HOMOLOGY OF GAUSSIAN GROUPS
HOMOLOGIE DES GROUPES GAUSSIENS

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Abstract. We describe new combinatorial methods for constructing explicit free resolutions of $\mathbf{Z}$ by $\mathbf{Z}G$-modules when G is a group of fractions of a monoid where enough least common multiples exist ("locally Gaussian monoid"), and, therefore, for computing the homology of G. Our constructions apply in particular to all Artin–Tits groups of finite Coxeter type. Technically, the proofs rely on the properties of least common multiples in a monoid.

Résumé. Nous décrivons de nouvelles méthodes combinatoires fournissant des résolutions explicites du module trivial par des $\mathbf{Z}G$-modules libres lorsque G est le groupe de fractions d'un monoïde possédant suffisamment de ppcm ("monoïde localement Gaussien"), et, donc, permettant de calculer l'homologie de G. Nos constructions s'appliquent en particulier à tous les groupes d'Artin–Tits de type de Coxeter fini. D'un point de vue technique, les démonstrations reposent sur les propriétés des ppcm dans un monoïde.

Introduction

The (co)homology of Artin's braid groups B_n has been computed by methods of differential geometry and algebraic topology in the beginning of the 1970's [3, 4, 29, 16], and the results have then been extended to Artin–Tits groups of finite Coxeter type [8, 31, 46], see also [17, 18, 19, 38, 39, 47]. A purely algebraic and combinatorial approach was developed by C. Squier in his unpublished PhD thesis of 1980—see [42]—relying both on the fact that these groups are groups of fractions of monoids admitting least common multiples and on the particular form of the Coxeter relations involved in their standard presentation.

On the other hand, it has been observed in recent years that most of the algebraic results established for the braid groups and, more generally, the Artin–Tits groups of finite Coxeter type ("spherical Artin–Tits groups") by Garside, Brieskorn, Saito, Adyan, Thurston among others, extend to a wider class of so-called Garside groups. A Gaussian group is defined to be the group of fractions of a monoid in which left and right division make a well-founded lattice, *i.e.*, in which we have a good theory of least common multiples, and a Garside group is a Gaussian group that satisfies an additional finiteness condition analogous to sphericality (see the precise definition in Section 1 below). In some sense, such an extension is natural, as the role of least common multiples (lcm's for short) in some associated monoid had already been emphasized and proved to be crucial in the study of the braid groups, in particular in the solution of the conjugacy problem by Garside [30] and the construction of an automatic structure by Thurston [45], see also [28, 12, 13]. However, the family

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of Garside groups includes new groups defined by relations quite different from Coxeter relations, such as $\langle a, b, c, \dots ; a^p = b^q = c^r = \dots \rangle$, $\langle a, b, c; abc = bca = cab \rangle$, or $\langle a, b; ababa = b^2 \rangle$—see [36] for many examples—and, even if the fundamental Kürzungslemma of [9] remains valid in all Gaussian monoids, many technical results about spherical Artin–Tits groups fail for general Gaussian groups, typically all results relying on the symmetry of the Coxeter relations, like the preservation of the length by the relations or the result that the fundamental element Δ is squarefree. Thus, the extension from spherical Artin–Tits groups to general Gaussian groups or, at least, Garside groups is not trivial, and, in most cases, it requires finding new arguments: see [25] for the existence of a quadratic isoperimetric inequality, [21] for torsion freeness, [23] for the existence of a bi-automatic structure, [37] for the existence of a decomposition into a crossed product of groups with a monogenic center, [40] for the decidability of the existence of roots.

According to this program, it is natural to look for a possible extension of Squier's approach to arbitrary Gaussian groups (or to even more general groups). Such an idea is already present in Squier's paper, whose first part addresses general groups and monoids which are essentially the Gaussian groups we shall consider here. However, in the second part of his paper, he can complete the construction only in the special case of Artin–Tits groups. Roughly speaking, what we do in the current paper is to develop new methods so as to achieve the general program sketched in the first part of [42].

As in [42], we observe that the homology of a group of fractions coincides with that of the involved monoid, so our aim will be to construct a resolution of the trivial module $\mathbf{Z}$ by free $\mathbf{Z}M$-modules when M is a monoid with good lcm properties. We start with the natural idea of constructing an explicit simplicial complex where the n cells correspond to n-tuples of elements $(\alpha_1, \dots, \alpha_n)$ of M, and, in order to obtain reasonable (finite type) modules, we assume in addition that the α_i's are taken in some fixed set of generators of M. The idea, which is already present in [42] even if not stated explicitly, is that the cell $[\alpha_1, \dots, \alpha_n]$ represents the computation of the left lcm of $\alpha_1, \dots, \alpha_n$. The core of the problem is to define the boundary operator and to construct a contracting homotopy. Here Squier uses a trick that allows him to avoid addressing the question directly. Indeed, he first defines by purely syntactical means a top degree approximation (in the sense of Stallings [44]) of the desired resolution, and then he introduces his resolution as a deformation of this abstract approximated version. The miraculous existence of this top approximation directly relies on the symmetry of the Coxeter relations that define Artin–Tits monoids. For more general relations, in particular for relations that do not preserve the length of the words, such as those mentioned above, even the notion of a top factor is problematic, and it is doubtful that Squier's construction can be extended—see Remark 4.11 for further comments about obstructions.

In this paper, we develop new solutions, which address the construction directly. We propose two methods, one more simple, and one more general. Our first solution is based on word reversing, a syntactic technique introduced in [20] for investigating those monoids admitting least common multiples. Starting with two words u, v that represent some elements x, y of our monoid, word reversing constructs (in good cases) two new words u', v' such that both $u'v$ and $v'u$ represent the left lcm of x and y, when the latter exists. The idea here is to use word reversing to fill the faces of the n-cubes we are about to construct. The resulting method turns out to be very simple, and we show that it leads to a free resolution of $\mathbf{Z}$ for every Gaussian monoid (and even for more general monoids called locally Gaussian) provided we start with a convenient family of generators, typically the divisors of the fundamental element Δ in the case of a Garside monoid. We also show that the resolution so obtained is connected with the one constructed by Charney, Meier, and Whittlesey in [14] (in the special cases considered in the latter paper), and with the Deligne–Salvetti resolution [26, 38, 18] (in the more special cases of Artin–Tits groups).

Our second solution is more general. It is reminiscent of work by Kobayashi [32] about the homology of rewriting systems—see also [34, 41]—and it relies on using a convenient linear ordering on the considered generators and an induction on some derived well-ordering of the cells. This second construction works for arbitrary generators in all Gaussian monoids, and, more generally, in so-called locally left Gaussian monoids where we only assume that any two elements that admit a common left multiple admit a left lcm (non-spherical Artin–Tits monoids are typical examples). The price to pay for the generality of the construction is that we have so far no explicit geometrical (or homotopical) interpretation for the boundary operator and the contracting homotopy, excepted in low degree.

With the previous tools, we reprove and extend the results about the homology of spherical Artin–Tits groups, and, more generally, of arbitrary Artin–Tits monoids. In particular, we prove

Theorem 0.1. *Assume that M is a finitely generated locally left Gaussian monoid. Then M is of type FL, in the sense that $\mathbf{Z}$ admits a finite free resolution over $\mathbf{Z}M$.*

(See Proposition 4.9 for an explicit bound for the length of the resolution in terms of the cardinality of a generating set.)

Corollary 0.2. *Every Garside group G is of type FL, i.e., $\mathbf{Z}$ admits a finite free resolution over $\mathbf{Z}G$.*

The paper is organized as follows. In Section 1, we list the needed basic properties of (locally) Gaussian and Garside monoids, and, in particular, we introduce word reversing. We also recall that the homology of a monoid satisfying Ore's embeddability conditions coincides with the one of its group of fractions. In Section 2, we consider a (locally) Gaussian monoid M and we construct an explicit resolution of $\mathbf{Z}$ by a graded free $\mathbf{Z}M$-module relying on word reversing and on the greedy normal form of [28]. We give a natural geometrical interpretation involving n-cubes in the Cayley graph of M. In Section 3, we extract from the resolution of Section 2 a smaller resolution, and we establish a precise connection between the latter and the resolution considered in [14]. Finally, in Section 4, we consider a locally left Gaussian monoid M (a weaker hypothesis), and we construct a new free resolution of $\mathbf{Z}$, relying on a well ordering of the cells. A few examples are investigated, including the first Artin and Birman-Ko-Lee braid monoids.

The results in Sections 2 and 3 are mainly due to the first author, while the results in Section 4 are mainly due to the second author. The authors thank Christian Kassel for his comments and suggestions, as well as Ruth Charney, John Meier, and Kim Whittlesey for interesting discussions about their independent approach [14]. They also thank the referee, who, by asking for a clarification of the connection with the latter paper, has induced the results of Section 3.

1. Gaussian and Garside monoids

The material in this section is mostly classical. However, the key result, namely Proposition 1.10 which connects the greedy normal form and the word reversing process, receives a new, slighty shorter proof than the one of [23], while the result is stated in a more general framework, namely locally Gaussian monoids instead of Garside monoids.

1.1. **Gaussian and locally Gaussian monoids.** Our notations follow those of [42] on the one hand, and those of [25] and [23] on the other hand. Let M be a monoid. We say that x is a *left divisor* (resp. a proper left divisor) of y in M, denoted $x \sqsubseteq y$ (resp. $x \sqsubset y$), if $y = xz$ holds for some z (resp. for some z with $z \neq 1$). Alternatively, we say that y is a right multiple of x. Right divisors and left multiples are defined symmetrically (but we introduce no specific notation).

Definition. We say that a monoid M is *left Noetherian* if left divisibility is well-founded in M, *i.e.*, there exists no infinite descending sequence $\cdots \sqsubset x_2 \sqsubset x_1$.

Note that, if M is a left Noetherian monoid, there is no invertible element in M but 1, and, therefore, the relation $\sqsubset$ is a strict ordering on M (and so is the symmetric right divisibility relation). For x, y in M, we say that z is a *least common left multiple*, or left lcm, of x and y, if z is a left multiple of x and y, and every common left multiple of x and y is a left multiple of z. If z and z' are two left lcm's for x and y, then we have $z \sqsubseteq z'$ and $z' \sqsubseteq z$ by definition, hence $z = z'$ whenever M is left Noetherian. Thus, in a left Noetherian monoid, left lcm's are unique when they exist.

Definition. We say that a monoid M is *left Gaussian* if it is right cancellative (*i.e.*, $zx = zy$ implies $x = y$), left Noetherian, and any two elements of M admit a left lcm. We say that M is *locally left Gaussian* if it satisfies the first two conditions above, but the third one is relaxed into: any two elements that admit a common left multiple admit a left lcm.

If M is a locally left Gaussian monoid, and x, y are elements of M that admit at least one common left multiple, we denote by $x \vee y$ the left lcm of x and y, and by $x_{/y}$ the unique element z satisfying $zy = x \vee y$; the latter is called the *left complement* of x in y. Thus we have

$$x_{/y} \cdot y = x \vee y = y_{/x} \cdot x$$

whenever x and y have a common left multiple. Observe that, if y happens to be a right divisor of x, then $x_{/y}$ is the corresponding quotient, *i.e.*, we have $x = x_{/y} \cdot y$: this should make the notation natural. It is easy to see that, in a locally left Gaussian monoid M, any two elements x, y admit a right gcd, *i.e.*, a common right divisor z such that every common right divisor of x and y is a right divisor of z; then M equipped with right division is an inf-semi-lattice with least element 1.

The notion of a (locally) right Gaussian monoid is defined symmetrically in terms of right Noetherianity, left cancellativity and existence of right lcm's. If M is a (locally) right Gaussian monoid, and x, y are elements of M that admit a common right multiple, we denote by ${}_{x\backslash}y$ the unique element of M such that $x\,{}_{x\backslash}y$ is the right lcm of x and y, and call it the *right complement* of x in y (we shall need no specific notation for the right lcm in this paper).

Finally, we introduce Gaussian monoids as those monoids satisfying the previous conditions on both sides:

Definition. We say that a monoid M is *(locally) Gaussian* if it is both (locally) left Gaussian and (locally) right Gaussian.

Roughly speaking, Gaussian monoids are those monoids where a good theory of divisibility exists, with in particular left and right lcm's and gcd's for every finite family of elements. Locally Gaussian monoids are similar, with the exception that the lcm's operations, and, therefore, the associated complements operations, are only partial operations. The Artin–Tits monoid associated with an arbitrary Coxeter matrix is a typical example of a locally Gaussian monoid [9]; such an Artin–Tits monoid is Gaussian if and only if the associated Coxeter group is finite, *i.e.*, in the so-called spherical case. We refer to [36] and [24] for many more examples of (locally) Gaussian monoids. Let us just still mention here the Baumslag-Solitar monoid $\langle a, b\,;\, ba = ab^2\rangle^+$, another typical example of a locally left Gaussian monoid that is not Gaussian, as the elements ab and a have no common left multiple.

If M is a Gaussian monoid, it satisfies Ore's conditions [15] and, therefore, it embeds in a group of fractions. We say that a group G is *Gaussian* if there exists at least one Gaussian monoid M such that G is the group of fractions of M. The example of Artin's braid groups B_n, which is both the group of fractions of the monoid B_n^+ [30] and of the Birman-Ko-Lee monoid BKL_n^+ [7] shows that a given Gaussian group may be the group of fractions of several non-isomorphic Gaussian monoids—as well as of many more monoids that need not be Gaussian [36].

1.2. **Garside and locally Garside monoids.** In the sequel, we shall be specially interested in finitely generated (locally) Gaussian monoids. Actually, we shall consider a stronger condition, namely admitting a finite generating subset that is closed under some operations.

Definition. We say that a monoid M is *(locally) Garside* [1] if it is (locally) Gaussian and it admits a finite generating subset $\mathcal{X}$ that is closed under left and right lcm, and under left and right complements, this meaning that, if x, y belong to $\mathcal{X}$ and they admit a common left multiple, then the left lcm $x \vee y$ and the left complement $x_{/y}$, if the latter is not 1, still belong to $\mathcal{X}$, and a similar condition holds with right multiples.

As is shown in [23], Garside monoids may be characterized by weaker assumptions: for instance, a sufficient condition for a Gaussian monoid to be Garside is to admit a finite generating subset closed under left complement. Another equivalent condition is the existence of a Garside element, defined as an element Δ such that the left and right divisors of Δ coincide, they are finite in number and they generate M. In this case, the family $\mathcal{D}_\Delta$ of all divisors of Δ is a finite generating set that is closed under left and right complement, left and right lcm, and left and right gcd. In particular, $\mathcal{D}_\Delta$ equipped with the operation of left lcm and right gcd (or of right lcm and left gcd) is a finite lattice, with minimum 1 and maximum Δ, and this lattice completely determines the monoid M. It is also known that every Gaussian monoid admits a unique minimal generating family, which implies that it admits a unique minimal Garside element, for instance the fundamental element Δ_n in the case of the monoid B_n^+ of positive braids. Let us mention that no example of a Gaussian non-Garside monoid of finite type is known.

[1] Garside monoids as defined above are called Garside monoids in [23] and [14], but they were called "small Gaussian" or "thin Gaussian" in previous papers [25, 37], where a more restricted notion of a Garside monoid was also considered.

Locally Garside monoids need not possess a Garside element Δ in general. Typical examples are free monoids and, more generally, FC-type Artin–Tits monoids [2]. In the case of a free monoid $\mathcal{X}^*$ (the set of all words over the alphabet $\mathcal{X}$), the set $\mathcal{X}$ is a generating set that is trivially closed under lcm and complement: any two distinct elements x, y of $\mathcal{X}$ admit no common multiple, so $x \vee y$ and $x_{/y}$ trivially belong to $\mathcal{X}$ when they exist, *i.e.*, never.

1.3. **Identities for the complement.** In the sequel we need a convenient lcm calculus. As already pointed out in [25, 23], the main object here is not the lcm operation, but rather the derived complement operation and the algebraic identities it satisfies.

Notation. For $n \geq 2$, we write $x_{/y_1,\dots,y_n}$ for $x_{/(y_1 \vee \dots \vee y_n)}$.

Thus, the iterated complement operation is defined by the equality

$$x_{/y_1,\dots,y_n} \cdot (y_1 \vee \dots \vee y_n) = x \vee y_1 \vee \dots \vee y_n. \tag{1.1}$$

Observe that (1.1) remains true for $n = 0$ provided we define $x_{/Y}$ to be x if Y is the empty sequence.

Lemma 1.1. *The following identities hold:*

$$x_{/y,z} \cdot y_{/z} = (x \vee y)_{/z} = x_{/z} \vee y_{/z}, \tag{1.2}$$

$$(x_{/y})_{/(z_{/y})} = x_{/y,z} = (x_{/z})_{/(y_{/z})}, \tag{1.3}$$

$$(xy)_{/z} = x_{/(z_{/y})} \cdot y_{/z}, \tag{1.4}$$

$$z_{/(xy)} = (z_{/y})_{/x}. \tag{1.5}$$

Proof. Using the associativity of the lcm, we obtain

$$x_{/y,z} \cdot y_{/z} \cdot z = x_{/y,z} \cdot (y \vee z) = x \vee (y \vee z) = (x \vee y) \vee z = (x \vee y)_{/z} \cdot z,$$

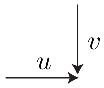
and we deduce the first equality in (1.2) by cancelling z on the right. The proof of (1.3) is similar, as multiplying both $(x_{/y})_{/(z_{/y})}$ and $x_{/y,z}$ by $y \vee z$ on the right gives $x \vee y \vee z$. Then one deduces the second equality in (1.2) easily. Formulas (1.4) and (1.5) are proved by expressing in various ways the lcm of xy and z. □

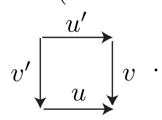
1.4. **Word reversing.** The constructions we shall describe in Sections 2 and, partly, 4, rely on a word process called *word reversing*. It was introduced in [20], and investigated more systematically in Chapter II of [22]—see also [24] for further generalizations.

If $(\mathcal{X}, \mathcal{R})$ is a monoid presentation, *i.e.*, a set of letters plus a list of relations $u = v$ with u, v words over $\mathcal{X}$, we denote by $\langle \mathcal{X} ; \mathcal{R} \rangle^+$ the associated monoid, and by $\langle \mathcal{X} ; \mathcal{R} \rangle$ the associated group. If u, v are words over $\mathcal{X}$, we shall denote by $\overline{u}$ the element of the monoid $\langle \mathcal{X} ; \mathcal{R} \rangle^+$ represented by u, and we write $u \equiv v$ for $\overline{u} = \overline{v}$. We use $\mathcal{X}^*$ for the free monoid generated by $\mathcal{X}$, *i.e.*, the set of all words over $\mathcal{X}$; we use ε for the empty word. We also introduce $\mathcal{X}^{-1}$ as a disjoint copy of $\mathcal{X}$ consisting of one letter α^{-1} for each letter α of $\mathcal{X}$. Finally, we say that the presentation $(\mathcal{X}, \mathcal{R})$ is *positive* if all relations in $\mathcal{R}$ have the form $u = v$ with u, v nonempty, and that it is *complemented* if it is positive and, for each pair of letters α, β in $\mathcal{X}$, there exists at most one relation of the form $v\alpha = u\beta$ in $\mathcal{R}$, and no relation $u\alpha = v\alpha$ with $u \neq v$.

Definition. Assume that $(\mathcal{X}, \mathcal{R})$ is a positive monoid presentation. For w, w' words over $\mathcal{X} \cup \mathcal{X}^{-1}$, we say that w is $\mathcal{R}$*-reversible* to w' (on the left) if we can transform w to w' by iteratively deleting subwords uu^{-1} where u is a word over $\mathcal{X}$, and replacing subwords of the form uv^{-1} with $v'^{-1}u'$, where u, v are nonempty words over $\mathcal{X}$ and $u'v = v'u$ is one of the relations of $\mathcal{R}$.

For further intuition, it is important to associate with every reversing sequence starting with a word w a labelled planar graph defined inductively and analogous to a van Kampen diagram: first we associate with w a path labelled by the successive letters of w, in which the positive letters (those in $\mathcal{X}$) are given horizontal right-oriented edges and the negative letters (those in $\mathcal{X}^{-1}$) are given vertical down-oriented edges. Then, word reversing consists in inductively completing the diagram by using a relation $v'u = u'v$ of $\mathcal{R}$ (or a trivial relation $u = u$) to close a pattern of

the form 

into 

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Example 1.2. Let us consider the standard presentation of the braid monoid B_4^+, namely $\langle \sigma_1, \sigma_2, \sigma_3 \,;\, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2,\ \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3,\ \sigma_1\sigma_3 = \sigma_3\sigma_1 \rangle^+$. Then

$$\sigma_2\underline{\sigma_1\sigma_3^{-1}}\sigma_2^{-1} \to \underline{\sigma_2\sigma_3^{-1}}\sigma_1\sigma_2^{-1} \to \sigma_3^{-1}\sigma_2^{-1}\sigma_3\sigma_2\underline{\sigma_1\sigma_2^{-1}} \to$$
$$\sigma_3^{-1}\sigma_2^{-1}\sigma_3\underline{\sigma_2\sigma_2^{-1}}\sigma_1^{-1}\sigma_2\sigma_1 \to \sigma_3^{-1}\sigma_2^{-1}\underline{\sigma_3\sigma_1^{-1}}\sigma_2\sigma_1 \to \sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_3\sigma_2\sigma_1$$

is a maximal reversing sequence (the pattern that is reversed is underlined at each step), and the associated diagram is displayed in Figure 1.

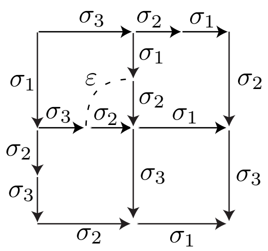


FIGURE 1. Word reversing diagram for the standard presentation of B_4^+

In general, word reversing is not a deterministic process: starting with one word may lead to various sequences of words, various diagrams, and, in particular, to several terminal words, the latter being those words that contain no pattern $\alpha\alpha^{-1}$ or $\alpha\beta^{-1}$ such that there exists at least one relation $v\alpha = u\beta$ in $\mathcal{R}$. However, it is easily shown (see Chapter II of [22]) that, if $\mathcal{R}$ is a complemented presentation, then there exists a unique maximal reversing diagram starting with a given word w, and w is reversible to at most one terminal word, so, in particular, at most one word of the form $u^{-1}v$ with u, v words over $\mathcal{X}$.

Definition. Assume that $(\mathcal{X}, \mathcal{R})$ is a complemented presentation, and u, v are words over $\mathcal{X}$. Then we denote by $u_{/^*v}$ and $v_{/^*u}$ the unique words over $\mathcal{X}$ such that uv^{-1} is reversible to $(v_{/^*u})^{-1}u_{/^*v}$, if such words exist.

(The symmetry of the reversing process guarantees that, if $u^{-1}v$ is reversible to $v'u'^{-1}$, then $v^{-1}u$ is reversible to $u'v'^{-1}$, so there is no ambiguity in the previous definition.) Observe that, if α and β are letters in $\mathcal{X}$, then $\alpha_{/^*\beta}$ and $\beta_{/^*\alpha}$ are the (unique) words u, v such that $v\alpha = u\beta$ is a relation in $\mathcal{R}$, if such a relation exists.

By definition, each step of $\mathcal{R}$-reversing consists in replacing a subword with another word that represents the same element of the group $\langle \mathcal{X} ; \mathcal{R} \rangle$, so an induction shows that, if w is reversible to w', then w and w' represent the same element of $\langle \mathcal{X} ; \mathcal{R} \rangle$. A slightly more careful argument gives the following result, which is stronger in general as it need not be true that the monoid congruence $\equiv$ is the restriction to positive words of the associated group congruence, *i.e.*, that the monoid $\langle \mathcal{X} ; \mathcal{R} \rangle^+$ embeds in the group $\langle \mathcal{X} ; \mathcal{R} \rangle$.

Lemma 1.3. [24] *Assume that u, v, u', v' are words in $\mathcal{X}^*$ and uv^{-1} is $\mathcal{R}$-reversible to ${v'}^{-1}u'$. Then we have $\overline{v'u} = \overline{u'v}$, i.e., $v'u$ and $u'v$ represent the same element in the monoid $\langle \mathcal{X} ; \mathcal{R} \rangle^+$. In particular, if $(\mathcal{X}, \mathcal{R})$ is complemented and u, v are words in $\mathcal{X}^*$ such that $u_{/^*v}$ exists, we have $\overline{v_{/^*u}} \cdot \overline{u} = \overline{u_{/^*v}} \cdot \overline{v}$.*

Thus, we see that (left) reversing constructs common left multiples. The question is whether *all* common left multiples are obtained in this way. The answer is not always positive, but the nice point is that there exists an effective criterion for recognizing when this happens—and that every locally left Gaussian monoid admits presentations for which this happens.

Proposition 1.4. [25] *(i) Assume that $(\mathcal{X}, \mathcal{R})$ is a complemented presentation satisfying the following conditions:*

(I) There exists a map ν of $\mathcal{X}^*$ to the ordinals, compatible with $\equiv$, and satisfying $\nu(uv) \geq \nu(u) + \nu(v)$ for all u, v and $\nu(\alpha) > 0$ for α in $\mathcal{X}$;

(II) We have $(\alpha_{/^*\beta})_{/^*(\gamma_{/^*\beta})} \equiv (\alpha_{/^*\gamma})_{/^*(\beta_{/^*\gamma})}$ for all α, β, γ in $\mathcal{X}$, this meaning that both sides exist and are equivalent, or that neither exists;

Then the monoid $\langle \mathcal{X} ; \mathcal{R} \rangle^+$ is locally left Gaussian, and, for all u, v in $\mathcal{X}^2$, the word $u_{/^*v}$ exists if and only if the elements $\overline{u}$ and $\overline{v}$ admit a common left multiple, and, in this case, $u_{/^*v}$ represents $\overline{u}_{/\overline{v}}$; Moreover, for all words u, v, w, we have

$$(u_{/^*v})_{/^*(w_{/^*v})} \equiv (u_{/^*w})_{/^*(v_{/^*w})}. \tag{1.6}$$

(ii) Conversely, assume that M is a locally left Gaussian monoid, and $\mathcal{X}$ is an arbitrary set of generators for M. Let $\mathcal{R}$ consist of one relation $v\alpha = u\beta$ for each pair of letters α, β in $\mathcal{X}$ such that α and β have a common left multiple, where u and v are chosen (arbitrary) representatives of $\alpha_{/\beta}$ and $\beta_{/\alpha}$ respectively. Then $(\mathcal{X}, \mathcal{R})$ is a complemented presentation of M that satisfies Conditions I and II.

Thus, Proposition 1.4 tells us that, in good cases, left word reversing computes the left complement operation (and, therefore, the left lcm) in the associated monoid. If M is a locally left Gaussian monoid, and $(\mathcal{X}, \mathcal{R})$ is a presentation of M as in Proposition 1.4(ii), then, if α and β belong to $\mathcal{X}$ and admit a common left multiple, the *word* $\alpha_{/^*\beta}$ of $\mathcal{X}^*$ represents the *element* $\alpha_{/\beta}$ of M. In particular, if $\mathcal{X}$ happens to be closed under left complement, the word $\alpha_{/^*\beta}$ has length 1, and it consists of the unique letter $\alpha_{/\beta}$. Thus, the operation $/^*$ can be seen as an extension of operation $/$ to words—as the notation suggests. However, it should be kept in mind that $u_{/^*v}$ is a word (not an element of the monoid), and that computing it depends not only on u, v, and M, but also on a particular presentation.

When M is a Gaussian monoid, then, for every set of generators $\mathcal{X}$, Proposition 1.4(ii) provides us with a good presentation of M, one for which lcm's can be computed using word reversing. In this case, the lcm always exists, the complement operation is everywhere defined, and, therefore, the operation $/^*$ on words is everywhere defined as well, which easily implies that word reversing from an arbitrary word over $\mathcal{X} \cup \mathcal{X}^{-1}$ always terminates with a word $v^{-1}u$ with u, v words over $\mathcal{X}$.

Example 1.5. The standard presentation of the braid monoid B_n^+, and, more generally, the Coxeter presentation of all Artin–Tits monoids, are eligible for Proposition 1.4: with a different setting, verifying that Conditions I and II are satisfied is the main technical task of [30, 9], as well as it is the task of [7] in the case of the Birman-Ko-Lee monoid BKL_n^+.

Assume that M is a locally left Gaussian monoid and $\mathcal{X}$ is a generating subset of M that is closed under left complement (a typical example is when M is a Garside monoid, and $\mathcal{X}$ the set of all nontrivial divisors of some Garside element Δ). Then, when applying Proposition 1.4(ii), we can choose for each pair α, β of letters, the relation

$$\beta_{/\alpha}\, \alpha = \alpha_{/\beta}\, \beta : \tag{1.7}$$

so, here, $\alpha_{/\beta}$ and $\beta_{/\alpha}$ are words of length 1 or 0, *i.e.*, letters or ε. The set of these relations, which depends only on M and on the choice of $\mathcal{X}$, will be denoted $\mathcal{R}_{\mathcal{X}}$ in the sequel. As the left and the right hand sides of every relation in $\mathcal{R}_{\mathcal{X}}$ have length 2 or 1, $\mathcal{R}_{\mathcal{X}}$-reversing does not increase the length of the words: for all words u, v in $\mathcal{X}^*$, the length of the word $u_{/^*v}$ is at most the length of the word u; in particular, for every letter α and every word v, the word $\alpha_{/^*v}$ has length 1 or 0, so it is either an element of $\mathcal{X}$ or the empty word. Another technically significant consequence is:

Lemma 1.6. *Assume that M is a locally left Gaussian monoid, and $\mathcal{X}$ is a generating subset of M that is closed under left complement. Then the following strenghtening of Relation* (1.6) *is satisfied by $\mathcal{R}_{\mathcal{X}}$-reversing: for all words u, v, w in $\mathcal{X}^*$, we have*

$$(u_{/^*v})_{/^*(w_{/^*v})} = (u_{/^*w})_{/^*(v_{/^*w})}. \tag{1.8}$$

Proof. Condition II gives an equivalence for the words in (1.8); now, if u has length 1, these words have length 1 at most, *i.e.*, they belong to $\mathcal{X}$ or are empty, and equivalence implies equality for such words. The general case follows using an induction. □

1.5. **The greedy normal form.** If M is a locally Gaussian monoid, and $\mathcal{X}$ is a generating subset of M that is closed enough, we can define a unique distinguished decomposition for every element x of M by considering the maximal left divisor of x lying in $\mathcal{X}$ and iterating the process. This construction is well known in the case of Artin–Tits monoids [26, 28, 45, 27], where it is known as the (left) greedy normal form, and it extends without change to all Garside monoids [23]. The case of locally Gaussian monoids is not really more complicated: the only point that could possibly fail is the existence of a maximal divisor of x belonging to $\mathcal{X}$; we shall see below that this existence is guaranteed by the Noetherianity condition. Here we describe the construction in the case of a locally *right* Gaussian monoid, *i.e.*, we use right lcm's, and not left lcm's as in most parts of this paper: Proposition 1.10 below will explain this choice.

Lemma 1.7. *Assume that M is a locally right Gaussian monoid, and $\mathcal{X}$ is a generating subset of M that is closed under right lcm. Then every nontrivial element x of M admits a unique greatest divisor lying in $\mathcal{X}$.*

Proof. Let $x = yz$ be a decomposition of x with $y \in \mathcal{X}$ and z minimal with respect to right division among all z' such that $x = y'z'$ holds for some y' in $\mathcal{X}$: such an element z exists since M is right Noetherian. Let y' be an arbitrary left divisor of x lying in $\mathcal{X}$. By construction, y and y' admit a common right multiple, namely x, hence they admit a right lcm y'' which belongs to $\mathcal{X}$, and we have $x = y''z''$ for some z''. Write $y'' = yt$. Then we have $x = yz = y''z'' = ytz''$, hence, by cancelling y on the left, $z = tz''$. The minimality hypothesis on z implies $t = 1$, hence $y'' = y$, *i.e.*, $y' \sqsubseteq y$. So every left divisor of x lying in $\mathcal{X}$ is a left divisor of y. The uniqueness of y then follows from 1 being the only invertible element of M, hence the relation $\sqsubseteq$ being an ordering. □

We deduce that, under the assumptions of Lemma 1.7, every nontrivial element x of M admits a unique decomposition $x = x_1 \cdots x_p$ such that, for each i, x_i is the greatest left divisor of $x_i \cdots x_p$ lying in $\mathcal{X}$. Indeed, if x_1 is the greatest left divisor of x lying in $\mathcal{X}$, we have $x = x_1x'$, and the hypothesis that $\mathcal{X}$ generates M guarantees that x_1 is not 1, hence x' is a proper right divisor of x, so the hypothesis that M is right Noetherian implies that the iteration of the process terminates in a finite number of steps.

What makes the distinguished decomposition constructed in this way interesting is the fact that it can be characterized using a purely local criterion, involving only two factors at one time. This criterion is crucial in the existence of an automatic structure [28], and it will prove crucial in our current development as well.

Definition. Assume that M is a monoid, and $\mathcal{X}$ is a subset of M. For x, y in M, we define the relation $x \triangleright_{\mathcal{X}} y$ to be true if every left divisor of xy lying in $\mathcal{X}$ is a left divisor of x.

Lemma 1.8. *Assume that M is a locally right Gaussian monoid, and $\mathcal{X}$ is a generating subset of M that is closed under right lcm and right complement. Then $x \triangleright_{\mathcal{X}} y \triangleright_{\mathcal{X}} z$ implies $x \triangleright_{\mathcal{X}} yz$.*

Proof. Let t be an element of $\mathcal{X}$ dividing xyz on the left. Let $x = x_1 \cdots x_p$ be a decomposition of x as a product of elements of $\mathcal{X}$. By hypothesis, t and x_1 have a common right multiple, namely xyz, hence a right lcm, say x_1t_1, and t_1, which is the right complement of t in x_1, belongs to $\mathcal{X}$ by hypothesis. Now we have $x_1t_1 \sqsubseteq x_1x_2 \cdots x_pyz$, hence $t_1 \sqsubseteq x_2 \cdots x_pyz$. By the same argument, t_1 and x_2 have a right lcm, say x_2t_2, with $t_2 \in \mathcal{X}$, and we have $t_2 \sqsubseteq x_3 \cdots x_pyz$. After p steps, we obtain t_p in $\mathcal{X}$ satisfying $t \sqsubseteq xt_p$, and $t_p \sqsubseteq yz$. The hypothesis $y \triangleright_{\mathcal{X}} z$ implies $t_p \sqsubseteq y$, hence $t \sqsubseteq xt_p \sqsubseteq xy$, and the hypothesis $x \triangleright_{\mathcal{X}} y$ then implies $t \sqsubseteq x$. So we proved that $t \sqsubseteq xyz$ implies $t \sqsubseteq x$ for $t \in \mathcal{X}$, *i.e.*, we proved $x \triangleright_{\mathcal{X}} xyz$. □

Definition. Assume that M is a monoid, and $\mathcal{X}$ is a subset of M. We say that a finite sequence $(x_1, \ldots, x_p)$ in $\mathcal{X}^p$ is $\mathcal{X}$-*normal* if, for $1 \leq i < p$, we have $x_i \triangleright_{\mathcal{X}} x_{i+1}$.

Proposition 1.9. *Assume that M is a locally right Gaussian monoid, and $\mathcal{X}$ is a generating subset of M that is closed under right lcm and right complement. Then every nontrivial element x of M admits a unique decomposition $x = x_1 \cdots x_p$ such that $(x_1, \ldots, x_p)$ is a $\mathcal{X}$-normal sequence.*

Proof. We have already seen that every element of M admits a unique decomposition of the form $x_1 \cdots x_p$ with $x_1, \ldots, x_p$ in $\mathcal{X}$ satisfying $x_i \triangleright_{\mathcal{X}} x_{i+1} \cdots x_p$ for each i. Clearly, $x_i \triangleright_{\mathcal{X}} x_{i+1} \cdots x_p$ implies $x_i \triangleright_{\mathcal{X}} x_{i+1}$, so the only problem is to show that, conversely, if we have $x_1 \triangleright_{\mathcal{X}} x_2 \triangleright_{\mathcal{X}} \cdots \triangleright_{\mathcal{X}} x_p$, then we have $x_i \triangleright_{\mathcal{X}} x_{i+1} \cdots x_p$ for each i: this follows from Lemma 1.8 using an induction on p. $\square$

In the sequel, we shall denote by $\mathrm{NF}(x)$ the $\mathcal{X}$-normal form of x. For our problem, the main property of the $\mathcal{X}$-normal form is the following connection between the normal forms of x and of $x\alpha$, established in [23] in the case of a Garside monoid:

Proposition 1.10. *Assume that M is a locally Gaussian monoid and $\mathcal{X}$ is generating subset of M that is closed under right lcm, and left and right complement. Then, for every x in M and every β in $\mathcal{X}$, we have*

$$\mathrm{NF}(x) = \mathrm{NF}(x\beta)_{/^{*}\beta}, \tag{1.9}$$

i.e., the $\mathcal{X}$-normal form of x is obtained by reversing the word $\mathrm{NF}(x\beta)\beta^{-1}$ on the left.

Proof. By hypothesis, the elements $x\beta$ and β admit a common left multiple, namely $x\beta$ itself, so reversing the word $\mathrm{NF}(x\beta)\beta^{-1}$ on the left must succeed with an empty denominator. Let $(\gamma_1, \dots, \gamma_p)$ be the $\mathcal{X}$-normal form of $x\beta$. Let us define the elements α_i and β_i by $\beta_p = \beta$, and, using descending induction,

$$\beta_{i-1} = \beta_{i/\gamma_i}, \qquad \alpha_i = \gamma_{i/\beta_i}$$

(Figure 2). The hypothesis that the elements $x\beta$ and β admit a common left multiple, namely $x\beta$ itself, in M guarantees that β_i and γ_i admit a common left multiple, and, therefore, the inductive definition leads to no obstruction, and, in addition, we must have $\beta_0 = 1$. By definition, the result of reversing $\gamma_1 \cdots \gamma_p\beta^{-1}$ to the left is the word $\alpha_1 \cdots \alpha_p$, so the question is to prove that $(\alpha_1, \dots, \alpha_p)$ is the $\mathcal{X}$-normal form of x. First, in M, we have $\alpha_1 \cdots \alpha_p = \gamma_1 \cdots \gamma_p\beta^{-1} = x\beta\beta^{-1} = x$, so the only question is to prove that the sequence $(\alpha_1, \dots, \alpha_p)$ is $\mathcal{X}$-normal.

We shall prove that, for each i, the relation $\gamma_i \triangleright_{\mathcal{X}} \gamma_{i+1}$, which is true as, by hypothesis, the sequence $(\gamma_1, \dots, \gamma_p)$ is $\mathcal{X}$-normal, implies $\alpha_i \triangleright_{\mathcal{X}} \alpha_{i+1}$.

So, let us assume that some element δ of $\mathcal{X}$ is a left divisor of $\alpha_i\alpha_{i+1}$. Then we have $\delta \sqsubseteq \alpha_i\alpha_{i+1}\beta_{i+1} = \beta_{i-1}\gamma_i\gamma_{i-1}$. Let $\beta_{i-1}\delta'$ be the right lcm of δ and β_{i-1}, which exists as $\beta_{i-1}\gamma_i\gamma_{i+1}$ is a common right multiple of δ and β_{i-1}. Then δ' belongs to $\mathcal{X}$, and we have $\delta' \sqsubseteq \gamma_i\gamma_{i+1}$, hence $\delta' \sqsubseteq \gamma_i$ as $\gamma_i \triangleright_{\mathcal{X}} g_{i+1}$ holds by hypothesis. Hence δ is a left divisor of $\beta_{i-1}\gamma_i$, *i.e.*, of $\alpha_i\beta_i$. Let $\alpha_i\delta''$ be the right lcm of δ and α_i. Then $\delta \sqsubseteq \alpha_i\alpha_{i+1}$ implies $\delta'' \sqsubseteq \alpha_{i+1}$, and $\delta \sqsubseteq \alpha_i\beta_i$ implies $\delta'' \sqsubseteq \beta_i$. Now, by construction, the only common left divisor of α_{i+1} and β_i is 1, for, otherwise, $\alpha_{i+1}\beta_{i+1}$ would not be the left lcm of β_{i+1} and γ_{i+1}. So we have $\delta'' = 1$, *i.e.*, δ is a left divisor of α_i, and $\alpha_i \triangleright_{\mathcal{X}} \alpha_{i+1}$ is true. $\square$

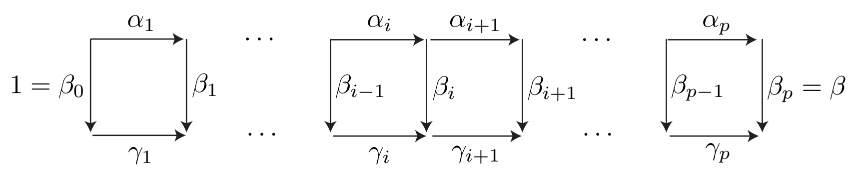


FIGURE 2. Computing the normal form using reversing

1.6. **Group of fractions vs. monoid.** Our purpose in the sequel is to compute the homology of a (semi)-Gaussian monoid starting from a presentation. When the considered monoid M satisfies Ore's conditions on the left, *i.e.*, when M is cancellative and any two elements of M admit a common left multiple, then M embeds in a group of left fractions G, and every presentation of M as a monoid is a presentation of G as a group. By tensorizing by $\mathbf{Z}G$ over $\mathbf{Z}M$ we can extend every (left) $\mathbf{Z}M$-module into a $\mathbf{Z}G$-module. As in in [42], we shall use the following result:

Proposition 1.11. [11] *Assume that M is a monoid satisfying the Ore conditions on the left. Let G be the group of fractions of M. Then the functor $R \to \mathbf{Z}G \otimes_{\mathbf{Z}M} R$ is exact.*

Corollary 1.12. *Under the above hypotheses, we have $H_*(G, \mathbf{Z}) = H_*(M, \mathbf{Z})$.*

So, from now on, we shall consider monoids exclusively. When the monoid happens to be an Ore monoid, the homology of the monoid automatically determines the homology of the associated group of fractions, but the case is not really specific.

2. The reversing resolution

In this section, we assume that M is a locally Gaussian monoid, *i.e.*, M is cancellative, left and right Noetherian, and every two elements of M admitting a common left (resp. right) multiple admits a left (resp. right) lcm. Next we assume that $\mathcal{X}$ is a generating subset of M not containing 1 that is closed under left and right lcm, and such that $\mathcal{X} \cup \{1\}$ is closed under left and right complement. Special cases are M being Gaussian (in this case, lcm's always exist), M being locally Garside (in this case, $\mathcal{X}$ can be assumed to be finite), and M being Garside (both conditions simultaneously: then, we can take for $\mathcal{X}$ the divisors of some Garside element Δ).

Our aim is to construct a resolution by free $\mathbf{Z}M$-modules for $\mathbf{Z}$, made into a trivial $\mathbf{Z}M$-module by putting $x \cdot 1 = 1$ for every x in M.

2.1. **The chain complex.** We shall consider in the sequel a cubical complex associated with finite families of distinct elements of $\mathcal{X}$ that admit a left lcm. To avoid redundant cells, we fix a linear ordering $<$ on $\mathcal{X}$.

Definition. For $n \geq 0$, we denote by $\mathcal{X}^{[n]}$ the family of all strictly increasing n-tuples $(\alpha_1, \ldots, \alpha_n)$ in $\mathcal{X}$ such that $\alpha_1, \ldots, \alpha_n$ admit a left lcm. We denote by C_n the free $\mathbf{Z}M$-module generated by $\mathcal{X}^{[n]}$. The generator of C_n associated with an element A of $\mathcal{X}^{[n]}$ is denoted $[A]$, and it is called an *n-cell*; the left lcm of A is then denoted by $\ulcorner A \lrcorner$. The unique 0-cell is denoted $[\emptyset]$.

The elements of C_n will be called *n-chains*. As a $\mathbf{Z}$-module, C_n is generated by the elements of the form $x[A]$ with $x \in M$; such elements will be called *elementary* n-chains.

The leading idea in the sequel is to associate to each n-cell an oriented n-cube reminiscent of a van Kampen diagram in M and constructed using the $\mathcal{R}_{\mathcal{X}}$-reversing process of Section 1. The vertices of that cube are elements of M, while the edges are labelled by elements of $\mathcal{X}$. The n-cube associated with $[\alpha_1, \ldots, \alpha_n]$ starts from the vertex 1 and ends at the vertex $\alpha_1 \vee \cdots \vee \alpha_n$, so the lcm of the generators $\alpha_1, \ldots, \alpha_p$ is the main diagonal of the cube, as the notation $\ulcorner A \lrcorner$ would suggest. We start with n edges labelled $\alpha_1, \ldots, \alpha_n$ pointing to the final vertex, and we construct the other edges backwards using left reversing, *i.e.*, we inductively close every pattern consisting of two converging edges α, β with two diverging edges $\beta_{/\alpha}$, $\alpha_{/\beta}$. The construction terminates with 2^n vertices. Finally, we associate with the elementary n-chain $x[A]$ the image of the n-cube (associated with) $[A]$ under the left translation by x: the cube starts from x instead of starting from 1.

Example 2.1. Let BKL_3^+ denote the Birman-Ko-Lee monoid for 3-strand braids, *i.e.*, the monoid $\langle a, b, c\,;\, ab = bc = ca\rangle^+$. Then BKL_3^+ is a Gaussian monoid, the element Δ defined by $\Delta = ab = bc = ca$ is a Garside element, and the non-trivial divisors of Δ are a, b, c, and Δ. Thus, we can take for $\mathcal{X}$ the 4-element set $\{a, b, c, \Delta\}$. The construction of the cube associated with the 3-cell $[a, b, c]$ is illustrated on Figure 3; the main diagonal happens to be Δ.

Similarly, the monoid B_4^+ of Example 1.2 is a Gaussian monoid, and the minimal Garside element is $\Delta_4 = \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1$; in this case, we can take for $\mathcal{X}$ the set of the 23 ($= 4!-1$) nontrivial divisors of Δ_4. The 3-cube associated with the cell $[\sigma_1, \sigma_2, \sigma_3]$ is displayed on Figure 4 (left).

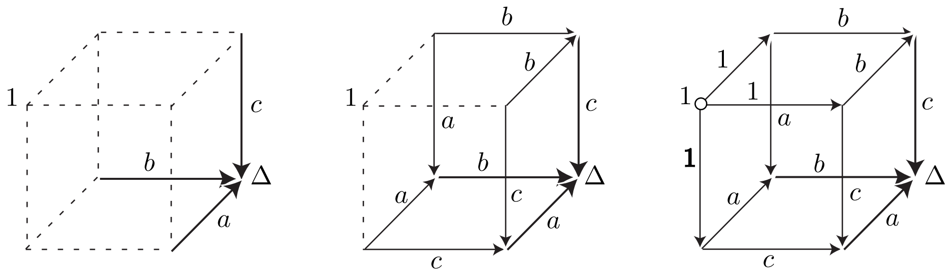


Figure 3. The 3-cube associated with a 3-cell, case of BKL_3^+

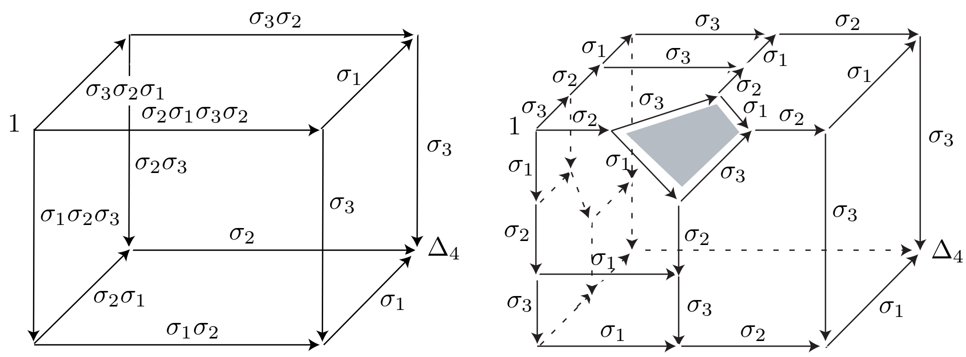


Figure 4. The 3-cube associated with the 3-cell $[\sigma_1, \sigma_2, \sigma_3]$ in B_4^+ when the generators are the divisors of Δ_4 (left) and when they are the σ_i's (right)

Remark 2.2. A similar construction can be made even if we do not assume our set of generators to be closed under left complement: once a complemented presentation has been chosen, we can associate with every n-tuple of generators $(\alpha_1, \dots, \alpha_n)$ the n-dimentional simplex obtained by starting with n terminal edges labeled $\alpha_1, \dots, \alpha_n$ and completing each open pattern consisting of two converging edges α, β with edges labeled $f(\alpha, \beta)$ and $f(\beta, \alpha)$, where f is the involved complement. The construction terminates when all open patterns have been closed, and the cube condition, as defined in [23], is the technical condition that guarantees that this hapens. When the set of generators is closed under left complement, the construction adds single edges at each step, and we finish with an n-cube. In the general case, the construction may add sequences of edges of length greater than 1, and, as a result, the final simplex may be more complicated than an n-cube, although it remains the skeleton of an n-ball. We display on Figure 4 (right) the 3-dimensional simplex associated with the 3-cell $[\sigma_1, \sigma_2, \sigma_3]$ in the standard presentation of the braid monoid B_4^+. Observe the grey facet $[\sigma_1, \sigma_3]$ starting at σ_2: its existence corresponds to the fact that the *words* $(\sigma_{2/^*\sigma_1})_{/^*(\sigma_{3/^*\sigma_1})}$ and $(\sigma_{2/^*\sigma_3})_{/^*(\sigma_{1/^*\sigma_3})}$, namely $\sigma_2\sigma_1\sigma_3\sigma_2$ and $\sigma_2\sigma_3\sigma_1\sigma_2$, are equivalent, but not equal.

With the previous intuition at hand, the definition of a boundary map is clear: for A an n-cell, we define $\partial_n[A]$ to be the $(n-1)$-chain obtained by enumerating the $(n-1)$-faces of the n-cube (associated with) $[A]$, which are $2n$ in number, with a sign corresponding to their orientation, and taking into account the vertex they start from. In order to handle such enumerations, we need to extend our notations.

Notation. (i) For $\alpha_1, \dots, \alpha_n$ in $\mathcal{X} \cup \{1\}$, we define $[\alpha_1, \dots, \alpha_n]$ to be

$$\varepsilon(\pi)[\alpha_{\pi(1)}, \dots, \alpha_{\pi(n)}]$$

if the α_i's are not equal to 1 and pairwise distinct, $\alpha_{\pi(1)}, \dots, \alpha_{\pi(n)}$ is their $<$-increasing enumeration, and $\varepsilon(\pi)$ is the signature of π, and to be 0_{C_n} in all other cases.

(ii) For A a cell, say $A = [\alpha_1, \dots, \alpha_n]$, and α an element of $\mathcal{X}$, we denote by $A_{/\alpha}$ the sequence $(\alpha_{1/\alpha}, \dots, \alpha_{n/\alpha})$; we denote by A^i (resp. $A^{i,j}$) the sequence obtained by removing the i-th term of A (resp. the i-th and the j-th terms).

Definition. (Figure 5) For $n \geq 1$, we define a $\mathbf{Z}M$-linear map $\partial_n : C_n \to C_{n-1}$ by

$$\partial_n[A] = \sum_{i=1}^{n} (-1)^i [A^i{}_{/\alpha_i}] - \sum_{i=1}^{n} (-1)^i \alpha_{i/A^i}[A^i], \tag{2.1}$$

for $A = (\alpha_1, \dots, \alpha_n)$; we define $\partial_0 : C_0 \to \mathbf{Z}$ by $\partial_0[\emptyset] = 1$.

So, in low degrees, the formulas take the following form:

$$\partial_1[\alpha] = \alpha[\emptyset] - [\emptyset], \qquad \partial_2[\alpha, \beta] = [\alpha_{/\beta}] + \alpha_{/\beta}[\beta] - [\beta_{/\alpha}] - \beta_{/\alpha}[\alpha]. \tag{2.2}$$

Example 2.3. For the Birman-Ko-Lee monoid $BKL3^+$, we read both on the above definition and on Figure 3 the value

$$\partial_3[a, b, c] = [b, c] - [a, c] + [a, b].$$

Here the coefficients are ± 1 as the labels of the three initial edges of the cube are empty words, thus representing 1 in M; the three missing factors are $[a, a]$, $[b, b]$, $[c, c]$, which are null by definition.

We suggest the reader to check on Figure 4 (left) the formula

$$\begin{aligned}\partial_3[\sigma_1, \sigma_2, \sigma_3] = -[\sigma_1\sigma_2, \sigma_3] + [\sigma_2\sigma_1, \sigma_2\sigma_3] - [\sigma_1, \sigma_3\sigma_2] \\ + \sigma_3\sigma_2\sigma_1[\sigma_2, \sigma_3] - \sigma_2\sigma_1\sigma_3\sigma_2[\sigma_1, \sigma_3] + \sigma_1\sigma_2\sigma_3[\sigma_1, \sigma_2]\end{aligned}$$

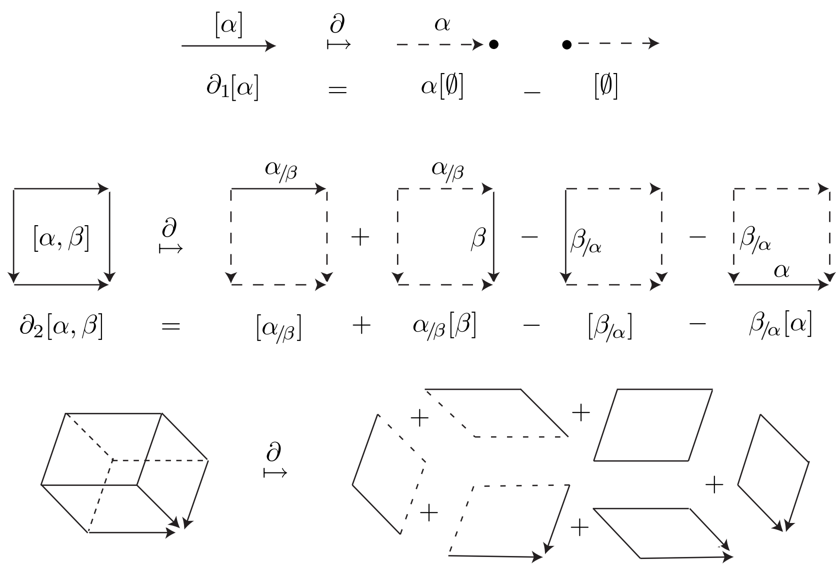


FIGURE 5. The boundary operator ∂

when we consider the monoid B_4^+ and take for $\mathcal{X}$ the divisors of the minimal Garside element Δ_4.

Proposition 2.4. *The module (C_*, ∂_*) is a complex: for $n \geq 1$, we have $\partial_{n-1}\partial_n = 0$.*

Proof. First, we have $\partial_1[\alpha] = \alpha[\emptyset] - [\emptyset]$, hence $\partial_0\partial_1[\alpha] = \alpha \cdot 1 - 1_M \cdot 1 = 0$.

Assume now $n \geq 2$. For $A = (\alpha_1, \dots, \alpha_n)$ with $\alpha_1, \dots, \alpha_n \in \mathcal{X}$, we obtain

$$\begin{aligned}
\partial_{n-1}\partial_n[A] &= \sum_i (-1)^i \partial_{n-1}[A^i{}_{/\alpha_i}] - \sum_i (-1)^i \alpha_{i/A^i}\, \partial_{n-1}[A^i] \\
&= \sum_{i \neq j} (-1)^{i+j+e(i,j)} [(A^{i,j}{}_{/\alpha_i})_{/(\alpha_{j/\alpha_i})}] \\
&\quad - \sum_{i \neq j} (-1)^{i+j+e(j,i)} (\alpha_{j/\alpha_i})_{/(A^{i,j}{}_{/\alpha_i})} [A^{i,j}{}_{/\alpha_i}] \\
&\quad - \sum_{i \neq j} (-1)^{i+j+e(j,i)} \alpha_{i/A^i}\, [A^{i,j}{}_{/\alpha_j}] \\
&\quad + \sum_{i \neq j} (-1)^{i+j+e(i,j)} \alpha_{i/A^i}\, \alpha_{j/A^{i,j}}\, [A^{i,j}],
\end{aligned} \tag{2.3}$$

with $e(i,j) = +1$ for $i < j$, and $e(i,j) = 0$ otherwise.

First, applying (1.3) to α_k, α_i, and α_j, we obtain $[(A^{i,j}{}_{/\alpha_i})_{/(\alpha_{j/\alpha_i})}] = [A^{i,j}{}_{/\alpha_i,\alpha_j}]$, where α_i and α_j play symmetric roles, and the first sum in (2.3) becomes

$$\sum_{i \neq j} (-1)^{i+j+e(i,j)} [A^{i,j}{}_{/\alpha_i,\alpha_j}].$$

Now, each factor $[A^{i,j}{}_{/\alpha_i,\alpha_j}]$ appears twice, with coefficients $(-1)^{i+j}$ and $(-1)^{i+j+1}$ respectively, so the sum vanishes.

When applied to α_j, α_i, and $A^{i,j}$, (1.3) gives $(\alpha_{j/\alpha_i})_{/(A^{i,j}{}_{/\alpha_i})} = \alpha_{j/A^j}$. It follows that the second and the third sum in (2.3) contain the same factors, but, as $e(i,j)+e(j,i)=1$ always holds, the signs are opposite, and the global sum is 0.

Finally, applying (1.2) to α_i, α_j, and $\ulcorner A^{i,j}\lrcorner$ gives $\alpha_{i/A^i}\,\alpha_{j/A^{i,j}} = (\alpha_i \vee \alpha_j)_{/A^{i,j}}$, in which α_i and α_j play symmetric roles. So, as for the first sum, every factor in the fourth sum appears twice with opposite signs, and the sum vanishes.

Observe that the case of null factors is not a problem above, as we always have $1_{/\alpha} = 1$ and $\alpha_{/1} = \alpha$, and, therefore, Formula (2.1) is true for degenerate cells. $\square$

It will be convenient in the sequel to extend the notation $[\alpha_1, \dots, \alpha_n]$ to the case when the letters α_i are replaced by words, *i.e.*, by finite sequences of letters. Actually, it will be sufficient here to consider the case when the first letter only is replaced by a word, *i.e.*, to consider extended cells of the form $[w, A]$ where w is a word over the alphabet $\mathcal{X}$ and A is a finite sequence of letters in $\mathcal{X}$.

Definition. For w a word over $\mathcal{X}$ and A in $\mathcal{X}^{[n]}$, the $(n+1)$-chain $[w, A]$ is defined inductively by

$$(2.4) \qquad [w, A] = \begin{cases} 0_{C_{n+1}} & \text{if } w \text{ is the empty word } \varepsilon, \\ [v, A_{/\alpha}] + \overline{v}_{/(A_{/\alpha})}\ [\alpha, A] & \text{for } w = v\alpha \text{ with } \alpha \in \mathcal{X}. \end{cases}$$

If w has length 1, *i.e.*, if v is empty in the inductive clause of (2.4) gives $[vA_{/\alpha}] = 0$ and $\overline{v}_{/(A_{/\alpha})} = 1$, so our current definition of $[w, A]$ is compatible with the previous one. Our extended notation should appear natural when one keeps in mind the geometrical intuition that the cell $[w, A]$ is to be associated with a $(n+1)$-parallelotope computing the left lcm of $\overline{w}$ and A using left reversing: in order to compute the left lcm of $\overline{v}\alpha$ and A, we first compute the left lcm of α and A, and then compute the left lcm of $\overline{v}$ and the complement of A in α, *i.e.*, of $A_{/\alpha}$. However, the rightmost cell does not start from 1, but from $\overline{v}_{/(A_{/\alpha})}$ as shown in Figure 6.

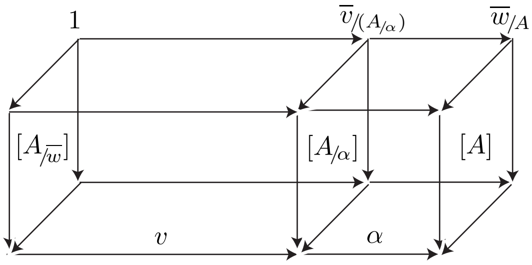


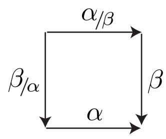
FIGURE 6. The chain $[w, A]$ for $w = v\alpha$

An easy induction shows that, for $w = \alpha_1 \dots \alpha_k$, we have for $[w]$ the simple expression

$$(2.5) \qquad [w] = \sum_{i=1}^{k} \overline{\alpha_1 \cdots \alpha_{i-1}}\ [\alpha_i].$$

Also observe that Formula (2.1) for ∂_2 can be rewritten as

$$(2.6) \qquad \partial_2[\alpha, \beta] = [\alpha_{/\beta}\, \beta] - [\beta_{/\alpha}\, \alpha],$$

according to the intuition that ∂_2 enumerates the boundary of  .

The following computational formula, which extends and generalizes (2.1) describes the boundary of the parallelotope associated with $[w, A]$ taking into account the specific role of the w-labelled edge: as shown in Figure 7, there is the right face $[A]$ at $\overline{w}_{/A}$, the left face $[A_{/\overline{w}}]$, the n lower faces $[w, A^i]$ at $\alpha_{i/A^i,w}$, and, finally, the n upper faces $[w_{/^*\alpha_i}, {A^i}_{/\alpha_i}]$.

Lemma 2.5. *For every word w, we have*

$$\partial_1[w] = -[\emptyset] + \overline{w}\,[\emptyset]$$

and, for $n \geq 1$ and every n-cell A,

$$\partial_{n+1}[w, A] = -[A_{/\overline{w}}] - \sum(-1)^i[w_{/^*\alpha_i}, {A^i}_{/\alpha_i}] + \sum(-1)^i \alpha_{i/\overline{w},A^i}[w, A^i] + \overline{w}_{/A}[A].$$

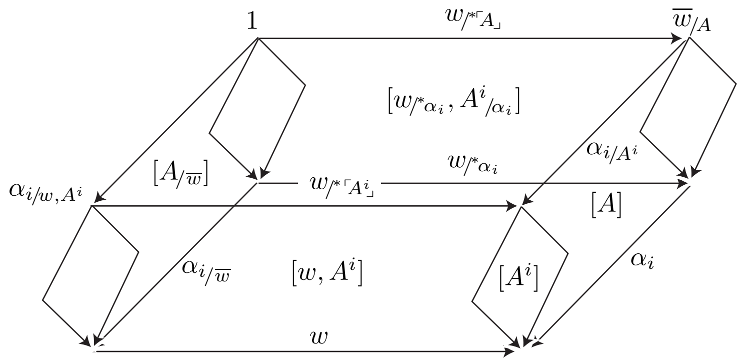


FIGURE 7. Decomposition of $[w, A]$

Proof. The case $n = 0$ is obvious, so assume $n \geq 1$. We use induction on the length of the word w. If w is empty, the factors $[w, A]$, $[w, A^i]$, $[w_{/^*\alpha_i}, {A^i}_{/\alpha_i}]$ vanish, we have $A_{/\overline{w}} = A$, and the right hand side reduces to $[A] - [A]$, hence to 0, and the equality holds. Otherwise, assume $w = v\alpha$. By definition, we have

$$\partial_{n+1}[w, A] = \partial_{n+1}[v, B] + \overline{v}_{/B}\,\partial_{n+1}[\alpha, A]$$

with $B = (\beta_1, \dots, \beta_n) = A_{/\alpha}$. Applying the induction hypothesis for $\partial_{n+1}[v, B]$ and the definition for $\partial_{n+1}[\alpha, A]$, which reads

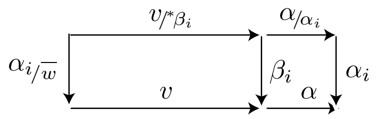
$$\partial_{n+1}[\alpha, A] = -[B] - \sum(-1)^i[\alpha_{/\alpha_i}, {A^i}_{/\alpha_i}] + \sum(-1)^i \alpha_{i/\alpha,A^i}[\alpha, A^i] + \alpha_{/A}[A],$$

we obtain

$$\begin{aligned} \partial_{n+1}[w, A] = &- [B_{/\overline{v}}] - \sum(-1)^i[v_{/^*\beta_i}, {B^i}_{/\beta_i}] \\ &+ \sum(-1)^i \beta_{i/\overline{v},B^i}[v, B^i] + \overline{v}_{/B}\,[B] \\ &- \overline{v}_{/B}\,[B] - \sum(-1)^i\,\overline{v}_{/B}\,[\alpha_{/\alpha_i}, {A^i}_{/\alpha_i}] \\ &+ \sum(-1)^i\,\overline{v}_{/B}\,\alpha_{i/\alpha,A^i}[\alpha, A^i] + \overline{v}_{/B}\,\alpha_{/A}[A]. \end{aligned} \tag{2.7}$$

We have $\beta_{i/\overline{v}} = \alpha_{i/\overline{w}}$ by (1.5), so the first factor in (2.7) is $-[A_{/\overline{w}}]$. Then, the two medial factors vanish, and, by construction again, we have $v_{/^*B}\,\alpha_{/A} = w_{/^*A}$, so the last factor is $\overline{w}_{/A}[A]$. There remains the two negative sums, and the two positive ones. The i-th factors in the negative sums are

$$[v_{/^*\beta_i}, {B^i}_{/\beta_i}] + \overline{v}_{/B}\,[\alpha_{/\alpha_i}, {A^i}_{/\alpha_i}],$$

and we claim that this is $[w_{/^*\alpha_i}, {A^i}_{/\alpha_i}]$. Indeed, we have $w_{/^*\alpha_i} = v_{/^*\beta_i}\,\alpha_{/\alpha_i}$ as can be read on , so (2.4) gives

$$[w_{/^*\alpha_i}, {A^i}_{/\alpha_i}] = [v_{/^*\beta_i}, ({A^i}_{/\alpha_i})_{/(\alpha_{/\alpha_i})}] + (\overline{v}_{/\beta_i})_{/(({A^i}_{/\alpha_i})_{/(\alpha_{/\alpha_i})})}\,[\alpha_{/\alpha_i}, {A^i}_{/\alpha_i}].$$

By (1.3), we have first $({A^i}_{/\alpha_i})_{/(\alpha_{/\alpha_i})} = ({A^i}_{/\alpha})_{/(\alpha_{i/\alpha})} = {B^i}_{/\beta_i}$, and, then,

$$(\overline{v}_{/\beta_i})_{/(({A^i}_{/\alpha_i})_{/(\alpha_{/\alpha_i})})} = (\overline{v}_{/\beta_i})_{/({B^i}_{/\beta_i})} = \overline{v}_{/\beta_i, B^i} = \overline{v}_{/B},$$

which proves the claim.

The argument for the positive factors in (2.7) is similar. The i-th factors are

$$\beta_{i/\overline{v},B^i}[v, B^i] + \overline{v}_{/B}\,\alpha_{i/\alpha,A^i}[\alpha, A^i],$$

which we claim is $\alpha_{i/\overline{w},A^i}[w, A^i]$. Indeed, (2.4) gives

$$[w, A^i] = [v, B^i] + \overline{v}_{/B^i}\,[\alpha, A^i],$$

and it remains to check the equalities

$$\beta_{i/\overline{v},B^i} = \alpha_{i/\overline{w},A^i}, \quad \text{and} \quad \overline{v}_{/B}\cdot\alpha_{i/\alpha,A^i} = \alpha_{i/\overline{w},A^i}\cdot\overline{v}_{/B^i} :$$

both can be read on the diagram of Figure 8, whose commutativity directly follows from the associativity of the lcm operation. ☐

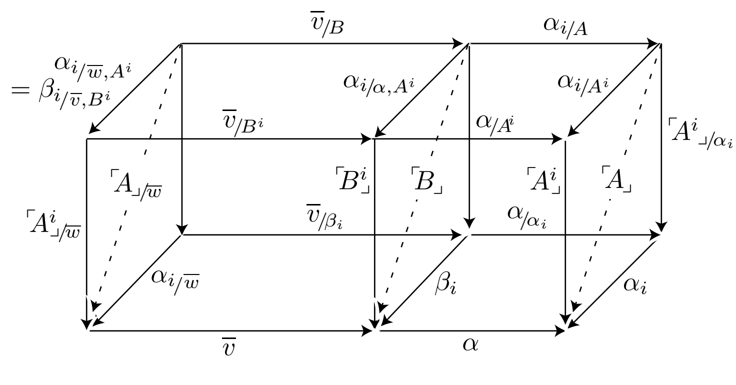


FIGURE 8. Computation of $\overline{w} \vee \ulcorner A \lrcorner$ with $w = v\alpha$

2.2. **A contracting homotopy.** Our aim is to prove

Proposition 2.6. *For each locally Gaussian monoid M, the complex (C_*, ∂_*) is a resolution of the trivial $\mathbf{Z}M$-module $\mathbf{Z}$ by free $\mathbf{Z}M$-modules.*

To this end, it is sufficient to construct a contracting homotopy for (C_*, ∂_*), *i.e.*, a family of $\mathbf{Z}$-linear maps $s_n : C_n \to C_{n+1}$ satisfying $\partial_{n+1}s_n + s_{n-1}\partial_n = \mathrm{id}_{C_n}$ for each degree n. We shall do it using the $\mathcal{X}$-normal form. Once again, the geometric intuition is simple: as the chain $x[A]$ represents the cube $[A]$ with origin translated to x, we shall define $s_n(x[A])$ to be an $(n+1)$-parallelotope whose terminal face is $[A]$ starting at x. To specify this simplex, we have to describe its $n+1$ terminal edges: n of them are the elements of A; the last one must force the main diagonal to be $x\ulcorner A\lrcorner$: the most obvious choice is to take the normal form of $x\ulcorner A\lrcorner$ itself, which guarantees in addition that the initial face will contain only trivial labels, *i.e.*, labels equal to 1.

Definition. The $\mathbf{Z}$-linear mapping $s_n : C_n \to C_{n+1}$ is defined for x in M by

$$s_n(x[A]) = [\mathrm{NF}(x\ulcorner A\lrcorner), A] \tag{2.8}$$

(Figure 9); we define $s_{-1} : \mathbf{Z} \to C_0$ by $s_{-1}(1) = [\emptyset]$.

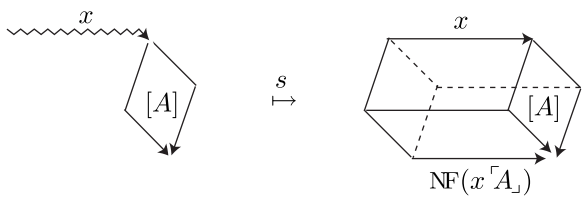


FIGURE 9. The contracting homotopy s

So we have in particular

$$s_0(x[\emptyset]) = [\mathrm{NF}(x)], \quad \text{and} \quad s_1(x[\alpha]) = [\mathrm{NF}(x\alpha), \alpha] \tag{2.9}$$

for every x in M and every α in $\mathcal{X}$.

Lemma 2.7. *For $n \geq 0$, we have $\partial_{n+1}s_n + s_{n-1}\partial_n = \mathrm{id}_{C_n}$.*

Proof. Assume first $n = 0$, and $x \in M$. Let $w = \mathrm{NF}(x)$. We have $s_0(x[\emptyset]) = [w]$, hence $\partial_1 s_0(x[\emptyset]) = \partial_1[w] = -[\emptyset] + x[\emptyset]$, and, on the other hand, $\partial_0(x[\emptyset]) = x \cdot 1 = 1$, hence $s_{-1}\partial_0(x[\emptyset]) = [\emptyset]$, and $(\partial_1 s_0 + s_{-1}\partial_0)(x[\emptyset]) = x[\emptyset]$.

Assume now $n \geq 1$ (see Figure 10 for the case $n = 2$). Let $w = \mathrm{NF}(x\ulcorner A\lrcorner)$. Applying the definition of s_n and Lemma 2.5, we find

$$\begin{aligned}\partial_{n+1}s_n(x[A]) = -[A_{/\overline{w}}] &- \sum(-1)^i[w_{/^*\alpha_i}, A^i{}_{/\alpha_i}] \\ &+ \sum(-1)^i\alpha_{i/\overline{w},A^i}[w, A^i] + \overline{w}_{/A}[A].\end{aligned}$$

By construction, each α_i is a right divisor of $\overline{w}$, *i.e.*, of $x\ulcorner A\lrcorner$, so we have $[A_{/\overline{w}}] = [\varepsilon, \dots, \varepsilon] = 0$. At the other end, we have $\overline{w}_{/A} = (x\ulcorner A\lrcorner)_{/A} = x$. Then α_i is a right divisor of $\overline{w}$, so we have $\alpha_{i/\overline{w},A^i} = 1$, and it remains

$$\partial_{n+1}s_n(x[A]) = -\sum(-1)^i[w_{/^*\alpha_i}, A^i{}_{/\alpha_i}] + \sum(-1)^i[w, A^i] + x[A].$$

On the other hand, we have by definition

$$\partial_n(x[A]) = \sum_i (-1)^i x[A^i{}_{/\alpha_i}] - \sum_i (-1)^i x\alpha_{i/A^i}[A^i].$$

Now we have $x\ulcorner A^i{}_{/\alpha_i}\lrcorner \cdot \alpha_i = x\ulcorner A\lrcorner$, which, by Proposition 1.10, implies that the $\mathcal{X}$-normal form of $x\ulcorner A^i{}_{/\alpha_i}\lrcorner$ is $w_{/^*\alpha_i}$. Then $x\alpha_{i/A^i}\ulcorner A^i\lrcorner$ is equal to $x\ulcorner A\lrcorner$, and, therefore, its normal form is w. Applying the definition of s_{n-1}, we deduce

$$s_{n-1}\partial_n(x[A]) = \sum(-1)^i[w_{/^*\alpha_i}, A^i{}_{/\alpha_i}] - \sum(-1)^i[w, A^i],$$

and, finally, $(\partial_{n+1}s_n + s_{n-1}\partial_n)(x[A]) = x[A]$. □

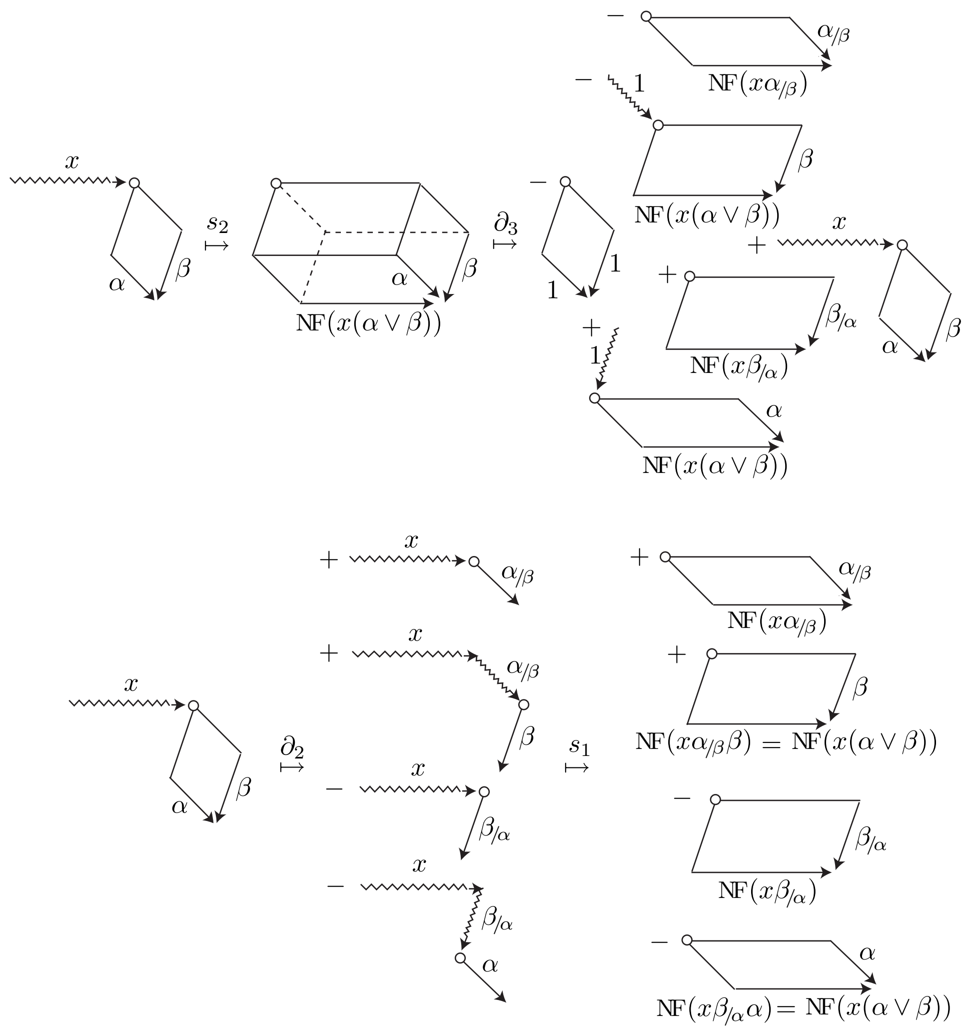


FIGURE 10. Comparing $\partial_3 s_2(x[\alpha,\beta])$ and $s_1\partial_2(x[\alpha,\beta])$: in the sum, there remain only the trivial left face, and $x[\alpha,\beta]$ itself.

Thus the sequence s_* is a contracting homotopy for the complex (C_*, ∂_*), and Proposition 2.6 is established.

Remark 2.8. The point in the previous argument and, actually, in the whole construction, is the fact that the normal form is computed by left reversing: this is what makes the explicit direct definition of the contracting homotopy possible. There is no need that the normal form we use be exactly the $\mathcal{X}$-normal form of Section 1: the only required property is that stated in Proposition 1.10, namely that, if w is the normal form of $x\beta$, then the normal form of x is obtained from w and β by left reversing.

2.3. **Applications.** By definition, the set $\mathcal{X}^{[n]}$ is a basis for the degree n module C_n in our resolution of $\mathbf{Z}$ by free $\mathbf{Z}M$-modules. If the set $\mathcal{X}$ happens to be finite, then $\mathcal{X}^{[n]}$ is empty for n larger than the cardinality of $\mathcal{X}$, and the resolution is finite. By definition, choosing a finite set $\mathcal{X}$ with the required closure properties is possible in those monoids we called locally Garside monoids in Section 1, so we may state:

Proposition 2.9. *Every locally Garside monoid is of type FL.*

Every Garside monoid admits a group of fractions, so, using Proposition 1.11, we deduce

Corollary 2.10. *Every Garside group is of type FL.*

As our constructions are explicit, they can be used to practically compute the homology of the considered monoid (or group). Indeed, let d_n be the $\mathbf{Z}$-linear mapping on C_n such that $\mathrm{d}_n[A]$ is obtained from $\partial_n[A]$ by collapsing all M-coefficients to 1. Then we have

$$H_n(M, \mathbf{Z}) = \mathrm{Ker}\,\mathrm{d}_n / \mathrm{Im}\,\mathrm{d}_{n+1}.$$

Below is an example of such computations.

Example 2.11. Let us consider the Birman-Ko-Lee monoid BKL_3^+ of Example 2.1 with $\mathcal{X} = \{a, b, c, \Delta\}$. We recall that, by Proposition 2.3, the homology of BKL_3^+ is also that of its group of fractions, here the braid group B_3.

First, we find $\partial_1[a] = (a-1)[\emptyset]$, hence $\mathrm{d}_1[a] = 0$. The result is similar for all 1-cells, and $\mathrm{Ker}\,\mathrm{d}_1$ is generated by $[a]$, $[b]$, $[c]$, and $[\Delta]$.

Then, we find $\partial_2[a,b] = [a]+a[b]-[c]-c[a]$, hence $\mathrm{d}_2[a,b] = [b]-[c]$, and, similarly, $\mathrm{d}_2[b,c] = [c]-[a]$, $\mathrm{d}_2[a,c] = [b]-[a]$, $\mathrm{d}_2[a,\Delta] = [\Delta]-[a]-[c]$, $\mathrm{d}_2[b,\Delta] = [\Delta]-[b]-[a]$, and $\mathrm{d}_2[c,\Delta] = [\Delta]-[c]-[b]$. It follows that $\mathrm{Im}\,\mathrm{d}_2$ is generated by the images of $[a,b]$, $[a,c]$, and $[b,\Delta]$, namely $[b]-[a]$, $[c]-[b]$, and $[\Delta]-[b]-[a]$, and we deduce

$$H_1(B_3, \mathbf{Z}) = H_1(BKL_3^+, \mathbf{Z}) = \mathrm{Ker}\,\mathrm{d}_1/\mathrm{Im}\,\mathrm{d}_2 = \mathbf{Z}.$$

Then, it is easy to check that $\mathrm{Ker}\,\mathrm{d}_2$ is generated by $[b,c]+[a,b]-[a,c]$, $[a,\Delta]-[b,\Delta]-[a,b]$, and $[c,\Delta]-[a,\Delta]+[a,c]$. Next, from the value of $\partial_3[a,b,c]$ computed in Example 2.3, we deduce $\mathrm{d}_3[a,b,c] = [b,c]-[a,c]+[a,b]$, and, similarly, $\mathrm{d}_3[a,b,\Delta] = [b,\Delta]-[a,\Delta]+[a,b]$, $\mathrm{d}_3[b,c,\Delta] = [c,\Delta]-[b,\Delta]+[b,c]$, and $\mathrm{d}_3[a,c,\Delta] = [c,\Delta]-[a,\Delta]+[a,c]$. Therefore $\mathrm{Im}\,\mathrm{d}_3$ is generated by $[b,c]+[a,b]-[a,c]$, $[a,\Delta]-[b,\Delta]-[a,b]$, and $[c,\Delta]-[a,\Delta]+[a,c]$, so it coincides with $\mathrm{Ker}\,\mathrm{d}_2$, and we conclude

$$H_2(B_3, \mathbf{Z}) = H_2(BKL_3^+, \mathbf{Z}) = \mathrm{Ker}\,\mathrm{d}_2/\mathrm{Im}\,\mathrm{d}_3 = 0.$$

We also see that $\mathrm{Ker}\,\mathrm{d}_3$ is generated by $[a,b,c]-[a,b,\Delta]-[b,c,\Delta]+[a,c,\Delta]$.

Finally, we compute

$$\begin{aligned}\partial_4[a,b,c,\Delta] &= -[c,c,c] + [a,a,a] - [b,b,b] + [\varepsilon,\varepsilon,\varepsilon] + a_{/b,c,\Delta}[b,c,\Delta] \\ &\qquad - b_{/a,c,\Delta}[a,c,\Delta] + c_{/a,b,\Delta}[a,b,\Delta] - \Delta_{/a,b,c}[a,b,c] \\ &= [b,c,\Delta] - [a,c,\Delta] + [a,b,\Delta] - [a,b,c].\end{aligned}$$

So we have $\mathrm{d}_4[a,b,c,\Delta] = [b,c,\Delta] - [a,c,\Delta] + [a,b,\Delta] - [a,b,c]$, $\operatorname{Im}\mathrm{d}_4$ coincides with $\operatorname{Ker}\mathrm{d}_3$, and $H_3(BKL_3^+, \mathbf{Z})$ is trivial (as will be obvious in the next sections).

Remark 2.12. As was observed in Remark 2.2 and illustrated in Figure 4 (right), it is still possible to associate with every n-tuple of generators an n-dimensional simplex by using reversing when we consider an arbitrary set of generators $\mathcal{X}$ instead of the divisors of some Garside element Δ, provided Conditions I and II of Proposition 1.4 is satisfied. We can construct in this way a complex C_*, and use reversing to define the boundary: the formulas are not so simple as in (2.1) because the simplex is not a cube in general, but the principle remains the same, and a precise definition can be given using induction of $\nu(\ulcorner A \lrcorner)$, where ν is a mapping satisfying Condition I. For instance, we obtain with the standard generators of B_4^+

$$\begin{aligned}\partial_3[\sigma_1,\sigma_2,\sigma_3] = (-1 &+ \sigma_1 - \sigma_2\sigma_1 + \sigma_3\sigma_2\sigma_1)[\sigma_2,\sigma_3] \\ &+ (-1 + \sigma_2 - \sigma_1\sigma_2 - \sigma_3\sigma_2 + \sigma_1\sigma_3\sigma_2 - \sigma_2\sigma_1\sigma_3\sigma_2)[\sigma_1,\sigma_3] \\ &+ (-1 + \sigma_3 - \sigma_2\sigma_3 + \sigma_1\sigma_2\sigma_3)[\sigma_1,\sigma_2]\end{aligned}$$

where the term $\sigma_2[\sigma_1,\sigma_3]$ corresponds to the grey facet on Figure 4 (right). The question of whether this complex is exact will be left open here (see the end in Section 3 for further discussion).

3. A simplicial resolution

In general, the resolution constructed in Section 2 is far from minimal. In this section, we show how to deduce a shorter resolution by decomposing each n-cube into $n!$ n-simplexes. In the special case of Garside monoids, the resolution so obtained happens to be the one considered by Charney, Meier, and Whittlesey in [14].

3.1. Descending cells.

We keep the hypotheses of Section 2, *i.e.*, we assume that M is a locally Gaussian monoid, and that $\mathcal{X}$ is a fixed set of generators of M not containing 1 that is closed under left and right lcm, and such that $\mathcal{X} \cup \{1\}$ is closed under left and right complement. We start from the complex (C_*, ∂_*), and extract a subcomplex which is still a resolution of $\mathbf{Z}$.

The point is to distinguish those cells in C_* that are decreasing with respect to right divisibility. In order that our definitions make sense, we shall assume in the sequel that the linear order on $\mathcal{X}$ used to enumerate the cells is chosen so that $\alpha < \beta$ holds whenever β is a proper right divisor of α: this is possible, as we assume that right division in M has no cycle.

Definition. We say that an n-cell $[\alpha_1, \ldots, \alpha_n]$ is *descending* if α_{i+1} is a proper right divisor of α_i for each i. The submodule of C_n generated by descending n-cells will be denoted by C'_n.

According to our intuition that the cell $[\alpha_1, \ldots, \alpha_n]$ is associated with an n-cube representing the computation of the lcm of $\alpha_1, \ldots, \alpha_n$, a descending n-cell is associated with a special n-cube with many edges labelled 1, and it is accurately associated with an n-simplex, as shown in Figure 11.

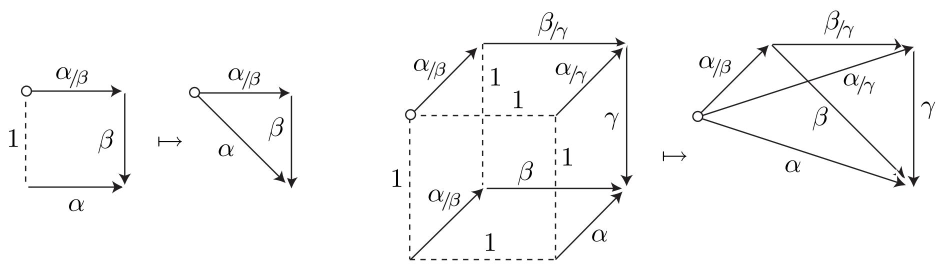


FIGURE 11. The n-simplex associated with a descending n-cell

The first, easy remark is that the boundary of a descending cell consists of descending cells exclusively.

Lemma 3.1. *The differential ∂_* maps C'_* to itself; more precisely, if $[A]$ is a descending n-cell, say $A = (\alpha_1, \dots, \alpha_n)$, we have*

$$\partial_n[A] = \alpha_{1/\alpha_2}[A^1] - \sum_{i=2}^{n}(-1)^i[A^i] + (-1)^n[A^n{}_{/\alpha_n}]. \tag{3.1}$$

Proof. If α_j is a right divisor of α_i, we have $\alpha_{j/\alpha_i} = 1$ by definition. So, when (2.1) is applied to compute $\partial_n[A]$, each factor $[A^i{}_{/\alpha_i}]$ with $i < n$ contains α_{n/α_i}, which is 1, so this factor vanishes, and the only remaining factor from the first sum is $[A^n{}_{/\alpha_n}]$, *i.e.*, $[\alpha_{1/\alpha_n}, \dots, \alpha_{n-1/\alpha_n}]$, a descending cell as, by Formula (1.3), x being a right divisor of y implies $x_{/z}$ being a right divisor of $y_{/z}$ for every z.

Next, the hypothesis that $[A]$ is descending implies that the lcm of A^1 is α_2, while, for $i \geq 2$, the lcm of A_i is α_1, of which each α_i is a right divisor, and (3.1) follows. ☐

In particular, we obtain $\partial_2[\alpha, \beta] = \alpha_{/\beta}[\beta] - [\alpha] + [\alpha_{/\beta}]$ for $[\alpha, \beta]$ in C'_2, and $\partial_3[\alpha, \beta, \gamma] = \alpha_{/\beta}[\beta, \gamma] - [\alpha, \gamma] + [\alpha, \beta] - [\alpha_{/\gamma}, \beta_{/\gamma}]$ for $[\alpha, \beta, \gamma]$ in C'_3, as can be read on Figure 11.

So it makes sense to consider the restriction ∂'_* of ∂_* to C'_*, and we obtain in this way a new complex. Our aim in this section is to prove

Proposition 3.2. *For each locally Gaussian monoid M, the subcomplex (C'_*, ∂'_*) of (C_*, ∂_*) is a finite resolution of the trivial $\mathbf{Z}M$-module $\mathbf{Z}$ by free $\mathbf{Z}M$-modules.*

In order to prove Proposition 3.2, we shall construct a contracting homotopy. The section s_* considered in Section 2 cannot be used, as s_n does not map C'_n to C'_{n+1} in general. However, it is easy to construct the desired section by introducing a convenient $\mathbf{Z}M$-linear mapping of C_n into C'_n. The idea is to partition each n-cube into the union of $n!$ disjoint n-simplexes.

Starting from an arbitrary n-cell $[\alpha_1, \dots, \alpha_n]$, one can obtain a descending n-cell by taking lcm's: indeed, by construction, the n-cell

$$[\alpha_1 \vee \cdots \vee \alpha_n, \alpha_2 \vee \cdots \vee \alpha_n, \dots, \alpha_{n-1} \vee \alpha_n, \alpha_n] \tag{3.2}$$

is descending. The n-cell in (3.2) will be denoted $[\![\alpha_1, \dots, \alpha_n]\!]$ is the sequel. Note that $[\![A]\!] = [A]$ is true whenever $[A]$ is descending.

If π is a permutation of $\{1, \dots, n\}$, and A is an n-sequence, say $A = (\alpha_1, \dots, \alpha_n)$, we shall denote by A^π the sequence $(\alpha_{\pi(1)}, \dots, \alpha_{\pi(n)})$. So, for each n-cell $[A]$ in C_n, we have a family of $n!$ descending n-cells $[\![A^\pi]\!]$ in C'_n. The simplexes associated with the descending cells $[\![A^\pi]\!]$ make a partition of the cube associated with $[A]$. For instance, in dimension 2, we have the decomposition of the square $[\alpha, \beta]$ into the two triangles $[\alpha \vee \beta, \beta]$ and $[\alpha \vee \beta, \alpha]$. Similarly, in dimension 3, we have the decomposition of the cube $[\alpha, \beta, \gamma]$ into the six tetrahedra shown in Figure 12.

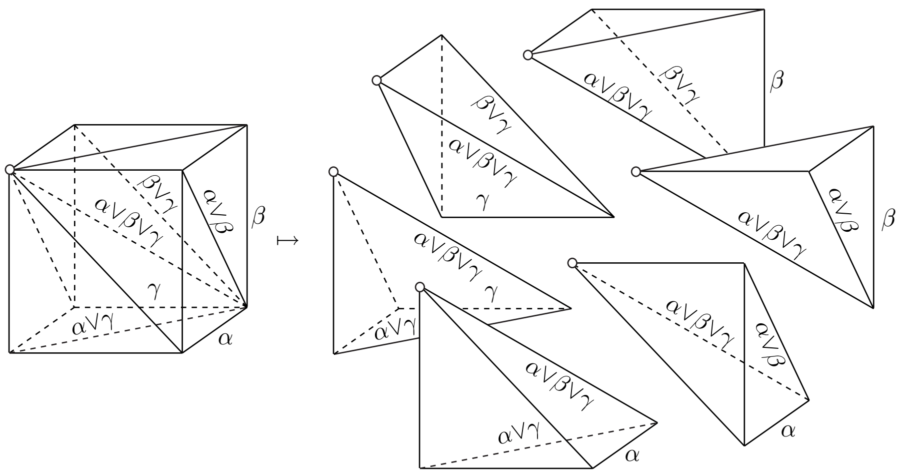


Figure 12. Decomposition of a cube into six tetrahedra

Definition. For each n, we define a $\mathbf{Z}M$-linear map $f_n : C_n \to C'_n$ by

$$f_n[A] = \sum_{\pi \in \mathfrak{S}_n} \varepsilon(\pi)[\![A^\pi]\!]. \tag{3.3}$$

The following observation is straightforward:

Lemma 3.3. *The map f_n is the identity on C'_n.*

Proof. Assume that $[A]$ is a descending n-cell, say $[A] = [\alpha_1, \dots, \alpha_n]$. Let π be a permutation not equal to identity. Then there exists a least integer i such that $\pi(i) > i$ is true. Then, by construction, we have

$$\alpha_{\pi(i)} \vee \dots \vee \alpha_{\pi(n)} = \alpha_i = \alpha_{\pi(i+1)} \vee \dots \vee \alpha_{\pi(n)},$$

which shows that the cell $[\![A^\pi]\!]$ contains twice α_i, and, therefore, it is trivial. Thus the only nontrivial factor in $f_n[A]$ is $[\![A]\!]$ (*i.e.*, $[A]$). □

The point now is that the boundary operator ∂_* happens to be compatible with the decomposition map f_* in the following sense:

Lemma 3.4. *For each n-cell $[A]$, we have $\partial_n f_n[A] = f_{n-1}\partial_n[A]$.*

Proof. (Figure 12 for the case $n = 3$.) By definition, we have

$$\partial_n f_n[A] = \sum_{\pi \in \mathfrak{S}_n} \varepsilon(\pi)\partial_n[\![A^\pi]\!].$$

According to Lemma 3.1, the contribution of the descending cell $[\![A^\pi]\!]$ consists of:

$$\alpha_{\pi(1)}\vee\cdots\vee\alpha_{\pi(n)}{}_{/\alpha_{\pi(2)}\vee\cdots\vee\alpha_{\pi(n)}}[\alpha_{\pi(2)}\vee\cdots\vee\alpha_{\pi(n)},\ldots,\alpha_{\pi(n)}], \tag{3.4}$$

$$-\sum_{i=2}^{n}[\alpha_{\pi(1)}\vee\cdots\vee\alpha_{\pi(n)},\ldots,\alpha_{\pi(i)}\widehat{\vee\cdots\vee}\alpha_{\pi(n)},\ldots,\alpha_{\pi(n)}], \tag{3.5}$$

$$+(-1)^n[\alpha_{\pi(1)}\vee\cdots\vee\alpha_{\pi(n)}{}_{/\alpha_{\pi(n)}},\ldots,\alpha_{\pi(n-1)}\vee\alpha_{\pi(n)}{}_{/\alpha_{\pi(n)}}]. \tag{3.6}$$

When π ranges over $\mathfrak{S}_n$ and $\varepsilon(\pi)$ is added, the sum of the factors (3.4) with $\pi(1)=i$ is

$$\sum_{\pi(1)=i}\varepsilon(\pi)\alpha_{i/A^i}[\alpha_{\pi(2)}\vee\cdots\vee\alpha_{\pi(n)},\ldots,\alpha_{\pi(n)}],$$

which, by definition, is equal to $(-1)^{i+1}\alpha_{i/A^i}f_{n-1}[A^i]$. Then each factor in (3.5) appears twice, with opposite signs due to $\varepsilon(\pi)$, and the global contribution of these factors is null. Finally, observing that $\alpha_{\pi(k)}\vee\cdots\vee\alpha_{\pi(n)}{}_{/\alpha_{\pi(n)}}$ is always equal to $\alpha_{\pi(k)}\vee\cdots\vee\alpha_{\pi(n-1)}{}_{/\alpha_{\pi(n)}}$, and using Formula (1.2), we see that the sum of the factors (3.6) with $\pi(n)=i$ is

$$\sum_{\pi(n)=i}\varepsilon(\pi)[\alpha_{\pi(1)/\alpha_i}\vee\cdots\vee\alpha_{\pi(n-1)/\alpha_i},\ldots,\alpha_{\pi(n-1)/\alpha_i}],$$

which, by definition, is equal to $(-1)^i f_{n-1}[A^i{}_{/\alpha_i}]$. Summing up, and applying the definition of $\partial_n[A]$, we conclude that $\partial_n f_n[A]$ is equal to the image of $\partial_n[A]$ under f_{n-1}. □

Proposition 3.2 now follows immediately, as the conjunction of Lemma 3.3 and Lemma 3.4 shows that f_* is a retraction of C_* to C'_*: defining $s'_n : C'_n \to C'_{n+1}$ by $s'_n = f_{n+1}s_n$, we obtain that s'_* is a contracting homotopy for (C'_*, ∂'_*), and the latter is therefore an exact complex.

Example 3.5. The interest of restricting to descending cells is clear: first, the length of the resolution, and the dimensions of the modules are drastically reduced; secondly, the boundary operator is now given by Formula (3.1)—or, equivalently, (3.7) below—which has $n+1$ degree n terms only, instead of the $2n$ terms of Formula (2.1).

Let us for instance consider the computations of Examples 2.11 again. As the norm of Δ, *i.e.*, the maximal length of a decomposition as a product of nontrivial elements, is 2, there exist no descending 3-cell, and the triviality of $H_3(BKL_3^+, \mathbf{Z})$ is now obvious. As for $H_2(BKL_3^+, \mathbf{Z})$, it is easy to check that ∂'_2 is injective, so $H_2(BKL_3^+, \mathbf{Z})$ is trivial.

By construction, the maximal length of a nontrivial descending n-cell is the maximal length of a decomposition of an element of $\mathcal{X}$ as a product of nontrivial elements. So, on the model of BKL_3^+ above, we can state

Corollary 3.6. *Assume that M is a locally Gaussian monoid admitting a generating set $\mathcal{X}$ closed under left and right complement and lcm and such that the norm of every element in $\mathcal{X}$ is bounded above by n. Then the (co)homological dimension of M is at most n.*

The resolution of Proposition 3.2 is both smaller and simpler than the one of Proposition 2.6, so that one could wonder whether the latter is still useful. We claim it is, as the construction of the contracting homotopy really relies on the intuition of filling n-cubes: using the decomposition f_n, we could certainly restrict to descending cells from the beginning, but, then, introducing s'_* would be quite artificial. Also, we hope that our approach can be extended in the future as outlined in Remark 2.12, and considering descending cells in this extended framework remains unclear.

Remark 3.7. Instead of considering the subcomplex C'_* of C_* obtained by restricting to descending sequences, we could also consider the quotient $\overline{C}_*$ of C_* obtained by identifying $[A]$ with $f_n[A]$ for every n-cell $[A]$. It is easy to check that both ∂_* and s_* induce well-defined maps on $\overline{C}_*$, so that $(\overline{C}_*, \overline{\partial}_*)$ is also a resolution of $\mathbf{Z}$. This resolution is equivalent to (C'_*, ∂'_*) as the classes of descending cells generate $\overline{C}_*$ and we can use descending cells as distinguished representatives for the classes in $\overline{C}_*$ (this is standard as we have a retraction of C_* to C'_*).

3.2. **Connection with the $K(\pi, 1)$ approach.** Building on Bestvina's paper [6], R. Charney, J. Meier, and K. Whittlesey developed in [14] an alternative approach. We shall now establish a precise connection, actually an equivalence, between their approach and ours, in the common cases.

The hypotheses of [14] are more restrictive than the ones we consider here, as they cover the case of Garside groups and monoids only: in comparison to our current framework, the additional hypotheses are that common multiples are always assumed to exist in the monoid, and that there exists a finite generating set closed under lcm and complement. Assuming that G is a Garside group, and Δ is a Garside element in some Garside monoid M of which G is a group of fractions, the study of [14] consists in constructing a finite $K(\pi, 1)$ for G by introducing a flag complex whose 1-skeleton is the fragment of the Cayley graph of G associated with the divisors of some fixed Garside element Δ in M. The main point is that this flag complex is contractible, which follows from its being the product of some real line $\mathbf{R}$ corresponding to the powers of Δ and of a more simple flag complex corresponding to the monoid M. Considering the action of G on the flag complex leads to an explicit free resolution of $\mathbf{Z}$ by $\mathbf{Z}G$-modules.

Proposition 3.8. *Assume that M is a Garside monoid, Δ is a Garside element in M, and $\mathcal{X}$ is the set of all divisors of Δ. Then the resolution of $\mathbf{Z}$ constructed in* [14] *is isomorphic to the resolution of Proposition 3.2.*

Technically, the connection between the cells considered in [14] and ours is analogous to what happens when one goes from a standard resolution to a bar resolution [10]—so it is just a change of variables.

Proof. By definition, the n-cells considered in [14] are of the form $(\beta_1, \dots, \beta_n)$ with $\beta_1, \dots, \beta_n$ in M such that the product $\beta_1 \cdots \beta_n$ belongs to $\mathcal{X}$ (which implies that each β_j belongs to $\mathcal{X}$). We map such a cell to C'_n by

$$\phi((\beta_1, \dots, \beta_n)) = [\beta_1 \cdots \beta_n, \beta_2 \cdots \beta_n, \dots, \beta_{n-1}\beta_n, \beta_n].$$

The map ϕ is injective as the monoid M is right cancellative, and it is surjective as, if $[\alpha_1, \dots, \alpha_n]$ is a descending cell, we have

$$[\alpha_1, \dots, \alpha_n] = \phi((\alpha_{1/\alpha_2}, \alpha_{2/\alpha_3}, \dots, \alpha_{n-1/\alpha_n}, \alpha_n)).$$

It remains to check that the differentials are homomorphic. The formula for $\partial(\beta_1, \dots, \beta_n)$ in [14] is that of a classical bar resolution, namely

$$\beta_1(\beta_2, \dots, \beta_n) + \sum_{i=1}^{n-1} (-1)^i (\beta_1, \dots, \beta_i\beta_{i+1}, \dots, \beta_n) + (-1)^n (\beta_1, \dots, \beta_{n-1}), \tag{3.7}$$

and we leave it to the reader to check that applying ϕ yields (3.1). □

Thus the results of the current sections 2 and 3 may be seen as an extension of the results of [14] to the framework of locally Gaussian monoids.

3.3. **Topological interpretation.** As mentioned above, the resolution constructed in [14] and, therefore, the isomorphic resolution C'_* (or $\overline{C}_*$) defined here are associated with a topological space (in the case of a Garside group G), namely some flag complex $\mathcal{T}'$ whose 1-skeleton is the Cayley graph of the lattice of divisors of some Garside element Δ (in the particular case of an Artin–Tits group, this graph is isomorphic to the Cayley graph of the associated Coxeter group).

Similarly, a topological space $\mathcal{T}$ can be associated with the resolution C_* of Section 2. Considering the way $\overline{C}_*$ is constructed from C_* makes it natural to introduce $\mathcal{T}$ (in the general case of a locally Gaussian monoid M) as the topological space admitting the Cayley graph of the set $\mathcal{X}$ (*i.e.*, the subgraph of the Cayley graph of M corresponding to vertices in $\mathcal{X}$) as a 1-skeleton, but containing in addition all the n-cubes of C_* for $n \geq 2$. The difference between $\mathcal{T}$ and the flag complex $\mathcal{T}'$ is that, typically, if $\alpha, \beta, \alpha', \beta'$ are generators in $\mathcal{X}$ satisfying $\alpha \vee \beta = \alpha' \vee \beta'$, then the associated squares $[\alpha, \beta]$ and $[\alpha', \beta']$ only share the initial and the final ends of their main diagonal, namely the two vertices $[\emptyset]$ and $(\alpha \vee \beta)[\emptyset]$, while, after quotienting to $\overline{C}_*$, *i.e.*, after decomposing the squares into triangles, they share the whole diagonal $[\alpha \vee \beta]$ (Figure 13).

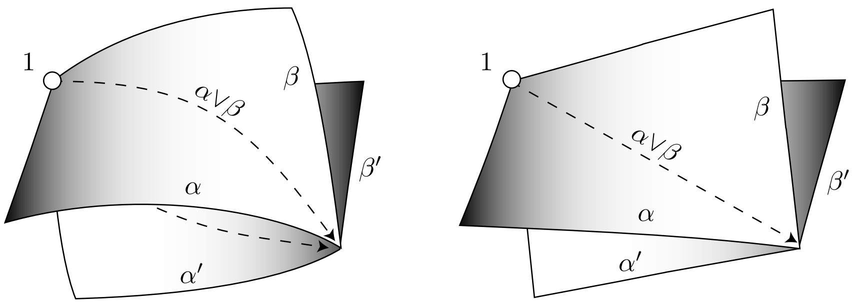


FIGURE 13. Going from $\mathcal{T}$ to $\mathcal{T}'$ by pinching common edges

If M is a spherical Artin-Tits monoid, and G is the corresponding group of fractions, the quotient of $\mathcal{T}$ obtained by identifying homonymous n-faces is the classifying space of G, and the associated resolution is similar to the Deligne–Salvetti resolution for G [26, 38, 18] (see also [35] for a description in the case of standard braids), with the difference that, here, we consider a family of generators $\mathcal{X}$ that is supposed to be closed under complement and lcm. If our construction could be extended to an arbitrary family of generators, hence, in particular, to the family of atoms in M, then we would obtain the Salvetti complex, and deduce in this way a purely algebraic proof of the exactness of this complex.

4. The order resolution

The construction of Sections 2 and 3 is simple and convenient, but it requires using a particular set of generators, namely one that is closed under several operations. We shall now develop another construction, which is more general, as it starts with an arbitrary set of generators and does not require the considered monoid to be locally Gaussian both on the left and on the right. The price to pay for the extension is that the construction of the boundary operator and of the contracting homotopy is more complicated; in particular, it is an inductive definition and not a direct one as in Sections 2 and 3.

In the sequel, we assume that M is a locally left Gaussian monoid, *i.e.*, that M admits right cancellation, that left division in M has no infinite descending chain, and that any two elements of M that admit a common left multiple admit a left lcm. We start with an *arbitrary* set of generators $\mathcal{X}$ of M that does not contain 1.

4.1. **Cells and chains.** Our first step is to fix a linear ordering $<$ on $\mathcal{X}$ with the property that, for each x in M, the set of all right divisors of x in $\mathcal{X}$ is well-ordered by $<$. At the expense of using the axiom of choice, we can always find such an ordering; practically, we shall be mostly interested in the case when $\mathcal{X}$ is finite, or, more generally, when $\mathcal{X}$ is possibly infinite but every element of M can be divised by finitely many elements of $\mathcal{X}$ only, as is the case for the direct limit B_∞^+ of the braid monoids B_n^+: in such cases, any linear ordering on $\mathcal{X}$ is convenient.

Notation. For $\mathcal{X}$ and $<$ as above, and x a nontrivial (*i.e.*, not equal to 1) element of M, we denote by $\mathrm{mindiv}(x)$ the $<$-least right divisor of x in $\mathcal{X}$.

As in Section 2, the simplicial complexes we construct are associated with finite increasing families of generators, but we introduce additional restrictions.

Definition. For $n \geq 0$, we denote by $\mathcal{X}^{[\underset{\sim}{n}]}$ the family of all n-tuples $(\alpha_1, \dots, \alpha_n)$ with $\alpha_1 < \dots < \alpha_n \in \mathcal{X}$ such that $\alpha_1, \dots, \alpha_n$ admit a common left multiple (hence a left lcm), and, in addition, $\alpha_i = \mathrm{mindiv}(\alpha_i \vee \dots \vee \alpha_n)$ holds for each i. We let $\underset{\sim}{C}_n$ denote the free $\mathbf{Z}M$-module generated by $\mathcal{X}^{[\underset{\sim}{n}]}$.

As above, the generator of $\underset{\sim}{C}_n$ associated with an element A of $\mathcal{X}^{[\underset{\sim}{n}]}$ is denoted $[A]$, and it is called an *n-cell*; the left lcm of A is then denoted by $\ulcorner A \lrcorner$.

Example 4.1. In some cases, all increasing sequences of generators satisfy our current additional hypotheses. For instance, if we consider the braid monoid B_∞^+ and the standard generators σ_i ordered by $\sigma_i < \sigma_{i+1}$, then there exists an n-cell $[\sigma_{i_1}, \dots, \sigma_{i_n}]$ for each increasing sequence $i_1 < \dots < i_n$, as left lcm always exist in B_n^+ and σ_{i_1} is the right divisor with least index of $\sigma_{i_1} \vee \dots \vee \sigma_{i_n}$.

On the other hand, if we consider the Birman-Ko-Lee monoid BKL_3^+ of Example 2.1, with the ordering $a < b < c$, we see that there are 3 increasing sequences of length 2, namely (a, b), (a, c), and (b, c), but there are two 2-cells only, namely $[a, b]$ and $[a, c]$, as we have $a = \mathrm{mindiv}(b \vee c)$, which discards $[b, c]$.

As in Section 2, we can think of associating with every elementary n-chain $x[\alpha_1, \dots, \alpha_n]$ an n-dimensional oriented simplex originating at x, ending at $x(\alpha_1 \vee \dots \vee \alpha_n)$, and containing n terminal edges labelled α_1, …, α_n, but the way of filling the picture will be different, and, in particular, the simplex is not a cube in general, and it seems not to be very illuminating. The main tool here is the following preordering on elementary chains:

Definition. For A a nonempty sequence, we denote by $A_{(1)}$ the first element of A. Then, if $x[A]$, $y[B]$ are elementary n-chains, we say that $x[A] \prec y[B]$ holds if we have either $x\ulcorner A\lrcorner \sqsubset y\ulcorner B\lrcorner$, or $n > 0$, $x\ulcorner A\lrcorner = y\ulcorner B\lrcorner$ and $A_{(1)} < B_{(1)}$. If $\sum x_i[A_i]$ is an arbitrary n-chain, we say that $\sum x_i[A_i] \prec y[B]$ holds if $x_i[A_i] \prec y[B]$ holds for every i.

Lemma 4.2. *For every n, the relation $\prec$ on n-dimensional elementary chains is compatible with multiplication on the left, and it has no infinite decreasing sequence.*

Proof. Assume $x[A] \prec y[B]$, and let z be an arbitrary element of M. Then $x\ulcorner A\lrcorner \sqsubset y\ulcorner B\lrcorner$ implies $zx\ulcorner A\lrcorner \sqsubset zy\ulcorner B\lrcorner$, and $x\ulcorner A\lrcorner = y\ulcorner B\lrcorner$ implies $zx\ulcorner A\lrcorner = zy\ulcorner B\lrcorner$, so we have $zx[A] \prec zy[B]$ in all cases.

Assume now $\cdots \prec x_2[A_2] \prec x_1[A_1]$. First, we deduce $\cdots \sqsubseteq x_2\ulcorner A_2\lrcorner \sqsubseteq x_1\ulcorner A_1\lrcorner$. As M is left Noetherian, this decreasing sequence is eventually constant, *i.e.*, for some i_0, we have $x_i\ulcorner A_i\lrcorner = x_{i+1}\ulcorner A_{i+1}\lrcorner$ for $i \geq i_0$. Then, for $i \geq i_0$, we must have $A_{i+1(1)} < A_{i(1)}$. Now, by construction, $A_{i(1)}$ is a right divisor of $\ulcorner A_i\lrcorner$, hence of $x_i\ulcorner A_i\lrcorner$, and, therefore, of $x_{i_0}\ulcorner A_{i_0}\lrcorner$ provided $i \geq i_0$ is true. But, then, the hypothesis that the right divisors of $x_{i_0}\ulcorner A_{i_0}\lrcorner$ are well-ordered by $<$ contradicts the fact that the elements $A_{i(1)}$ make a decreasing sequence. ☐

4.2. **Reducible chains.** We shall now construct simultaneously the boundary maps $\utilde{\partial}_n : \utilde{C}_n \to \utilde{C}_{n-1}$ together with a contracting homotopy $\utilde{s}_n : \utilde{C}_n \to \utilde{C}_{n+1}$ and a so-called reduction map $r_n : \utilde{C}_n \to \utilde{C}_n$. The map $\utilde{\partial}_n$ is $\mathbf{Z}M$-linear, while $\utilde{s}_n$ and r_n are $\mathbf{Z}$-linear.

Definition. Assume that $x[A]$ is an elementary chain. We say that $x[A]$ is *irreducible* if either A is empty and x is 1, *i.e.*, we have $x\ulcorner A\lrcorner = 1$, or the first element of A is the $<$-least right divisor of $x\ulcorner A\lrcorner$, *i.e.*, we have $A_{(1)} = \mathrm{mindiv}(x\ulcorner A\lrcorner)$; otherwise, we say that $x[A]$ is reducible.

Our construction uses induction on n. The induction hypothesis, denoted (H_n), is the conjunction of the following two statements, where r_n stands for $\utilde{s}_{n-1} \circ \utilde{\partial}_n$:

$$(P_n) \quad \utilde{\partial}_n(r_n(x[A])) = \utilde{\partial}_n(x[A]),$$

$$(Q_n) \quad r_n(x[A]) \begin{cases} = x[A] \text{ if } x[A] \text{ is irreducible,} \\ \prec x[A] \text{ if } x[A] \text{ is reducible} \end{cases}$$

(observe that (Q_n) makes our terminology for reducible chains coherent).

In degree 0, the construction is the same as in Section 2: we define $\utilde{\partial}_0 : \utilde{C}_0 \to \mathbf{Z}$ and $\utilde{s}_{-1} : \mathbf{Z} \to \utilde{C}_0$ by

$$\utilde{\partial}_0([\emptyset]) = 1, \qquad \utilde{s}_{-1}(1) = [\emptyset]. \tag{4.1}$$

Lemma 4.3. *Property (H_0) is satisfied.*

Proof. The mapping r_0 is $\mathbf{Z}$-linear and we have

$$r_0(x[\emptyset]) = \utilde{s}_{-1}(\utilde{\partial}_0([\emptyset])) = [\emptyset]$$

for every x in M. Hence, we obtain

$$\utilde{\partial}_0(r_0(x[\emptyset])) = \utilde{\partial}_0([\emptyset]) = 1, \qquad \utilde{\partial}_0(x[\emptyset]) = x \cdot 1 = 1$$

owing to the trivial structure of $\mathbf{Z}M$-module of $\mathbf{Z}$. Thus (P_0) holds. Then, by definition, $x[\emptyset]$ is irreducible if and only if x is 1. In this case, we have $r_0(x[\emptyset]) = [\emptyset]$. Otherwise, we have $r_0(x[\emptyset]) = [\emptyset] \prec x[\emptyset]$ by definition of $\prec$, and (Q_0) holds. $\square$

We assume now that $\underset{\sim}{\partial}_n$ and r_n have been constructed so that (H_n) is satisfied. We aim at defining

$$\underset{\sim}{\partial}_{n+1} : \underset{\sim}{C}_{n+1} \to \underset{\sim}{C}_n, \quad \underset{\sim}{s}_n : \underset{\sim}{C}_n \to \underset{\sim}{C}_{n+1}, \quad r_{n+1} = \underset{\sim}{s}_n \circ \underset{\sim}{\partial}_{n+1} : \underset{\sim}{C}_{n+1} \to \underset{\sim}{C}_{n+1}$$

so that (H_{n+1}) is satisfied. In the sequel, we use the notation $[\alpha, A]$ for displaying the first element of an $(n+1)$-cell; we simply write $\ulcorner a, A \lrcorner$ for the associated lcm, *i.e.*, for $a \vee \ulcorner A \lrcorner$. Thus we always have

$$\ulcorner \alpha, A \lrcorner = \alpha_{/A} \cdot \ulcorner A \lrcorner. \tag{4.2}$$

Definition. (Figure 14) We define the $\mathbf{Z}M$-linear map $\underset{\sim}{\partial}_{n+1} : \underset{\sim}{C}_{n+1} \to \underset{\sim}{C}_n$ by

$$\underset{\sim}{\partial}_{n+1}([\alpha, A]) = \alpha_{/A}[A] - r_n(\alpha_{/A}[A]); \tag{4.3}$$

We inductively define the $\mathbf{Z}$-linear map $\underset{\sim}{s}_n : \underset{\sim}{C}_n \to \underset{\sim}{C}_{n+1}$ by

$$\underset{\sim}{s}_n(x[A]) = \begin{cases} 0 & \text{if } x[A] \text{ is irreducible,} \\ y[\alpha, A] + \underset{\sim}{s}_n(y r_n(\alpha_{/A}[A])) & \text{otherwise,} \\ & \text{with } \alpha = \operatorname{mindiv}(x \ulcorner A \lrcorner) \text{ and } x = y\alpha_{/A}. \end{cases} \tag{4.4}$$

Finally, we define $r_{n+1} : \underset{\sim}{C}_{n+1} \to \underset{\sim}{C}_{n+1}$ by $r_{n+1} = \underset{\sim}{s}_n \circ \underset{\sim}{\partial}_{n+1}$.

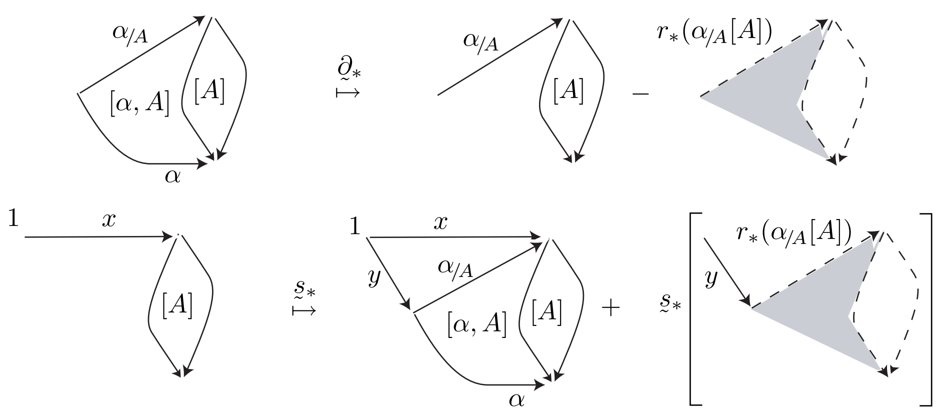


Figure 14. The boundary operator $\underset{\sim}{\partial}_*$ and the section $\underset{\sim}{s}_*$

The definition of $\underset{\sim}{\partial}_{n+1}$ is direct (once r_n has been constructed). That of $\underset{\sim}{s}_n$ is inductive, and we must check that it is well-founded. Now, we observe that, in (4.4), the chain $\alpha_{/A}[A]$ is reducible, as $\alpha < A_{(1)}$ holds by definition, so (Q_n) gives $r_n(\alpha_{/A}[A]) \prec \alpha_{/A}[A]$, and, therefore,

$$y r_n(\alpha_{/A}[A]) \prec y\alpha_{/A}[A] = x[A]. \tag{4.5}$$

Thus, our inductive definition of $\underset{\sim}{s}_n$ makes sense, and so does that of r_{n+1}.

Our aim is to prove that the sequence $(\underset{\sim}{C}_*, \underset{\sim}{\partial}_*)$ is a free resolution of $\mathbf{Z}$. First, we observe that

$$\underset{\sim}{\partial}_n \circ \underset{\sim}{\partial}_{n+1} = 0 \tag{4.6}$$

automatically holds, as, using (P_n), we obtain

$$\partial_n \partial_{n+1}[\alpha, A] = \partial_n(\alpha_{/A}[A]) - \partial_n(r_n(\alpha_{/A}[A])) = 0.$$

Lemma 4.4. *Assuming (H_n), we have for every elementary n-chain $x[A]$*

$$\partial_{n+1} s_n(x[A]) = x[A] - r_n(x[A]). \tag{4.7}$$

Proof. We use $\prec$-induction on $x[A]$. If $x[A]$ is irreducible, applying (Q_n), we find

$$\partial_{n+1} s_n(x[A]) = 0 = x[A] - r_n(x[A])$$

directly. Assume now that $x[A]$ is reducible. With the notation of (4.4), we obtain

$$\partial_{n+1} s_n(x[A]) = y\partial_{n+1}[\alpha, A] - \partial_{n+1} s_n(yr_n(\alpha_{/A}[A])).$$

By (Q_n), we have $yr_n(\alpha_{/A}[A]) \prec x[A]$, so the induction hypothesis gives

$$\partial_{n+1} s_n(yr_n(\alpha_{/A}[A])) = yr_n(\alpha_{/A}[A]) - r_n(yr_n(\alpha_{/A}[A])).$$

Applying (P_n), we deduce

$$\begin{aligned} r_n(yr_n(\alpha_{/A}[A])) &= s_{n-1}(y\partial_n(r_n(\alpha_{/A}[A]))) \\ &= s_{n-1}(y\partial_n(\alpha_{/A}[A])) = r_n(y\alpha_{/A}[A]) = r_n(x[A]), \end{aligned}$$

hence

$$\partial_{n+1} s_n(x[A]) = y\alpha_{/A}[A] - yr_n(\alpha_{/A}[A]) + yr_n(\alpha_{/A}[A]) - r_n(x[A]),$$

i.e., $\partial_{n+1} s_n(x[A]) = x[A] - r_n(x[A])$, as was expected. □

Lemma 4.5. *Assuming (H_n), (P_{n+1}) is satisfied.*

Proof. Assume that $x[A]$ is an elementary $n+1$-chain. We find

$$\begin{aligned} \partial_{n+1}(r_{n+1}(x[A])) &= \partial_{n+1} s_n \partial_{n+1}(x[A]) \\ &= \partial_{n+1}(x[A]) - r_n(\partial_{n+1}(x[A])) \\ &= \partial_{n+1}(x[A]) - s_{n-1}\partial_n \partial_{n+1}(x[A]) = \partial_{n+1}(x[A]), \end{aligned}$$

by applying Lemma 4.4 and Formula (4.6). □

Lemma 4.6. *Assume that $x[\alpha, A]$ is a reducible chain. Then, for each reducible chain $y[B]$ satisfying $y\ulcorner B\lrcorner \sqsubseteq x\ulcorner \alpha, A\lrcorner$, we have $s_n(y[B]) \prec x[\alpha, A]$.*

Proof. We use $\prec$-induction on $y[B]$. By definition, we have

$$s_n(y[B]) = z[\gamma, B] + s_n(\sum z_i[C_i]), \tag{4.8}$$

with $\gamma = \mathrm{mindiv}(y\ulcorner B\lrcorner)$, $y\ulcorner B\lrcorner = z\ulcorner \gamma, B\lrcorner$, and $\sum z_i[C_i] = zr_n(\gamma_{/B}[B])$. By (4.5), we always have $z_i[C_i] \prec y[B]$, hence, in particular, $z_i\ulcorner C_i\lrcorner \sqsubseteq y\ulcorner B\lrcorner \sqsubseteq x\ulcorner \alpha, A\lrcorner$. So, the induction hypothesis gives $s_n(z_i[C_i]) \prec x[\alpha, A]$ if $z_i[C_i]$ is reducible. If $z_i[C_i]$ is irreducible, there is no contribution of $s_n(z_i[C_i])$ to the sum in (4.8), so, in both cases, it only remains to compare $z[\gamma, B]$ and $x[\alpha, A]$.

Two cases are possible. Assume first $y\ulcorner B\lrcorner \sqsubset x\ulcorner \alpha, A\lrcorner$. By construction, we have $z\ulcorner \gamma, B\lrcorner = y\ulcorner B\lrcorner$, so we deduce $z\ulcorner \gamma, B\lrcorner \sqsubset x\ulcorner \alpha, A\lrcorner$, and therefore $z[\gamma, B] \prec x[\alpha, A]$.

Assume now $y\ulcorner B\lrcorner = x\ulcorner \alpha, A\lrcorner$. By construction, γ is the least right divisor of $y\ulcorner B\lrcorner$, hence of $x\ulcorner \alpha, A\lrcorner$, and the hypothesis that $x[\alpha, A]$ is reducible means that α is a right divisor of the latter element, but is not its least right divisor, so we must have $\gamma < \alpha$. This, by definition, gives $z[\gamma, B] \prec x[\alpha, A]$. □

Lemma 4.7. *Assuming (H_n), (H_{n+1}) is satisfied.*

Proof. Owing to Lemma 4.5, it remains to prove (Q_{n+1}). Let $x[\alpha, A]$ be an $(n+1)$-dimensional elementary chain. By definition, we have

$$r_{n+1}(x[\alpha, A]) = \underset{\sim}{s}_n(x\,\alpha_{/A}[A]) - \underset{\sim}{s}_n(\sum y_i[B_i]). \tag{4.9}$$

with $\sum y_i[B_i] = x\, r_n(\alpha_{/A}[A])$ If $x[\alpha, A]$ is irreducible, α is the least right divisor of $x\ulcorner\alpha, A\lrcorner$, the definition of $\underset{\sim}{s}_n$ gives

$$\underset{\sim}{s}_n(x\alpha_{/A}[A]) = x[\alpha, A] + \underset{\sim}{s}_n(\sum y_i[B_i]),$$

and we deduce $r_{n+1}(x[\alpha, A]) = x[\alpha, A]$.

Assume now that $x[\alpha, A]$ is reducible. First, we have $x\alpha_{/A}\ulcorner A\lrcorner = x\ulcorner\alpha, A\lrcorner$, so applying Lemma 4.6 to $x\alpha_{/A}[A]$ gives $\underset{\sim}{s}_n(x\alpha_{/A}[A]) \prec x[\alpha, A]$. Then, by hypothesis, the chain $\alpha_{/A}[A]$ is reducible, so Property (Q_n) gives $r_n(\alpha_{/A}[A]) \prec \alpha_{/A}[A]$, hence, by Lemma 4.2, $x\, r_n(\alpha_{/A}[A]) \prec x\,\alpha_{/A}[A]$, *i.e.*, $y_i[B_i] \prec x\,\alpha_{/A}[A]$, which implies in particular $y_i\ulcorner B_i\lrcorner \sqsubseteq x\alpha_{/A}\ulcorner A\lrcorner = x\ulcorner\alpha, A\lrcorner$. Applying Lemma 4.6 to $y_i[B_i]$ gives $\underset{\sim}{s}_n(y_i[B_i]) \prec x[\alpha, A]$. Putting this in (4.9), we deduce $r_{n+1}(x[\alpha, A]) \prec x[\alpha, A]$, which is Property (Q_{n+1}). □

Thus the induction hypothesis is maintained, and the construction can be carried out. We can now state:

Proposition 4.8. *For M a locally left Gaussian monoid, the complex $(\underset{\sim}{C}_*, \underset{\sim}{\partial}_*)$ is a resolution of the trivial $\mathbf{Z}M$-module $\mathbf{Z}$ by free $\mathbf{Z}M$-modules.*

Proof. First, Formula (4.6) shows that $(\underset{\sim}{C}_*, \underset{\sim}{\partial}_*)$ is a complex in each degree. Then Formula (4.7) rewrites into

$$\underset{\sim}{\partial}_{n+1} \circ \underset{\sim}{s}_n + \underset{\sim}{s}_{n-1} \circ \underset{\sim}{\partial}_n = \mathrm{id}_{\underset{\sim}{C}_n}, \tag{4.10}$$

which shows that $\underset{\sim}{s}_*$ is a contracting homotopy. □

An immediate corollary is the following precise version of Theorem 0.1:

Proposition 4.9. *Assume that M is a locally left Gaussian monoid admitting a linearly ordered set of generators $(\mathcal{X}, <)$ such that n is the maximal size of an increasing sequence $(\alpha_1, \dots, \alpha_n)$ in $\mathcal{X}$ such that $\alpha_1 \vee \cdots \vee \alpha_n$ exists and α_i is the least right divisor of $\alpha_i \vee \cdots \vee \alpha_n$ for each i. Then $\mathbf{Z}$ admits a finite free resolution of length n over $\mathbf{Z}M$; so, in particular, M is of type FL.*

Example 4.10. We have seen that the Birman-Ko-Lee monoid BKL_3^+ has a presentation with 3 generators $a < b < c$, but 2 is the maximal cardinality of a family as in Proposition 4.9, since (a, b, c) is not eligible. We conclude that $\mathbf{Z}$ admits a free resolution of length 2 over $\mathbf{Z}BKL_3^+$ (as already seen in Example 3.5).

Remark 4.11. Squier's approach in [42] has in common with the current approach to use the modules C_n (or $\underset{\sim}{C}_n$ with order assumptions dropped). However, the boundary operators he considers is different from $\underset{\sim}{\partial}_n$ (and from ∂_n). Roughly speaking, Squier uses an induction on $\sqsubset$ and not on $\prec$. This means that he guesses the exact form of all top degree factors in $\underset{\sim}{\partial}_n[A]$, while we only guess one of these factors, namely the least one. Technically, the point is that, in the case of [42], *i.e.*, of Artin–Tits monoids, the length of the words induces a well defined grading on the monoid. Squier starts with a (very elegant) combinatorial construction capturing the symmetries of the Coxeter relations, uses it to define a first sketch of the differential, and then he defines his final differential as a deformation of the latter. It seems quite problematic to extend this approach to our general framework, because there need not exist any length grading, and we do not assume our defining relations to admit any symmetry. Due to this lack of symmetry, Theorem 6.10 of [42], which is instrumental in his construction, fails in general: a typical example is the monoid $\langle a, b\,;\, aba = b^2\rangle^+$, which is Gaussian— the associated group of fractions is the braid group B_3—and we have $\{a\} \subseteq \{a, b\}$, and $a \vee b = uv$ with $u = v = b$, but there is no way to factor $u = u_1u_2$, $v = v_1v_2$ in such a way that u_2v_1 is equal to a.

4.3. **Geometrical interpretation.** We have seen that the construction of Section 2 admits a simple geometrical interpretation in terms of greedy normal forms and word reversing. Here we address the question of finding a similar geometrical interpretation for the current construction. The answer is easy in low degree, but quite unclear in general.

The first step is to introduce a convenient normal form for the elements of our monoid M. This is easy: as in the case of the $\mathcal{X}$-normal form, every nontrivial element x of M has a distinguished right divisor, namely its least right divisor $\mathrm{mindiv}(x)$.

Definition. We say that a word w over $\mathcal{X}$, say $w = \alpha_1 \cdots \alpha_p$, is the *ordered normal form* of x, denoted $w = \underset{\sim}{\mathrm{NF}}(x)$, if we have $x = \overline{w}$, and $\alpha_i = \mathrm{mindiv}(\overline{\alpha_1 \cdots \alpha_i})$ for each i.

Once again, an easy induction on $\sqsubset$ shows that every element of M admits a unique ordered normal form: indeed, the empty word is the unique normal form of 1, and, for $x \neq 1$, we write $x = y \cdot \mathrm{mindiv}(x)$, and the ordered normal form of x is obtained by appending $\mathrm{mindiv}(x)$ to the ordered normal form of y.

Example 4.12. Assume that M is a Garside group and $\mathcal{X}$ is the set of all divisors of some Garside element Δ of M. If $<$ is any linear ordering on $\mathcal{X}$ that extends the opposite of the partial ordering given by right divisibility, then the ordered normal form associated with $<$ is the right greedy normal form, *i.e.*, the normal form constructed as the $\mathcal{X}$-normal form of Section 2 exchanging left and right divisors: indeed, for every nontrivial element x of M, the rightmost factor in the right greedy normal form of x is the right gcd of x and Δ, hence it is a left multiple of every right divisor of x lying in $\mathcal{X}$, and, therefore, it is the $<$-least such divisor.

The question now is whether there exist global expressions for $\underset{\sim}{\partial}_*$ and $\underset{\sim}{s}_*$ in the spirit of those of Section 2, *i.e.*, involving the normal form and a word reversing process. We still use the notation of Formula 2.4, *i.e.*, we write $[w]$ for the chain inductively defined by (2.4) or (2.5).

Lemma 4.13. *For every x in M and α, β in $\mathcal{X}$, we have*

$$\underset{\sim}{\partial}_1[\alpha] = (\alpha - 1)[\emptyset], \quad \underset{\sim}{s}_0(x[\emptyset]) = [\underset{\sim}{\mathrm{NF}}(x)], \tag{4.11}$$

$$r_1(x[\alpha]) = [\underset{\sim}{\mathrm{NF}}(x\alpha)] - [\underset{\sim}{\mathrm{NF}}(x)], \quad \underset{\sim}{\partial}_2\underset{\sim}{s}_1(x[\alpha]) = [\underset{\sim}{\mathrm{NF}}(x)\,\alpha] - [\underset{\sim}{\mathrm{NF}}(x\alpha)], \tag{4.12}$$

$$\underset{\sim}{\partial}_2[\alpha,\beta] = [\underset{\sim}{\mathrm{NF}}(\alpha_{/\beta})\,\beta] - [\underset{\sim}{\mathrm{NF}}(\beta_{/\alpha})\,\alpha] = [\underset{\sim}{\mathrm{NF}}(\alpha_{/\beta})] + \alpha_{/\beta}[\beta] - [\underset{\sim}{\mathrm{NF}}(\beta_{/\alpha})] - \beta_{/\alpha}[\alpha]. \tag{4.13}$$

Proof. The definition gives

$$\underset{\sim}{\partial}_1[\alpha] = a_{/\emptyset}[\emptyset] - r_0(a_{/\emptyset}[\emptyset]) = a[\emptyset] - [\emptyset].$$

For $\underset{\sim}{s}_0$, we use $\sqsubset$-induction on x. For $x = 1$, $x[\emptyset]$ is irreducible, so $\underset{\sim}{s}_0(x[\emptyset]) = 0$ holds, while $\underset{\sim}{\mathrm{NF}}(x)$ is empty, and we find $[\underset{\sim}{\mathrm{NF}}(x)] = 0$. Otherwise, let $\alpha = \mathrm{mindiv}(x)$ and $x = ya$. We have $\underset{\sim}{\mathrm{NF}}(x) = \underset{\sim}{\mathrm{NF}}(y) \cdot a$, hence $[\underset{\sim}{\mathrm{NF}}(x)] = [\underset{\sim}{\mathrm{NF}}(y)] + y[\alpha]$. By definition, we have $\underset{\sim}{s}_0(x[\emptyset]) = y[\alpha] + \underset{\sim}{s}_0(y)$, hence $\underset{\sim}{s}_0(x[\emptyset]) = y[\alpha] + [\underset{\sim}{\mathrm{NF}}(y)]$ by induction hypothesis, and comparing the expressions gives $\underset{\sim}{s}_0(x[\emptyset]) = [\underset{\sim}{\mathrm{NF}}(x)]$.

Next, we obtain

$$r_1(x[\alpha]) = \underset{\sim}{s}_0(x\underset{\sim}{\partial}_1[\alpha]) = \underset{\sim}{s}_0(xa[\emptyset]) - \underset{\sim}{s}_0(x[\emptyset]) = [\underset{\sim}{\mathrm{NF}}(xa)] - [\underset{\sim}{\mathrm{NF}}(x)]$$

The second relation in (4.12) follows from (4.11) using $\underset{\sim}{\partial}_2\underset{\sim}{s}_1(x[\alpha]) = x[\alpha] - \underset{\sim}{s}_0\underset{\sim}{\partial}_1(x[\alpha])$.

Assume now that $[\alpha, \beta]$ is a 2-cell, *i.e.*, that $\alpha < \beta$ holds, $\alpha \vee \beta$ exists, and $\alpha = \mathrm{mindiv}(\alpha \vee \beta)$ holds. Applying (4.12), we find

$$\begin{aligned} \underset{\sim}{\partial}_2[\alpha, \beta] &= \alpha_{/\beta}[\beta] - r_1(\alpha_{/\beta}[\beta]) \\ &= \alpha_{/\beta}[\beta] - [\underset{\sim}{\mathrm{NF}}(\alpha_{/\beta}\beta)] + [\underset{\sim}{\mathrm{NF}}(\alpha_{/\beta})] = [\underset{\sim}{\mathrm{NF}}(\alpha_{/\beta})\,\beta] - [\underset{\sim}{\mathrm{NF}}(\alpha \vee \beta)]. \end{aligned}$$

The hypothesis $\alpha = \mathrm{mindiv}(\alpha \vee \beta)$ implies that the normal form of $\alpha \vee \beta$ is $\underset{\sim}{\mathrm{NF}}(\beta_{/\alpha})\alpha$, and we obtain

$$\underset{\sim}{\partial}_2[\alpha, \beta] = [\underset{\sim}{\mathrm{NF}}(\alpha_{/\beta})\,\beta] - [\underset{\sim}{\mathrm{NF}}(\beta_{/\alpha})\,\alpha] = [\underset{\sim}{\mathrm{NF}}(\alpha_{/\beta})] + \alpha_{/\beta}[\beta] - [\underset{\sim}{\mathrm{NF}}(\beta_{/\alpha})] - \beta_{/\alpha}[\alpha],$$

as was expected. □

So, we see that the counterparts of Formulas (2.1) and (2.6), for $\underset{\sim}{\partial}_1$ and $\underset{\sim}{\partial}_2$ and of (2.9) for $\underset{\sim}{s}_0$ are valid: as for $\underset{\sim}{\partial}_2$, the counterpart of (2.1) has to include normal forms since, in general, the elements $\alpha_{/\beta}$ and $\beta_{/\alpha}$ do not belong to $\mathcal{X}$, as they did in the framework of Section 2. Observe that (4.13) would fail in general if we did not restrict to cells $[\alpha, \beta]$ such that α is the least right divisor of $\alpha \vee \beta$: this is for instance the case of the pseudo-cell $[b, c]$ in the monoid BKL_3^+ with $a < b < c$.

The next step is to interpret $\underset{\sim}{s}_1(x[\alpha])$. Here, we need to define a 2-chain $[u, v]$ for all word u, v over $\mathcal{X}$. To this end, we keep the intuition of Formula (2.4) and use word reversing. First, we introduce the presentation $(\mathcal{X}, \mathcal{R}_<)$ of M by using the method of Proposition 1.4(ii) and choosing, for every pair of letters α, β in $\mathcal{X}$, the unique relation $\underset{\sim}{\mathrm{NF}}(\alpha_{/\beta})\,\beta = \underset{\sim}{\mathrm{NF}}(\beta_{/\alpha})\,\alpha$. This presentation is uniquely determined once $\mathcal{X}$ and $<$ have been chosen.

Definition. We define the 2-chain $[u, v]$ so that the following rules are obeyed for all u, v, w: $[u, \varepsilon] = 0$, $[v, u] = -[u, v]$, and

$$[uv, w] = [u, w_{/^*v}] + \overline{u}_{/(\overline{v}_{/\overline{w}})}\,[v, w]. \tag{4.14}$$

The Noetherianity of left division in M implies that $[u, v]$ is well defined for all u, v; the induction rules mimic those of word reversing, and the idea is that $[u, v]$ is the sum of all elementary chains corresponding to the reversing diagram of uv^{-1}.

Question 4.14. *Is the following equality true:*

$$\underset{\sim}{s}_1(x[\alpha]) = [\underset{\sim}{\mathrm{NF}}(x\alpha), \underset{\sim}{\mathrm{NF}}(x)\,\alpha]? \tag{4.15}$$

In the framework of Section 2, the definition gives $s_1(x[\alpha]) = [\mathrm{NF}(x\alpha), \alpha]$. On the other hand, Formula 4.14 yields

$$[\mathrm{NF}(x\alpha), \mathrm{NF}(x)\,\alpha] = [\mathrm{NF}(x), \mathrm{NF}(x)] + \overline{\varepsilon}[\mathrm{NF}(x\alpha), \alpha] = [\mathrm{NF}(x\alpha), \alpha],$$

as, by Proposition 1.10, $\mathrm{NF}(x\alpha)_{/^*\alpha}$ is equal to $\mathrm{NF}(x)$, and, therefore, (4.15) is true. It is not hard to extend the result to our current general framework provided the extension of Proposition 1.10 is still valid, *i.e.*, provided $\underset{\sim}{\mathrm{NF}}(x\alpha)_{/^*\alpha} = \underset{\sim}{\mathrm{NF}}(x)$ holds for every x in M and α in $\mathcal{X}$. Now, it is easy to see that this extension is not true in general, for instance by using the monoid B_4^+ and the generators $\sigma_2 < \sigma_1 < \sigma_3$.

However, even if the argument sketched above fails, Equality (4.15) remains true in all cases we tried. This suggests that the considered geometrical interpretation could work further.

4.4. **Examples.** Let us conclude with a few examples of our construction. We shall successively consider the 4-strand braid monoid, the 3-strand Birman-Ko-Lee monoid, and the torus knot monoids. We use $\underset{\sim}{d}_n$ for the $\mathbf{Z}$-linear map obtained from $\underset{\sim}{\partial}_n$ by trivializing M, so, again, $\mathrm{Ker}\,\underset{\sim}{d}_{n+1}/\mathrm{Im}\,\underset{\sim}{d}_n$ is $H_n(M,\mathbf{Z})$—as well as $H_n(G,\mathbf{Z})$ if M is a Ore monoid and G is the associated group of fractions.

Example 4.15. Let us consider the standard presentation of B_4^+. To obtain shorter formulas, we write a, b, c instead of σ_1, σ_2, σ_3. We choose $a < b < c$. From $\underset{\sim}{\partial}_1[a] = (a-1)[\emptyset]$ we deduce $\underset{\sim}{d}_1[a] = 0$, and $\mathrm{Ker}\,\underset{\sim}{d}_1$ is generated by $[a]$, $[b]$, $[c]$. Then (4.13) applies, and we find

$$\underset{\sim}{\partial}_2[a,b] = [bab] - [aba] = (-1 + b - ab)[a] + (1 - a + ba)[b],$$
$$\underset{\sim}{\partial}_2[b,c] = [cbc] - [bcb] = (-1 + c - bc)[b] + (1 - b + cb)[c],$$
$$\underset{\sim}{\partial}_2[a,c] = [ac] - [ca] = (1 - c)[a] + (-1 + a)[c],$$

hence $\underset{\sim}{d}_2([a,b] = -[a]+[b]$, $\underset{\sim}{d}_2[b,c] = -[b]+[c]$, $\underset{\sim}{d}_2[a,c] = 0$. So $\mathrm{Im}\,\underset{\sim}{d}_2$ is generated by $-[a]+[b]$ and $-[b]+[c]$, and we deduce $H_1(B_4^+,\mathbf{Z}) = H_1(B_4,\mathbf{Z}) = \mathrm{Ker}\,\underset{\sim}{d}_1/\mathrm{Im}\,\underset{\sim}{d}_2 = \mathbf{Z}$.

Next, we have $\ulcorner a,b,c\lrcorner = cbacbc = cba\ulcorner b,c\lrcorner$, hence

$$\underset{\sim}{\partial}_3[a,b,c] = cba[b,c] - r_2(cba[b,c]),$$

and $r_2(cba[b,c])$, *i.e.*, $\underset{\sim}{s}_1\underset{\sim}{\partial}_2(cba[b,c])$, evaluates to

$$-\underset{\sim}{s}_1(cba[b]) + \underset{\sim}{s}_1(cbac[b]) - \underset{\sim}{s}_1(cbabc[b]) + \underset{\sim}{s}_1(cba[c]) - \underset{\sim}{s}_1(cbab[c]) + \underset{\sim}{s}_1(cbacb[c]).$$

None of the previous six chains $x[A]$ is irreducible as, in each case, a is a right divisor of $x\ulcorner A\lrcorner$. We have $a = \mathrm{mindiv}(cbab)$ and $cbab = c(a \vee b)$, hence

$$\underset{\sim}{s}_1(cba[b]) = c[a,b] + \underset{\sim}{s}_1(c\,r_1(ba[b])).$$

Using (4.12), we find

$$r_1(ba[b]) = [\underset{\sim}{\mathrm{NF}}(bab)] - [\underset{\sim}{\mathrm{NF}}(ba)] = [aba] - [ba] = (1 - b + ab)[a] + (-1 + a)[b],$$

hence

$$\underset{\sim}{s}_1(cba[b]) = c[a,b] + \underset{\sim}{s}_1(c[a]) - \underset{\sim}{s}_1(cb[a]) + \underset{\sim}{s}_1(cab[a]) - \underset{\sim}{s}_1(c[b]) + \underset{\sim}{s}_1(ca[b]).$$

Every chain $x[a]$ is irreducible, and so are $c[b]$ and $ca[b]$, as we have $\mathrm{mindiv}(cb) = \mathrm{mindiv}(cab) = b$. We deduce $\underset{\sim}{s}_1(cba[b]) = c[a,b]$. Similar computations give $\underset{\sim}{s}_1(cbac[b]) = bc[a,b]$, $\underset{\sim}{s}_1(cbabc[b]) = abc[a,b]$, $\underset{\sim}{s}_1(cba[c]) = [b,c]+cb[a,c]$, $\underset{\sim}{s}_1(cbab[c]) = a[b,c] + (-1 + cab)[a,c]$, $\underset{\sim}{s}_1(cbacb[c]) = [a,b] + ba[b,c] + (-b + ab + bcab)[a,c]$. We deduce the value of $r_2(cba[b,c])$, and, finally,

$$\underset{\sim}{\partial}_3[a,b,c] = (-1 + c - bc + abc)[a,b] + (-1 + a - ba + cba)[b,c] + (-1 + b - ab - cb + cab - bcab)[a,c].$$

Triviliazing B_4^+ gives $\underset{\sim}{d}_3[a,b,c] = -2[a,c]$. So $\mathrm{Im}\,\underset{\sim}{d}_3$ is generated by $2[a,c]$, while $\mathrm{Ker}\,\underset{\sim}{d}_2$ is generated by $[a,c]$. We deduce $H_2(B_4^+,\mathbf{Z}) = H_2(B_4,\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$.

It can be observed that the values obtained above for $\underset{\sim}{\partial}_*$ coincide with those of [42]—more precisely, the formulas of [42] correspond to what we would obtain here starting with the initial ordering $a > b > c$: this is natural as the presentation has the property that, for each finite sequence of generators A in the considered presentation, we have $\inf A = \mathrm{mindiv}(\ulcorner A\lrcorner)$.

Example 4.16. Let us consider the Birman-Ko-Lee monoid BKL_3^+ of Example 2.1. As the minimal divisor of the lcm of α and β need not be α or β, the computations are slightly more complicated. The reader can check that $\operatorname{Ker} \utilde{\mathrm{d}}_1$ is generated by $[a]$, $[b]$, and $[c]$, and that we have $\utilde{\mathrm{d}}_2[a,b] = [b] - [c]$, $\utilde{\mathrm{d}}_2[b,c] = \utilde{\mathrm{d}}_2[a,c] = -[a] + [b]$, so $\operatorname{Im} \utilde{\mathrm{d}}_2$ is generated by $[a] - [b]$ and $[b] - [c]$, and we have $H_1(M, \mathbf{Z}) = \mathbf{Z}$.

For degree 2, the definition gives

$$\utilde{\partial}_3[a,b,c] = [b,c] - r_2[b,c].$$

Then we have

$$r_2[b,c] = \utilde{s}_1 \utilde{\partial}_2[b,c] = -\utilde{s}_1(c[a]) + \utilde{s}_1([b]) - \utilde{s}_1([c]) + \utilde{s}_1(b[c]) = \utilde{s}_1(b[c]),$$

as $c[a]$, $[b]$, and $[c]$ are irreducible chains. Now we obtain

$$\utilde{s}_1(b[c]) = [a,c] + \utilde{s}_1(r_1(b[c]))$$

and, by (4.12),

$$r_1(b[c]) = [\utilde{\mathrm{NF}}(bc)] - [\utilde{\mathrm{NF}}(b)] = [ca] - [b] = [c] + c[a] - [b],$$

hence $\utilde{s}_1(r_1(b[c])) = 0$, and $\utilde{s}_1(b[c]) = [a,c]$. Finally, we obtain $r_2[b,c] = [a,c]$, and $\utilde{\partial}_3[a,b,c] = [b,c] - [a,c]$. We deduce $\utilde{\mathrm{d}}_3[a,b,c] = [b,c] - [a,c]$, so $\operatorname{Im} \utilde{\mathrm{d}}_3$ is generated by $[b,c] - [a,c]$, as is $\operatorname{Ker} \utilde{\mathrm{d}}_2$, and, therefore, $H_2(M, \mathbf{Z}) = 0$, as could be expected since the group of fractions of M is B_3.

Example 4.17. Finally, let M be the monoid $\langle a, b\,;\, a^p = b^q \rangle^+$ with $(p,q) = 1$. Then M is locally left Gaussian, even Gaussian, and the associated group of fractions is the group of the torus knot $K_{p,q}$. One obtains

$$\utilde{\partial}_2[a,b] = (1 + \cdots + a^{p-1})[a] + (1 + \cdots + b^{q-1})[b],$$

whence $\utilde{\mathrm{d}}_2\,[a,b] = -p[a] + q[b]$, and $H_1(M, \mathbf{Z}) = \mathbf{Z}$.

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