg(u)\lg(v)$ when it is finite. In particular, the complement is convergent if and only if $f(y, x)$ exists for every pair (x, y) in S^2. From these remarks, we deduce:

Proposition 3.4. *Assume that f is an everywhere defined coherent complement on S, and there exists a finite set of words S' that includes S and is closed under $\backslash_f$, i.e., $u\backslash_f v$ belongs to S' when u and v do. Then there exists a constant C such that the inequality*

$$c_f(u, v) \leq C \lg(u) \lg(v) \tag{3.9}$$

holds for all words u, v in S^*.

Proof. Assume that u and v are words in S^* such that u can be decomposed into the product of p words in S' and v can be decomposed into the product of q words in S'. Using Lemma 1.2 and the previous argument, we see that the f-reversing of a word $u^{-1}v$ can be decomposed into pq elementary reversings of words of the form $u'^{-1}v'$ with u', v' in S. Then define C to be the supremum of the integers $c_f(u', v')$ for u', v' in S'. ∎

Under the previous hypotheses, we deduce from (1.5) the bound

$$d_f(u,v) \le C\,\lg(u)\,\lg(v) \tag{3.10}$$

when u and v are f-equivalent, which shows that the monoid $\langle S\,;\,R_f\rangle$ satisfies a quadratic isoperimetric inequality. We refer to [13] and [12] for the further study of this important special case, which includes in particular all Artin monoids associated with finite Coxeter groups.

Finally, let us mention that, even in the simple case of the braid monoids (which are associated with a complement of degree 4 and deviation 2), many simple questions about the complexity of word reversing remain unsolved. For instance, we have no upper bound on $c_f(u,v)$ in terms of the lengths of u and v in the non finitely generated case of the monoid B_∞^+: in this case, there exist sequences of pairs of words (u_ℓ, v_ℓ) of length ℓ such that $c_f(u_\ell, v_\ell)$ grows like ℓ^3, so (3.9) is false, and no upper bound is known.

4. Low degrees

Proposition 2.4 applies to every Noetherian complement of degree $k \ge 1$, so, in particular it applies in the case of degree 1, and we can see that Inequalities (3.2) and (3.3) take in this case a more simple form: $F_1(x)$ is 1, and $G_1(x)$ is 2^x-1. In particular, the rank no longer occurs in Formula (3.2). This suggests that the Noetherianity hypothesis is perhaps not needed in this case. This turns out to be true.

Proposition 4.1. *Assume that f is a complement on S with degree at most 1. Then f is coherent, and word reversing is complete for f.*

More precisely, let u, v, u', v' be words in S^* such that $u\backslash_f v$ exists, u' is f-equivalent to u and v' is f-equivalent to v. Then $u'\backslash_f v'$ exists as well, and we have

$$d_f(u'\backslash_f v', u\backslash_f v) + d_f(v'\backslash_f u', v\backslash_f u) \le d_f(u', u) + d_f(v', v), \tag{4.1}$$

and

$$c_f(u', v') \le c_f(u,v) + \delta\cdot(2^{c_f(u,v)}-1)\cdot(d_f(u',u) + d_f(v',v)) \tag{4.2}$$

where δ is the deviation of f.

Proof. We go back to the proof of Proposition 2.4, but reverse the order of the inductions, and use $c_f(u,v)$ instead of $e_f(u,v)$, which is possible as the successive new words that appear while reversing $u^{-1}v'$ remain always at distance at most 1 from the corresponding words that appear while reversing $u^{-1}v$.

So, the main induction involves $m = d_f(u', v') + d_f(u,v)$. As always, the case $m = 0$ is trivial, so we consider the case $m = 1$. Everything is obvious for $n = 0$. Otherwise, we are again in the situation of Figure 2.1, of which we use the notations once more. By hypothesis, we have $d_f(u_2', u_2) + d_f(v_2', v_2) \le 1$. So at most one of $u_2' \ne u_2$, $v_2' \ne v_2$ may hold. Assume for instance $v_2' = v_2$. Then we have $u_3' = u_3$ and $v_3' = v_3$. By construction, we have $d_f(u_2', u_2) \le 1$, and $c_f(u_2, v_1) < n$. So, by induction hypothesis, we deduce

$$d_f(u_4', u_4) + d_f(v_4', v_4) \le 1, \quad \text{and} \quad c_f(u_2', v_1) \le c_f(u_2, v_1) + \delta\cdot(2^{n-1}-1).$$

Two cases are possible. Assume first $d_f(u_4', u_4) = 1$. Then we have $v_4' = v_4$, $u_5' = u_5$, and $v_5' = v_5$. So $u\backslash_f v'$ and $v'\backslash_f u$ exist, $d_f(u\backslash_f v', u\backslash_f v) + d_f(v'\backslash_f u, v\backslash_f u)$ is 1, and, by summing up the terms, we find

$$c_f(u, v') \leq n + \delta + \delta \cdot (2^{n-1} - 1) \leq n + \delta \cdot (2^n - 1).$$

Assume now $d_f(u'_4, u_4) = 0$, *i.e.*, $u'_4 = u_4$. Then we have $d_f(v'_4, v_4) \leq 1$, and we are again in the same induction position for the pairs (u_3, v_4) and (u_3, v'_4). So we deduce as above

$$d_f(u'_5, u_5) + d_f(v'_5, v_5) \leq 1, \quad \text{and} \quad c_f(u'_3, v_4) \leq c_f(u_3, v_4) + \delta \cdot (2^{n-1} - 1).$$

Again, we conclude that $u\backslash_f v'$ and $v'\backslash_f u$ exist, we find

$$d_f(u\backslash_f v', u\backslash_f v) + d_f(v'\backslash_f u, v\backslash_f u = 1,$$

and, by summing up the terms,

$$c_f(u, v') \leq n + \delta + \delta \cdot (2^{n-1} - 1) + \delta \cdot (2^{n-1} - 1) = n + \delta \cdot (2^n - 1).$$

Finally, the induction on m for $m \geq 2$ is straightforward. ■

Corollary 4.2. *Assume that f is a complement on S with degree at most* 1, *and deviation δ. Assume that u is a word in S^* with length ℓ, and that v is f-equivalent to u. Then the word $u^{-1}v$ is f-reversible to the empty word, and we have*

$$d_f(u, v) \leq c_f(u, v) \leq \ell + \delta \cdot (2^\ell - 1) \cdot d_f(u, v).$$

The previous results are optimal. Indeed, we have seen that the complement of Example 1.6 has degree 2, so it is locally coherent, but it is not coherent: indeed, if we consider

$$u = abc, \quad v = c, \quad v' = bc,$$

we see that v and v' are f-equivalent words at distance 1, and $u\backslash_f v$ exists, while $u\backslash_f v'$ does not. The failure of coherence for the complement f implies that word reversing is not complete for f: we have already seen that the words abc and ac are f-equivalent at distance 3, but the reversing of the word $(abc)^{-1}(ac)$ never terminates.

In the previous counter-example, the lack of completeness for word reversing involves a word that is not reversible. One could suspect that the failure of completeness originates from the lack of convergence, *i.e.*, the existence of words whose f-reversing never terminates. Actually, this is *not* the case: we give below the example of a locally coherent complement of degree 3 that is convergent but not coherent—let us mention that we could not find any degree 2 example.

Example 4.3. Let M be the monoid with presentation

$$\langle a, b, c \ ; \ a = bbc, a = ca, ba = c \rangle.$$

This monoid is associated with the complement f defined by

$$f(a, b) = \varepsilon, \ f(b, a) = bc, \ f(c, a) = a, \ f(a, c) = f(c, b) = \varepsilon, \ f(b, c) = a.$$

Here f is locally coherent of degree 3, and the set $S' = \{a, b, c, bc, \varepsilon\}$ is closed under $\backslash_f$. So f-reversing always terminates, and f is convergent. We have $c \equiv_f baa \equiv_f bbbc$, hence f is not Noetherian. Now, the reader can check that we have

$$a\backslash_f bc = a, \quad bc\backslash_f a = \varepsilon, \quad \text{and} \quad ca\backslash_f bc = \varepsilon, \quad bc\backslash_f ca = a^2,$$

so f is not coherent. An example witnessing non-completeness is (bba, caa): these words are f-equivalent with distance 3, as we have $bba \equiv_f^1 bbca \equiv_f^1 aa \equiv_f^1 caa$. Now the word $(bba)^{-1}(caa)$ is f-reversible to aaa, and not to ε.

We finish this section with the special case of degree 0. By Proposition 4.1, every degree 0 complement is coherent, and we have upper bounds on the corresponding equivalences. However, we can still lower the bounds of Proposition 4.1 in this case.

Proposition 4.4. *Assume that f is a complement on S with degree 0 and deviation δ. Let u, v, u', v' be words in S^* such that $u\backslash_f v$ exists, u' is f-equivalent to u and v' is f-equivalent to v. Then we have*

$$c_f(u', v') \leq c_f(u, v) + \delta \cdot (d_f(u', u) + d_f(v', v)).$$

Proof. We follow the same scheme as in the case of degree 1. Again we use induction on $m = d_f(u', u) + d_f(v', v)$. Let us consider the case $m = 1$, with $u' = u$ and $d_f(v', v) = 1$. We use induction on $n = c_f(u, v)$. The case $n = 0$ is trivial, so we assume $n > 0$. We follow once more the notations of Figure 2.1. Now, in the current case, we must have $u_2' = u_2$ and $v_2' = v_2$, and we deduce $u\backslash_f v' = u\backslash_f v$. Moreover, the only difference between $c_f(u, v')$ and $c_f(u, v)$ comes from the initial reversings of $z^{-1}(x \vee_f y)$ and $z^{-1}(y \vee_f x)$, so it is bounded above by δ, and we obtain

$$c_f(u', v') \leq n + \delta.$$

The induction on m is straightforward. ∎

Corollary 4.5. *Assume that f is a complement on S with degree 0 and deviation δ. Assume that u is a word in S^* with length ℓ, and that v is f-equivalent to u. Then the word $u^{-1}v$ is f-reversible to the empty word, and we have*

$$d_f(u, v) \leq c_f(u, v) \leq \ell + \delta \cdot d_f(u, v).$$

By Lemma 3.1, the previous result applies in particular to *every* complement on a two elements set. Remember that these results do not require any hypothesis about the convergence of the complement: for instance, the monoid $\langle a, b \; ; \; a = bba \rangle$, and, more generally, all monoids of the form $\langle a, b \; ; \; au = bbav \rangle$, which are associated with non-convergent complements, are eligible.

5. Embeddings

We finish the paper with an application of the possible completeness of word reversing to the existence of embeddings between monoids with a complemented presentation. The criterion we obtain is very simple to verify in practice, and, in good cases, it gives us a nice method for proving the existence of such embeddings.

Assume that f_1 and f_2 are complements on S_1 and S_2 respectively. Let φ be a mapping of S_1 into S_2^*. We denote by φ^* the alphabetical extension of φ into a homomorphism of S_1^* into S_2^*, and wonder whether φ^* induces a homomorphism, and, possibly, an embedding, of the monoid $\langle S_1 \; ; \; R_{f_1} \rangle$ into the monoid $\langle S_2 \; ; \; R_{f_2} \rangle$.

Definition. Let f_1 be a complement on S_1, f_2 be a complement on S_2 and φ be a mapping of S_1 into S_2^*. We say that φ is a f_1-f_2-*morphism* if, for every pair of letters (x, y) in the domain of f_1, the word $\varphi(x)\backslash_{f_2}\varphi(y)$ exists, and it is f_2-equivalent to $\varphi^*(f_1(x, y))$.

As should be clear on Figure 5.1, the previous notion is what is needed for φ^* to define, *up to f_2-equivalence*, a morphism of every f_1-reversing graph into a φ_2-reversing graph.

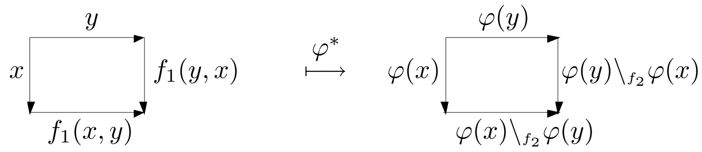


Figure 5.1. Morphism of complements

Lemma 5.1. *Assume that f_1 be a complement on S_1, f_2 is a complement on S_2 and φ is a f_1-f_2-morphism. Then φ^* induces a homomorphism of the monoid $\langle S_1\,;\,R_{f_1}\rangle$ into the monoid $\langle S_2\,;\,R_{f_2}\rangle$.*

Proof. By definition, $\langle S_1\,;\,R_{f_1}\rangle$ is the quotient of S_1^* under the congruence relation generated by the pairs $(xf_1(x,y), yf_1(y,x))$ with $x, y \in S_1$. So, it suffices to show that the images of such pairs are pairs of f_2-equivalent words. By construction, we have

$$\begin{aligned}\varphi^*(xf_1(x,y)) &= \varphi(x)\varphi^*(f_1(x,y)) \equiv_{f_2} \varphi(x)(\varphi(x)\backslash_{f_2}\varphi(y))\\ &\equiv_{f_2} \varphi(y)(\varphi(y)\backslash_{f_2}\varphi(x)) \equiv_{f_2} \varphi(y)\varphi^*(f_1(y,x)) = \varphi^*(yf_1(y,x)),\end{aligned}$$

which completes the proof. ■

The point is that a complement gives information not only about the possible equivalence of two given words, but also about their non-equivalence.

Lemma 5.2. *Assume that f_1 be a complement on S_1, f_2 is a coherent complement on S_2 and φ is a f_1, f_2-morphism. Assume that u, v are words on S_1 and $u\backslash_{f_1}v$ exists. Then $\varphi^*(u)\backslash_{f_2}\varphi^*(v)$ exists, and we have*

$$\varphi^*(u)\backslash_{f_2}\varphi^*(v) \equiv_{f_2} \varphi^*(u\backslash_{f_1}v). \tag{5.1}$$

Proof. The result is obvious if u or v is empty. Now we use induction on the integer $n = c_{f_1}(u, v)$. By the previous remark, the result is clear for $n = 0$ for, in this case, u or v must be empty. Assume $n \geq 1$. We argue inductively on the integer $\ell = \lg(u) + \lg(v)$. By the previous remark again, the first case to consider is $\ell = 2$ with $u, v \in S_1$, and, then, the result is the hypothesis that φ is a morphism with respect to f_1 and f_2. Assume now $\ell \geq 3$. We may assume $u = u_1u_2$ with $1 \leq \lg(u_1) < \lg(u)$. By Lemma 1.3, we have $v\backslash_{f_1}u = (v\backslash_{f_1}u_1)((u_1\backslash_{f_1}v)\backslash_{f_1}u_2)$. By induction hypothesis, $\varphi^*(v)\backslash_{f_2}\varphi^*(u_1)$ and $\varphi^*(u_1)\backslash_{f_2}\varphi^*(v)$ exist, and we have

$$\varphi^*(v)\backslash_{f_2}\varphi^*(u_1) \equiv_{f_2} \varphi^*(v\backslash_{f_1}u_1), \tag{5.2}$$

$$\varphi^*(u_1)\backslash_{f_2}\varphi^*(v) \equiv_{f_2} \varphi^*(u_1\backslash_{f_1}v). \tag{5.3}$$

We have $c_{f_1}(u_2, u_1\backslash_{f_1}v) \leq n - 1$, so, by induction hypothesis, $\varphi^*(u_1\backslash_{f_1}v)\backslash_{f_2}\varphi^*(u_2)$ exists and we have

$$\varphi^*(u_1\backslash_{f_1}v)\backslash_{f_2}\varphi^*(u_2) \equiv_{f_2} \varphi^*((u_1\backslash_{f_1}v)\backslash_{f_1}u_2). \tag{5.4}$$

Now, f_2 is coherent, so we deduce from (5.3) that $(\varphi^*(u_1)\backslash_{f_2}\varphi^*(v))\backslash_{f_2}\varphi^*(u_2)$ exists and that it is f_2-equivalent to the second term of (5.4). Using this and (5.2), we see that $\varphi^*(v)\backslash_{f_2}\varphi^*(u)$, *i.e.*, $\varphi^*(v)\backslash_{f_2}((\varphi^*(u_1)\varphi^*(u_2))$, exists and that we have

$$\begin{aligned}\varphi^*(v)\backslash_{f_2}\varphi^*(u) &\equiv_{f_2} \varphi^*(v\backslash_{f_1}u_1)\varphi^*((u_1\backslash_{f_1}v)\backslash_{f_1}u_2)\\ &= \varphi^*((v\backslash_{f_1}u_1)((u_1\backslash_{f_1}v)\backslash_{f_1}u_2)) = \varphi^*(v\backslash_{f_1}(u_1u_2)) = \varphi^*(v\backslash_{f_1}u),\end{aligned}$$

which completes the proof. ■

Lemma 5.3. *Assume that f_1 be a complement on S_1, f_2 is a coherent Noetherian complement on S_2, and φ is a f_1-f_2-morphism such that, for all x, y in S_1, $\varphi(x)$ is nonempty and $f_1(x,y)$ exists whenever $\varphi(x)\backslash_{f_2}\varphi(y)$ does. Then f_1 is Noetherian as well, and, for all words u, v in S_1^*, we have*

$$\rho_{f_1}(u) \le \rho_{f_2}(\varphi^*(u)),$$

and $u\backslash_{f_1}v$ exists whenever $\varphi^*(u)\backslash_{f_2}\varphi^*(v)$ does.

Proof. Define a mapping ρ of S_1^* into the ordinals by $\rho(u) = \rho_{f_2}(\varphi^*(u))$. Then ρ satisfies all requirements of Lemma 2.3(ii), hence f_1 is Noetherian, and $\rho_{f_1}(u) \le \rho(u)$ holds for every word u.

As for the existence of $u\backslash_{f_1}v$ when $\varphi^*(u)\backslash_{f_2}\varphi^*(v)$ exists, the result is obvious if at least one of the words $\varphi^*(u)$, $\varphi^*(v)$ is empty, *i.e.*, if u or v is empty. Now we use induction on the ordinal $\alpha = e_{f_2}(\varphi^*(u), \varphi^*(v))$. For $\alpha = 0$, we have $\varphi^*(u) = \varphi^*(v) = \varepsilon$, so u or v is empty, and we are done. Otherwise, we may assume $u, v \ne \varepsilon$. We use the notations of Figure 1.3. So, we write $u = xu_0$, $v = yv_0$ with x, y in S_1. By construction, we have

$$e_{f_2}(\varphi^*(u_0), \varphi^*(f_1(x,y))) < \alpha,$$

and $\varphi^*(u_0)\backslash_{f_2}(\varphi(x)\backslash_{f_2}\varphi(y))$ exists. As $\varphi(x)\backslash_{f_2}\varphi(y)$ exists, $f_1(x,y)$ does, and, since f_2 is coherent, $\varphi^*(u_0)\backslash_{f_2}\varphi^*(f_1(x,y))$ exists as well. Hence, by induction hypothesis, $u_1 = f_1(x,y)\backslash_{f_1}u_0$ exists, and, by the previous lemma, its image under φ^* is f_2-equivalent to the word $\varphi^*(f_1(x,y))\backslash_{f_2}\varphi^*(u_0)$. A similar argument shows that the word $v_2 = f_1(y,x)\backslash_{f_1}v_0$ exists and its image under φ^* is f_2-equivalent to the word $\varphi^*(f_1(y,x))\backslash_{f_2}\varphi^*(v_0)$. Finally, we have also

$$e_{f_2}(\varphi^*(u_1), \varphi^*(v_2)) < \alpha,$$

and $\varphi^*(u_1)\backslash_{f_2}\varphi^*(v_2)$ exists since f_2 is coherent. By induction hypothesis, we conclude that $u_1\backslash_{f_1}v_2$ exists as well, and so does $u\backslash_{f_1}v$. ■

Remark. The hypothesis we use for the converse Lemma 5.3 is stronger than the one used for Lemma 5.2, as we require Noetherianity. This is because we only require that the words $\varphi^*(f_1(x,y))$ and $\varphi(x)\backslash_{f_2}\varphi(y)$ are f_2-equivalent in the definition of a morphism. Actually, if we required equality instead of f_2-equivalence, we could simply use an induction on $c_{f_2}(\varphi^*(u), \varphi^*(v))$ in the proof of Lemma 5.3 and avoid requiring explicit Noetherianity. Observe that there exists a close connection between the present arguments and the proofs of coherence, which amount to using the identity mapping as a morphism. Actually, we could define the degree of a f_1-f_2-morphism φ as the supremum of the quantities

$$d_{f_2}(\varphi^*(f_1(x,y)), \varphi(x)\backslash_{f_2}\varphi(y)) + d_{f_2}(\varphi^*(f_1(y,x)), \varphi(y)\backslash_{f_2}\varphi(x)),$$

and Noetherianity of f_2 is then necessary only for a morphism of degree at least 2. Similarly, we could obtain upper bounds for $c_{f_1}(u,v)$ in terms of $c_{f_2}(\varphi^*(u),\varphi^*(v))$ in the spirit of the formulas of Section 3.

Proposition 5.4. *Assume that f_1 is a complement on S_1, f_2 is a Noetherian coherent complement on S_2 and φ is a f_1-f_2-morphism such that, for all x, y in S_1, $\varphi(x)$ is nonempty and $f_1(x,y)$ exists whenever $\varphi(x)\backslash_{f_2}\varphi(y)$ does. Then φ^* induces an embedding of the monoid $\langle S_1\,;\,R_{f_1}\rangle$ into the monoid $\langle S_2\,;\,R_{f_2}\rangle$, and f_1 is Noetherian and coherent.*

Proof. We know that φ^* induces a homomorphism of the monoid $\langle S_1\,;\,R_{f_1}\rangle$ into the monoid $\langle S_2\,;\,R_{f_2}\rangle$. So, assume that u, v are words in S_1^* and that $\varphi^*(u)$ and $\varphi^*(v)$ are f_2-equivalent. Since f_2 is coherent, word reversing is complete for f_2, and, therefore, the words

$$\varphi^*(u)\backslash_{f_2}\varphi^*(v) \quad \text{and} \quad \varphi^*(v)\backslash_{f_2}\varphi^*(u)$$

exist and are empty. By Lemma 5.3, the words $u\backslash_{f_1}v$ and $v\backslash_{f_1}u$ exist, and, by Lemma 5.2, we have

$$\varphi^*(u\backslash_{f_1}v) \equiv_{f_2} \varphi^*(u)\backslash_{f_2}\varphi^*(v) = \varepsilon.$$

The hypothesis that $\varphi(x)$ is nonempty for every x in S_1 implies $u\backslash_{f_1}v = \varepsilon$. Similarly, the word $v\backslash_{f_1}u$ is empty, and u and v are f_1-equivalent.

By Lemma 5.3, f_1 is Noetherian, so it remains to prove coherence. Assume that u, v are f_1-equivalent words in S_1. The hypothesis that φ is a morphism implies that $\varphi^*(u)$ and $\varphi^*(v)$ are f_2-equivalent, and the previous argument shows that $u\backslash_{f_1}v$ and $v\backslash_{f_1}u$ are empty. This means that word reversing is complete for f_1, hence f_1 is coherent. ■

A typical framework where the previous embedding criterion applies is that of Artin monoids associated with finite Coxeter groups. We have seen above that the standard presentation of these monoids is a complemented presentation. Noetherianity is obvious as the defining relations preserve the length, and, therefore, coherence follows from local coherence, which has been established in [4] and [14]. Here we mention two easy results.

Proposition 5.5. *(i) Let φ be the mapping of $\{\sigma_1,\ldots,\sigma_n\}$ into $\{\sigma_1,\ldots,\sigma_{2n-1}\}$ defined by*

$$\varphi(\sigma_i) = \sigma_i\sigma_{2n-i} \text{ for } i<n, \quad \varphi(\sigma_n) = \sigma_n.$$

Then φ induces an embedding of the Artin monoid of type B_n into the Artin monoid of type A_{2n-1}, i.e., into the monoid of $2n$ strand braids.

(ii) Assume $n \geq 4$. Let φ be the mapping of $\{\sigma_1,\sigma_2\}$ into $\{\sigma_1,\ldots,\sigma_{2n-5}\}$ defined by

$$\varphi(\sigma_1) = \sigma_1\sigma_3\cdots\sigma_{2n-5}, \quad \varphi(\sigma_2) = \sigma_2\sigma_4\cdots\sigma_{2n-4}.$$

Then φ induces an embedding of the Artin monoid of type $I_2(n)$ into the Artin monoid of type A_{2n-5}.

Proof. It suffices to show that the mappings are morphisms with respect to the considered complements, an easy verification. For instance, we see on Figure 5.2 that, if f is the complement associated with the Coxeter type A_3, then we have

$$\sigma_1\sigma_3\backslash_f\sigma_2 \equiv_f (\sigma_2)(\sigma_1\sigma_3)(\sigma_2), \quad \sigma_2\backslash_f\sigma_1\sigma_3 \equiv_f (\sigma_1\sigma_3)(\sigma_2)(\sigma_1\sigma_3),$$

i.e., the braids σ_2 and $\sigma_1\sigma_3$ behave, as far as complement is concerned, as two generators connected by a weight 4 edge in a Coxeter graph. Thus $\varphi(\sigma_1) = \sigma_1\sigma_3$, $\varphi(\sigma_2) = \sigma_2$ defines an embedding of the monoid of type B_2 into the monoid of type A_3. ■

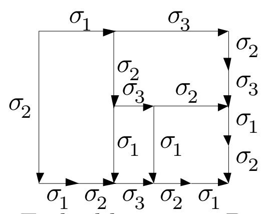


Figure 5.2. Embedding type B_2 in type A_3

By standard results [4], Artin monoids associated with finite Coxeter groups embed in their groups of fractions, and it is then obvious to extend the previous embeddings to the corresponding groups. For instance, we obtain an embedding of the Artin group of type B_n in the Artin group of type A_{2n-1}—as was already established in [13] by another method.

Let us conclude with a corollary.

Proposition 5.6. *Every Artin group of type B or $I_2(n)$ is left orderable, i.e., there exists a linear ordering on the group that is compatible with multiplication on the left.*

Proof. By [7] and [9], the braid groups, *i.e.*, Artin groups of type A, are left orderable. We can then use the previous embeddings to define an order on each embedded group. ■

Corollary 5.7. *If G is a Artin group of type B or $I_2(n)$, the group algebra $\mathbf{C}[G]$ has no zero divisor.*

The linear ordering of the braid group constructed in [7] is characterized by the fact that an element a is bigger than 1 if and only if it admits a decomposition where σ_1 occurs but σ_1^{-1} does not, or a decomposition where σ_2 occurs, but none of σ_1, σ_1^{-1}, σ_2^{-1} does, *etc.* The explicit definition of the embedding shows that the same characterization holds for the linear orderings on Artin groups of type B or $I_2(p)$ constructed above. Let us mention that nothing seems to be known about the possible orderings of Artin groups of type D to H.

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