

COMPLETE POSITIVE GROUP PRESENTATIONS

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ABSTRACT. A combinatorial property of positive group presentations, called completeness, is introduced, with an effective criterion for recognizing complete presentations, and an iterative method for completing an incomplete presentation. We show how to directly read several properties of the associated monoid and group from a complete presentation: cancellativity or existence of common multiples in the case of the monoid, or isoperimetric inequality in the case of the group. In particular, we obtain a new criterion for recognizing that a monoid embeds in a group of fractions. Typical presentations eligible for the current approach are the standard presentations of the Artin–Tits groups and the Heisenberg group.

INTRODUCTION

This paper is about monoids and groups defined by a presentation. As is well-known, it is hopeless to directly read from a presentation the properties of a group or a monoid: even recognizing whether the group is trivial is undecidable in general [31]. However, partial results may exist when one restricts to presentations of a special form: a typical example is the small cancellation theory, in which a number of properties are established for those groups or monoids defined by presentations satisfying some conditions about subword overlapping in the relations [24, 25, 30, 38]. Another example is Adyan’s criterion [1, 37] which shows that a presented monoid embeds in the corresponding group if there is no cycle in some graph associated with the presentation. The aim of this paper is to study a combinatorial property of positive group presentations (*i.e.*, of presentations where all relations are of the form $u = v$ with only positive exponents in u and v) that we call completeness, and to show that several non-trivial properties of the associated monoid and group can be read directly when a complete presentation is known: the properties we shall investigate here are cancellativity, existence of common multiples, embeddability in a group of fractions in the case of the monoid, solution for the word problem, and isoperimetric inequality in the case of the group. What we do in each case is to give sufficient conditions for the monoid or the group defined by a supposedly complete presentation to satisfy the considered property. A typical example is Prop. 6.1, which states that, if $(\mathcal{S}, \mathcal{R})$ is a complete presentation, then a sufficient condition for the associated monoid to be cancellative is that \mathcal{R} contains no relation of the form $su = sv$ or $us = vs$ with $u \neq v$: thus, if there is no obvious counter-example to cancellativity, then there is no hidden counter-example either.

The interest of such results could be void if complete presentations did not exist. Actually, they do: it is even trivial that every group admits complete

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presentations—as the name suggests, a complete presentation is one with enough relations, and the full presentation consisting of all relations is always complete. The interesting case is when there exists a finite (or, at least, simple) complete presentation: we shall see that this happens for a number of groups, such as many generalized braid groups (in particular some of those associated with complex reflection groups [5]), more generally all Garside groups of [15], but also quite different groups, such as the Heisenberg group, which is nilpotent.

The main technical ingredient we shall use is a combinatorial transformation called *word reversing*. It is a refinement of the monoid congruence, in the sense that applying reversing to a word gives an equivalent word, but, in general, the converse is not true, *i.e.*, it is not true that any pair of equivalent words can be produced (or, better, detected) using reversing. Essentially, we say that a presentation is complete when the latter occurs, in which case the difficult study of word equivalence can be replaced with the easier study of reversing.

It seems that the word reversing process has been first considered in [9], and it has been investigated—and in particular some notion of completeness has been considered—in several papers [10, 17, 13, 14], but so far always in the particular case of presentations with few relations, namely the so-called complemented presentations where there exists at most one relation $s \cdots = t \cdots$ for each pair of letters $\{s, t\}$. K. Tatsuoka in [40] (in the case of Artin–Tits groups) and R. Corran in [8] (in the case of singular Artin–Tits monoids) independently developed equivalent processes in slightly different frameworks, but always with equally or more restricted initial assumptions. Let us mention that, in the context of presentations where all relations preserve the length of the words, our completeness condition is merely equivalent to Corran’s reduction property (see Remark 2.2 below). Also, a construction similar to word reversing is considered by M. Droste and D. Kuske in [19]—where it is called a complete grid—in their study of divisibility monoids (see also [28]): the latter are similar to the locally Gaussian monoids of [15, 16] and to the chainable monoids of [8], but with an additional distributivity assumption about the involved divisibility lattices which discards the braid monoids and, more generally, all monoids mentioned in the current work except those of Example 3.2.

In contrast to all previous works involving word reversing, we address arbitrary positive presentations in this paper. The advantage of such a generalization—which forces us to renew the technical framework—does not lie only in the new groups and monoids that become eligible, but rather in the underlying change of viewpoint. Previously, the principle was to study the possible completeness of a (complemented) presentation: in good cases, the presentation was complete and one could deduce consequences—as in the case of the standard presentation of the braid groups [21] or of their alternative presentation of [3]—otherwise, if the presentation was not complete, one could say nothing. Our current approach enables us not only to establish the completeness of a presentation, but also, if needed, to complete an initially incomplete presentation. This completion process may require an infinite number of steps, but, in good cases, it is a finite procedure, and we shall see on examples how it enables us to investigate some monoids or groups that remained outside the range of all previously known methods. In particular, we obtain a new method for proving that a monoid embeds in a group of fractions, and apply it to answer a question of [22] about a nonstandard presentation of Artin’s braid group B_3 introduced by V. Sergiescu in [39].

One of the applications of word reversing is (in good cases) a solution of the word problem. Let us mention here some similarity between this solution and Dehn’s algorithm for hyperbolic groups: in both cases, the idea is to decide whether a word represents 1 without introducing any new pair of generators ss^{-1} or $s^{-1}s$. However, contrary to Dehn’s algorithm, the reversing algorithm may increase the length of the words, and it is not linear in general, but, on the other hand, it works for groups that are not word hyperbolic, such as the braid groups, or even the Heisenberg group, whose isoperimetric function is known to be cubic.

The rather vague description above might remind the reader of the Knuth-Bendix completion method [26, 7], which also consists in starting with a group presentation, possibly adding some consequences of the initial relations, and obtaining a so-called complete rewrite-system that enables one to solve the word problem—see [23] for examples in the case of spherical Artin–Tits groups. The similarity with the current approach is superficial only: our method also possibly provides a solution to the word problem by means of rewriting rules, and the rôle of the cube conditions in our completion procedure is analogous to that of critical pairs in [26], but there seems to be no more precise connection in general, and we do not see how to attach any confluent rewrite-system to the combinatorial word transformations we consider, in particular because we simultaneously use positive and arbitrary words, *i.e.*, we work both with the monoid and the group. Actually, more than in the Knuth-Bendix method, our approach originates in Garside’s analysis of the braid monoids [21]: with our current definitions, the proof of Prop. H in [21], as well as that of the Kürzungslemma of [4] is a proof that the standard presentations of the (generalized) braid groups is complete.

The paper is organized as follows. In Sec. 1, we define the general reversing process and establish its basic properties. Then, in Sec. 2, we introduce completeness, and, again, establish basic results, in particular that every monoid admits a complete presentation. In Sec. 3, we introduce the cube condition, a technical property which we show is equivalent to completeness. We use it to establish our main criterion for recognizing completeness in Sec. 4 and, in Sec. 5, to complete initially incomplete presentations. The rest of the paper is devoted to studying monoids and groups from a complete presentation. In Sec. 6, we consider properties of the monoid: cancellativity, word problem, common multiples. Finally, in Sec. 7, we investigate similar questions for the group: recognizing groups of fractions, solving the word problem, computing bounds for the isoperimetric function.

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Convention. A number of notions will appear with a right and a left version. We shall use r - for “right” and l - for “left”: r -reversing, r -completeness, etc.

1. REVERSING

Our aim is to study groups and monoids from a presentation. Here we consider positive group presentations, defined as those presentations where all relations have the form $u = v$, where u and v are nonempty positive words, *i.e.*, inverses of the chosen generators do not occur in u or v . At the expense of adding new generators, this is not a restriction in the case of groups, but this means that we restrict to monoids with non non-trivial units. Our notation will be as follows. If \mathcal{S} is a nonempty set, we denote by \mathcal{S}^* the free monoid generated by \mathcal{S} , *i.e.*, the set of

all words on \mathcal{S} equipped with concatenation; we use ε for the empty word. A positive group presentation is then a pair $(\mathcal{S}, \mathcal{R})$ where \mathcal{R} is a family of pairs of nonempty words in \mathcal{S}^* , the relations of the presentation. As usual, we shall often write $u = v$ instead of $\{u, v\}$ for a relation. We denote by $\langle \mathcal{S}; \mathcal{R} \rangle^+$ the monoid associated with the presentation $(\mathcal{S}, \mathcal{R})$, *i.e.*, the monoid \mathcal{S}^*/\equiv , where \equiv is the smallest congruence on \mathcal{S}^* that includes \mathcal{R} . Then, we denote by $\langle \mathcal{S}; \mathcal{R} \rangle$ the associated group: introducing for each letter s in \mathcal{S} a disjoint copy s^{-1} of s , and using \mathcal{S}^{-1} for the set of all s^{-1} 's, the group $\langle \mathcal{S}; \mathcal{R} \rangle$ is $(\mathcal{S} \cup \mathcal{S}^{-1})^*/\equiv^\pm$, where \equiv^\pm is the smallest congruence on $(\mathcal{S} \cup \mathcal{S}^{-1})^*$ that includes \mathcal{R} (hence \equiv) and contains all pairs $\{ss^{-1}, \varepsilon\}$, $\{s^{-1}s, \varepsilon\}$, *i.e.*, all relations $ss^{-1} = s^{-1}s = \varepsilon$, for s in \mathcal{S} . For \mathbf{w} a word on $\mathcal{S} \cup \mathcal{S}^{-1}$, we denote by \mathbf{w}^{-1} the word obtained from \mathbf{w} by exchanging s and s^{-1} everywhere and reversing the order of the letters: if \mathbf{w} represents x in $\langle \mathcal{S}; \mathcal{R} \rangle$, then \mathbf{w}^{-1} represents x^{-1} .

Convention. In the previous framework, we reserve s, t for letters in \mathcal{S} , and u, v, w for words in \mathcal{S}^* . We use bold letters $\mathbf{u}, \mathbf{v}, \mathbf{w}$ for words on the symmetrized alphabet $\mathcal{S} \cup \mathcal{S}^{-1}$. For u a word in \mathcal{S}^* , we shall use \bar{u} for the element of the considered monoid $\langle \mathcal{S}; \mathcal{R} \rangle^+$ represented by u .

Our main tool in the sequel is a combinatorial transformation of words called reversing.

Definition 1.1. Assume that $(\mathcal{S}, \mathcal{R})$ is a positive group presentation, and \mathbf{w}, \mathbf{w}' are words on $\mathcal{S} \cup \mathcal{S}^{-1}$. We say that $\mathbf{w} \curvearrowright_r^{(1)} \mathbf{w}'$ is true if \mathbf{w}' is obtained from \mathbf{w}

- either by deleting some subword $u^{-1}u$ where u is a nonempty word on \mathcal{S} ,
- or by replacing some subword $u^{-1}v$ where u, v are nonempty words on \mathcal{S} with a word $v'u'^{-1}$ such that $uv' = vu'$ is a relation of \mathcal{R} .

Defining an r -reversing sequence to be a (finite or infinite) sequence of words $\mathbf{w}_0, \mathbf{w}_1, \dots$ satisfying $\mathbf{w}_i \curvearrowright_r^{(1)} \mathbf{w}_{i+1}$ for every i , we write $\mathbf{w} \curvearrowright_r^{(k)} \mathbf{w}'$ if there exists a length k r -reversing sequence from \mathbf{w} to \mathbf{w}' , and we say that \mathbf{w} is r -reversible (*i.e.*, right reversible) to \mathbf{w}' —or that \mathbf{w} reverses to \mathbf{w}' on the right—denoted $\mathbf{w} \curvearrowright_r \mathbf{w}'$ if $\mathbf{w} \curvearrowright_r^{(k)} \mathbf{w}'$ holds for some nonnegative integer k .

Symmetrically, we say that \mathbf{w} is l -reversible to \mathbf{w}' , denoted $\mathbf{w} \curvearrowleft_l \mathbf{w}'$, if \mathbf{w}' is obtained from \mathbf{w} by repeatedly deleting subwords uu^{-1} and replacing subwords uv^{-1} with words $v'^{-1}u'$ such that $v'u = u'v$ is a relation of \mathcal{R} .

Fig. 1.1 illustrates reversing in the Cayley graph of $\langle \mathcal{S}; \mathcal{R} \rangle$: a relation $uv' = vu'$ corresponds to an oriented cell, and the words \mathbf{w}, \mathbf{w}' correspond to paths; then saying that $\mathbf{w} \curvearrowright_r^{(1)} \mathbf{w}'$ is true means that the path associated with \mathbf{w}' is obtained from that associated with \mathbf{w} by reversing the way the cell $uv' = vu'$ is crossed, namely going through the final vertex instead of through the initial one. The case when we delete $u^{-1}u$ is not particular provided we assume that the trivial relation $u = u$ is added to the presentation.

The study of l -reversing is of course similar to that of r -reversing. However the reader should keep in mind that $u^{-1}v \curvearrowright_r v'u'^{-1}$ *does not imply* $v'u'^{-1} \curvearrowleft_l u^{-1}v$: deleting $s^{-1}s$ is not a reversible process, and we always have $s^{-1}s \curvearrowright_r \varepsilon$, but never $\varepsilon \curvearrowleft_l s^{-1}s$.

Example 1.2. Consider the presentation $(a, b; ab = ba, a^2 = b^2)$, and let $\mathbf{w} = a^{-1}bab^{-1}$. By using the first relation, we find $\mathbf{w} \curvearrowright_r^{(1)} ba^{-1}ab^{-1} \curvearrowright_r^{(1)} bb^{-1}$, hence

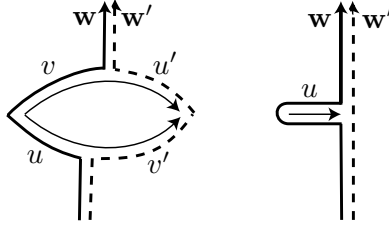


FIGURE 1.1. Right reversing in the Cayley graph

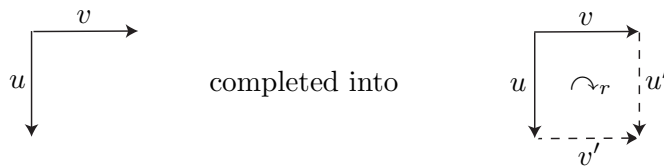
$\mathbf{w} \curvearrow_r^{(2)} bb^{-1}$, and no further r -reversing is possible. By using the second relation first, we can construct a different r -reversing sequence, for instance $\mathbf{w} \curvearrow_r^{(1)} ab^{-1}ab^{-1} \curvearrow_r^{(1)} a^2b^{-2}$. Observe that the previous sequences are maximal in the sense that they end up with a word of the form vu^{-1} with u, v in \mathcal{S}^* , and no further r -reversing is possible as such a word contains no subword of the form $u'^{-1}v'$ with $u', v' \neq \varepsilon$. An example of a (maximal) l -reversing sequence is $\mathbf{w} \curvearrow_l^{(1)} a^{-1}bb^{-1}a \curvearrow_l^{(1)} a^{-1}a$.

As the previous example shows, reversing is not a deterministic process in general: there can exist many ways of reversing one word. The only case where r -reversing is certainly deterministic is the case of complemented presentations:

Definition 1.3. A positive presentation $(\mathcal{S}, \mathcal{R})$ is said to be r -complemented if, for all letters s, t in \mathcal{S} , there is at most one relation of the type $s \cdots = t \cdots$ in \mathcal{R} , and no relation of the type $s \cdots = s \cdots$. We say that $(\mathcal{S}, \mathcal{R})$ is *complemented* if it is both r - and l -complemented, the latter being defined symmetrically.

Reversing has been investigated in the complemented case in [10] and [13]. The purpose of our current study is to extend the results to the general case, *i.e.*, to non necessarily complemented presentations. We hope to convince the reader that this extension is not trivial and that the general case is actually the most convenient one, in particular because it forces us to carefully choose the right technical conditions whereas an additional superfluous hypothesis like complementedness left some misleading flexibility.

It is convenient to associate with every r -reversing sequence $\mathbf{w}_0, \mathbf{w}_1, \dots$ a labelled planar graph as follows. First, we associate with \mathbf{w}_0 a path labelled with the successive letters of \mathbf{w}_0 : we associate to every positive letter s an horizontal right-oriented edge labelled s , and to every negative letter s^{-1} a vertical down-oriented edge labelled s . Then we by and by represent the words $\mathbf{w}_1, \mathbf{w}_2, \dots$ as follows: if \mathbf{w}_{i+1} is obtained from \mathbf{w}_i by replacing $u^{-1}v$ with $v'u'^{-1}$ (such that $uv' = vu'$ is a relation of our presentation), then the involved factor $u^{-1}v$ is associated with a diverging pair of edges in a path labelled \mathbf{w}_i and we complete our graph by closing the open pattern $u^{-1}v$ using horizontal edges labelled v' and vertical edges labelled u' :



The case of the empty word ε , which appears when a factor $u^{-1}u$ is deleted or some relation $uv' = v$ is used, is treated similarly: we introduce ε -labelled edges and use them according to the conventions $\varepsilon^{-1}u \curvearrowright_r u\varepsilon^{-1}$, $u^{-1}\varepsilon \curvearrowright_r \varepsilon u^{-1}$, and $\varepsilon^{-1}\varepsilon \curvearrowright_r \varepsilon\varepsilon^{-1}$. A symmetric construction is associated with l -reversing. With these conventions, the graphs associated with the reversing sequences of Example 1.2 are those represented in Fig. 1.2. Notice that the reversing graphs, which are reminiscent of van Kampen diagrams, need not be fragments of the Cayley graph: several vertices may represent the same element of the group, and they are not identified.

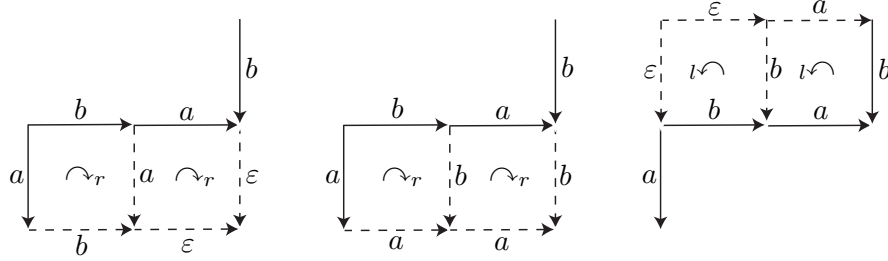


FIGURE 1.2. Two r - and one l -reversing sequences from $a^{-1}bab^{-1}$

Let us turn to the technical study of reversing. First, we observe that we can restrict without loss of generality to reversing transformations of a particular type, namely those involving length 2 initial factors, *i.e.*, to the case when u and v are single letters.

Lemma 1.4. *Let $\curvearrowright_r^{[1]}$ be the binary relation defined as $\curvearrowright_r^{(1)}$ excepted that we require that the words u and v have length 1 exactly. Then the relation \curvearrowright_r is the transitive closure of $\curvearrowright_r^{[1]}$, *i.e.*, for all words \mathbf{w}, \mathbf{w}' , we have $\mathbf{w} \curvearrowright_r^{(k)} \mathbf{w}'$ if and only if, writing $\curvearrowright_r^{[n]}$ for the n th power of $\curvearrowright_r^{[1]}$, we have $\mathbf{w} \curvearrowright_r^{[k']} \mathbf{w}'$ for some k' (not necessarily equal to k).*

Proof. (Fig. 1.3) By definition, $\curvearrowright_r^{[1]}$ is included in \curvearrowright_r , so it suffices to prove that $\mathbf{w} \curvearrowright_r^{(1)} \mathbf{w}'$ implies $\mathbf{w} \curvearrowright_r^{[k]} \mathbf{w}'$ for some k . Assume that $uv' = vu'$ is a relation of \mathcal{R} , with $u, v \neq \varepsilon$. Let s and t be the first letters of u and v , say $u = su_0$ and $v = tv_0$, where u_0 and v_0 have length p and q , respectively. By hypothesis, $su_0v' = tv_0u'$ is a relation of \mathcal{R} , so we find

$$u^{-1}v = u_0^{-1}s^{-1}tv_0 \curvearrowright_r^{[1]} u_0^{-1}u_0v'u'^{-1}v_0^{-1}v_0 \curvearrowright_r^{[p]} v'u'^{-1}v_0^{-1}v_0 \curvearrowright_r^{[q]} v'u'^{-1},$$

as, by construction, $w^{-1}w \curvearrowright_r^{[\ell]} \varepsilon$ holds for every word w of length ℓ in \mathcal{S}^* . \square

Remark 1.5. Instead of restricting the definition of reversing by considering particular subwords $u^{-1}v$, we can extend it by relaxing the assumption that u and v are nonempty. Merely dropping the assumption would allow one to replace ε by any word uv^{-1} such that $u = v$ is a relation of \mathcal{R} , which contradicts the implicit underlying principle that reversing should not increase complexity. But an interesting notion is obtained when we allow u to be empty provided u' is empty as well, *i.e.*, we allow replacing v with v' when $v = v'$ is a relation of \mathcal{R} , and, symmetrically, we allow v to be empty provided v' is, *i.e.*, we allow replacing u^{-1} with u'^{-1}

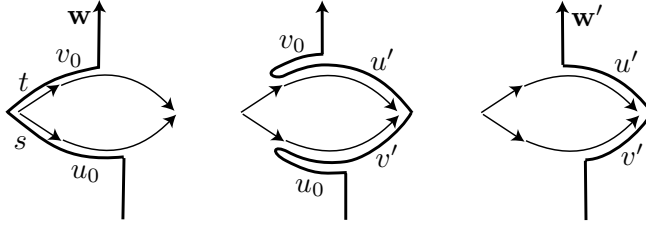


FIGURE 1.3. Restricted reversing

when $u = u'$ is a relation of \mathcal{R} . Most of the subsequent study of \curvearrowright_r remains valid when the extended relation $\curvearrowright_r^\sharp$ so defined replaces \curvearrowright_r . However, in practice, in particular when implementations are concerned, using $\curvearrowright_r^\sharp$ instead of \curvearrowright_r makes the verifications longer, as more transformations have to be considered.

We establish now some general properties of (right) reversing. Owing to Lemma 1.4, we can always assume without loss of generality that the basic right reversing steps corresponds to what we denoted $\curvearrowright_r^{[1]}$, *i.e.*, they involve factors of the form $s^{-1}t$ where s and t are single letters.

Lemma 1.6. *For all words \mathbf{w}, \mathbf{w}' on $\mathcal{S} \cup \mathcal{S}^{-1}$, $\mathbf{w} \curvearrowright_r \mathbf{w}'$ implies $\mathbf{w} \equiv^\pm \mathbf{w}'$ and $\mathbf{w}^{-1} \curvearrowright_r \mathbf{w}'^{-1}$.*

Proof. It suffices to prove the result for $\mathbf{w} \curvearrowright_r^{[1]} \mathbf{w}'$. The case when some factor $s^{-1}s$ has been deleted is obvious. Otherwise, assume that \mathbf{w}' has been obtained from \mathbf{w} by substituting $s^{-1}t$ with vu^{-1} where $sv = tu$ is one of the relations of the considered presentation. Then we have $sv \equiv tu$, and, a fortiori, $sv \equiv^\pm tu$, hence $s^{-1}t \equiv^\pm vu^{-1}$, and, therefore, $\mathbf{w} \equiv^\pm \mathbf{w}'$. On the other hand, \mathbf{w}'^{-1} is obtained from \mathbf{w}^{-1} by replacing $t^{-1}s$ with uv^{-1} , which is also an r -reversing. \square

Lemma 1.7. *(Fig. 1.4) Assume $\mathbf{w} \curvearrowright_r^{[k]} vu^{-1}$ with $u, v \in \mathcal{S}^*$. Then, for every decomposition $\mathbf{w} = \mathbf{w}_1\mathbf{w}_2$, there exist in \mathcal{S}^* decompositions $u = u_1u_2$, $v = v_1v_2$, and u_0, v_0 satisfying $\mathbf{w}_1 \curvearrowright_r^{[k_1]} v_1u_0^{-1}$, $\mathbf{w}_2 \curvearrowright_r^{[k_2]} v_0u_1^{-1}$, and $u_0^{-1}v_0 \curvearrowright_r^{[k_0]} v_2u_2^{-1}$ with $k = k_1 + k_2 + k_0$.*

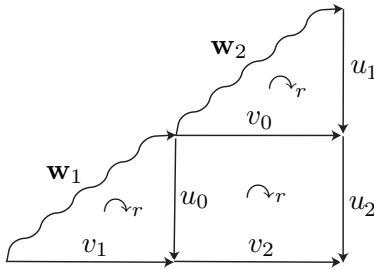


FIGURE 1.4. Reversing a product (general case)

Proof. We use induction on k . For $k = 0$, the hypothesis means that the word \mathbf{w} is equal to vu^{-1} for some words u, v in \mathcal{S}^* . Two cases may occur, according to whether \mathbf{w}_1 is shorter than v or not. Assume for instance that \mathbf{w}_1 is a prefix of v ,

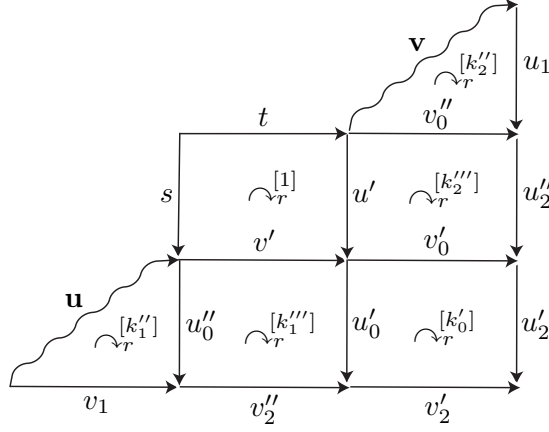


FIGURE 1.5. Proof of Lemma 1.7

say $v = \mathbf{w}v_2$. We put $v_0 = v_2$, $v_1 = \mathbf{w}_1$, $u_0 = u_1 = \varepsilon$, and $u_2 = u$. Then, trivially, we have $\mathbf{w}_1 = v_1 \curvearrowright_r^{[0]} v_1 u_0^{-1}$, $\mathbf{w}_2 = v_2 u^{-1} \curvearrowright_r^{[0]} v_0 u_1^{-1}$, and $u_0^{-1} v_0 \curvearrowright_r^{[0]} v_2 u_2^{-1}$, and the expected result is true. The argument is equally trivial when v is a prefix of \mathbf{w}_1 .

Assume now $k \geq 1$, and let \mathbf{w}' be the second word in a shortest reversing sequence from \mathbf{w} to vu^{-1} : by definition, we have $\mathbf{w} = \mathbf{u}s^{-1}t\mathbf{v}$ and $\mathbf{w}' = \mathbf{u}v'u'^{-1}\mathbf{v}$, with s, t in \mathcal{S} and $sv' = tu'$ in \mathcal{R} . Let us consider a decomposition $\mathbf{w} = \mathbf{w}_1\mathbf{w}_2$. Three cases may occur.

If $\mathbf{u}s^{-1}t$ is a prefix of \mathbf{w}_1 , say $\mathbf{w}_1 = \mathbf{u}s^{-1}t\mathbf{v}_1$, then we have $\mathbf{w}_1 \curvearrowright_r^{[1]} \mathbf{w}'_1$ with $\mathbf{w}'_1 = \mathbf{u}v'u'^{-1}\mathbf{v}_1$. By construction, we have $\mathbf{w}' = \mathbf{w}'_1\mathbf{w}'_2$. Applying the induction hypothesis to $\mathbf{w}' \curvearrowright_r^{[k-1]} vu^{-1}$, we find u_0, \dots, v_2 satisfying $u = u_1u_2$, $v = v_1v_2$, and $\mathbf{w}'_1 \curvearrowright_r^{[k_1]} v_1u_0^{-1}$, $\mathbf{w}'_2 \curvearrowright_r^{[k_2]} v_0u_1^{-1}$, $u_0^{-1}v_0 \curvearrowright_r^{[k_0]} v_2u_2^{-1}$ with $k_1 + k_2 + k_0 = k - 1$. Then $\mathbf{w}_1 \curvearrowright_r^{[1]} \mathbf{w}'_1$ implies $\mathbf{w}_1 \curvearrowright_r^{[k_1+1]} v_1u_0^{-1}$, and we are done.

The case when $s^{-1}t\mathbf{v}$ is a suffix of \mathbf{w}_2 is symmetric. So we are left with the non-trivial case, namely $\mathbf{w}_1 = \mathbf{u}s^{-1}$ and $\mathbf{w}_2 = t\mathbf{v}$ (Fig. 1.5). We apply the induction hypothesis to $\mathbf{w}' \curvearrowright_r^{[k-1]} vu^{-1}$ with the decomposition $\mathbf{w}' = \mathbf{w}'_1\mathbf{w}'_2$ with $\mathbf{w}'_1 = \mathbf{u}v'$ and $\mathbf{w}'_2 = u'^{-1}\mathbf{v}$. In this way, we find words $u'_0, u'_1, u'_2, v'_0, v'_1, v'_2$ in \mathcal{S}^* satisfying $u = u'_1u'_2$, $v = v'_1v'_2$, $\mathbf{w}'_1 \curvearrowright_r^{[k_1]} v'_1u'_0^{-1}$, $\mathbf{w}'_2 \curvearrowright_r^{[k_2]} v'_0u'_1^{-1}$, $u'_0^{-1}v'_0 \curvearrowright_r^{[k_0]} v'_2u'_2^{-1}$ with $k_1 + k_2 + k_0 = k - 1$. Now, applying the induction hypothesis to $\mathbf{w}'_1 \curvearrowright_r^{[k_1]} v'_1u'_0^{-1}$ with the decomposition $\mathbf{w}'_1 = \mathbf{u}v'$ gives us words u''_0, v_1, v'_2 in \mathcal{S}^* satisfying $v'_1 = v_1v'_2$, $\mathbf{u} \curvearrowright_r^{[k''_1]} v_1u''_0^{-1}$, and $u''_0^{-1}v' \curvearrowright_r^{[k'''_1]} v'_2u''_0^{-1}$ with $k''_1 + k'''_1 = k_1$: indeed, the hypothesis that v' belongs to \mathcal{S}^* implies that $v' \curvearrowright_r^{[0]} v'\varepsilon^{-1}$ is the only possible reversing from v' . Similarly, applying the induction hypothesis to $\mathbf{w}'_2 \curvearrowright_r^{[k_2]} v'_0u'_1^{-1}$ with the decomposition $\mathbf{w}'_2 = u'^{-1}\mathbf{v}$ gives us words v''_0, u_1, u''_2 in \mathcal{S}^* satisfying $u'_1 = u_1u''_2$, $\mathbf{v} \curvearrowright_r^{[k''_2]} v''_0u_1^{-1}$, and $u'^{-1}v''_0 \curvearrowright_r^{[k'''_2]} v''_0u''_2^{-1}$ with $k''_2 + k'''_2 = k_2$. Put $u_0 = su''_0$, $u_2 = u''_2u'_2$, $v_0 = tv''_0$, and $v_2 = v''_2v'_2$. By construction, we have $u = u_1u_2$ and $v = v_1v_2$. Then we find $\mathbf{w}_1 = \mathbf{u}s^{-1} \curvearrowright_r^{[k''_1]} v_1u_0^{-1}$, and $\mathbf{w}_2 = t\mathbf{v} \curvearrowright_r^{[k''_2]} v_0u_1^{-1}$.

Finally, we obtain

$$\begin{aligned} u_0^{-1}v_0 &= u_0''^{-1}s^{-1}tv_0'' \curvearrowright_r^{[1]} u_0''^{-1}v'u'^{-1}v_0'' \\ &\quad \curvearrowright_r^{[k_1''']} v_2''u_0'^{-1}u'^{-1}v_0'' \\ &\quad \curvearrowright_r^{[k_2''']} v_2''u_0'^{-1}v_0''u_2''^{-1} \curvearrowright_r^{[k_0']} v_2''v_2''u_2''^{-1}u_2''^{-1} = v_2u_2^{-1}, \end{aligned}$$

hence $u_0^{-1}v_0 \curvearrowright_r^{[k_0]} v_2u_2^{-1}$ with $k_0 = 1 + k_1''' + k_2''' + k_0'$. As we check $k_1'' + k_2'' + k_0 = k$, we are done. \square

Applying the previous result to the case when \mathbf{w}_1 has the form $u^{-1}v_1$ and \mathbf{w}_2 belongs to \mathcal{S}^* gives:

Lemma 1.8. (Fig. 1.6) Assume $u, v_1, v_2, u', v' \in \mathcal{S}^*$ and $u^{-1}v_1v_2 \curvearrowright_r v'u'^{-1}$. Then there exists in \mathcal{S}^* a decomposition $v' = v'_1v'_2$ and a word u_1 satisfying $u^{-1}v_1 \curvearrowright_r v'_1u_1^{-1}$ and $u_1^{-1}v_2 \curvearrowright_r v'_2u'^{-1}$.

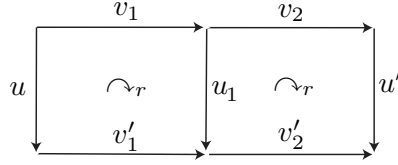


FIGURE 1.6. Reversing a product (positive case)

Proposition 1.9. Assume that $(\mathcal{S}, \mathcal{R})$ is a positive group presentation, and u, v, u', v' are words in \mathcal{S}^* . Then $u^{-1}v \curvearrowright_r v'u'^{-1}$ implies $uv' \equiv vu'$.

Proof. We use induction on the number of basic steps, say k , needed to reverse $u^{-1}v$ into $v'u'^{-1}$, i.e., we assume $u^{-1}v \curvearrowright_r^{[k]} v'u'^{-1}$. For $k = 0$, the only possibility is that u or v is empty, in which case we have $u' = u$ and $v' = v$, and the result is true.

Assume $k \geq 1$. In this case, the words u and v cannot be empty. Write $u = su_0$, $v = tv_0$, where s, t are letters in \mathcal{S} . The first step in the reversing process from $u^{-1}v$ to $v'u'^{-1}$ must involve the fragment $s^{-1}t$, so there exists some relation $sv_1 = tu_1$ in \mathcal{R} such that the reversing sequence decomposes into

$$u^{-1}v = u_0^{-1}s^{-1}tv_0 \curvearrowright_r^{[1]} u_0^{-1}v_1u_1^{-1}v_0 \curvearrowright_r^{[k-1]} v'u'^{-1}.$$

Applying Lemma 1.7 to $u_0^{-1}v_1u_1^{-1}v_0 \curvearrowright_r^{[k-1]} v'u'^{-1}$ with the decomposition into $u_0^{-1}v_1$ and $u_1^{-1}v_0$, we find words $u_2, v_2, u_3, v_3, u_4, v_4$ in \mathcal{S}^* and numbers k_2, k_3, k_4 with $k_2 + k_3 + k_4 = k - 1$, $u' = u_3u_4$, and $v' = v_2v_4$, such that the relations $u_0^{-1}v_1 \curvearrowright_r^{[k_2]} v_2u_2^{-1}$, $u_1^{-1}v_0 \curvearrowright_r^{[k_3]} v_3u_3^{-1}$, and $u_2^{-1}v_3 \curvearrowright_r^{[k_4]} v_4u_4^{-1}$ are satisfied (Fig. 1.7). Applying the induction hypothesis, which is legal as we have $k_i < k$ for $i = 2, 3, 4$, we obtain $u_0v_2 \equiv v_1u_2$, $u_1v_3 \equiv v_0u_3$, and $u_2v_4 \equiv v_3u_4$, hence

$$uv' = su_0v_2v_4 \equiv sv_1u_2v_4 \equiv tu_1u_2v_4 \equiv tu_1v_3u_4 \equiv tv_0u_3u_4 = vu',$$

which gives the expected equivalence. \square

For future use, let us state two applications of the previous result:

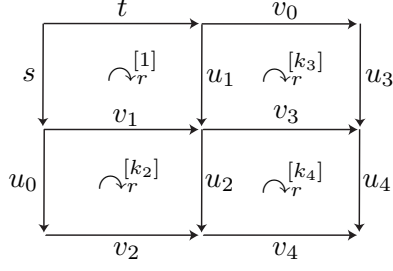


FIGURE 1.7. Reversing gives positive equivalence

Lemma 1.10. (i) The relation $u^{-1}ww^{-1}v \curvearrowright_r v'u'^{-1}$ implies $wv' \equiv vu'$.
(ii) The relation $(wv')^{-1}(vu') \curvearrowright_r \varepsilon$ implies that there exist u'', v'', w'' in \mathcal{S}^* satisfying $u^{-1}v \curvearrowright_r v''u''^{-1}$, $u' \equiv u''w''$, and $v' \equiv v''w''$.

Proof. (Fig. 1.8) (i) By Lemma 1.7, the relation $u^{-1}ww^{-1}v \curvearrowright_r v'u'^{-1}$ implies the existence of two decompositions $u' = u'_1u'_2$, $v' = v'_1v'_2$ and of u_0, v_0 satisfying $u^{-1}w \curvearrowright_r v'_1u_0^{-1}$, $w^{-1}v \curvearrowright_r v_0u_1^{-1}$, and $u_0^{-1}v_0 \curvearrowright_r v'_2u_2^{-1}$. Then, using Prop. 1.9, we obtain

$$wv' = wv'_1v'_2 \equiv wu_0v'_2 \equiv wv_0u'_2 \equiv vu'_1u'_2 = vu'.$$

(ii) By Lemma 1.8, the relation $(wv')^{-1}(vu') \curvearrowright_r \varepsilon$ implies the existence of words u'', v'', w', w'' satisfying $u^{-1}v \curvearrowright_r v''u''^{-1}$, $v'^{-1}v'' \curvearrowright_r w'^{-1}$, $u''^{-1}u' \curvearrowright_r w''$, and $w'^{-1}w'' \curvearrowright_r \varepsilon$. By Prop. 1.9, the latter relations imply $u' \equiv u''w''$, $w' \equiv w''$, and $v' \equiv v''w' \equiv v''w''$. \square

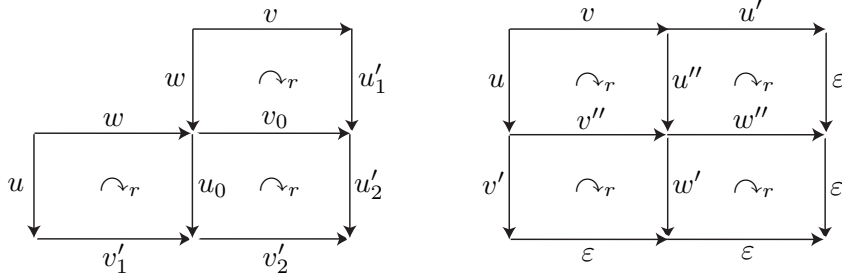


FIGURE 1.8. Proof of Lemma 1.10

The question of whether reversing converges, *i.e.*, the existence of an upper bound for the length of the reversing sequences starting from a given word, is difficult in general. It is easy to give examples of simple finite presentations, such as the Baumslag-Solitar presentation $(a, b; ba = a^2b)$, or the non-spherical Artin-Tits presentation $(a, b, c; aba = bab, bcb = cbc, aka = cac)$, where infinitely long reversing sequences exist: start for instance with $b^{-1}ab$ and with $a^{-1}bc$ in the examples above. Also, [14] contains an example of an infinite presentation where all reversing sequences are finite, but the only known bound on the length of a reversing sequence starting from a length n word is a tower of exponentials of height $O(2^n)$. Besides such complicated cases, easy upper bounds can be established when the closure of the initial alphabet under reversing happens to be known.

Definition 1.11. Assume that $(\mathcal{S}, \mathcal{R})$ is a positive group presentation. We say that a subset \mathcal{S}' of \mathcal{S}^* is *closed under r -reversing* if u' and v' lie in \mathcal{S}' whenever u and v do and $u^{-1}v \curvearrowright_r v'u'^{-1}$ holds. The *closure* of \mathcal{S} under r -reversing is defined to be the smallest subset of \mathcal{S}^* that includes \mathcal{S} and is closed under r -reversing.

Example 1.12. Let us consider the presentation $(a, b; a^2 = b^2, ab = ba)$ of Example 1.2. Then the set $\{\varepsilon, a, b\}$ is closed under r -reversing: up to a symmetry, the only possibilities are $a^{-1}a \curvearrowright_r \varepsilon$, $a^{-1}b \curvearrowright_r ba^{-1}$, $a^{-1}b \curvearrowright_r ab^{-1}$, and the only words of $\{a, b\}^*$ involved in the right hand sides are ε , a , and b . So $\{\varepsilon, a, b\}$ is the closure of $\{a, b\}$ under r -reversing.

Starting with a finite (or, simply, recursive) positive group presentation $(\mathcal{S}, \mathcal{R})$, determining the closure of \mathcal{S} under r -reversing is typically a recursively enumerable process: for each word \mathbf{w} on $\mathcal{S} \cup \mathcal{S}^{-1}$, we can enumerate all words \mathbf{w}' to which \mathbf{w} is r -reversible in 1, 2, etc. steps, and, each time we find a word of the form uv^{-1} with u, v in \mathcal{S}^* , add it to the current family. Provided we enumerate the words in a systematic way, each word in the closure of \mathcal{S} will appear at some finite step of the process, but, if we have no recursive upper bound for the lengths of the r -reversing sequences from \mathbf{w} in terms of the length of \mathbf{w} , we shall never know whether all words in the closure of \mathcal{S} have been found (even if the latter is finite). However, if we happen to find a finite set of words \mathcal{S}' that includes \mathcal{S} and we can prove that every r -reversing sequence from $u^{-1}v$ with $u, v \in \mathcal{S}'$ either ends up with a failure, *i.e.*, with a word containing some factor $s^{-1}t$ for which there is no relation $s \cdots = t \cdots$ in \mathcal{R} , or with a word $v'u'^{-1}$ with $u', v' \in \mathcal{S}'$, then we can claim that \mathcal{S}' includes the closure of \mathcal{S} under reversing. Example 1.12 provides a (trivial) instance of this situation.

Proposition 1.13. *Assume that $(\mathcal{S}, \mathcal{R})$ is a recursive positive presentation such that the closure $\widehat{\mathcal{S}}$ of \mathcal{S} under r -reversing and the restriction $\curvearrowright_r^\bullet$ of the relation $u^{-1}v \curvearrowright_r v'u'^{-1}$ to $\widehat{\mathcal{S}}^4$ are recursive. Then the relation $\mathbf{w} \curvearrowright_r vu^{-1}$ on $(\mathcal{S} \cup \mathcal{S}^{-1})^* \times (\mathcal{S}^*)^2$ is recursive; if \mathbf{w} is a word with p letters in \mathcal{S} and q letters in \mathcal{S}^{-1} , and $\mathbf{w} \curvearrowright_r vu^{-1}$ holds, then v belongs to $\widehat{\mathcal{S}}^p$, u belongs to $\widehat{\mathcal{S}}^q$, and the reversing of \mathbf{w} to vu^{-1} can be decomposed into at most pq reversings in $\curvearrowright_r^\bullet$.*

Proof. By hypothesis, the word \mathbf{w} is $w_1^{e_1} \cdots w_n^{e_n}$ with $w_i \in \widehat{\mathcal{S}}$ and $e_i = \pm 1$ for $i = 1, \dots, n$. Denote by $d(\mathbf{w})$ the number of pairs (i, j) with $i < j$, $e_i = -1$, and $e_j = 1$. By construction, we have $d(\mathbf{w}) \leq pq$. We prove the result using induction on $d(\mathbf{w})$. For $d(\mathbf{w}) = 0$, the word \mathbf{w} has the form vu^{-1} with $v \in \widehat{\mathcal{S}}^q$ and $u \in \widehat{\mathcal{S}}^p$, and it is reversed, so the result is true. Otherwise, there exist i satisfying $e_i = -1$ and $e_{i+1} = +1$. Using Lemma 1.7 twice, we see that there must exist w'_i, w'_{i+1} in \mathcal{S}^* such that $w_i^{-1}w_{i+1} \curvearrowright_r w'_iw'_{i+1}^{-1}$ holds and $\mathbf{w} \curvearrowright_r vu^{-1}$ may be decomposed into

$$\mathbf{w} = \mathbf{u}w_i^{-1}w_{i+1}\mathbf{v} \curvearrowright_r \mathbf{u}w'_iw'_{i+1}^{-1}\mathbf{v} \curvearrowright_r vu^{-1}$$

with $\mathbf{u} = w_1^{e_1} \cdots w_{i-1}^{e_{i-1}}$ and $\mathbf{v} = w_{i+2}^{e_{i+2}} \cdots w_n^{e_n}$. By construction, the words w'_i and w'_{i+1} belong to $\widehat{\mathcal{S}}$, and, letting $\mathbf{w}' = \mathbf{u}w'_iw'_{i+1}^{-1}\mathbf{v}$, we see that the word \mathbf{w}' satisfies the same requirements as \mathbf{w} with $d(\mathbf{w}') = d(\mathbf{w}) - 1$, so we can apply the induction hypothesis. \square

A favourable case is when all relations in the considered presentation involve words of length 2 at most: in this case, the closure $\widehat{\mathcal{S}}$ of \mathcal{S} under reversing is merely $\mathcal{S} \cup \{\varepsilon\}$, so we obtain:

Corollary 1.14. *Assume that $(\mathcal{S}, \mathcal{R})$ is a finite positive presentation and all relations in \mathcal{R} have the form $u = v$ with u and v of length 1 or 2. Then every r -reversing sequence starting with a length n word has length $n^2/4$ at most, and all words in such a sequence have length n at most.*

The case above is not the only one when the closure can be determined. For instance, in the case of the standard presentation of the braid group B_n , the closure of the generators $\sigma_1, \dots, \sigma_{n-1}$ under r -reversing is a set of positive words representing the $n! - 1$ proper divisors of Δ_n , so, in particular, it is a finite set. We refer to [33, 34, 35] for many other examples (in the complemented case).

Remark 1.15. It is proved in [13] that, if $(\mathcal{S}, \mathcal{R})$ is a finite complemented presentation and all relations in \mathcal{R} preserve the length, then there exists a constant C such that, if u and v are \equiv -equivalent length n words, then $u^{-1}v$ reverses to ε in at most $2^{2^{Cn}}$ steps. Whether this result extends to arbitrary finite presentations is unknown.

2. COMPLETE PRESENTATIONS

We introduce now our key notion, namely that of a complete presentation. The idea is that a presentation is complete if it contains enough relations to make reversing exhaustive.

Definition 2.1. Let $(\mathcal{S}, \mathcal{R})$ be a positive presentation. For u, v, u', v' in \mathcal{S}^* , we say that $(\mathcal{S}, \mathcal{R})$ is r -complete at u, v, u', v' if the following implication holds:

$$(2.1) \quad \begin{array}{l} \text{If } uv' \equiv vu' \text{ holds, then there exist } u'', v'', w \text{ in } \mathcal{S}^* \\ \text{satisfying } u^{-1}v \curvearrow_r v''u''^{-1}, u' \equiv u''w, \text{ and } v' \equiv v''w; \end{array}$$

we say that $(\mathcal{S}, \mathcal{R})$ is r -complete if (2.1) holds for all u, v, u', v' .

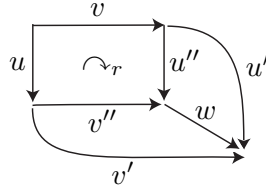
Symmetrically, we say that $(\mathcal{S}, \mathcal{R})$ is l -complete at u, v, u', v' if we have

$$(2.2) \quad \begin{array}{l} \text{If } v'u \equiv u'v \text{ holds, then there exist } u'', v'', w \text{ in } \mathcal{S}^* \\ \text{satisfying } uv^{-1} \curvearrow_l v''^{-1}u'', u' \equiv wu'', \text{ and } v' \equiv wv'; \end{array}$$

we say that $(\mathcal{S}, \mathcal{R})$ is l -complete if (2.2) holds for all u, v, u', v' , and that $(\mathcal{S}, \mathcal{R})$ is *complete* if it is both r - and l -complete.

Completeness says something non-trivial only for those 4-tuples that satisfy $uv' \equiv vu'$: for the other ones, the implications (2.1) and (2.2) are trivially true. By Prop. 1.9, $u^{-1}v \curvearrow_r v''u''^{-1}$ implies $uv'' \equiv vu''$, and, symmetrically, $uv^{-1} \curvearrow_l v''^{-1}u''$ implies $v''u \equiv u''v$, so the converse implications of (2.1) and (2.2) always hold. Completeness claims that these sufficient conditions also are necessary: it tells us that every common multiple relation $uv' \equiv vu'$ factors through some reversing, as illustrated in Fig. 2.1.

Remark 2.2. The statement of the completeness property and the picture in Fig. 2.1 are reminiscent of Prop. H in [21], or of the Kürzungslemma in [4]. However, the point here is not the factorization property for common multiples, but the fact that the square $(uv'', v'u'')$ corresponds to an r -reversing process. Completeness is also closely connected with Corran's reduction property of [8]. With our current


 FIGURE 2.1. The r -completeness condition

definitions, the latter corresponds to r -completeness at every 4-tuple u, v, u', v' with u, v in \mathcal{S} . So it is clear that r -completeness always implies the reduction property. In the other direction, it can be proved that the reduction property implies what will be called the r -cube condition on \mathcal{S} in Sec. 3 below, so, for r -homogeneous presentations (see Sec.4), it implies r -completeness, and both notions are equivalent in this case.

The following result is a straightforward consequence of the definition:

Lemma 2.3. *Assume that $(\mathcal{S}, \mathcal{R})$ is an r -complete positive presentation, and \mathcal{R}' includes \mathcal{R} and defines the same monoid as \mathcal{R} , i.e., every relation in $\mathcal{R}' \setminus \mathcal{R}$ is a consequence of \mathcal{R} . Then $(\mathcal{S}, \mathcal{R}')$ is r -complete as well.*

A natural question is whether complete presentations exist. The answer is trivial:

Proposition 2.4. *Every monoid with no non-trivial unit admits a complete presentation.*

Proof. Let M be a monoid, and \mathcal{S} be an arbitrary set of generators for M . Let \cong be the congruence on \mathcal{S}^* such that M is the quotient \mathcal{S}^*/\cong . Let \mathcal{R} consist of all relations $u = v$ with $u \cong v$ and $u, v \neq \varepsilon$. As $u \cong \varepsilon$ is supposed to hold for no nonempty word u , $(\mathcal{S}, \mathcal{R})$ is a presentation of M , which we claim is complete. Indeed, assume $uv' \equiv vu'$. If u or v is empty, the condition for completeness holds trivially. Otherwise, we write $u = su_0, v = tv_0$ with $s, t \in \mathcal{S}$. The hypothesis is $su_0v' \equiv tv_0u'$, i.e., $su_0v' \cong tv_0u'$, hence, by definition, the relation $su_0v' = tv_0u'$ belongs to \mathcal{R} as the considered words are nonempty. Then $s^{-1}t \curvearrowright_r u_0v'u'^{-1}v_0^{-1}$ holds by definition, which implies

$$u^{-1}v = u_0^{-1}s^{-1}tv_0 \curvearrowright_r u_0^{-1}u_0v'u'^{-1}v_0^{-1}v_0 \curvearrowright_r v'u'^{-1}.$$

Putting $u'' = u', v'' = v'$ and $w = \varepsilon$ gives (2.1), proving r -completeness at u, v, u', v' . The verification of l -completeness is similar. \square

The practical interest of the previous result is weak: the complete presentation given by Prop. 2.4 is infinite whenever the considered monoid is infinite, and, more important, writing such a presentation supposes knowing a solution to the word problem. As we shall see below, the interesting case is that of a finite complete presentation, about which Prop. 2.4 tells us nothing in general.

A more interesting method to possibly obtain complete presentations consists in considering minimal common multiples (when they exist).

Definition 2.5. Assume that M is a monoid. For $x, y, z \in M$, we say that z is a *minimal* common right multiple, or *r -mcm*, of x and y if z is a right multiple both of x and y , but no proper left divisor of z is.

The notion of a minimal common multiple is a generalization of that of a least common multiple: saying that two elements x, y admit a least common multiple amounts to saying that they admit a unique minimal common multiple. Mcm's need not exist in general, but they do in good cases, namely when the considered monoid is *Noetherian*. If x, y are elements of a monoid M , we write $x \prec_l y$ if $y = xz$ holds for some $z \neq 1$, and, symmetrically, $x \prec_r y$ if $y = zx$ holds for some $z \neq 1$.

Definition 2.6. We say that a monoid M is *l-Noetherian* if the relation \prec_l has no infinite descending chain, *i.e.*, there exists no infinite sequence $x_0 \succ_l x_1 \succ_l \dots$ in M . Symmetrically, we say that M is *r-Noetherian* if \prec_r has no infinite descending chain, and that M is *Noetherian* if it is both *l-* and *r-Noetherian*.

If M is an *l-Noetherian* monoid, the associated relation \prec_l must be irreflexive, so, in particular, M contains no non-trivial invertible element; more generally, the relation \prec_l is then a partial ordering on M , which is compatible with multiplication on the left, and for which 1 is a least element.

Lemma 2.7. *Assume that M is an l-Noetherian monoid. Then any common r -multiple of two elements x, y of M is an r -multiple of some r -mcm of x and y .*

Proof. Our hypothesis is that every nonempty subset of M contains an element which is minimal with respect to \prec_l . Applying this property to the set of all common right multiples of x and y which are left divisors of z gives the expected right mcm. \square

We shall now prove how to obtain complete presentations in the case of a Noetherian monoid by considering r -mcm relations.

Definition 2.8. Assume that M is a monoid, and \mathcal{S} is a set of generators for M . We say that a family of relations \mathcal{R} is an *r-selector* on \mathcal{S} in M if, for all s, t in \mathcal{S} and for each r -mcm x of s and t , there exist two words u, v in \mathcal{S}^* such that $sv = tu$ belongs to \mathcal{R} and both sv and tu represent x .

Thus, an r -selector is a family of relations that proves all equalities connected with right mcm's in the considered monoid M . Observe that r -selectors always exist, but an r -selector may be just empty when no right mcm exists. The following result shows that, in the case of a Noetherian monoid, each r -selector gives rise to a presentation, which moreover turns out to be r -complete.

Proposition 2.9. *Assume that M is a left cancellative Noetherian monoid, \mathcal{S} is a set of generators for M , and \mathcal{R} is an r -selector on \mathcal{S} in M . Then $(\mathcal{S}, \mathcal{R})$ is an r -complete presentation of M .*

Proof. As in the proof of Prop. 2.4, let \cong denote the congruence on \mathcal{S}^* such that M is isomorphic to \mathcal{S}^*/\cong . Let \equiv be the congruence associated with the selector \mathcal{R} . By definition, \mathcal{R} consists of pairs $\{u, v\}$ that satisfy $u \cong v$, so $u \equiv v$ implies $u \cong v$ trivially.

We shall now prove conversely that $u \cong v$ implies $u \equiv v$ for all u, v in \mathcal{S}^* using induction on \bar{u} with respect to \prec_r (we recall that \bar{u} denotes the element of M represented by u). As 1 is the least element relative to \prec_r , let us first assume $\bar{u} = \bar{v} = 1$, *i.e.*, $u \cong v \cong \varepsilon$. We have seen that 1 is the only invertible element in M , so, necessarily, u and v are empty, and we have $u = v$, hence $u \equiv v$.

Assume now $\bar{u} = \bar{v} \succ_r 1$. Then u and v are nonempty words, say $u = tu_0$, $v = sv_0$, with $s, t \in \mathcal{S}$. The hypothesis $u \cong v$ means that \bar{u} is a common r -multiple

of s and t in M . By Lemma 2.7, some left divisor z of \bar{u} has to be an r -mcm of s and t . So, by definition of \mathcal{R} , there must exist some relation $tu' = sv'$ in \mathcal{R} such that both tu' and sv' represent z in M , and the hypothesis that z is a left divisor of \bar{u} implies that some word w satisfies

$$tu_0 \cong tu'w \cong sv'w \cong sv_0.$$

Applying the hypothesis that M is left cancellative, we deduce $u_0 \cong u'w$ and $v_0 \cong v'w$. By construction, \bar{u}_0 and \bar{v}_0 are proper right divisors of \bar{u} , so the induction hypothesis allows us to deduce $u_0 \equiv u'w$ from $u_0 \cong u'w$ and $v_0 \equiv v'w$ from $v_0 \cong v'w$, and we obtain

$$u = tu_0 \equiv tu'w \equiv sv'w \equiv sv_0 = v.$$

It remains to prove that $(\mathcal{S}, \mathcal{R})$ is r -complete: we postpone the proof to Sec. 4, as the needed argument is similar to, but simpler than, the argument developed for Prop. 4.4. \square

In order to obtain a complete presentation (and not only an r -complete one), we can appeal to the symmetric obvious notion of an l -selector, and using Proposition 2.9, its left counterpart, and Lemma 2.3, we obtain

Proposition 2.10. *Assume that M is a cancellative Noetherian monoid, \mathcal{S} is a set of generators for M , \mathcal{R}_r is an r -selector on \mathcal{S} in M , and \mathcal{R}_l is an l -selector on \mathcal{S} in M . Then $(\mathcal{S}, \mathcal{R}_r \cup \mathcal{R}_l)$ is a complete presentation of M .*

Let us conclude this section with yet another way of constructing a complete presentation, even in a non-Noetherian case, when what is called a spanning subset in [16] happens to be known.

Proposition 2.11. *Assume that M is a left cancellative monoid with no non-trivial unit, and \mathcal{S} is a set of generators of M satisfying the following condition:*

(*) *For all x, y in \mathcal{S}' , if z is a common right multiple of x and y , then there exist x' and y' in $\mathcal{S} \cup \{1\}$ satisfying $xy' = yx' \preceq_l z$.*

Let \mathcal{R} be the set of all relations $xy' = yx'$ with x, y in \mathcal{S} and x', y' in $\mathcal{S} \cup \{1\}$ true in M . Then, $(\mathcal{S}, \mathcal{R})$ is an r -complete presentation of M .

That $(\mathcal{S}, \mathcal{R})$ is a presentation of M is proved in [16]. The argument is similar to that of Prop. 2.9, but it uses an induction on $\lg(u) + \lg(v)$ instead of an induction on \bar{u} relative to \prec_r , which need not be well-founded. The r -completeness of the presentation is then a direct translation of Condition (*). Because of this connection between Condition (*) and r -completeness, Prop. 2.11 is essentially trivial, and so is the converse statement that, if $(\mathcal{S}, \mathcal{R})$ is an r -complete presentation of some monoid M , then the subset of M consisting of those elements that can be represented by words in the closure of \mathcal{S} under r -reversing satisfies Condition (*).

In practice, it often happens that a set of (minimal) generators \mathcal{S}_0 is known, and some extension \mathcal{S} of \mathcal{S}_0 satisfies the assumptions of Prop. 2.11, thus giving rise to an r -complete presentation $(\mathcal{S}, \mathcal{R})$. We can then deduce an r -complete presentation in terms of the initial generators \mathcal{S}_0 as follows:

Proposition 2.12. *Assume that $(\mathcal{S}, \mathcal{R})$ is an r -complete presentation of a monoid M , and \mathcal{S}_0 is a subset of \mathcal{S} that generates M . For each x in \mathcal{S} , choose a word $f(x)$ in \mathcal{S}_0^* that represents x , with the requirement $f(x) = x$ for $x \in \mathcal{S}_0$, and let \mathcal{R}_0 be the image of \mathcal{R} under f . Then $(\mathcal{S}_0, \mathcal{R}_0)$ is an r -complete presentations of M .*

Proof. By construction, the monoid $\langle \mathcal{S}_0; \mathcal{R}_0 \rangle^+$ is a quotient of M . On the other hand, $u \equiv^{\mathcal{R}} f(u)$ is true for every word u on \mathcal{S} , so all relations in \mathcal{R}_0 are consequences of \mathcal{R} , and $(\mathcal{S}_0, \mathcal{R}_0)$ is a presentation of M .

First we claim that, if \mathbf{w}, \mathbf{w}' are words on $\mathcal{S} \cup \mathcal{S}^{-1}$, then $\mathbf{w} \curvearrow_r^{\mathcal{R}} \mathbf{w}'$ implies $f(\mathbf{w}) \curvearrow_r^{\mathcal{R}_0} f(\mathbf{w}')$. For the basic case, assume $\mathbf{w} = s^{-1}t$ and $\mathbf{w}' = vu^{-1}$ with $s, t \in \mathcal{S}$. Let $f(s) = s_1 \cdots s_p$, $f(t) = t_1 \cdots t_q$ with $s_1, \dots, t_q \in \mathcal{S}_0$. By hypothesis, $sv = tu$ is a relation in \mathcal{R} , so $s_1 \dots s_p f(v) = t_1 \dots t_q f(u)$ is a relation in \mathcal{R}_0 . We find

$$\begin{aligned} f(\mathbf{w}) &= s_p^{-1} \dots s_2^{-1} s_1^{-1} t_1 t_2 \dots t_q \\ \curvearrow_r^{\mathcal{R}_0} s_p^{-1} \dots s_2^{-1} s_2 \dots s_p f(v) f(u)^{-1} t_q^{-1} \dots t_2^{-1} t_2 \dots t_q \curvearrow_r^{\mathcal{R}_0} f(v) f(u)^{-1} &= f(\mathbf{w}'). \end{aligned}$$

The claim follows using an easy induction on the number of reversing steps.

Assume now $uv' \equiv^{\mathcal{R}_0} vu'$, where u, v, u', v' are words on \mathcal{S}_0 . As \mathcal{S}_0 is included in \mathcal{S} , the words u, v, u', v' are words on \mathcal{S} , they are fixed under f , and we have $uv' \equiv^{\mathcal{R}} vu'$ since $(\mathcal{S}, \mathcal{R})$ is a presentation of M . As $(\mathcal{S}, \mathcal{R})$ is r -complete, we must have $u^{-1}v \curvearrow_r^{\mathcal{R}} v''u''^{-1}$, $u' \equiv^{\mathcal{R}} u''w$, and $v' \equiv^{\mathcal{R}} v''w$ for some words u'', v'', w in \mathcal{S}^* . By the claim above, $u^{-1}v \curvearrow_r^{\mathcal{R}} v''u''^{-1}$ implies $u^{-1}v = f(u)^{-1}f(v) \curvearrow_r^{\mathcal{R}_0} f(v'')f(u'')^{-1}$. Next $u' \equiv^{\mathcal{R}} u''w$ implies $f(u') \equiv^{\mathcal{R}_0} f(u'')f(w)$ by definition of \mathcal{R}_0 , and, similarly, $v' \equiv^{\mathcal{R}} v''w$ implies $f(v') \equiv^{\mathcal{R}_0} f(v'')f(w)$. This shows that the words $f(u'')$, $f(v'')$, and $f(w)$ fulfill the requirements for $(\mathcal{S}_0, \mathcal{R}_0)$ to be r -complete at u, v, u', v' . \square

We do not know whether the previous result still holds when the set \mathcal{S}_0 is not supposed to be included in \mathcal{S} (but we have no counter-example either).

3. THE CUBE CONDITION

At this point, we know that every monoid M (with no non-trivial unit) and every group G admit complete presentations, but we are left with the question of recognizing that a given presentation is possibly complete. In every case, even for a finite presentation, the question is non-trivial, as checking r -completeness for one particular 4-tuple of words requires being able to decide \equiv -equivalence, and checking it for all 4-tuples is an infinite process.

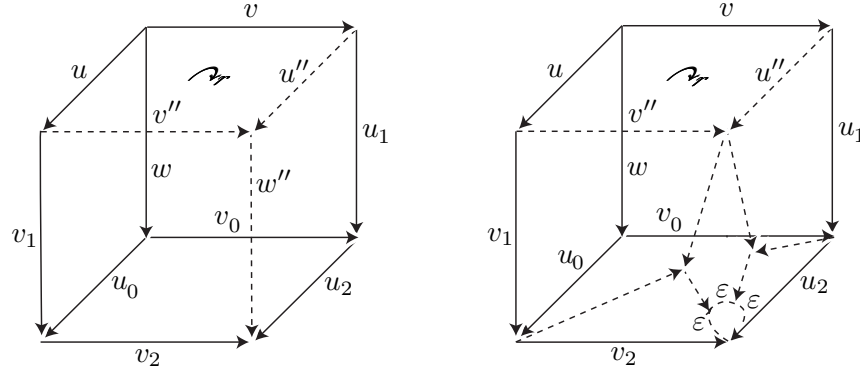
In this section, we introduce a new combinatorial condition involving reversing, the cube condition, and we prove that completeness is equivalent to that cube condition being satisfied. This is a first step toward an effective completeness criterion that will be established in the subsequent section.

Definition 3.1. Let $(\mathcal{S}, \mathcal{R})$ be a positive presentation. For u, v, w in \mathcal{S}^* , we say that $(\mathcal{S}, \mathcal{R})$ satisfies the r -cube condition (resp. the strong r -cube condition) at u, v, w if the following implication holds:

$$(3.1) \quad \begin{aligned} &\text{If we have } u^{-1}wv^{-1} \curvearrow_r v'u'^{-1} \text{ with } u', v' \in \mathcal{S}^*, \\ &\text{then there exist } u'', v'', w'' \text{ in } \mathcal{S}^* \text{ satisfying} \\ &\quad u^{-1}v \curvearrow_r v''u''^{-1}, u' \equiv u''w'', \text{ and } v' \equiv v''w''. \\ &\text{(resp. then we have } (uv')^{-1}(vu') \curvearrow_r \varepsilon \text{).} \end{aligned}$$

For $\mathcal{S}' \subseteq \mathcal{S}^*$, we say that $(\mathcal{S}, \mathcal{R})$ satisfies the (strong) r -cube condition on \mathcal{S}' if the (strong) r -cube condition holds for all u, v, w in \mathcal{S}' .

The cube conditions are illustrated in Fig. 3.1. We start with an incomplete cube consisting of three faces constructed on (u, w) , (w, v) , and (u_0, v_0) and correspond


 FIGURE 3.1. The r -cube and the strong r -cube conditions

to r -reversings, and the condition means that we can complete the cube with a top reversing face and a last edge. In the cube condition, we require that the last two faces correspond to equivalences, while, in the strong cube condition, we require that the last two faces correspond, in a slightly more complicated way, to reversings. As the name suggests, the strong cube condition implies the cube condition: indeed, Lemma 1.10 tells us that $(uv')^{-1}(vu') \curvearrow_r \varepsilon$ implies the existence of u'', v'', w'' satisfying $u^{-1}v \curvearrow_r v''u''^{-1}$, $u' \equiv u''w''$, and $v' \equiv v''w''$. We shall see below that both conditions actually are equivalent in the case of an r -complete presentation.

Example 3.2. Let \mathcal{S}_n be $\{a_1, \dots, a_n\}$, and \mathcal{R}_n be the family of all relations $a_i a_{i+p} = a_j a_{j+p}$ with $1 \leq i < j \leq n$ and $1 \leq p \leq n$, where $x \dot{+} y$ denotes the unique number in $\{1, \dots, n\}$ equal to $x + y$ modulo n . For instance, the monoid $\langle \mathcal{S}_2; \mathcal{R}_2 \rangle^+$ is (isomorphic to) $\langle a, b; a^2 = b^2, ab = ba \rangle^+$ considered in Example 1.2, while $\langle \mathcal{S}_3; \mathcal{R}_3 \rangle^+$ is (isomorphic to)

$$\langle a, b, c; a^2 = b^2 = c^2, ab = bc = ca, ac = ba = cb \rangle^+.$$

We claim that the (strong) r -cube condition is satisfied by $(\mathcal{S}_n, \mathcal{R}_n)$ for every triple of letters a_i, a_j, a_k . Indeed, the words to which $a_i^{-1}a_k$ reverses are the words $a_{i+p}a_{k+p}^{-1}$ with $1 \leq p \leq n$; similarly, the words to which $a_k^{-1}a_j$ reverses are the words $a_{k+q}a_{j+q}^{-1}$ with $1 \leq q \leq n$; finally, the words to which $a_{k+p}^{-1}a_{k+q}$ reverses are the words $a_{k+p+r}a_{k+q+r}^{-1}$ with $1 \leq r \leq n$. But, then, $a_i^{-1}a_j$ reverses to $a_i a_j^{-1}$, and we have $a_{i+p}a_{k+p+r} \equiv a_i a_{k+r}$ and $a_{j+q}a_{k+q+r} \equiv a_j a_{k+r}$ (Fig. 3.2), which is the r -cube condition at a_i, a_j, a_k . Moreover, we find

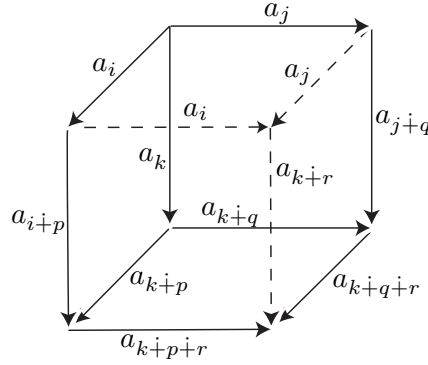
$$\begin{aligned} a_{k+p+r}^{-1} a_{i+p}^{-1} a_i a_{k+r} &\curvearrow_r a_{k+p+r}^{-1} a_{k+p+r} a_{k+r}^{-1} a_{k+r} \curvearrow_r \varepsilon, \\ a_{k+q+r}^{-1} a_{j+q}^{-1} a_j a_{k+r} &\curvearrow_r a_{k+q+r}^{-1} a_{k+q+r} a_{k+r}^{-1} a_{k+r} \curvearrow_r \varepsilon, \end{aligned}$$

which gives the strong r -cube condition.

A condition similar to the cube condition appears in [18]; however, the similarity seems to be superficial only, as the main point in the current condition is the specific use of the reversing relation.

The connection between completeness and cube condition is as follows:

Proposition 3.3. *A positive presentation $(\mathcal{S}, \mathcal{R})$ is r -complete if and only if any of the following four equivalent conditions is satisfied:*

FIGURE 3.2. The cube condition for $\langle \mathcal{S}_n; \mathcal{R}_n \rangle^+$

- (i) Equivalence is detected by r -reversing: $u \equiv v$ is equivalent to $u^{-1}v \curvearrowright_r \varepsilon$.
- (ii) The relation $u^{-1}v \curvearrowright_r \varepsilon$ is transitive.
- (iii) The strong r -cube condition is satisfied on \mathcal{S}^* .
- (iv) The r -cube condition is satisfied on \mathcal{S}^* .

Proof. Assume $u \equiv v$, i.e., $u\varepsilon \equiv v\varepsilon$. If $(\mathcal{S}, \mathcal{R})$ is r -complete, we obtain u'' , v'' , w satisfying $u^{-1}v \curvearrowright_r v''u''^{-1}$, $\varepsilon \equiv u''w$, and $\varepsilon \equiv v''w$. As $(\mathcal{S}, \mathcal{R})$ is positive, $\varepsilon \equiv u''w$ implies $u'' = w = \varepsilon$, and $\varepsilon \equiv v''w$ implies $v'' = \varepsilon$. This means that we have $u^{-1}v \curvearrowright_r \varepsilon$, and (i) is true.

Conversely, assume $uv' \equiv vu'$. If (i) holds, we have $(uv')^{-1}(vu') \curvearrowright_r \varepsilon$. By Lemma 1.10(ii), we obtain u'' , v'' , and w'' satisfying $u^{-1}v \curvearrowright_r v''u''^{-1}$, $u' \equiv u''w''$ and $v' \equiv v''w''$, i.e., the r -completeness condition for u , v , u' , v' is satisfied. So r -completeness is equivalent to (i).

Next, by definition, the relation \equiv is an equivalence relation, hence it is transitive, so (i) implies (ii). Conversely, by construction, the relation $u^{-1}v \curvearrowright_r \varepsilon$ is always reflexive, symmetric, and compatible with multiplication on both sides so, if (ii) holds, the relation is a congruence on the monoid \mathcal{S}^* . By Prop. 1.9, this congruence is included in \equiv . On the other hand, it contains all relations of \mathcal{R} , so it includes \equiv , and, finally, it coincides with the latter. So (ii) is equivalent to (i).

Assume now $u^{-1}wv \curvearrowright_r v'u'^{-1}$. By Lemma 1.7, there exist $u_i, v_i, i = 0, 1, 2$, satisfying $u^{-1}w \curvearrowright_r v_1u_0^{-1}$, $w^{-1}v \curvearrowright_r v_0u_1^{-1}$, and $u_0^{-1}v_0 \curvearrowright_r v_2u_2^{-1}$, and we have $u' = u_1u_2$ and $v' = v_1v_2$ (as in Fig. 3.1). We read

$$uv' = uv_1v_2 \equiv wu_0v_2 \equiv wv_0u_2 \equiv vu_1u_2 = vu',$$

hence $uv' \equiv vu'$. If the presentation is r -complete, this implies that there exist u'' , v'' , w'' satisfying $u^{-1}v \curvearrowright_r v''u''^{-1}$, $u' \equiv u''w''$, and $v' \equiv v''w''$, which gives the strong r -cube condition. So r -completeness, hence (ii) as well, implies (iii), hence (iv) by Lemma 1.10(i).

Finally, assume $u^{-1}w \curvearrowright_r \varepsilon$ and $w^{-1}v \curvearrowright_r \varepsilon$. As $\varepsilon^{-1}\varepsilon \curvearrowright_r \varepsilon$ trivially holds, we deduce $u^{-1}ww^{-1}v \curvearrowright_r \varepsilon$. If the r -cube condition is satisfied, we deduce that there exist u'' , v'' , w'' satisfying $u^{-1}v \curvearrowright_r v''u''^{-1}$, $\varepsilon \equiv u''w''$, and $\varepsilon \equiv v''w''$. The latter relations imply $u'' = v'' = w'' = \varepsilon$, hence $u^{-1}v \curvearrowright_r \varepsilon$. This shows that (iv) implies (ii), and, therefore, that (ii), (iii), and (iv) are equivalent. \square

By Prop. 3.3, establishing the possible completeness of a presentation reduces to establishing the (strong) cube condition for all triples of words. Observe that, in

practice, checking the strong cube condition is easier than checking the cube condition, as the former involves only reversing, while the latter involves the equivalence relation \equiv of which we have no control as long as the presentation is not known to be complete.

In the complemented case, *i.e.*, when r -reversing is a deterministic process, the cube condition takes special forms that have been considered in [10] and [15]. Indeed, in this case, there exists for each pair of words (u, v) at most one pair of words (u', v') satisfying $u^{-1}v \curvearrow_r v'u'^{-1}$. Let us define $(v \setminus u, u \setminus v)$ to be the unique such pair (u', v') when it exists—by Lemma 1.6, the symmetry of reversing makes the definition unambiguous.

Lemma 3.4. *Assume that $(\mathcal{S}, \mathcal{R})$ is a complemented presentation. Then a sufficient condition for the r -cube (resp. the strong r -cube) condition to be satisfied at u, v, w is that the relation*

$$(3.2) \quad (u \setminus v) \setminus (u \setminus w) \equiv (v \setminus u) \setminus (v \setminus w)$$

$$(3.3) \quad (\text{ resp. } ((u \setminus v) \setminus (u \setminus w)) \setminus ((v \setminus u) \setminus (v \setminus w)) = \varepsilon)$$

and the relations obtained by permutation of u, v, w are satisfied.

Proof. The only word of the form $v'u'^{-1}$ to which $u^{-1}wv^{-1}$ reverses is

$$(u \setminus w)((w \setminus u) \setminus (w \setminus v))((w \setminus v) \setminus (w \setminus u))^{-1}(v \setminus w)^{-1},$$

and the only word of this form to which $u^{-1}v$ may reverse is $(u \setminus v)(v \setminus u)^{-1}$. So the point for the cube condition is to find w' satisfying

$$(u \setminus w)((w \setminus u) \setminus (w \setminus v)) \equiv (u \setminus v)w' \text{ and } (v \setminus w)((w \setminus v) \setminus (w \setminus u)) \equiv (v \setminus u)w'.$$

Now, assuming (3.2) and its cyclic analogs, and using the identity $u_1(u_1 \setminus v_1) \equiv v_1(v_1 \setminus u_1)$, which is the form taken by Prop 1.9 in this context, we find

$$(u \setminus w)((w \setminus u) \setminus (w \setminus v)) \equiv (u \setminus w)((u \setminus w) \setminus (u \setminus v))$$

$$\equiv (u \setminus v)((u \setminus v) \setminus (u \setminus w)) \equiv (u \setminus v)((v \setminus u) \setminus (v \setminus w)),$$

$$(v \setminus w)((w \setminus v) \setminus (w \setminus u)) \equiv (v \setminus w)((v \setminus w) \setminus (v \setminus u)) \equiv (v \setminus u)((v \setminus u) \setminus (v \setminus w)),$$

the expected form with $w' = (v \setminus u) \setminus (v \setminus w)$.

As for the strong cube condition, we wish to prove the relation

$$((w \setminus u) \setminus (w \setminus v))^{-1}(u \setminus w)^{-1}u^{-1}v(v \setminus w)((w \setminus v) \setminus (w \setminus u)) \curvearrow_r \varepsilon.$$

Fig. 3.3 gives the result assuming (3.3) and its analogs. \square

It is not clear that the sufficient conditions of Lemma 3.4 are necessary for a given triple of words (u, v, w) , but they are globally necessary in that, if (3.2) or (3.3) is satisfied for all triples (u, v, w) , then, as is proved in [15], $u \equiv v$ is equivalent to $u^{-1}v \curvearrow_r \varepsilon$, so, in our current framework, the presentation is r -complete, and, therefore, the cube and the strong cube conditions are satisfied for all triples.

Remark 3.5. The most natural generalization of Condition (3.2) would be:

$$(3.4) \quad \text{Assume } u^{-1}w \curvearrow_r v_1u_0^{-1}, w^{-1}v \curvearrow_r v_0u_1^{-1}, \text{ and } u_0^{-1}v_0 \curvearrow_r v_2u_2^{-1};$$

$$\text{then there exist } u'', v'', u_2'', v_2'', w_1, w_2 \text{ in } \mathcal{S}^* \text{ satisfying}$$

$$u^{-1}v \curvearrow_r v''u''^{-1}, v_1^{-1}v'' \curvearrow_r v_2''w_1, u''^{-1}u_1 \curvearrow_r w_2u_2''^{-1},$$

$$\text{and } v_2 \equiv v_2'', u_2 \equiv u_2'', w_1 \equiv w_2 \text{ (Fig. 3.4 left).}$$

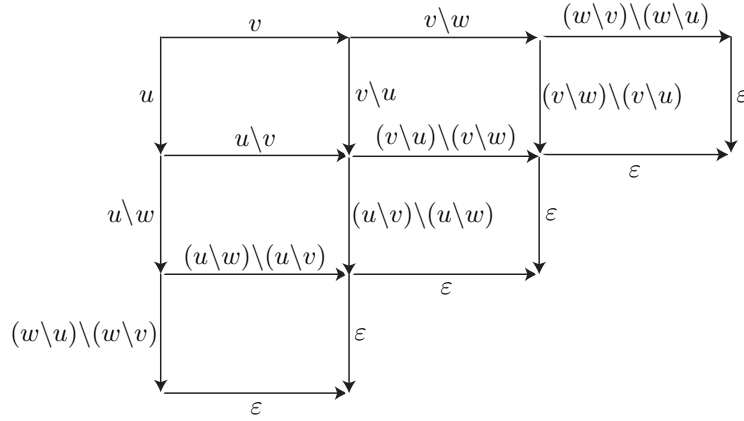


FIGURE 3.3. Strong cube condition in the complemented case

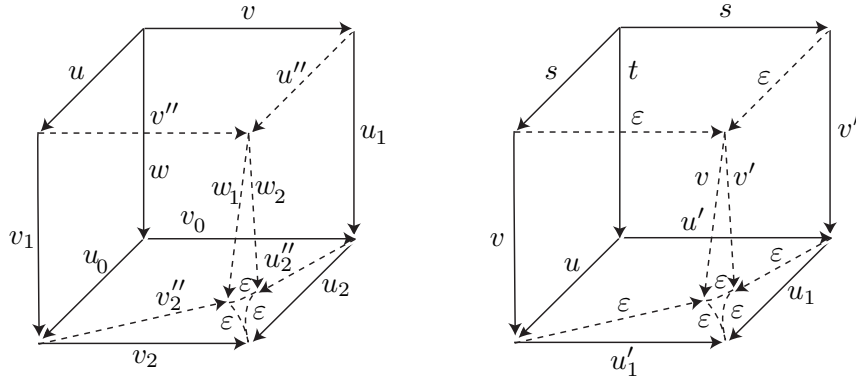


FIGURE 3.4. Variants of the cube conditions

However, Condition (3.4) is not suitable, as it may hold only if the considered presentation is equivalent to a complemented presentation, at least if there is no relation $s \cdots = s \cdots$ in \mathcal{R} and r -reversing is convergent, *i.e.*, every word $u^{-1}v$ reverses to at least one word $v'u'^{-1}$. Indeed, assume that $sv = tu$ and $sv' = tu'$ belong to \mathcal{R} . Then we have $s^{-1}t \curvearrow_r vu^{-1}$, $t^{-1}s \curvearrow_r u'v'^{-1}$, and there exist u_1, u'_1 satisfying $u^{-1}u' \curvearrow_r u'_1u_1^{-1}$ (Fig. 3.4 right). We apply (3.4): as $s^{-1}s \curvearrow_r \epsilon$, $v^{-1}\epsilon \curvearrow_r \epsilon v^{-1}$, and $\epsilon^{-1}v' \curvearrow_r v'\epsilon^{-1}$ are the only possibilities, and $u_1 \equiv \epsilon$ implies $u_1 = \epsilon$, we deduce $v \equiv v'$ and $u^{-1}u' \curvearrow_r \epsilon$, hence $u \equiv u'$, *i.e.*, the two relations $s \cdots = t \cdots$ are essentially one and the same relation.

The same remark applies to the most natural generalization of Condition (3.3), namely the following variant of (3.4) corresponding to a 6-face reversing cube:

$$(3.5) \quad \begin{aligned} &\text{Assume } u^{-1}w \curvearrow_r v_1u_0^{-1}, w^{-1}v \curvearrow_r v_0u_1^{-1}, \text{ and } u_0^{-1}v_0 \curvearrow_r v_2u_2^{-1}; \\ &\text{then there exist } u'', v'', w'' \text{ in } \mathcal{S}^* \text{ satisfying} \\ &u^{-1}v \curvearrow_r v''u''^{-1}, v_1^{-1}v'' \curvearrow_r v_2w''^{-1}, \text{ and } u''^{-1}u_1 \curvearrow_r w''u_2^{-1}. \end{aligned}$$

(Conditions (3.4) and (3.5) might make sense in a non-complemented case would the current relation \curvearrow_r be replaced with the extended relation \curvearrow_r^\sharp of Remark 1.5.)

Remark 3.6. If $(\mathcal{S}, \mathcal{R})$ is an r -complete complemented presentation, then r -reversing is compatible with \equiv in the sense that, if we have $u^{-1}v \curvearrowright_r v'u'^{-1}$ and $u_1 \equiv u$, then we have $u_1^{-1}v \curvearrowright_r v'_1u_1'^{-1}$ for some words u'_1, v'_1 satisfying $u'_1 \equiv u_1$ and $v'_1 \equiv v_1$. We have no such general result here. Indeed, with the previous hypotheses, r -completeness gives words u'_1, v'_1 , and w satisfying $u_1^{-1}v \curvearrowright_r v'_1u_1'^{-1}$, $u' \equiv u'_1w$ and $v' \equiv v'_1w$, but there is no general reason for w to be empty. Let us say that two words u_0, v_0 are co-prime if the conjunction of $u_0 \equiv u'_0w_0$ and $v_0 \equiv v'_0w_0$ implies $w_0 = \varepsilon$. Then, we could deduce $w = \varepsilon$ above if we knew that u'_1 and v'_1 are co-prime, *i.e.*, that reversing always produces co-prime words. This is true in the complemented case, but, not in general, even if u and v are co-prime for each relation $su = tv$ in \mathcal{R} , as shows the example developed in Remark 6.12 below.

4. RECOGNIZING COMPLETENESS

The characterizations of completeness given in Prop. 3.3 all are infinitary, in that they involve checking some condition on infinitely many words. They therefore give us no effective criterion for proving completeness. We shall establish now such a criterion in the case of certain presentations called homogeneous and connected with Noetherianity.

Definition 4.1. We say that a positive presentation $(\mathcal{S}, \mathcal{R})$ is r -homogeneous if the associated congruence \equiv preserves some r -pseudolength, the latter being defined as a map λ of \mathcal{S}^* to the ordinals satisfying, for every s in \mathcal{S} and every u in \mathcal{S}^* ,

$$(4.1) \quad \lambda(su) > \lambda(u).$$

We say that $(\mathcal{S}, \mathcal{R})$ is homogeneous if it preserves both an r -pseudolength and an l -pseudolength, the latter defined by the symmetric condition $\lambda(us) > \lambda(u)$.

By definition, the congruence \equiv associated with a presentation $(\mathcal{S}, \mathcal{R})$ is the equivalence relation generated by the pairs $(uvw, uv'w)$ such that $v = v'$ is a relation of \mathcal{R} , so saying that \equiv preserves λ is equivalent to saying that we have

$$(4.2) \quad \lambda(uvw) = \lambda(uv'w) \quad \text{for } v = v' \text{ in } \mathcal{R} \text{ and } u, w \text{ in } \mathcal{S}^*.$$

If all relations in \mathcal{R} consist of words of equal length, then the length is both an r - and an l -pseudolength, and the presentation is homogeneous. However, completely different types exist, as the following examples show.

Example 4.2. The presentation $(a, b; aba = b^2)$ is homogeneous. Indeed, the mapping λ defined by $\lambda(a) = 1$, $\lambda(b) = 2$, and $\lambda(uv) = \lambda(u) + \lambda(v)$ is both an r - and an l -pseudolength.

A slightly more complicated example is $(a, b, c; ab = bac, ac = ca, bc = cb)$, a presentation for the Heisenberg group. Here, no function λ satisfying $\lambda(uv) = \lambda(u) + \lambda(v)$ may be a pseudolength. However, if we define $\lambda(u)$ to be the length of u augmented by the number of pairs (i, j) with $i < j$ such that the i -th letter of u is a and the j -th letter is b —so, for instance, we have $\lambda(ab) = \lambda(bac) = 3$ —then λ is an r - and an l -pseudolength, and the presentation is homogeneous.

Finally, the presentation $(a, b; ab = a)$ is r -homogeneous, as shows the r -pseudolength λ defined by $\lambda(a) = 1$, $\lambda(b) = \omega$, and $\lambda(uv) = \lambda(v) + \lambda(u)$. As the monoid $\langle a, b; ab = a \rangle^+$ is not l -Noetherian since we have $a \prec_l a$, the next result shows that this presentation is not homogeneous.

Proposition 4.3. *The monoid $\langle \mathcal{S}; \mathcal{R} \rangle^+$ is r -Noetherian (resp. Noetherian) if and only if the presentation $(\mathcal{S}, \mathcal{R})$ is r -homogeneous (resp. homogeneous).*

Proof. If λ is an r -pseudolength on \mathcal{S}^* , it induces a well defined mapping $\bar{\lambda}$ on $\langle \mathcal{S}; \mathcal{R} \rangle^+$ such that, by definition, $x \prec_r y$ implies $\bar{\lambda}(x) < \bar{\lambda}(y)$. Since the ordinals are well ordered, the relation \prec_r may have no infinite descending chain.

Conversely, assume that M is an r -Noetherian monoid and $(\mathcal{S}, \mathcal{R})$ is a presentation for M . Standard arguments of basic set theory (see for instance [29]) show that there exists a map ρ of M to the ordinals such that $x \prec_r y$ implies $\rho(x) < \rho(y)$. Then the map λ defined by $\lambda(u) = \rho(\bar{u})$ is an r -pseudolength on \mathcal{S}^* . \square

Our main result now is that, when a presentation is r -homogeneous, then, in order to prove that the presentation is r -complete, it is sufficient to establish the r -cube condition for all triples of letters.

Proposition 4.4. *An r -homogeneous positive presentation is r -complete if and only if any one of the following equivalent conditions is satisfied:*

- (i) *The strong r -cube condition is satisfied on \mathcal{S}^* ;*
- (ii) *The strong r -cube condition is satisfied on \mathcal{S} ;*
- (iii) *The r -cube condition is satisfied on \mathcal{S} .*

We have already seen in Prop. 3.3 that r -completeness is equivalent to (i), it is clear that (i) implies (ii), and we have observed that the strong r -cube condition always implies the r -cube condition, so (ii) implies (iii). So, we are left with the question of proving that (iii) implies say r -completeness, which is the non-trivial point. The argument will be splitted into several intermediate statements. Until the end of the proof, we assume that $(\mathcal{S}, \mathcal{R})$ is an r -homogeneous presentation, and we wish to establish r -completeness for every 4-tuple of words, *i.e.*, we wish to prove that, if $uw' \equiv vu'$ holds, then there exist some words u'', v'', w'' satisfying $u^{-1}v \curvearrowright_r v''u''^{-1}$, $u' \equiv u''w''$, and $v' \equiv v''w''$. We fix an r -pseudolength λ on \mathcal{S}^* which is invariant under \equiv .

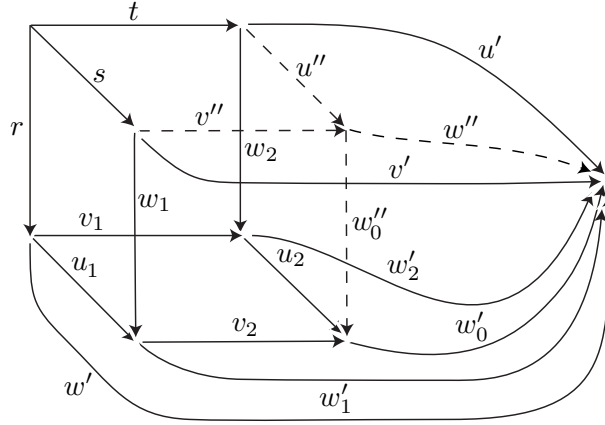
Lemma 4.5. *The r -completeness condition holds for all words u, v, u', v' satisfying $\lambda(uv') = 0$.*

Proof. The only possibility is $u = v' = v = u' = \varepsilon$, and taking $u'' = v'' = w'' = \varepsilon$ gives the result. \square

Lemma 4.6. *Assume that the r -cube condition holds on \mathcal{S} , and r -completeness holds for all u, v, u', v' with $\lambda(uv') < \alpha$. Then r -completeness holds for all u, v, u', v' with $u, v \in \mathcal{S}$ and $\lambda(uv') \leq \alpha$.*

Proof. Assume $sv' \equiv tu'$ with $s, t \in \mathcal{S}$ and $\lambda(sv') = \alpha$. We use induction on the minimal number of relations k needed to transform sv' into tu' . The case $k = 0$ corresponds to $sv' = tu'$, hence $s = t$ and $u' = v'$. In this case, taking $u'' = v'' = \varepsilon$, $w'' = u'$ gives the result. The case $k = 1$ divides into two subcases. Either the relation connecting sv' to tu' does not involve the initial letters: then we have $s = t$, and $u' \equiv v'$, and taking $u'' = v'' = \varepsilon$, $w'' = u'$ gives the result. Or the relation connecting sv' to tu' involves the initial letters: this means that there exists a relation $sv'' = tu''$ in \mathcal{R} and a word w'' satisfying $u' = u''w''$, and $v' = v''w''$: these words u'', v'', w'' give the result.

Assume now $k \geq 2$, and let rw' be an intermediate word in a shortest path from sv' to tu' (Fig. 4.1). We have $sv' \equiv rw'$ with less than k relations, so the induction

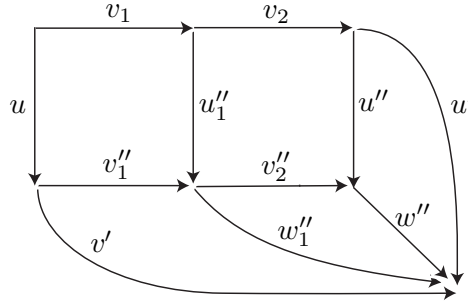

 FIGURE 4.1. Completeness on \mathcal{S}

hypothesis gives words u_1, w_1 and w'_1 satisfying $s^{-1}r \curvearrowright_r w_1u_1^{-1}$, $v' \equiv w_1w'_1$, and $w' \equiv u_1w'_1$. Similarly, we have $rw' \equiv tu'$ with less than k relations, so the induction hypothesis gives words v_1, w_2, w'_2 satisfying $r^{-1}t \curvearrowright_r v_1w_2^{-1}$, $w' \equiv v_1w'_2$, and $u' \equiv w_2w'_2$. Then, we have $u_1w'_1 \equiv v_1w'_2$, and, by definition of an r -pseudolength, $\lambda(u_1w'_1) < \lambda(ru_1w'_1) = \lambda(sw_1w'_1) = \lambda(sv') = \alpha$. Applying the hypothesis to u_1, v_1, w'_1, w'_2 , we obtain three words u_2, v_2 and w'_0 satisfying $u_1^{-1}v_1 \curvearrowright_r v_2u_2^{-1}$, $w'_1 \equiv v_2w'_0$, and $w'_2 \equiv u_2w'_0$. At this point, we have $s^{-1}rr^{-1}t \curvearrowright_r w_1v_2u_2^{-1}w_2^{-1}$, so the hypothesis that $(\mathcal{S}, \mathcal{R})$ satisfies the r -cube condition on $\{s, t, r\}$ gives three words u'', v'', w''_0 in \mathcal{S}^* satisfying $s^{-1}t \curvearrowright_r v''u''^{-1}$, $w_1v_2 \equiv v''w''_0$, and $w_2u_2 \equiv u''w''_0$. Put $w'' = w''_0w'_0$. Then we have $u' \equiv w_2u_2w'_0 \equiv u''w''$, and $v' \equiv w_1v_2w'_0 \equiv v''w''$, so the words u'', v'' , and w'' give the expected result. \square

Lemma 4.7. *Assume that r -completeness holds for all u, v, u', v' with $\lambda(uv') < \alpha$, and for all u, v, u', v' with $u, v \in \mathcal{S}$ and $\lambda(uv') \leq \alpha$. Then r -completeness holds for all u, v, u', v' with $\lambda(uv') \leq \alpha$.*

Proof. Assume $uv' \equiv vu'$ with $\lambda(uv') = \alpha$. We wish to prove that there exist u'', v'', w'' satisfying $u^{-1}v \curvearrowright_r v''u''^{-1}$, $u' \equiv u''w''$, and $v' \equiv v''w''$. If either u or v is empty, the result is obvious as, for $u = \varepsilon$, we can take $u'' = \varepsilon$, $v'' = v$, and $w'' = u'$. Now, we prove using induction on m that the result holds for $\lg(u) + \lg(v) \leq m$. By the previous remark, the first non-trivial case is $m = 2$ with both u and v in \mathcal{S} . Then the conclusion is our second hypothesis. Assume now $m \geq 3$, and v , say, has length at least 2. We write $v = v_1v_2$ with both v_1 and v_2 nonempty (Fig. 4.2). The hypothesis is $uv' \equiv v_1(v_2u')$ with $\lambda(uv') = \alpha$ and $\lg(u) + \lg(v_1) < m$. Applying the induction hypothesis to u, v_1, v_2u', v' , we obtain three words u''_1, v''_1 , and w''_1 in \mathcal{S}^* satisfying $u^{-1}v_1 \curvearrowright_r v''_1u''_1^{-1}$, $v_2u' \equiv u''_1w''_1$, and $v' \equiv v''_1w''_1$. Now, we have $\lambda(v_2u') < \lambda(v_1v_2u') = \alpha$, so applying the first hypothesis to u''_1, v_2, u', w''_1 , we obtain three words u'', v''_2 , and w'' satisfying $u''_1^{-1}v_2 \curvearrowright_r v''_2u''^{-1}$, $u' \equiv u''w''$, and $w''_1 \equiv v''_2w''$. Put $v'' = v''_1v''_2$. By construction, we have $u^{-1}v \curvearrowright_r v''_1u''_1^{-1}v_2 \curvearrowright_r v''_1v''_2u''^{-1}$, hence $u^{-1}v \curvearrowright_r v''u''^{-1}$, and we have $v' \equiv v''_1w''_1 \equiv v''_1v''_2w'' = v''w''$, the expected result. \square

It is now easy to complete the proof of Prop. 4.4.

FIGURE 4.2. Completeness on \mathcal{S}^*

Proof of Prop. 4.4. Assume that r -completeness fails for some u_0, v_0, u'_0, v'_0 . Let α be the minimal possible value of $\lambda(u_0 v'_0)$ for such a counter-example. By Lemma 4.5, α is not 0. Now, by construction, the presentation is r -complete for all u, v, u', v' with $\lambda(uv') < \alpha$, hence, by Lemma 4.6, it is r -complete for all u, v, u', v' with $\lambda(uv') \leq \alpha$ and $u, v \in \mathcal{S}$, hence, by Lemma 4.7, it is also r -complete for all u, v, u', v' with $\lambda(uv') \leq \alpha$, contradicting the definition of α . \square

We can also complete the proof of Prop. 2.9.

End of proof of Prop. 2.9. We assume that M is a Noetherian monoid, and \mathcal{R} is an r -selector on \mathcal{S} in M . We have already seen that $(\mathcal{S}, \mathcal{R})$ is a presentation for M , and we wish to prove that this presentation is r -complete. As M is Noetherian, we may use a pseudolength λ , and use the same inductive scheme as for Prop. 4.4. Here, by definition of an r -selector, the presentation is r -complete at u, v, u', v' whenever u and v are single letters, *i.e.*, the conclusion of Lemma 4.6 is true directly. Then it suffices to use Lemma 4.7 for going from $\lambda(uv') < \alpha$ to $\lambda(uv') \leq \alpha$ for every α , and deducing r -completeness for all u, v, u', v' . \square

Returning to the framework of this section, we deduce from Prop. 4.4 the following (necessary and sufficient) criterion for recognizing r -complete presentations:

Algorithm 4.8. *Let $(\mathcal{S}, \mathcal{R})$ be an r -homogeneous positive presentation. For each triple of letters s, t, r in \mathcal{S} :*

- (i) *Reverse $s^{-1} r r^{-1} t$ to all possible words of the form uv^{-1} ;*
 - (ii) *For each uv^{-1} so obtained, check $su \equiv tv$, or, alternatively, $(su)^{-1}(tv) \curvearrowright_r \varepsilon$.*
- Then $(\mathcal{S}, \mathcal{R})$ is r -complete if and only if the answer at Step (ii) is always positive.*

The theoretical interest of the previous result is to show that r -completeness, which is a priori a Σ_1^0 (*i.e.*, recursively enumerable, cf. [31]) property, actually is a Δ_1^0 (*i.e.*, recursive) property in good cases.

Proposition 4.9. *Assume that $(\mathcal{S}, \mathcal{R})$ is a finite homogeneous positive presentation such that, for some recursive function f , every r -reversing sequence from a length n word has length $f(n)$ at most. Then for $(\mathcal{S}, \mathcal{R})$ to be r -complete is a recursive property.*

Proof. Applying Algorithm 4.8 involves finitely many reversing processes, each of which is assumed to have a recursively bounded length. So the whole process has a recursively bounded length. \square

The main interest of the method presumably lies in its practical tractability. It can be implemented on a computer easily, and then be used to test concrete presentations (when the presentation contains several relations $su = tv$ with the same initial letters s and t , r -reversing is a non-deterministic process, and checking the cube condition by hand quickly becomes impossible). Observe that, for the computer approach, the strong cube condition is better suited than the cube condition, as the only practical way of proving $u \equiv v$ is to check that $u^{-1}v$ reverses to the empty word.

The completeness criterion of Proposition 4.4 applies in particular in the complemented case. In this special case, it had already been proved in [10] that the satisfaction of Condition (3.2) for u, v, w in \mathcal{S} , which we have seen is similar to the current cube condition, is a sufficient condition for r -completeness.

Example 4.10. Let us consider the standard presentation of Artin braid groups, or, more generally, of any Artin–Tits group with finite Coxeter type. This presentation is homogeneous, as all relations preserve the length of the words. Then the (strong) cube condition can be checked systematically. Observe that it suffices to consider the various possible types of relations only. For instance, in the case of the braid groups, there are only two types of relations, namely the length 2 relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ and the length 3 relations $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$, and, therefore, it is sufficient to consider one triple of generators for each possible triple of relations, so checking the cube condition for the three triples $(\sigma_1, \sigma_2, \sigma_3)$ for type 3, 3, 2, $(\sigma_1, \sigma_2, \sigma_4)$ for type 3, 2, 2, and $(\sigma_1, \sigma_3, \sigma_5)$ for type 2, 2, 2 is enough to claim that the standard presentation of *every* group B_n is complete. The verification is what Garside makes in his proof of Prop. H in [21]. Similarly, the standard presentation of every Artin–Tits group is complete, as shown in [4].

More recently, a new presentation of the braid group B_n has been proposed by Birman, Ko, and Lee in [3]. This presentation is homogeneous and complemented, and the cube condition is satisfied, as established in [3]. So the presentation is complete, as are more generally the so-called dual presentations of the Artin–Tits groups investigated in [2, 35].

Let us mention that other criteria have been established subsequently, always in the complemented case. In particular, it is proved in [15] that, if $(\mathcal{S}, \mathcal{R})$ is a complemented presentation (homogeneous or not), then the satisfaction of Condition (3.3) for u, v, w in the closure of \mathcal{S} under r -reversing is always a sufficient condition for r -completeness. This criterion does not seem to extend to the general case—nor does either the one established in [13]. The problem here is that the cube condition for letters does not imply the cube condition for words directly, because the elementary cubes cannot be stacked so as to give the desired cube. Such an approach can work only if we resort to the “superstrong” cube condition (3.4) where all faces are reversings.

5. COMPLETION OF A PRESENTATION

The criterion of Prop. 4.4 fails when we find a cube that cannot be completed using reversing. This means that some equivalence follows from the relations of the considered presentation, but that it cannot be proved using reversing. Now there always exists a way for forcing some relation $u \equiv v$ to be provable by reversing, namely adding it to the presentation. Of course, repairing one obstruction to completeness in this way may in turn introduce new obstructions. But we shall see now

that the completion process so sketched always comes to an end, thus yielding a complete presentation.

Let us begin with an example.

Example 5.1. (Fig. 5.1) Let us consider the presentation

$$(5.1) \quad (a, b, c, d; ab = bc = ca, ba = ad = db).$$

Presentation (5.1) is one of the nonstandard presentations of Artin's braid group B_3 introduced by V. Sergiescu in [39] and considered in [22]: the connection with the standard generators σ_1 and σ_2 is given by $a = \sigma_1$, $b = \sigma_2$, $c = \sigma_1\sigma_2\sigma_1^{-1}$, $d = \sigma_2\sigma_1\sigma_2^{-1}$. All relations involve words of equal length, so (5.1) is homogeneous, and Prop. 4.4 is relevant. Now, when checking the strong cube condition for (c, a, d) , we find that $c^{-1}aa^{-1}d$ reverses to a^2b^{-2} , while the presentation contains no relation of the form $c \cdots = d \cdots$. Here the strong cube condition fails, and the presentation (5.1) is not r -complete.

The previous failure is due to the relation $ca^2 = db^2$, which is a consequence of the relations in the presentation, but cannot be proved using reversing associated with (5.1). Now, if we add the above relation to the presentation, thus obtaining

$$(5.2) \quad (a, b, c, d; ab = bc = ca, ba = ad = db, ca^2 = db^2),$$

then (5.2) is equivalent to (5.1) in that the associated monoid and group are the same, and, by construction, the relation $ca^2 = db^2$ can now be proved by reversing. Of course, new obstructions could appear as introducing new relations produces new reversing sequences. However, this does not happen here, and the reader can check that the presentation (5.2) is r -complete.

A symmetric approach is possible for l -completeness using l -reversing and the l -strong cube condition. The reader can check that the presentation (5.2) is not l -complete: we have $ca^{-1}ad^{-1} \curvearrowright_l b^{-2}a^2$, and, again, no way for proving the relation $a^2d \equiv b^2c$ using (5.2)-reversing. Once more, the solution is to add the missing relation to the presentation, which becomes

$$(5.3) \quad (a, b, c, d; ab = bc = ca, ba = ad = db, ca^2 = db^2, a^2d = b^2c),$$

and the reader will now check that (5.3) is l -complete; it is also r -complete as it includes (5.2) which is r -complete, so, finally, (5.3) is a complete presentation.

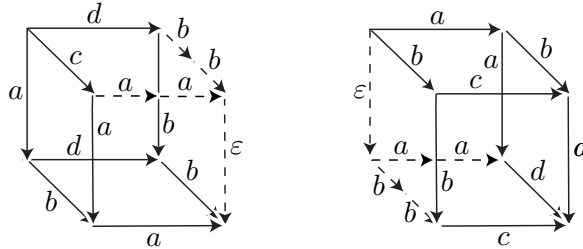


FIGURE 5.1. Completion of a presentation

The previous example gives a general method for constructing complete presentations.

Definition 5.2. Let $(\mathcal{S}, \mathcal{R})$ be a positive presentation.

(i) We say that $(\mathcal{S}, \mathcal{R}')$ is a 1-completion of $(\mathcal{S}, \mathcal{R})$ if there exist s, t, r in \mathcal{S} and u, v in \mathcal{S}^* satisfying $\mathcal{R}' = \mathcal{R} \cup \{sv = tu\}$, $s^{-1}rr^{-1}t \curvearrowright_r^{\mathcal{R}} vu^{-1}$ but $v^{-1}s^{-1}tu \not\curvearrowright_r^{\mathcal{R}} \varepsilon$.

(ii) We say that $(\mathcal{S}, \mathcal{R}_\xi)_{\xi \leq \theta}$ is an r -completing sequence if $(\mathcal{S}, \mathcal{R}_{\xi+1})$ is a 1-completion of $(\mathcal{S}, \mathcal{R}_\xi)$ for each ξ , and, for ξ limit, we have $\mathcal{R}_\xi = \bigcup_{\eta < \xi} \mathcal{R}_\eta$.

In other words, the presentation $(\mathcal{S}, \mathcal{R}')$ is a 1-completion of $(\mathcal{S}, \mathcal{R})$ if it is obtained by fixing one obstruction to the strong r -cube condition on \mathcal{S} for $(\mathcal{S}, \mathcal{R})$.

Proposition 5.3. Assume that $(\mathcal{S}, \mathcal{R})$ is a homogeneous positive presentation of cardinality κ . Then every maximal r -completing sequence from $(\mathcal{S}, \mathcal{R})$ has length less than $\sup(\kappa^+, \aleph_1)$ and it ends up with an r -complete presentation of the monoid $\langle \mathcal{S}; \mathcal{R} \rangle^+$.

Proof. Assume first that $(\mathcal{S}, \mathcal{R}')$ is a 1-completion for $(\mathcal{S}, \mathcal{R})$, say $\mathcal{R}' = \mathcal{R} \cup \{sv = tu\}$. By definition, we have $s^{-1}rr^{-1}t \curvearrowright_r^{\mathcal{R}} vu^{-1}$ for some r in \mathcal{S} , hence, by Lemma 1.10(i), $sv \equiv^{\mathcal{R}} tu$. Therefore, the congruence $\equiv^{\mathcal{R}'}$ coincides with $\equiv^{\mathcal{R}}$, and the presentations $(\mathcal{S}, \mathcal{R})$ and $(\mathcal{S}, \mathcal{R}')$ are equivalent. Any (r) -pseudolength that is preserved by $\equiv^{\mathcal{R}}$ is also preserved by $\equiv^{\mathcal{R}'}$, so $(\mathcal{S}, \mathcal{R})$ being (r) -homogeneous is equivalent to $(\mathcal{S}, \mathcal{R}')$ being (r) -homogeneous.

If \mathcal{S} has cardinality κ (finite or infinite), then \mathcal{S}^* has cardinality $\sup(\kappa, \aleph_0)$, and so does the set of all possible relations over \mathcal{S} . Then the length θ of a strictly increasing sequence of sets of relations on \mathcal{S} , say $(\mathcal{R}_\xi)_{\xi \leq \theta}$, is less than $\sup(\kappa, \aleph_0)^+$, i.e., than $\sup(\kappa^+, \aleph_1)$: otherwise, we would obtain an injective mapping f of the latter cardinal into $\mathcal{S}^* \times \mathcal{S}^*$ by defining $f(\xi)$ to be one element of $\mathcal{R}_{\xi+1} \setminus \mathcal{R}_\xi$. Finally, the hypothesis that $(\mathcal{S}, \mathcal{R}_\xi)_{\xi \leq \theta}$ is a maximal r -completing sequence implies that $(\mathcal{S}, \mathcal{R}_\theta)$ is r -complete by definition. \square

In particular, any finite presentation can be completed in a countable number of steps—but we do *not* claim that, starting from $(\mathcal{S}, \mathcal{R}_0)$ and defining $(\mathcal{S}, \mathcal{R}_{n+1})$ to be a 1-completion of \mathcal{R}_n implies that $(\mathcal{S}, \bigcup_n \mathcal{R}_n)$ is r -complete: the iteration might be longer than ω (but we have no example of such a situation so far). Actually, for practical examples, the interesting situation is when the possible completion requires a finite number of steps only, as was the case for the presentation of Example 5.1.

Example 5.4. Let us consider the standard presentation of the Heisenberg group

$$(5.4) \quad (a, b, c; ab = bac, ac = ca, bc = cb).$$

We have seen in Example 4.2 that it is homogeneous, and, therefore, eligible for our current approach. Now, we find $c^{-1}bb^{-1}a \curvearrowright_r bab^{-1}$, but $c^{-1}a$ only reverses to ac^{-1} , and $ba \equiv aw, b \equiv cw$ holds for no word w on $\{a, b, c\}$, so the cube condition at a, b, c fails. According to the scheme above, we add the missing relation $cba = ab$ (which is true, but not provable by reversing so far), getting the new presentation

$$(5.5) \quad (a, b, c; ab = bac, ac = ca, bc = cb, cba = ab).$$

The reader can check that, now, the strong r -cube condition holds on $\{a, b, c\}$, and, therefore, (5.5) is r -complete. The latter being symmetric, it is actually complete.

Example 5.5. Let us consider the presentation

$$(5.6) \quad (a, b, c; a^2 = b^2, ab = bc = ca).$$

One recognizes the Birman-Ko-Lee presentation of the braid group B_3 , augmented with the relation $a^2 = b^2$. Thus, the group defined by (5.6) is the quotient of B_3 under the relation $\sigma_1^2 = \sigma_2^2$. The reader can check that (5.6) is not r -complete, and that completing it leads (in 5 steps) to the presentation $(\mathcal{S}_3; \mathcal{R}_3)$ of Example 3.2.

Remark 5.6. Assume that $u = w$ and $w = v$ are two relations in the considered presentation. Then adding the relation $u = v$ is a special case of the completion procedure described above—which may suggest to call *transitive* a presentation satisfying the cube condition. Indeed, let us isolate the first letters in u, v, w , say $u = su', v = tv'$ and $w = rw'$. Then we have

$$s^{-1}rr^{-1}t \curvearrowright_r u'w'^{-1}w'v'^{-1} \curvearrowright_r u'v'^{-1},$$

and the completion procedure consists in adding the relation $su' = tv'$, *i.e.*, $u = v$, if we cannot obtain $u^{-1}v \curvearrowright_r \varepsilon$ using the current relations. (In the case of Example 5.4, the presentation (5.5) is r -complete although it contains $ab = bac$ and $ab = cba$ but not $bac = cba$ because the relation $(bac)^{-1}(cba) \curvearrowright_r \varepsilon$ is already true, and there is no need to add $bac = cba$.)

6. READING PROPERTIES OF THE MONOID

We enter now the second part of our study. Our aim is to show that, if $(\mathcal{S}, \mathcal{R})$ is a complete presentation, then several properties of the monoid $\langle \mathcal{S}; \mathcal{R} \rangle^+$ and of the group $\langle \mathcal{S}; \mathcal{R} \rangle$ can be read on the presentation. We begin with the monoid. We recall that, when u is a word in \mathcal{S}^* , then the element of $\langle \mathcal{S}; \mathcal{R} \rangle^+$ represented by u , *i.e.*, the \equiv -equivalence class of u , is denoted by \bar{u} .

Let us begin with cancellativity. As mentioned in the introduction, it is easy to recognize whether a monoid given by a complete presentation admits cancellation.

Proposition 6.1. *Assume that $(\mathcal{S}, \mathcal{R})$ is an r -complete presentation. Then the monoid $\langle \mathcal{S}; \mathcal{R} \rangle^+$ admits left cancellation if and only if $u^{-1}v \curvearrowright_r \varepsilon$ holds for every relation of the form $su = sv$ in \mathcal{R} with $s \in \mathcal{S}$. In particular, a sufficient condition for $\langle \mathcal{S}; \mathcal{R} \rangle^+$ to admit left cancellation is:*

$$(C_r) \quad \mathcal{R} \text{ contains no relation } su = sv \text{ with } u \neq v.$$

Proof. The condition is necessary, for $su = sv$ belonging to \mathcal{R} implies $su \equiv sv$, hence $u \equiv v$ if left cancellation is allowed, and, applying Prop. 3.3, $u^{-1}v \curvearrowright_r \varepsilon$ since the presentation is r -complete.

Conversely, assume that s belongs to \mathcal{S} and u', v' are words on \mathcal{S} satisfying $su' \equiv sv'$. By Prop. 3.3, we have $u'^{-1}s^{-1}sv' \curvearrowright_r \varepsilon$. By Lemma 1.7, this means that there exist words u, v, u'', v'' satisfying

$$(6.1) \quad s^{-1}s \curvearrowright_r vu^{-1}, u'^{-1}v \curvearrowright_r u''^{-1}, u^{-1}v' \curvearrowright_r v'', \text{ and } u''^{-1}v'' \curvearrowright_r \varepsilon.$$

By Prop. 1.9, this implies $u' \equiv vu'' \equiv vv''$ and $v' \equiv uv''$. Now, the first relation in (6.1) means that either $su = sv$ is a relation of \mathcal{R} , or we have $u = v = \varepsilon$. Since we assume that $u \equiv v$ (or, equivalently, $u^{-1}v \curvearrowright_r \varepsilon$) necessarily holds for each relation $su = sv$ in \mathcal{R} , we deduce $u' \equiv v'$ in each case, *i.e.*, left cancellation is allowed in $\langle \mathcal{S}; \mathcal{R} \rangle^+$. \square

Corollary 6.2. *Assume that $(\mathcal{S}, \mathcal{R})$ is a complete presentation. Then a sufficient condition for $\langle \mathcal{S}; \mathcal{R} \rangle^+$ to be cancellative is*

$$(C) \quad \mathcal{R} \text{ contains no relation } su = sv \text{ or } us = vs \text{ with } u \neq v.$$

Example 6.3. All presentations we have considered so far satisfy Condition (C), hence the corresponding monoids are cancellative. In particular, so is the monoid M_S of Example 5.1.

Remark 6.4. The criterion of Prop. 6.1, as well as the argument of the proof, was already used, in the complemented case, in earlier papers [9, 10]. Actually, the idea is already implicit in Garside's work [21]. Let us also mention that a very similar result is stated in [8] in the context of what is called there the reduction property (see Remark 2.2).

Let us consider now the word problem for the presentation $(\mathcal{S}, \mathcal{R})$, *i.e.*, the question of deciding whether two words u, v in \mathcal{S}^* represent the same element of the monoid $\langle \mathcal{S}; \mathcal{R} \rangle^+$, *i.e.*, whether $u \equiv v$ holds. By Prop. 3.3, if $(\mathcal{S}, \mathcal{R})$ is an r -complete presentation, then $u \equiv v$ is equivalent to $u^{-1}v \curvearrowright_r \varepsilon$, *i.e.*, word equivalence is always detected by r -reversing. As was observed in Sec. 1, this need not give a solution for the word problem if we have no bound on the length of the reversing sequences. However, Prop. 1.13 gives the following sufficient condition:

Proposition 6.5. *Assume that $(\mathcal{S}, \mathcal{R})$ is a finite r -complete presentation satisfying*

(F_r) *The closure of \mathcal{S} under r -reversing is finite.*

*Then the monoid $\langle \mathcal{S}; \mathcal{R} \rangle^+$ satisfies a quadratic isoperimetric inequality, *i.e.*, every relation $u \equiv v$ can be established using $O((\lg(u) + \lg(v))^2)$ relations of \mathcal{R} at most, and its word problem is solvable in quadratic time.*

Proof. Let k be the supremum of the number of r -reversing steps needed to reverse $u^{-1}v$ into $v'u'^{-1}$ for u, v, u', v' in the closure $\widehat{\mathcal{S}}$ of \mathcal{S} under r -reversing. Prop. 1.13 implies that, if u, v are words of length p and q respectively and $u \equiv v$ holds, then $u^{-1}v$ reverses to ε in kpq steps at most, hence in $O((p+q)^2)$ reversing steps. As each reversing step involves at most one relation of \mathcal{R} (reversing $s^{-1}s$ to ε requires none), we conclude that $u \equiv v$ can be proved using at most $O((p+q)^2)$ relations of \mathcal{R} . \square

Example 6.6. We already observed that Condition (F_r) applies to the monoids of Example 3.2: the latter therefore satisfy a quadratic isoperimetric inequality.

Let us consider now common (right) multiples. By Proposition 1.9, r -reversing computes common r -multiples in the considered monoid: $u^{-1}v \curvearrowright_r v'u'^{-1}$ implies $uv' \equiv vu'$, so the element of the monoid represented by uv' and vu' is a common right multiple of \bar{u} and \bar{v} . We can therefore expect properties involving common r -multiples to be easily recognized using r -reversing.

Proposition 6.7. *Assume that $(\mathcal{S}, \mathcal{R})$ is an r -complete presentation. Then a necessary and sufficient condition for any two elements of $\langle \mathcal{S}; \mathcal{R} \rangle^+$ to admit a common right multiple is*

(E_r) *There exists \mathcal{S}' satisfying $\mathcal{S} \subseteq \mathcal{S}' \subseteq \mathcal{S}^*$ and for all u, v in \mathcal{S}' , there exist u', v' in \mathcal{S}' satisfying $(uv')^{-1}(vu') \curvearrowright_r \varepsilon$.*

Proof. Assume that any two elements of $\langle \mathcal{S}; \mathcal{R} \rangle^+$ admit a common right multiple. This means that, for all words u, v in \mathcal{S}^* , there exist two words u', v' satisfying $uv' \equiv vu'$, *i.e.*, equivalently, $(uv')^{-1}(vu') \curvearrowright_r \varepsilon$, since $(\mathcal{S}, \mathcal{R})$ is r -complete. So $\mathcal{S}' = \mathcal{S}^*$ is convenient.

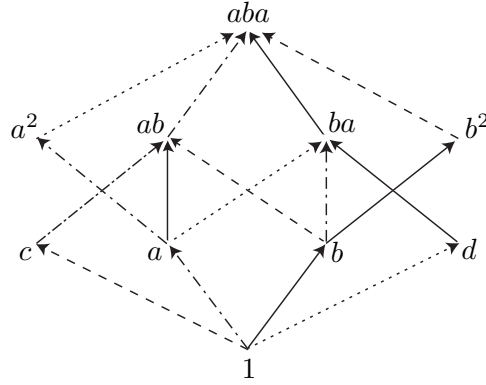


FIGURE 6.1. Cayley graph of Sergiescu's monoid M_S (fragment)

Conversely, assume that \mathcal{S}' satisfies Condition (E_r) . The latter implies that, for all u, v in \mathcal{S}' , there exist u', v' in \mathcal{S}' satisfying $uv' \equiv vu'$. Then, an easy induction on $p + q$ shows that, for u in \mathcal{S}'^p and v in \mathcal{S}'^q , there exist u' in \mathcal{S}'^p and v' in \mathcal{S}'^q satisfying $uv' \equiv vu'$, and, therefore, the elements of $\langle \mathcal{S}; \mathcal{R} \rangle^+$ represented by u and v admit a common r -multiple. \square

Example 6.8. Let us consider again the monoid M_S of Example 5.1. As shown in Fig. 6.1, the family $\{1, a, b, c, d, a^2, ab, ba, b^2, aba\}$ has the desired properties. So the monoid M_S admits common right multiples.

Remark 6.9. We may replace the relation $(uv')^{-1}(vu') \curvearrowright_r \varepsilon$ in Condition (E_r) by $u^{-1}v \curvearrowright_r v'u'^{-1}$, but the resulting condition (E'_r) is stronger, and, therefore, more difficult to check in practice. Indeed, $u^{-1}v \curvearrowright_r v'u'^{-1}$ implies $uv' \equiv vu'$, hence $(uv')^{-1}(vu') \curvearrowright_r \varepsilon$ for an r -complete presentation, so (E'_r) implies (E_r) . But, conversely, $(uv')^{-1}(vu') \curvearrowright_r \varepsilon$ implies that $u^{-1}v \curvearrowright_r v''u''^{-1}$ holds for some words u'', v'' , but the hypothesis that u', v' can be chosen in \mathcal{S}' need not imply that u'', v'' do.

As for the existence of least common multiples, we have the following criterion:

Proposition 6.10. *Assume that $(\mathcal{S}, \mathcal{R})$ is an r -complete presentation. Then a sufficient condition for any two elements of $\langle \mathcal{S}; \mathcal{R} \rangle^+$ admitting a common right multiple to admit a least one is that $(\mathcal{S}, \mathcal{R})$ is an r -complemented presentation, i.e., it satisfies the condition*

$$(U_r) \quad \begin{array}{l} \mathcal{R} \text{ contains no relation } su = sv \text{ with } u \neq v, \text{ and,} \\ \text{for } s \neq t, \text{ it contains at most one relation } s \cdots = t \cdots . \end{array}$$

In this case, the r -lcm of \bar{u} and \bar{v} is $\overline{uv'}$, where u' and v' are the unique words satisfying $u^{-1}v \curvearrowright_r v'u'^{-1}$.

Proof. If the presentation $(\mathcal{S}, \mathcal{R})$ is r -complemented, r -reversing is a deterministic process, so, for every pair of words u, v in \mathcal{S}^* , there exists at most one pair of words u'', v'' in \mathcal{S}^* satisfying $u^{-1}v \curvearrowright_r v''u''^{-1}$. Assume that uv' and vu' represent some common right multiple of \bar{u} and \bar{v} in $\langle \mathcal{S}; \mathcal{R} \rangle^+$. Then, by definition of r -completeness, there must exist w satisfying $u' \equiv u''w$ and $v' \equiv v''w$, where (u'', v'') is the unique pair satisfying $u^{-1}v \curvearrowright_r v''u''^{-1}$: this means that $\overline{uv''}$ is a right lcm of \bar{u} and \bar{v} . \square

Example 6.11. The criterion applies to the standard or dual presentations of the (generalized) braid groups, and to the many examples of [33], so, in each case, elements of the associated monoids that admit common multiples admit lcm's—as was already observed in previous papers dealing with reversing in the complemented case. In contradistinction, none of the presentations considered in Sec. 5 is complemented, and it is easy to check that lcm's do not exist there.

Observe that r -completeness is needed for Condition (U_r) to imply anything. For instance, (U_r) is true for the presentation (5.4) of the Heisenberg monoid of Example 5.4, though a and c have no r -lcm in the Heisenberg monoid: indeed, ac and bac are distinct r -mcm's of a and c , but neither is a multiple of the other. Now, of course, (U_r) fails for the r -complete presentation (5.5).

Remark 6.12. Prop. 6.10 tells us that, in the complemented case, r -reversing computes r -lcm's, and we could expect that, in the general case, it computes r -mcm's (minimal common multiples). This need not be the case, even for a homogeneous presentation. It is true that, if $(\mathcal{S}, \mathcal{R})$ is an r -complete presentation, then every possible r -mcm of \bar{u} and \bar{v} in $\langle \mathcal{S}; \mathcal{R} \rangle^+$ can be represented by uv' and vu' such that $u^{-1}v \curvearrowright_r v'u'^{-1}$ holds. Indeed, if $\overline{uv'}$ is an r -mcm of \bar{u} and \bar{v} , then r -completeness gives u'', v'', w satisfying $u^{-1}v \curvearrowright_r v''u''^{-1}$, $u' \equiv u''w$, and $v' = v''w$, and the minimality of $\overline{uv'}$ implies that w must be empty. But, conversely, it is not true in general that $u^{-1}v \curvearrowright_r v'u'^{-1}$ implies that uv' and vu' represent an r -mcm of \bar{u} and \bar{v} . A counter-example is given by the presentation $(a, b; ab = ba, a^2 = b^2)$. Indeed, we have seen that this presentation is homogeneous and complete, and, moreover, each relation represents an r -mcm. However, we have $a^{-1}b^2 \curvearrowright_r b^2a^{-1}$, but ab^2 is not an r -mcm of a and b^2 as a is a common right divisor of b^2 and a .

7. READING PROPERTIES OF THE GROUP

We turn to the question of reading properties of the group $\langle \mathcal{S}; \mathcal{R} \rangle$ when $(\mathcal{S}, \mathcal{R})$ is a complete (positive) presentation. Here we shall consider the question of whether $\langle \mathcal{S}; \mathcal{R} \rangle$ is a group of fractions, and, in this case, study its word problem.

Recognizing whether $\langle \mathcal{S}; \mathcal{R} \rangle$ is a group of fractions of the monoid $\langle \mathcal{S}; \mathcal{R} \rangle^+$ is easy. Indeed, it is well-known [6] that this happens if and only if $\langle \mathcal{S}; \mathcal{R} \rangle^+$ satisfies Ore's conditions, *i.e.*, it is cancellative and every two elements admit a common multiple. By gathering results from Sec. 6, we obtain directly:

Proposition 7.1. *Assume that $(\mathcal{S}, \mathcal{R})$ is a complete presentation. Then sufficient conditions for the monoid $\langle \mathcal{S}; \mathcal{R} \rangle^+$ to embed in a group of fractions are*

(C) \mathcal{R} contains no relation $su = sv$ or $us = vs$ with $u \neq v$,

(E_r) There exists \mathcal{S}' satisfying $\mathcal{S} \subseteq \mathcal{S}' \subseteq \mathcal{S}^*$ and such that, for all u, v in \mathcal{S}' , there exist u', v' in \mathcal{S}' satisfying $(uv')^{-1}(vu') \curvearrowright_r \varepsilon$.

Example 7.2. Typical presentations eligible for the previous criterion are the standard presentations of the spherical Artin–Tits groups, *i.e.*, those associated with a finite Coxeter group, or, more generally, all presentations of Gaussian groups investigated in [17, 15, 33]. All these presentations are complemented.

Now, also eligible are the presentations considered in Examples 3.2, 5.1, and 5.4. In each case, the conditions (C) and (E_r) are satisfied, and the associated monoid embeds in a group of fractions. This holds in particular for Sergiescu's monoid $M_{\mathcal{S}}$ of Example 5.1, of which the associated group of fractions is the braid group B_3 :

we thus obtain a new decomposition of B_3 as a group of fractions, besides the classical decomposition associated with the monoid B_3^+ and the Birman-Ko-Lee decomposition of [3] (this answers a question of [22]).

Under the hypotheses of Prop. 7.1, the congruence \equiv that defines the monoid $\langle \mathcal{S}; \mathcal{R} \rangle^+$ is the restriction of the congruence \equiv^\pm that defines the group $\langle \mathcal{S}; \mathcal{R} \rangle$, and standard arguments then imply that $vu^{-1} \equiv^\pm v'u'^{-1}$ is true if and only if there exist w and w' satisfying $uw \equiv u'w'$ and $vw \equiv v'w'$. We shall now reprove and extend this result by establishing a more precise connection between the congruences \equiv^\pm , \equiv and the r -reversing relation in the more general case when only (C_r) and (E_r) are assumed.

Proposition 7.3. *Assume that $(\mathcal{S}, \mathcal{R})$ is an r -complete presentation satisfying Conditions (C_r) and (E_r) .*

(i) *For all words \mathbf{w}, \mathbf{w}' on $\mathcal{S} \cup \mathcal{S}^{-1}$, the relation $\mathbf{w} \equiv^\pm \mathbf{w}'$ is true if and only if there exist u, v, w, u', v', w' in \mathcal{S}^* satisfying*

$$(7.1) \quad \mathbf{w} \curvearrow_r vu^{-1}, \quad \mathbf{w}' \curvearrow_r v'u'^{-1}, \quad uw \equiv u'w', \quad vw \equiv v'w'.$$

(ii) *In particular, for all words u, u' in \mathcal{S}^* , the relation $u \equiv^\pm u'$ is true if and only if there exists w in \mathcal{S}^* satisfying $uw \equiv u'w$.*

The proof will be split into several steps. We assume until the end of the proof of Prop. 7.3 that $(\mathcal{S}, \mathcal{R})$ is an r -complete presentation satisfying (C_r) and (E_r) . For \mathbf{w}, \mathbf{w}' words on $\mathcal{S} \cup \mathcal{S}^{-1}$, we say that $\mathbf{w} \cong \mathbf{w}'$ is true if there exist u, v, w, u', v', w' satisfying (7.1). Our aim is to prove that the relations \equiv^\pm and \cong coincide.

Lemma 7.4. *Assume $\mathbf{w} \curvearrow_r vu^{-1}$ and $\mathbf{w} \curvearrow_r v'u'^{-1}$ with $u, v, u', v' \in \mathcal{S}^*$. Then there exist w, w' in \mathcal{S}^* satisfying $uw = u'w'$ and $vw = v'w'$.*

Proof. The hypothesis that $(\mathcal{S}, \mathcal{R})$ satisfies (E_r) implies that there exist words w, w' satisfying $vw \equiv v'w'$, and the point is to prove that $vw \equiv v'w'$ implies $uw \equiv u'w'$ whenever some word \mathbf{w} reverses both to vu^{-1} and to $v'u'^{-1}$. We establish the latter implication using induction on the length of \mathbf{w} . The result is trivial if \mathbf{w} is empty. Assume that \mathbf{w} has length 1. If \mathbf{w} is a letter in \mathcal{S} , say s , the only possibility is $u = u' = \varepsilon, v = v' = s$, so the hypothesis is $sw \equiv sw'$, and the expected conclusion is $w \equiv w'$: so the implication is true provided $\langle \mathcal{S}; \mathcal{R} \rangle^+$ admits left cancellation. If \mathbf{w} is a letter in \mathcal{S}^{-1} , say s^{-1} , we have $u = u' = s, v = v' = \varepsilon$, the hypothesis is $w \equiv w'$, and the expected conclusion is $sw \equiv sw'$, so the implication is always true.

Assume now $\mathbf{w} = \mathbf{w}_1\mathbf{w}_2$ with $\text{lg}(\mathbf{w}_i) < \text{lg}(\mathbf{w})$. By Lemma 1.7, there exist words $u_i, v_i, u'_i, v'_i, i = 0, 1, 2$, satisfying $\mathbf{w}_1 \curvearrow_r v_1u_0^{-1}, \mathbf{w}_2 \curvearrow_r v_0u_1^{-1}$ and $u_0^{-1}v_0 \curvearrow_r v_2u_2^{-1}$, and similar dashed relations (see Fig. 7.1). By hypothesis, we have $v_1v_2w \equiv v'_1v'_2w'$ and \mathbf{w}_1 reverses both to $v_1u_0^{-1}$ and $v'_1u'_0{}^{-1}$, so applying the induction hypothesis to \mathbf{w}_1 gives $u_0v_2w \equiv u'_0v'_2w'$, hence $v_0u_2w \equiv v'_0u'_2w'$. Now \mathbf{w}_2 reverses both to $v_0u_1^{-1}$ and $v'_0u'_1{}^{-1}$, so applying the induction hypothesis to \mathbf{w}_2 gives $u_1u_2w \equiv u'_1u'_2w'$, i.e., $uw \equiv u'w'$, as was expected. \square

Lemma 7.5. *For s in \mathcal{S} , $\mathbf{w} \cong \mathbf{w}'$ implies $\mathbf{w}s \cong \mathbf{w}'s$ and $\mathbf{w}s^{-1} \cong \mathbf{w}'s^{-1}$.*

Proof. Assume

$$\mathbf{w} \curvearrow_r vu^{-1}, \quad \mathbf{w}' \curvearrow_r v'u'^{-1}, \quad uw \equiv u'w', \quad vw \equiv v'w'.$$

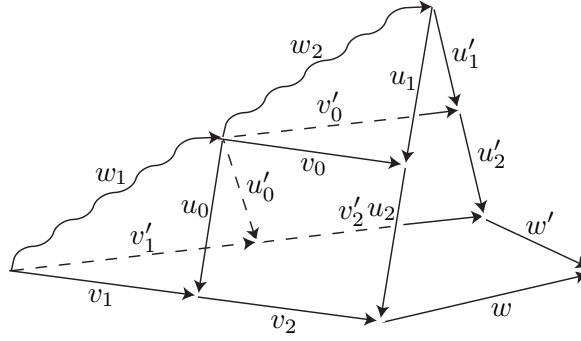


FIGURE 7.1. Several reversings

As Condition (E_r) is satisfied, there exist u_0, v_0 , and v_1, w_0 in \mathcal{S}^* satisfying $u^{-1}s \curvearrowright_r v_0u_0^{-1}$ and $w^{-1}v_0 \curvearrowright_r v_1w_0^{-1}$ (Fig. 7.2). So, by construction, we have $\mathbf{w}s \curvearrowright_r (vv_0)u_0^{-1}$ and $s^{-1}(uw) \curvearrowright_r u_0w_0v_1^{-1}$. As the presentation is r -complete, $uw \equiv u'w'$ implies $(uw)^{-1}(u'w') \curvearrowright_r \varepsilon$, and, by definition, we have $v_1^{-1}\varepsilon \curvearrowright_r \varepsilon v_1^{-1}$, hence $s^{-1}(uw)(uw)^{-1}(u'w') \curvearrowright_r u_0w_0v_1^{-1}$. The cube condition for $s, uw, u'w'$ is true, so there must exist words u'', v'', w'' in \mathcal{S}^* satisfying $s^{-1}u'w' \curvearrowright_r u''v''^{-1}$, $u''w'' \equiv u_0w_0$, and $v''w'' \equiv v_1$. By Lemma 1.7, there exist u'_0, v'_0, w'_0 and v'_1 satisfying $s^{-1}u' \curvearrowright_r u'_0v'_0^{-1}$, $v'_0^{-1}w' \curvearrowright_r w'_0v'_1^{-1}$, $u'' = u'_0w'_0$, and $v'' = v'_1$, hence $u_0w_0 \equiv u'_0w'_0w''$ and $v_1 = v'_1w''$. So, we have

$$(7.2) \quad \mathbf{w}s \curvearrowright_r (vv_0)u_0^{-1}, \quad \mathbf{w}'s \curvearrowright_r (v'v'_0)u'_0{}^{-1}.$$

Now we check

$$(7.3) \quad su_0w_0 \equiv uvv_1 \equiv u'w'v'_1w'' \equiv su'_0w'_0w'',$$

$$(7.4) \quad vv_0w_0 \equiv vvw_1 \equiv v'w'v'_1w'' \equiv v'v'_0w'_0w''.$$

As left cancellation is possible, (7.3) implies $u_0w_0 \equiv u'_0(w'_0w'')$, while (7.4) reads $(vv_0)w_0 \equiv (v'v'_0)(w'_0w'')$, which, together with (7.2), gives $\mathbf{w}s \cong \mathbf{w}'s$.

The case of s^{-1} is trivial: with the same notation, we have

$$\mathbf{w}s^{-1} \curvearrowright_r v(su)^{-1}, \quad \mathbf{w}'s^{-1} \curvearrowright_r v'(su')^{-1}, \quad (su)w \equiv (su')w', \quad vw \equiv v'w',$$

so $\mathbf{w}s^{-1} \cong \mathbf{w}'s^{-1}$ holds as well. \square

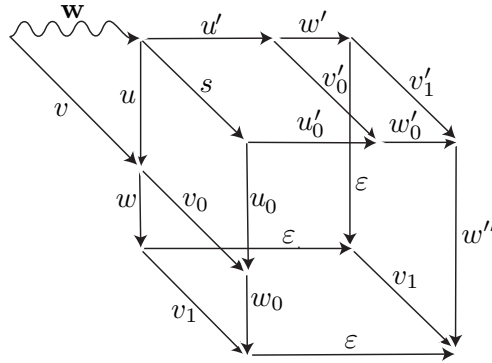


FIGURE 7.2. Compatibility with multiplication

Proof of Prop. 7.3. (i) Assume $\mathbf{w} \cong \mathbf{w}'$. With the notation of (7.1), we find

$$\mathbf{w} \equiv^{\pm} vu^{-1} \equiv^{\pm} vvw^{-1}u^{-1} \equiv^{\pm} v'w'w'^{-1}u'^{-1} \equiv^{\pm} v'u'^{-1} \equiv^{\pm} \mathbf{w}',$$

so $\mathbf{w} \cong \mathbf{w}'$ implies $\mathbf{w} \equiv^{\pm} \mathbf{w}'$.

Conversely, we shall prove that \cong is a congruence that contains pairs generating \equiv^{\pm} . By definition, the relation \cong is reflexive and symmetric. Assume $\mathbf{w} \cong \mathbf{w}' \cong \mathbf{w}''$. This means that there exist words u, \dots, w'' in \mathcal{S}^* satisfying

$$\begin{aligned} \mathbf{w} \curvearrow_r vu^{-1}, \quad \mathbf{w}' \curvearrow_r v'_1u_1'^{-1}, \quad uw \equiv u'_1w'_1, \quad vw \equiv v'_1w'_1, \\ \mathbf{w}' \curvearrow_r v'_2u_2'^{-1}, \quad \mathbf{w}'' \curvearrow_r v''u''^{-1}, \quad u'_2w'_2 \equiv u''w'', \quad v'_2w'_2 \equiv v''w''. \end{aligned}$$

By Lemma 7.4, there exist w_1, w_2 in \mathcal{S}^* satisfying $u'_1w_1 \equiv u'_2w_2$ and $v'_1w_1 \equiv v'_2w_2$. Now, as common right multiples exist in the monoid $\langle \mathcal{S}; \mathcal{R} \rangle^+$, we can find w_0, w_1'', w_2'' in \mathcal{S}^* satisfying $w_1w_0 \equiv w_1''w_1''$ and $w_2w_0 \equiv w_2''w_2''$, and we find

$$\begin{aligned} uww_1'' \equiv u'_1w_1''w_1'' \equiv u'_1w_1w_0 \equiv u'_2w_2w_0 \equiv u'_2w_2''w_2'' \equiv u''w''w_2'', \\ vww_1'' \equiv v'_1w_1''w_1'' \equiv v'_1w_1w_0 \equiv v'_2w_2w_0 \equiv v'_2w_2''w_2'' \equiv v''w''w_2'', \end{aligned}$$

so the words ww_1'' and $w''w_2''$ witness for $\mathbf{w} \cong \mathbf{w}''$. So \cong is an equivalence relation.

We claim now that \cong is a congruence, *i.e.*, it is compatible with multiplication on both sides. It suffices to consider the case of right or left multiplication by a single positive or negative letter. Lemma 7.5 gives the result for right multiplication, and we observe that $\mathbf{w} \cong \mathbf{w}'$ is equivalent to $\mathbf{w}^{-1} \cong \mathbf{w}'^{-1}$, so the result for left multiplication follows.

By definition, \equiv^{\pm} is the congruence on $(\mathcal{S} \cup \mathcal{S}^{-1})^*$ generated by all pairs $\{u, v\}$ in \mathcal{R} , augmented with all pairs $\{ss^{-1}, \varepsilon\}$ and $\{s^{-1}s, \varepsilon\}$ with $s \in \mathcal{S}$. Writing

$$\begin{aligned} u \curvearrow_r u, \quad v \curvearrow_r v, \quad \varepsilon\varepsilon \equiv \varepsilon\varepsilon, \quad u\varepsilon \equiv v\varepsilon, \\ ss^{-1} \curvearrow_r ss^{-1}, \quad \varepsilon \curvearrow_r \varepsilon\varepsilon^{-1}, \quad s\varepsilon \equiv \varepsilon s, \quad s\varepsilon \equiv \varepsilon s, \\ s^{-1}s \curvearrow_r \varepsilon\varepsilon^{-1}, \quad \varepsilon \curvearrow_r \varepsilon\varepsilon^{-1}, \quad \varepsilon\varepsilon \equiv \varepsilon\varepsilon, \quad \varepsilon\varepsilon \equiv \varepsilon\varepsilon, \end{aligned}$$

we see that $u \cong v$, $ss^{-1} \cong \varepsilon$, and $s^{-1}s \cong \varepsilon$ hold, and we conclude that \equiv^{\pm} is included in \cong , *i.e.*, that $\mathbf{w} \equiv^{\pm} \mathbf{w}'$ implies $\mathbf{w} \cong \mathbf{w}'$, which completes the proof of (i).

(ii) As \equiv is included in \equiv^{\pm} , the existence of a word w satisfying $uw \equiv u'w$ is a sufficient condition for $u \equiv^{\pm} u'$. Conversely, assume $u \equiv^{\pm} u'$. By (i), u and u' have to reverse to fractions satisfying (7.1). As u and u' belong to \mathcal{S}^* , the only possibilities are $u \curvearrow_r u\varepsilon^{-1}$ and $u' \curvearrow_r u'\varepsilon^{-1}$, so (7.1) reduces to the existence of w, w' in \mathcal{S}^* that satisfy $uw \equiv u'w'$ and $\varepsilon w \equiv \varepsilon w'$: this implies $uw \equiv u'w$. \square

Let us now return to the hypotheses of Prop. 7.1, *i.e.*, to the case when the group $\langle \mathcal{S}; \mathcal{R} \rangle$ is a group of fractions for the monoid $\langle \mathcal{S}; \mathcal{R} \rangle^+$. The following result shows that the word problem can always be solved by a double r -reversing, or, alternatively, an r -reversing followed with an l -reversing.

Proposition 7.6. *Assume that $(\mathcal{S}, \mathcal{R})$ is a complete presentation satisfying Conditions (C) and (E_r) . Then, for every word \mathbf{w} on $\mathcal{S} \cup \mathcal{S}^{-1}$, the following are equivalent:*

- (i) *We have $\mathbf{w} \equiv^{\pm} \varepsilon$;*
- (ii) *There exist u, v in \mathcal{S}^* satisfying $\mathbf{w} \curvearrow_r vu^{-1}$ and $u^{-1}v \curvearrow_r \varepsilon$;*
- (iii) *There exist u, v in \mathcal{S}^* satisfying $\mathbf{w} \curvearrow_r vu^{-1} \curvearrow_l \varepsilon$.*

Proof. Under the hypotheses, we know that, for every word \mathbf{w} , there exist positive words u, v satisfying $\mathbf{w} \curvearrowright_r vu^{-1}$. Then $\mathbf{w} \equiv^\pm \varepsilon$ is equivalent to $u \equiv^\pm v$, hence to $u \equiv v$ by Prop. 7.1, and, therefore, both to $u^{-1}v \curvearrowright_r \varepsilon$ and to $vu^{-1} \curvearrowright_l \varepsilon$ by Prop. 3.3. \square

Proposition 7.7. *Assume that $(\mathcal{S}, \mathcal{R})$ is a complete presentation satisfying Conditions (F_r) , (C) and (E_r) , i.e.,*

- (F_r) *The closure of \mathcal{S} under r -reversing is finite,*
- (C) *The presentation \mathcal{R} contains no relation $su = sv$ or $us = vs$ with $u \neq v$,*
- (E_r) *There exists \mathcal{S}' satisfying $\mathcal{S} \subseteq \mathcal{S}' \subseteq \mathcal{S}^*$ and such that, for all u, v in \mathcal{S}' , there exist u', v' in \mathcal{S}' satisfying $(uv')^{-1}(vu') \curvearrowright_r \varepsilon$.*

Then the group $\langle \mathcal{S}; \mathcal{R} \rangle$ satisfies a quadratic isoperimetric inequality.

Proof. We gather Prop. 7.6, which reduces the word problem in $\langle \mathcal{S}; \mathcal{R} \rangle$ to a double reversing process, and Prop. 6.5, which gives a bound on the complexity of the latter process. \square

Example 7.8. The previous criterion applies to the groups defined by the complemented presentations of Example 7.2. But it also applies to the groups associated with the presentations of Example 3.2, thus typically to the groups

$$(a, b; a^2 = b^2, ab = ba)$$

$$(a, b, c; a^2 = b^2 = c^2, ab = bc = ca, ac = ba = cb).$$

(We recall that the latter is the quotient of B_3 under the additional relation $\sigma_1^2 = \sigma_2^2$.) These groups therefore satisfy a quadratic isoperimetric inequality. So does the group associated with the monoid of Example 5.1, but we saw that the latter group is B_3 , and that result is well known.

Let us consider now the Heisenberg group H . The closure of $\{a, b, c\}$ under r -reversing with respect to the (incomplete) presentation (5.4) is the infinite set $\{\varepsilon, a, b, c\} \cup \{ac^n; n \geq 1\}$, and, using the latter, we easily conclude that common right multiples exist in the associated monoid, of which H is the group of fractions. Then Prop. 7.6 shows how to solve the word problem using a double reversing with respect to the complete presentation

$$(7.5) \quad (a, b, c; ab = bac, ac = ca, bc = cb, cba = ab).$$

It can be checked that the complexity of the procedure is cubic, which could be expected as H is known to admit a cubic isoperimetric function [20].

In the complemented case, the study proceeds farther, and it is known that, under the hypotheses of Prop. 7.7, the group $\langle \mathcal{S}; \mathcal{R} \rangle$ is a Garside group and, in particular, it is torsion-free [12] and admits a bi-automatic structure [15]. The question of whether the latter result extends to the general case of non necessarily complemented presentations seems to be difficult, as the automatic structures known in the complemented case rely on the uniqueness of the gcd's. In any case, the answer is connected with the fine structure of divisibility in the monoid $\langle \mathcal{S}; \mathcal{R} \rangle^+$, and the importance of words and reversing becomes secondary. So we shall not discuss the question here, but refer to [16] where the question is investigated directly. Let us mention that the groups of Example 7.8 turn out to be automatic.

The above study has led to results about the group $\langle \mathcal{S}; \mathcal{R} \rangle$ only in the case when the latter happens to be a group of fractions for the monoid $\langle \mathcal{S}; \mathcal{R} \rangle^+$. The main open question now is to determine to which extent word reversing can be used to prove results about the group $\langle \mathcal{S}; \mathcal{R} \rangle$ in the general case. In particular, it would be interesting to know whether reversing techniques can be used to study the possible embeddability of the monoid $\langle \mathcal{S}; \mathcal{R} \rangle^+$ in the group $\langle \mathcal{S}; \mathcal{R} \rangle$. Let us observe here that the presentation

$$(7.6) \quad (a, b, c, d, a', b', c', d'; aa' = bb', ca' = db', ac' = bd')$$

introduced in [25] is complete and it satisfies Condition (C), so the associated monoid is cancellative, but the latter does not embed in the corresponding group, as $cc' = dd'$ holds in the group (we have there $c^{-1}d = a'b'^{-1} = a^{-1}b = c'd'^{-1}$) but not in the monoid (we do not have $c'^{-1}c^{-1}dd' \curvearrowright_r \varepsilon$). Can this negative result be read directly from Presentation 7.6? Similarly, but in the other direction, it is known that every Artin–Tits monoid embeds in the corresponding group [32], but the remarkable proof of the result uses an indirect approach via a linear representation (inspired by [27]). Could reversing be used here?

We shall conclude this paper with a more precise question. Assume that $(\mathcal{S}, \mathcal{R})$ is a positive group presentation, and let \curvearrowright denote the union of the relations $\curvearrowright_r^\sharp$ and $\curvearrowright_l^\sharp$, *i.e.*, the extended r -reversing considered in Remark 1.5 and its left counterpart. Prop. 7.6 tells us that, if $(\mathcal{S}, \mathcal{R})$ is a complete presentation such that the monoid $\langle \mathcal{S}; \mathcal{R} \rangle^+$ is cancellative and admits common right multiples, then a word \mathbf{w} represents 1 in the group $\langle \mathcal{S}; \mathcal{R} \rangle$ if and only if $\mathbf{w} \curvearrowright \varepsilon$ holds. If common multiples do not exist in $\langle \mathcal{S}; \mathcal{R} \rangle^+$, the proof is no longer valid. However, the above result, namely that $\mathbf{w} \equiv^\pm \varepsilon$ is equivalent to $\mathbf{w} \curvearrowright_r \varepsilon$, also holds in the case of a free group *i.e.*, when \mathcal{R} is empty: in this case, reversing coincides with free reduction, and it is true that \mathbf{w} represents 1 in a free group if and only if it freely reduces to ε (with an unbounded number of alternations between r - and l -reversing, contrary to the case of Prop. 7.6 where one alternation is enough). Similarly, in the case of Presentation (7.6), the key relation $cc' = dd'$, which we have seen holds in the group but not in the monoid, can be proved using reversing, *i.e.*, $(cc')^{-1}(dd') \curvearrowright \varepsilon$ holds, as we find

$$c'^{-1}c^{-1}dd' \curvearrowright_r c'^{-1}a'b'^{-1}d' \curvearrowright_l c'^{-1}a^{-1}bd' \curvearrowright_r c'^{-1}c'd'^{-1}d' \curvearrowright_r \varepsilon$$

(with two alternations between r - and l -reversing). This leads to the general problem of whether the word problem of the group can be solved using reversing. Simple counter-examples, such as the presentation $(a, b, c; ab = ac)$ suggested by S. Lee, show that some assumptions have to be satisfied, but the following question is open:

Question 7.9. Let $(\mathcal{S}, \mathcal{R})$ be a complete presentation satisfying Condition (C) (so the monoid $\langle \mathcal{S}; \mathcal{R} \rangle^+$ is cancellative). Is $\mathbf{w} \curvearrowright \varepsilon$ a necessary (and sufficient) condition for a word \mathbf{w} on $\mathcal{S} \cup \mathcal{S}^{-1}$ to represent 1 in the group $\langle \mathcal{S}; \mathcal{R} \rangle$?

A positive answer would imply that we can prove $\mathbf{w} \equiv^\pm \varepsilon$ by introducing no new factor ss^{-1} or $s^{-1}s$, so, in some sense, by always going from one word to another that is not more complicated (if not shorter, in general). In this sense, solutions for the word problem based on word reversing are reminiscent of Dehn’s algorithm for hyperbolic groups, but their range includes more complicated groups, such as braid groups, or, more generally, Garside groups (which admit a quadratic isoperimetric function), or even more complicated groups like the nilpotent Heisenberg group

(which admits a cubic isoperimetric function). The underlying question is whether one can prove that a word \mathbf{w} is trivial by remaining not too far from \mathbf{w} in the Cayley graph of the considered group (a precise meaning was given in [11]), and reversing gives a positive answer for many particular groups. The general case is open, but we conjecture that completeness is relevant.

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