THE GROUP OF PARENTHESIZED BRAIDS

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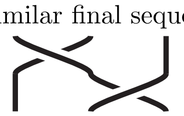
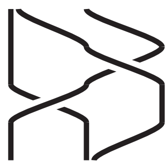
ABSTRACT. We investigate a group $B_\bullet$ that includes Artin's braid group B_∞ and Thompson's group F. The elements of $B_\bullet$ are represented by braids diagrams in which the distances between the strands are not uniform and, besides the usual crossing generators, new rescaling operators shrink or strech the distances between the strands. We prove that $B_\bullet$ is a group of fractions, that it is orderable, admits a non-trivial self-distributive structure, *i.e.*, one involving the law $x(yz) = (xy)(xz)$, it embeds in the mapping class group of a sphere with a Cantor set of punctures, and that Artin's representation of B_∞ into the automorphisms of a free group extends to $B_\bullet$.

The aim of this paper is to study a certain group, denoted $B_\bullet$, which includes both Artin's braid group B_∞ [3, 9, 15] and Thompson's group F [32, 28, 10]. The group $B_\bullet$ is generated by (the copies of) B_∞ and F, and its seemingly rich and deep properties appear to be a mixture of those of B_∞ and F. Here, starting from a geometric approach in terms of parenthesized braid diagrams, we give an explicit presentation of $B_\bullet$ that extends the standard presentations of B_∞ and F, we prove that $B_\bullet$ is a group of fractions, is an orderable group, and embeds into the mapping class group of a sphere with a Cantor set of punctures and into the automorphisms of a free group. Besides its group multiplication, $B_\bullet$ is also equipped with a second binary operation satisfying the self-distributivity law $x(yz) = (xy)(xz)$. We prove that every element of $B_\bullet$ generates a free subsystem with respect to that second operation—which shows that the self-distributive structure of $B_\bullet$ is highly non-trivial—and we deduce canonical decompositions for the elements of $B_\bullet$. The self-distributive structure is instrumental in proving most of the above results about the group structure of $B_\bullet$.

Here the elements of $B_\bullet$ are seen as *parenthesized braids*, which are braids in which the distances between the strands are not uniform. An ordinary braid diagram connects an initial sequence of equidistant positions to a similar final sequence, as for instance in



where the initial and final set of positions can be denoted $\bullet\bullet\bullet$. A parenthesized braid diagram connects a parenthesized sequence of positions to another possibly different parenthezied sequence of positions, the intuition being that grouped positions are (infinitely) closer than ungrouped ones. An example is



where the initial positions are $(\bullet\bullet)\bullet$ and the final positions are $\bullet(\bullet\bullet)$. Arranging such objects into a group leads to introducing, besides the usual braid generators σ_i that create crossings, new rescaling generators a_i that shrink the distances between the strands in the vicinity of

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position i: as one can expect, the σ_i's generate the copy of B_∞, while the a_i's generate the copy of Thompson's group F.

Parenthesized braids have been considered by D. Bar Natan in [1, 2] in connection with Vassiliev's invariants of knots and the computation of a Drinfeld associator. In these papers, parenthesized braids, and more generally parenthesized tangles, are studied as categories, and the question of finding presentations is not addressed.

The realization of $B_\bullet$ as a group of parenthesized braids is not the only possible one, and this group recently appeared in various frameworks. In [5, 6], M. Brin investigates a certain group BV introduced as a torsion-free version of Thompson's group V, and which admits a subgroup $\widehat{BV}$ that is isomorphic to $B_\bullet$. In [18], an independent approach leads to introducing $B_\bullet$ as the so-called geometry group for the associativity law together with a twisted version of the semi-commutativity law. All these approaches are more or less equivalent, but we think that parenthesized braids provide an especially intuitive and natural description. Larger groups extending both braid groups and Thompson's groups appear in [23, 21, 24].

The current paper is self-contained in that it requires no knowledge of the above mentioned papers (by contrast, [18] resorts to results from the current paper). As for results, the only overlap with other papers is the result that $B_\bullet$ is a group of fractions, which is established using Zappa-Szép products of monoids in [5], while we deduce it from general results involving the word reversing technique.

Remark on notation. We follow the usual braid conventions: our generators σ_i are numbered from 1, and the product corresponds to an action on the right (xy means x followed by y). For coherence, we adopt a similar notation for Thompson's group F, thus shifting indices and reversing products: what we denote a_i is x_{i-1}^{-1} (or X_{i-1}^{-1}) in the standard presentation of F [10]. An index of terms and notation is given at the end of the paper.

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1. Parenthesized braids

Throughout the paper, $\mathbf{N}$ denotes the set of all positive integers (0 excluded).

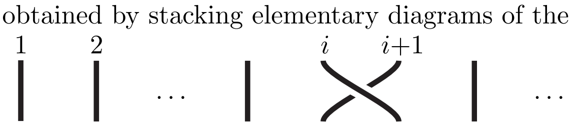
We construct a new group $B_\bullet$ using the approach that is standard for braids, namely starting with isotopy classes of braid diagrams. The difference is that we consider diagrams in which the distances between the endpoints of the strands need not be uniform. Such sets of positions can be specified using parenthesized expressions, like $\bullet((\bullet\bullet)\bullet)$, where grouped positions are to be seen as infinitely closed than the adjacent ones. This principle is implemented by considering positions that are indexed by finite sequence of integers.

The current construction of $B_\bullet$ is exactly as simple as that of B_∞. Although making it precise requires some notation, needed in particular in subsequent proofs, the ideas should be clear, and many details can be skipped.

1.1. **An intuitive description.** A braid diagram consists of curves that connect an initial sequence of positions to a similar final sequence of positions. In an ordinary braid diagram, the positions are indexed by positive integers

$$\overset{1}{\bullet}\quad \overset{2}{\bullet}\quad \bullet\quad \bullet\quad \bullet\quad \bullet\quad \bullet\quad \cdots$$

and a generic diagram is obtained by stacking elementary diagrams of the type

σ_i : 

or their reflections in a horizontal mirror.

Here we consider braid diagrams in which the initial and final positions need not be equidistant, but instead the distances may be $1, \epsilon, \epsilon^2, \ldots$ with $\epsilon \ll 1$. This leads to considering that, between the positions 1 and 2, infinitely many new positions $1+\epsilon$, $1+2\epsilon$, ... are possible, and so on iteratively. Thus $2+3\epsilon+\epsilon^2$ or $1+\epsilon^3$ are typical positions (Figure 1).

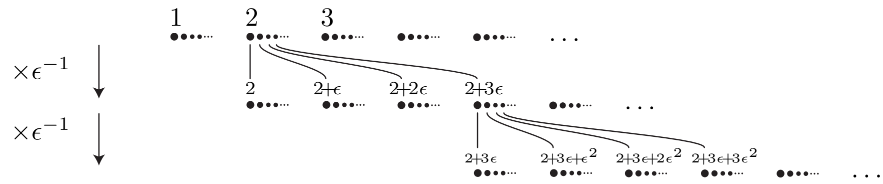
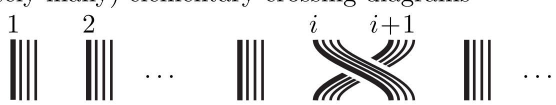
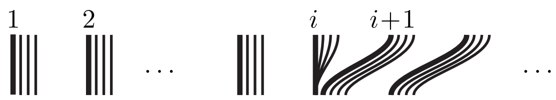


FIGURE 1. The set of all positions realized using infinitesimal distances

Then, as in the case of ordinary braid diagrams, we can consider generalized braid diagrams obtained by stacking (finitely many) elementary crossing diagrams

σ_i : 

in which all strands near position i cross over all strands near position $i+1$, and rescaling diagrams

a_i : 

in which the strands near position i are shrinked by a factor ϵ and all strands on the right are translated to fill the gaps. We also allow the mirror images of the above diagrams. Our claim is that such diagrams up to isotopy form a group, and this group is the object we investigate in this paper.

Though intuitive, the previous informal description is partly misleading in that it involves diagrams with infinitely many strands. The objects we really wish to consider are finite subdiagrams obtained by restricting to a finite set of positions. In this way, one exactly obtains the diagrams that are arranged into a small category in [1, 2], the objects being the possible sets of positions—which we shall see can be specified by parenthesized expressions or, equivalently, finite binary trees—and the morphisms being the isotopy classes of braid diagrams.

A (minor) problem arises when we wish to make a group out of the previous objects. In ordinary braid diagrams, the initial and final positions coincide, so, for each n, concatenating n strand diagrams is always possible, which leads to the braid group B_n. In our extended framework, concatenating two diagrams $\mathcal{D}_1, \mathcal{D}_2$ is possible only when the final set of positions in $\mathcal{D}_1$ coincides with the initial set of positions in $\mathcal{D}_2$, and an everywhere defined product appears only when we consider infinite completions, a situation similar to that of B_∞: to make a group out of all ordinary diagrams, independently on the number of strands, one embeds B_n into $B_{n'}$ for $n < n'$ and the elements of B_∞ are then represented by infinite diagrams.

1.2. **Sets of positions, parenthesized expressions and trees.** For a more formal construction, we first define the convenient sets of positions. Infinitesimal distances are intuitive, but there Is no need to use them: the infinitesimals we consider are polynomials in ϵ, and the simplest solution is to index positions by polynomials, *i.e.*, by finite sequences of nonnegative integers. To make explicit geometric constructions easier, we also embed positions into the unit interval using a dyadic expansion.

Definition 1.1. A finite sequence of nonnegative integers is called a *position* if it does not begin or finish with 0. The set of all positions is denoted by $\mathbf{N}_\bullet$. For s a position—or, more generally, any finite sequence of nonnegative integers not beginning with 0—say $s = (i_1, \dots, i_p)$, the *dyadic realization* of s is the rational number $s^\#$ with dyadic expansion $0.1^{i_1-1}01^{i_2}0\dots1^{i_p}$.

Intuitively, $(i_1, \dots, i_p)$ corresponds to what is denoted $i_1{+}i_2\epsilon{+}\cdots{+}i_p\epsilon^{p-1}$ in Figure 1. Under the dyadic realization, we find $(1)^\# = 0$, $(2)^\# = \frac{1}{2}$, $(3)^\# = \frac{3}{4}$, ... and $(1, 2, 1)^\# = 0.01101 = \frac{13}{32}$ (Figure 2). The requirement that positions do not finish with 0 is needed to guarantee that both the infinitesimal and the dyadic realizations be injective on $\mathbf{N}_\bullet$—alternatively, we can allow final 0's at the expense of identifying s and $(s, 0)$.

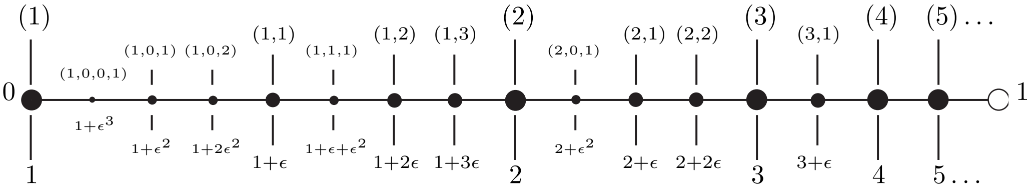


FIGURE 2. Realization of positions by dyadic numbers in the unit interval $[0, 1]$, and the corresponding infinitesimal numbers as in Figure 1

The set of positions involved in an ordinary braid diagram is an initial interval $\{1, 2, \dots, n\}$ of $\mathbf{N}$. When we turn to $\mathbf{N}_\bullet$, the role of such an interval is played by a finite binary tree—simply called a tree in the sequel. We denote by $\bullet$ the tree consisting of a single vertex and by t_1t_2 the tree with left subtree t_1 and right subtree t_2. Every tree has a unique decomposition in terms of $\bullet$, so we can identify trees and parenthesized expressions (Figure 3). The *right height* of a tree is defined to be the length of its righmost branch.

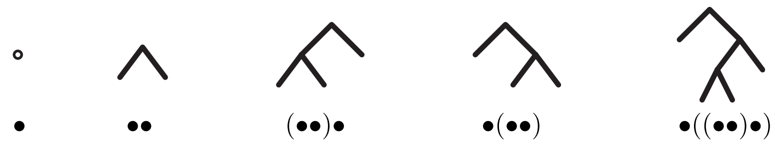


FIGURE 3. Typical trees and the corresponding parenthesized expressions

Then we associate with every tree a finite set of positions as follows:

Definition 1.2. For t a tree, we define a finite set of dyadic numbers $\mathrm{Dyad}(t)$ by the following rules: $\mathrm{Dyad}(\bullet)$ is $\{0, 1\}$, and $\mathrm{Dyad}(t_1t_2)$ is the union of $\mathrm{Dyad}(t_1)$ contracted from $[0, 1]$ to $[0, \frac{1}{2}]$ and of $\mathrm{Dyad}(t_2)$ contracted to $[\frac{1}{2}, 1]$. Then $\mathrm{Pos}(t)$ is defined to be the set of all positions s such that $s^\#$ belongs to $\mathrm{Dyad}(t)$ with the largest two elements removed.

Example 1.3. Let c_n denote the size $n+1$ right vine $\bullet(\dots(\bullet(\bullet\bullet))\dots)$, $n+1$ times $\bullet$. Then $\mathrm{Dyad}(c_n)$ is $\{0, \frac{1}{2}, \frac{3}{4}, \dots, 1{-}\frac{1}{2^n}, 1\}$, *i.e.*, $\{(1)^\#, (2)^\#, \dots, (n{+}1)^\#, 1\}$, and $\mathrm{Pos}(c_n)$ is $\{(1), \dots, (n)\}$. For $t = \bullet((\bullet\bullet)\bullet)$ (the last tree in Figure 3), we find $\mathrm{Dyad}(t) = \{0, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, 1\}$, hence $\mathrm{Dyad}(t) = \{(1)^\#, (2)^\#, (2, 1)^\#, (3)^\#, 1\}$, and $\mathrm{Pos}(t) = \{(1), (2), (2, 1)\}$.

Lemma 1.4. *Every tree t is determined by the set of positions* $\mathrm{Pos}(t)$.

Proof. An obvious induction shows that t is determined by $\mathrm{Dyad}(t)$. So the only problem is that, in $\mathrm{Pos}(t)$, the last two elements of $\mathrm{Dyad}(t)$ are forgotten. Now the last element of $\mathrm{Dyad}(t)$ is always 1, and an induction shows that the forelast one is $(n + 1)^\#$, where n is maximal such that (n) belongs to $\mathrm{Pos}(t)$ (*e.g.*, $(3)^\#$, *i.e.*, $\frac{3}{4}$, in the example above). □

Remark 1.5. Instead of using $\mathrm{Dyad}(t)$, we can attribute to each node in a binary tree an address that is a sequence of positive integers as in Figure 16 below; then $\mathrm{Pos}(t)$ consists of the addresses of the leaves in t, up to removing the last address, diminishing by 1 all non-initial factors and removing the final 0's in each sequence. Our notational convention may seem strange at first, because the initial and non-initial entries in a position are not treated similarly in the dyadic realization: the former is diminished by 1, the latter are not. A more homogeneous definition would force either to index positions starting from 0—and therefore numbering the braid generators σ_i from 0, which is unusual—or to identify s with $(s, 1)$ and not with $(s, 0)$—which is not intuitive.

1.3. **Parenthesized braid diagrams.** The diagrams we consider are constructed from two series of elementary diagrams indexed by letters $\sigma_i^{\pm 1}$ and $a_i^{\pm 1}$, and, therefore, a diagram will be specified using a word on these letters. In the sequel, such a word is called a σ,a-word, or, simply, a *word*. A word containing only letters $\sigma_i^{\pm 1}$ (*resp.* $a_i^{\pm 1}$) will be called a σ-*word* (*resp.* an a-*word*). Our aim is now to construct a parenthesized diagram $\mathcal{D}_t(w)$ for w a word and t a large enough tree, exactly as the ordinary diagram $\mathcal{D}_n(w)$ is defined for w a word in the letters $\sigma_i^{\pm 1}$ and n a large enough integer. For t of size $n + 1$, hence defining n positions, $\mathcal{D}_t(w)$ consists of n strands that connect the positions of $\mathrm{Pos}(t)$, considered as embedded in the unit interval, to n new positions.

If $[x, y)$ and $[x', y')$ are subintervals of $[0, 1)$, we say that we connect $[x, y)$ to $[x', y')$ *homothetically* to mean that each point $(z, 0)$ in $[x, y) \times \{0\}$ is connected to the point $(z', 1)$ of $[x', y') \times \{1\}$ that satisfies $(z' - x')/(z' - y') = (z - x)/(z - y)$.

Definition 1.6. (Figure 4) For t a tree of right height at least $i + 1$, the diagram $\mathcal{D}_t(\sigma_i)$ homothetically connects $[(i)^\#, (i + 1)^\#)$ with $[(i + 1)^\#, (i + 2)^\#)$, then $[(i + 1)^\#, (i + 2)^\#)$ with $[(i)^\#, (i + 1)^\#)$ with strands crossing under those of the previous family, and, finally, $[(k)^\#, (k + 1)^\#)$ with itself for $k \neq i, i + 1$.

The diagram $\mathcal{D}_t(a_i)$ homothetically connects $[(k)^\#, (k + 1)^\#)$ with itself for $k < i$, then $[(i)^\#, (i + 1)^\#)$ with $[(i)^\#, (i, 1)^\#)$, next $[(i + 1)^\#, (i + 2)^\#)$ with $[(i, 1)^\#, (i + 1)^\#)$, and, finally, $[(k)^\#, (k + 1)^\#)$ with $[(k - 1)^\#, (k)^\#)$ for $k > i + 1$.

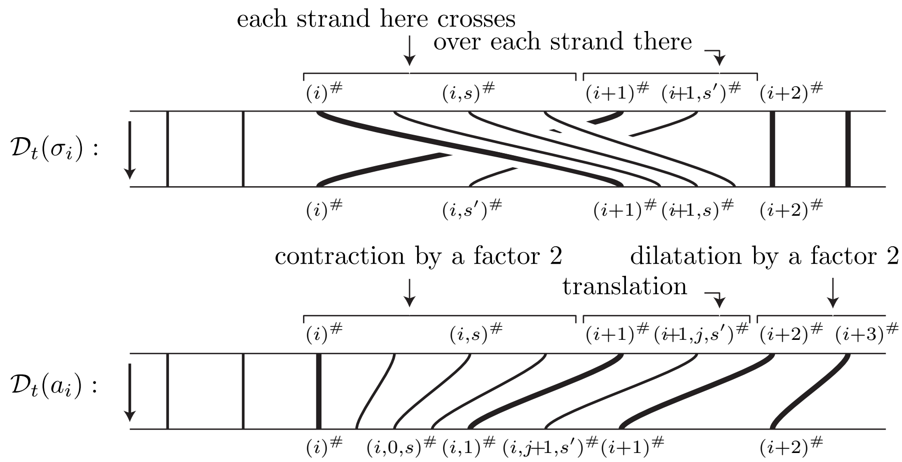


FIGURE 4. The diagrams $\mathcal{D}_t(\sigma_i)$ and $\mathcal{D}_t(a_i)$: in $\mathcal{D}_t(\sigma_i)$, the positions coming from $[(i)^\#, (i{+}1)^\#)$ and from $[(i{+}1)^\#, (i{+}2)^\#)$ are exchanged, with a contraction/dilatation factor 2 due to the dyadic realization; in $\mathcal{D}_t(a_i)$, the positions in $[(i)^\#, (i + 1)^\#)$ are contracted by 2, those in $[(i + 1)^\#, (i + 2)^\#)$ are translated to the left, and those in $[(k)^\#, (k + 1)^\#)$ are translated to the left and dilated by 2. In terms of positions, $\mathcal{D}_t(\sigma_i)$ exchanges (i, s) and $(i{+}1, s)$ for every s, while $\mathcal{D}_t(a_i)$ connects (i, s) to $(i, 0, s)$, then $(i + 1, j, s)$ to $(i, j + 1, s)$, and (k, s) to $(k - 1, s)$ for $k \geqslant i + 2$.

In contrast to the case of B_∞, the diagrams $\mathcal{D}_t(\sigma_i)$ or $\mathcal{D}_t(a_i)$ so defined cannot be carelessly stacked since the final positions of the strands need not coincide with the initial ones. Now, the changes correspond to an easily described (partial) action on trees.

Definition 1.7. (Figure 5) For t a tree, the unique sequence of trees $(t_1, \dots, t_n)$ such that t factorizes as $t_1(t_2(\dots(t_n\bullet)\dots))$ is called the *(right) decomposition* of t, and denoted by $\mathrm{dec}(t)$. For t a tree with $\mathrm{dec}(t) = (t_1, \dots, t_n)$ with $n > i$, we define the trees $t \bullet \sigma_i$ and $t \bullet a_i$ by:

$$\mathrm{dec}(t \bullet \sigma_i) = (t_1, \dots, t_{i-1}, t_{i+1}, t_i, t_{i+2}, \dots, t_n), \tag{1}$$

$$\mathrm{dec}(t \bullet a_i) = (t_1, \dots, t_{i-1}, t_i t_{i+1}, t_{i+2}, \dots, t_n). \tag{2}$$

Then, one inductively defines $t \bullet w$ for w a word so that $t \bullet w^{-1} = t'$ is equivalent to $t' \bullet w = t$ and $t \bullet (w_1 w_2)$ is equal to $(t \bullet w_1) \bullet w_2$.

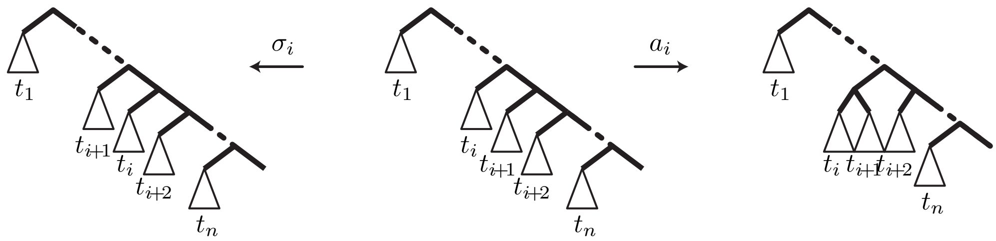


FIGURE 5. Action of σ_i and a_i on a tree: σ_i switches the ith and the $(i+1)$st factors in the right decomposition, while a_i glues them.

The definition implies that the final positions of the strands in $\mathcal{D}_t(\sigma_i)$ and $\mathcal{D}_t(a_i)$ are $\mathrm{Pos}(t \bullet \sigma_i)$ and $\mathrm{Pos}(t \bullet a_i)$, respectively. Completing the construction of the diagram $\mathcal{D}_t(w)$ is now obvious.

Definition 1.8. The diagrams $\mathcal{D}_t(\sigma_i^{-1})$ and $\mathcal{D}_t(a_i^{-1})$ are defined to be the mirror images of $D_{t\bullet\sigma_i}(\sigma_i)$ and $D_{t\bullet a_i}(a_i)$, respectively. Then, for w a word and t a binary tree such that $t \bullet w$ is defined, the *parenthesized braid diagram* $\mathcal{D}_t(w)$ is inductively defined by the rule that, if w is xw' where x is one of $\sigma_i^{\pm 1}$, $a_i^{\pm 1}$, then $\mathcal{D}_t(w)$ is obtained by stacking $\mathcal{D}_t(x)$ over $D_{t\bullet x}(w')$.

An example is displayed in Figure 6. Ordinary braid diagrams are special cases of parenthesized braid diagrams: an n strand braid diagram is a diagram of the form $\mathcal{D}_t(w)$ where t is the right vine of size $n+1$ and w is a σ-word.

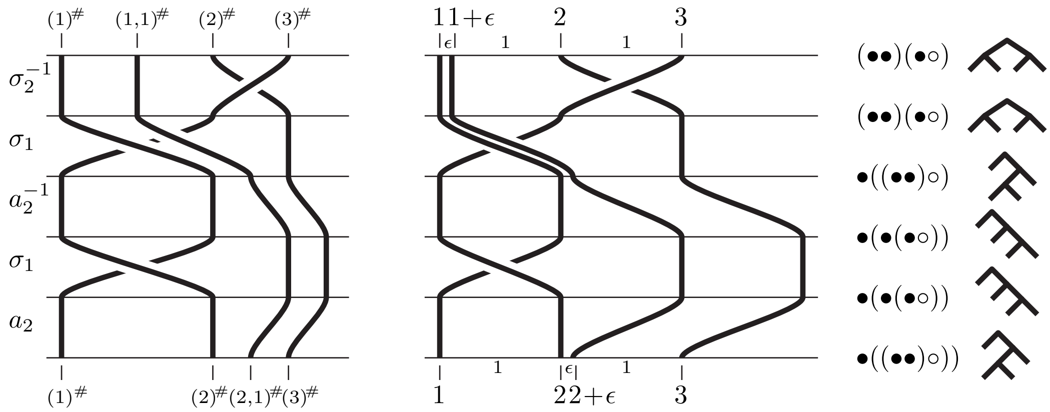


FIGURE 6. The dyadic realization of the diagram $\mathcal{D}_{(\bullet\bullet)(\bullet\bullet)}(\sigma_2^{-1}\sigma_1 a_2^{-1}\sigma_1 a_2)$ and its infinitesimal version, which (of course) is topologically equivalent; at each step, the corresponding set of positions is displayed, both as a parenthesized expression (the last node is marked $\circ$ because it contributes no position) and as a binary tree.

An easy induction gives:

Lemma 1.9. *For every tree t and every word w, the diagram $\mathcal{D}_t(w)$ is defined if and only if the tree $t \bullet w$ is, and, in this case, the final positions in $\mathcal{D}_t(w)$ are* $\mathrm{Pos}(t \bullet w)$.

1.4. **The group of parenthesized braids.** According to Artin's original construction, braids can be introduced as equivalence classes of braid diagrams. Viewing a diagram as the projection of a 3D-figure, one considers the equivalence relation corresponding to ambient isotopy of 3D-figures. As is well-known, this amounts to declaring equivalent those diagrams that can be connected by a finite sequence of Reidemeister moves of types II and III (Figure 7).

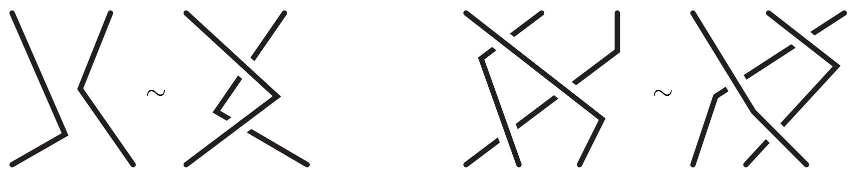


FIGURE 7. Reidemeister moves of type II (left) and III (right); the only requirement is that the endpoints remain fixed

From a topological point of view, parenthesized braid diagrams are just ordinary diagrams, so they are eligible for the same notion of equivalence:

Definition 1.10. Two parenthesized braid diagrams are declared *equivalent* if and only if they can be transformed one into the other by using Reidemeister moves of types II and III (and keeping the endpoints fixed).

Our aim is to make a group out of parenthesized braids—not only a groupoid, *i.e.*, a category, as in [1, 2]. As mentioned above, the problem is that we cannot compose arbitrary diagrams. It can be solved easily by introducing a completion procedure and defining the group operation on the completed objects. In the case of ordinary braids, the only parameter is the number of strands, and, in order to compose two diagrams $\mathcal{D}_{n_1}(w_1)$, $\mathcal{D}_{n_2}(w_2)$ with, say, $n_2 > n_1$, one first completes $\mathcal{D}_{n_1}(w_1)$ into the n_2-diagram $\mathcal{D}_{n_2}(w_1)$ obtained from $\mathcal{D}_{n_1}(w_1)$ by adding $n_2 - n_1$ unbraided strands on the right. The previous construction amounts to working with infinite diagrams. For each braid word w, the diagrams $\mathcal{D}_n(w)$ make an inductive system when n varies, and, defining $\mathcal{D}_\infty(w)$ to be the limit of this system, we obtain a well-defined product on infinite diagrams. Moreover, as the completion preserves equivalence, the product so defined induces a group structure, namely that of B_∞.

The procedure is similar for parenthesized braid diagrams, the appropriate ordering being the inclusion of trees viewed as sets of nodes.

Definition 1.11. For t, t' trees with $t \subseteq t'$, we denote by $c_{t,t'}$ the *completion* that maps $\mathcal{D}_t(w)$ to $\mathcal{D}_{t'}(w)$ whenever $\mathcal{D}_t(w)$ exists.

The explicit construction of parenthesized braid diagrams makes the completion procedure easy: as shown on Figure 8, the diagram $\mathcal{D}_{t'}(w)$ for $t' \supseteq t$ is obtained by keeping the existing strands, and adding new strands in $\mathcal{D}_t(w)$ that always lie half-way between their left and right neighbours—or 1 if there is no right neighbour. The only difference with ordinary diagrams is that there is in general more than one basic extension: the only way to extend the interval $\{1, 2, \dots, n\}$ into a bigger interval is to add $n+1$ while, in a tree t, each leaf can be split into a caret with two leaves, so there are $n+1$ basic extensions when t specifies n positions. As an induction shows, splitting the kth leaf amounts to doubling the kth strand.

The following observations gather what is needed for mimicking the construction of B_∞:

Lemma 1.12. *(i) For each word w, the system $(\mathcal{D}_t(w), c_{t,t'})$ is directed;*
(ii) Diagram concatenation induces a well-defined product on direct limits;
(iii) The completion maps are compatible with diagram equivalence.

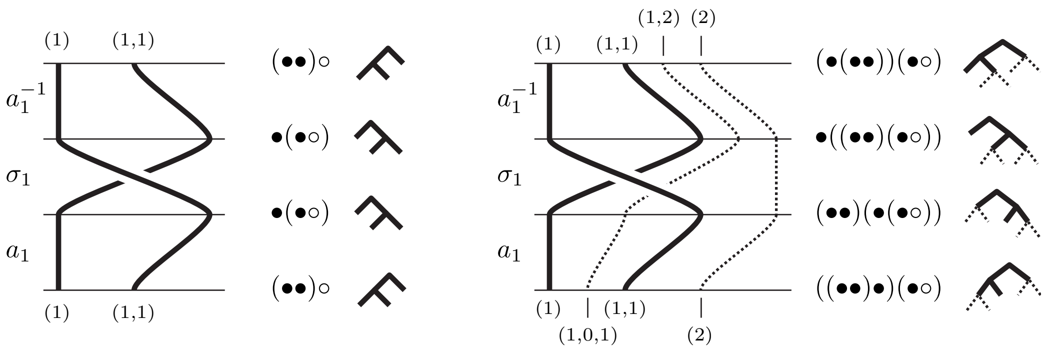


FIGURE 8. Completion of $\mathcal{D}_{(\bullet\bullet)\bullet}(a_1^{-1}\sigma_1 a_1)$ into $\mathcal{D}_{(\bullet(\bullet\bullet))(\bullet\bullet)}(a_1^{-1}\sigma_1 a_1)$: two more leaves in the tree, two more strands in the braid

Proof. For (i), for any two trees t_1, t_2, there exists a tree t that includes both t_1 and t_2, for instance the tree whose nodes are the union of the nodes in t_1 and t_2. For (ii), the completion $c_{t,t'}$ is compatible with the product in that, if $\mathcal{D}_t(w_1)$ and $\mathcal{D}_{t\bullet w_1}(w_2)$ exist so that $\mathcal{D}_t(w_1w_2)$ is defined, then, for each tree t' including t, the diagram $\mathcal{D}_{t'}(w_1w_2)$ exists and we have

$$\mathcal{D}_{t'}(w_1w_2) = \mathcal{D}_{t'}(w_1) \cdot \mathcal{D}_{t'\bullet w_1}(w_2).$$

Finally, (iii) follows from the description of completion in terms of strand addition. □

For each word w, let us define $\mathcal{D}_\bullet(w)$ to be the direct limit—actually, by construction, just the union—of the inductive system of all $\mathcal{D}_t(w)$'s. We call it an *infinite parenthesized braid diagram*. Then concatenation induces an everywhere defined product on infinite parenthesized braid diagrams, and isotopy induces a well-defined equivalence relation that is compatible with the previous product. Then the same argument as for ordinary braid diagrams gives:

Proposition 1.13. *Isotopy classes of infinite parenthesized braid diagrams make a group.*

Definition 1.14. The group of isotopy classes of infinite parenthesized braid diagrams is called the *group of parenthesized braids*, and denoted $B_\bullet$; its elements are called *parenthesized braids*.

1.5. **Relations in $B_\bullet$.** By construction, the group $B_\bullet$ is generated by the elements σ_i and a_i. An obvious task is to look for a presentation in terms of these elements. For the moment, we just observe that certain relations are satisfied in $B_\bullet$. That these relations make a presentation of $B_\bullet$ will be established in Section 3 below.

Lemma 1.15. *For $i \geqslant 1$ and $j \geqslant i+2$, the following relations induce diagram isotopies, hence equalities in $B_\bullet$:*

$$(3)\qquad \begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i, \quad \sigma_i a_j = a_j\sigma_i, \quad a_i a_{j-1} = a_j a_i, \quad a_i\sigma_{j-1} = \sigma_j a_i, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \sigma_{i+1}\sigma_i a_{i+1} = a_i\sigma_i, \quad \sigma_i\sigma_{i+1}a_i = a_{i+1}\sigma_i. \end{cases}$$

Proof. The graphical verification is given in Figure 9. □

Relations (3) include the standard braid relations, as well as the relations $a_ia_j = a_{j-1}a_i$ for $j \geqslant i+2$, which correspond to the standard presentation of Thompson's group F up to the change of name $a_i = x_{i-1}^{-1}$. In order to subsequently prove that (3) gives a presentation of $B_\bullet$, it is convenient to introduce the abstract group presented by these relations.

Definition 1.16. We denote by σ_* and a_* the families of all σ_i's and of all a_i's, and by $R_\bullet$ the relations (3). We define $\widetilde{B}_\bullet$ to be the group $\langle a_*, \sigma_* \,;\, R_\bullet\rangle$.

Lemma 1.15 states that the identity mapping on σ_* and a_* induces a surjective morphism of $\widetilde{B}_\bullet$ onto $B_\bullet$. One of our aims will be to prove that this morphism is an isomorphism.

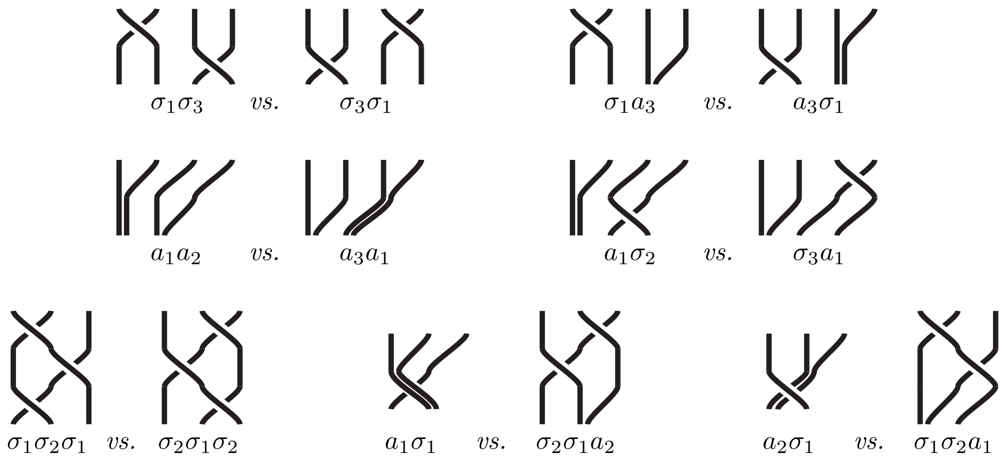


FIGURE 9. The relations of $R_\bullet$ and the corresponding diagrams isotopies (here in infinitesimal realization)

2. Algebraic properties of the group $\widetilde{B}_\bullet$

A number of algebraic properties of the group $\widetilde{B}_\bullet$ can be deduced from its explicit presentation, as we shall easily see using a specific combinatorial method called word reversing. The main results we prove are that $\widetilde{B}_\bullet$ is a group of left fractions, that it is torsion-free, and that it contains copies of the braid group B_∞ as well as of Thompson's group F.

2.1. The word reversing technique. In order to study the group $\widetilde{B}_\bullet$, we resort to general algebraic tools developed in [14, 16] and connected with Garside's seminal work [22]. This combinatorial method applies to monoid presentations and it is relevant for establishing properties like cancellativity or embeddability in a group of fractions.

For X a nonempty set (of letters), we call X*-word* a word made of letters from X, and $X^{\pm}$*-word* a word made of letters from $X \cup X^{-1}$, where X^{-1} is a disjoint copy of X containing one letter x^{-1} for each x in X. Then X-words are called positive, and we say that a group presentation (X, R) is *positive* if R exclusively consists of relations $u = v$ with u, v nonempty positive words. We use $\langle X\,; R\rangle$ for the group and $\langle X\,; R\rangle^+$ for the monoid defined by (X, R). Note that the presentation $(a_*, \sigma_*, R_\bullet)$ is positive.

Definition 2.1. [14, 16] Let (X, R) be a positive group presentation, and w, w' be $X^{\pm}$-words. We say that w is *right R-reversible* to w', denoted $w \curvearrowright_R w'$, if w' can be obtained from w using finitely many steps consisting either in deleting some length 2 subword $x^{-1}x$, or in replacing a length 2 subword $x^{-1}y$ by a word vu^{-1} such that $xv = yu$ is a relation of R.

Right R-reversing uses the relations of R to push the negative letters (those in X^{-1}) to the right and the positive letters (those in X) to the left by iteratively reversing $-+$ patterns into $+-$ patterns. Note that deleting $x^{-1}x$ enters the general scheme if we assume that, for every letter x in X, the trivial relation $x = x$ belongs to R.

Left R-reversing is defined symmetrically: the basic step consists in deleting a subword xx^{-1}, or replacing a subword xy^{-1} with $v^{-1}u$ such that $vx = uy$ is a relation of R.

Example 2.2. Let us consider the presentation $(a_*, \sigma_*, R_\bullet)$, and let w be the word $\sigma_4^{-1}a_2\sigma_2^{-1}a_1$. Then w contains two $-+$-subwords, namely $\sigma_4^{-1}a_2$ and $\sigma_2^{-1}a_1$. So there are two ways of starting a right reversing from w: replacing $\sigma_4^{-1}a_2$ with $a_2\sigma_3^{-1}$, which is legal as $\sigma_4 a_2 = a_2\sigma_3$ is a relation of $R_\bullet$, or replacing $\sigma_2^{-1}a_1$ with $\sigma_1 a_2\sigma_1^{-1}$, owing to the relation $\sigma_2(\sigma_1 a_2) = a_1\sigma_1$. The reader can check that, in any case, iterating the process leads in four steps to $a_2\sigma_1\sigma_2 a_3\sigma_2^{-1}\sigma_1^{-1}$. The latter word is terminal since it contains no $-+$ subword. It is helpful to visualize the process using a planar diagram similar to a Van Kampen diagram as shown in Figure 10.

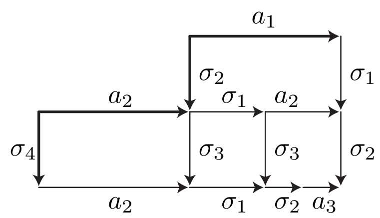


FIGURE 10. Right reversing diagram for $\sigma_4^{-1}a_2\sigma_2^{-1}a_1$: one starts with a staircase labelled $\sigma_4^{-1}a_2\sigma_2^{-1}a_1$ by drawing a vertical x-labelled arrow for each letter x^{-1}, and an horizontal y-labelled arrow for each positive letter y. Then, when $x^{-1}y$ is replaced with vu^{-1}, we complete the open pattern corresponding to $x^{-1}y$ into a square by adding horizontal v-labelled arrows and vertical u-labelled arrows.

If $xu = yv$ is a relation of R, then $x^{-1}y$ and vu^{-1} are R-equivalent, hence $w \curvearrowright_R w'$ implies that w and w' represent the same element of $\langle X\,;R\rangle$. A slightly more careful argument shows that, if u, v, u', v' are positive words, then $u^{-1}v \curvearrowright_R v'u'^{-1}$ implies that uv' and vu' represent the same element of $\langle X\,;R\rangle^+$. So, in particular, if u, v are positive words, $u^{-1}v \curvearrowright_R \varepsilon$ (the empty word) implies that u and v represent the same element of $\langle X\,;R\rangle^+$. The converse need not be true in general, but the interesting case is when this happens:

Definition 2.3. [16] A positive presentation (X, R) is said to be *complete for right reversing* if right reversing always detects positive equivalence, in the sense that, for all X-words u, v, one has $u^{-1}v \curvearrowright_R \varepsilon$ whenever u and v represent the same element of $\langle X\,;R\rangle^+$.

Symmetrically, we say that $(X; R)$ is complete for left reversing if uv^{-1} is left R-reversible to ε whenever u and v represent the same element of $\langle X\,;R\rangle^+$. The point is that there exists a tractable criterion for recognizing whether a given presentation is complete for reversing—or for adding new relations if it is not.

Definition 2.4. A positive presentation (X, R) is said to be *homogeneous* if there exists a R-invariant mapping λ from X-words to nonnegative integers such that $\lambda(x) \geqslant 1$ holds for every x in X, and $\lambda(uv) \geqslant \lambda(u) + \lambda(v)$ holds for all X-words u, v.

If all relations in R preserve the length of the words, then the length satisfies the requirements for the function λ and the presentation is homogeneous.

Proposition 2.5. [16] *A homogeneous positive presentation (X, R) is complete for right reversing if and only if the following condition holds for each triple (x, y, z) of letters:*

$$x^{-1}yy^{-1}z \curvearrowright_R vu^{-1} \quad \text{with } u, v \text{ positive implies} \quad v^{-1}x^{-1}zu \curvearrowright_R \varepsilon. \tag{4}$$

Condition (4) is called the *right cube condition for* (x, y, z). Of course, a symmetric left cube condition guarantees completeness for left reversing. We shall see now that the presentation $(a_*, \sigma_*; R_\bullet)$ is eligible for the previous criterion.

Lemma 2.6. *The presentation $(a_*, \sigma_*; R_\bullet)$ is homogeneous.*

Proof. The relations $\sigma_i\sigma_{i+1}a_i = a_{i+1}\sigma_i$ and $\sigma_{i+1}\sigma_i a_{i+1} = a_i\sigma_i$ do not preserve the length, so the latter cannot be used directly. Instead we construct a twisted length function λ so that, in $\lambda(w)$, each letter a_i contributes 1, but σ_i contributes nn', where n and n' are the numbers of strands involved in the diagram $\mathcal{D}_c(w)$ for c a sufficiently large right vine. Formally, we first define an action of positive words on sequences of integers by:

$$(\dots, n_{i-1}, n_i, n_{i+1}, n_{i+2}, \dots) \bullet a_i = (\dots, n_{i-1}, n_i + n_{i+1}, n_{i+2}, \dots),$$
$$(\dots, n_{i-1}, n_i, n_{i+1}, n_{i+2}, \dots) \bullet \sigma_i = (\dots, n_{i-1}, n_{i+1}, n_i, n_{i+2}, \dots).$$

Then n_i is the number of strands near position i, *i.e.*, corresponding to positions (i, s), in $\mathcal{D}_c(w)$, and the action is compatible with the relations of $R_\bullet$. Then, for w a positive word, we put $\lambda_\bullet(a_i, w) = 1$ and $\lambda_\bullet(\sigma_i, w) = n_i n_{i+1}$ for $(1, 1, \dots) \bullet w = (n_1, \dots, n_p)$. Finally, we define $\lambda(w) = \sum_k \lambda_\bullet(w(k), w_k)$, where $w(k)$ denotes the kth letter in w and w_k denotes the length $k-1$ prefix of w. Then λ witnesses that $(a_*, \sigma_*, R_\bullet)$ is homogeneous. □

Lemma 2.7. *The presentation $(a_*, \sigma_*; R_\bullet)$ satisfies the right and the left cube conditions for each triple of letters.*

Proof. As there are infinitely many letters, infinitely many cases are to be considered. However, it is clear that only the mutual distance of the indices matter, and, therefore, only finitely many types occur. The verification is easy, and we postpone it to an appendix. □

Applying the criterion of Proposition 2.5, we deduce:

Proposition 2.8. *The presentation $(a_*, \sigma_*; R_\bullet)$ is complete for both right and left reversing.*

2.2. **The monoid $\widetilde{B}_\bullet^+$.** Once the presentation $(a_*, \sigma_*, R_\bullet)$ is known to be complete for reversing, a number of results can be established easily. We begin with results involving the monoid presented by the relations $R_\bullet$.

Definition 2.9. We denote by $\widetilde{B}_\bullet^+$ the monoid $\langle a_*, \sigma_* \,; R_\bullet \rangle^+$.

The elements of the monoids $\widetilde{B}_\bullet^+$ are represented by positive words, and, by definition of completeness, two such words u, v represent the same element in $\widetilde{B}_\bullet^+$ if and only $u^{-1}v$ is right $R_\bullet$-reversible to the empty word, if and only if uv^{-1} is left $R_\bullet$-reversible to the empty word. Let us begin with cancellativity. The following criterion tells us that, whenever the presentation is complete, the monoid is cancellative provided there is no obvious obstruction.

Lemma 2.10. [16] *Assume that (X, R) is a positive presentation that is complete for right reversing. Then $\langle X \,; R \rangle^+$ is left cancellative whenever R contains no relation of the form $xu = xv$.*

There is no relation of the form $a_i u = a_i v$, $\sigma_i u = \sigma_i v$, $u a_i = v a_i$, $u \sigma_i = v \sigma_i$ in $R_\bullet$, so, using the previous criterion and its symmetric counterpart, we deduce:

Proposition 2.11. *The monoid $\widetilde{B}_\bullet^+$ admits left and right cancellation.*

Let us now consider common multiples. Say that z is a least common right multiple, or right lcm, of two elements x, y in a monoid M if z is a right multiple of x and y, *i.e.*, $z = xx' = yy'$ holds for some x', y', and every common right multiple of x and y is a right multiple of z.

Lemma 2.12. [16] *Assume that (X, R) is a positive presentation that is complete for right reversing. Then a sufficient condition for any two elements admitting a common right multiple to admit a right lcm is that, for all x, y in X, there is at most one relation of the form $xu = yv$ in R. In that case, for all X-words u, v, the word $u^{-1}v$ is right reversible to a word of the form $v'u'^{-1}$ with u', v' positive if and only if the elements represented by u and v in $\langle X \,; R \rangle^+$ admit a common right multiple, and then uv' represents the right lcm of these elements.*

The presentation $(a_*, \sigma_*, R_\bullet)$ is eligible for the previous criterion, and we deduce:

Proposition 2.13. *Any two elements of the monoid $\widetilde{B}_\bullet^+$ that admit a common right (resp. left) multiple admit a right (resp. left) lcm.*

Standard arguments imply:

Corollary 2.14. *Any two elements of the monoid $\widetilde{B}_\bullet^+$ admit a left and a right gcd.*

It remains to study whether common multiples do exist in $\widetilde{B}_\bullet^+$. For right multiples, the answer is negative: Lemma 2.12 tells us that the elements a_1 and a_2 admit a common right multiple in $\widetilde{B}_\bullet^+$ if and only if the right reversing of the word $a_1^{-1}a_2$ leads in a finite number of steps to some positive–negative word. As there is no relation of the form $a_1u = a_2v$ in $R_\bullet$, this cannot happen, and, therefore, a_1 and a_2 have no common right multiple in $\widetilde{B}_\bullet^+$. The situation is different for left multiples. In order to describe it, we need some notation.

Definition 2.15. For w a σ-word and k a positive integer, we denote by $w[k]$ the initial position of the strand that finishes at position k in the braid diagram $\mathcal{D}(w)$, and by $\mathrm{db}_k(w)$ the braid word that encodes the diagram obtained from $\mathcal{D}(w)$ by doubling the strand starting at position k. Similar notations are used for braids, which is legal as the needed compatibilities are satisfied.

Thus we have $\varepsilon[k] = k$ and $\mathrm{db}_k(\varepsilon) = \varepsilon$ for every k, and

$$\sigma_i[k] = \begin{cases} k & \text{for } k \neq i, i+1, \\ i+1 & \text{for } k = i, \\ i & \text{for } k = i+1, \end{cases} \qquad \mathrm{db}_k(\sigma_i) = \begin{cases} \sigma_{i+1} & \text{for } k < i, \\ \sigma_{i+1}\sigma_i & \text{for } k = i, \\ \sigma_i\sigma_{i+1} & \text{for } k = i+1, \\ \sigma_i & \text{for } k > i+1, \end{cases} \tag{5}$$

$$w[k] = w_1[w_2[k]], \qquad \mathrm{db}_k(w) = \mathrm{db}_k(w_1) \cdot \mathrm{db}_{w_1^{-1}[k]}(w_2) \qquad \text{for } w = w_1w_2. \tag{6}$$

Lemma 2.16. *Left $R_\bullet$-reversing always terminates in finitely many steps.*

Proof. The result is not *a priori* obvious as the length of the words appearing during the reversing may increase. By Garside's theory, any two elements in the braid monoid B_∞^+ admit a common left multiple, and, therefore, the left reversing of any word uv^{-1} with u, v positive σ-words terminates in finitely many steps. The same is true for a-words, since, in this case, the length cannot increase. The only remaining case is that of mixed words involving both types of letters. Now, in this case, we can describe the result of reversing explicitly. Indeed, we claim that, for every positive σ-word w and every positive integer k,

$$w \cdot a_k^{-1} \text{ is left } R_\bullet\text{-reversible to } a_{w[k]}^{-1} \cdot \mathrm{db}_{w[k]}(w). \tag{7}$$

We use induction on w. For $w = \sigma_i$, one easily checks (7) in the various cases. For instance, $\sigma_1 a_1^{-1}$ is left reversible to $a_2^{-1}\sigma_2\sigma_1$, and we have $\sigma_1[1] = 2$ and $\mathrm{db}_2(\sigma_1) = \sigma_1\sigma_2$. Then, for $w = w_1w_2$, using the definition of left reversing and the hypothesis that (7) holds for w_1 and w_2, we obtain that $w_1w_2a_k^{-1}$ is left reversible to $w_1a_{w_2[k]}^{-1}\mathrm{db}_{w_2[k]}(w_2)$, and then to $a_{w_1[w_2[k]]}^{-1}\mathrm{db}_{w_1[w_2[k]]}(w_1)\mathrm{db}_{w_2[k]}(w_2)$, which, by (6), is $a_{w[k]}^{-1}\mathrm{db}_{w[k]}(w)$. ☐

Applying Lemma 2.12, we deduce:

Proposition 2.17. *Any two elements in the monoid $\widetilde{B}_\bullet^+$ admit a left lcm.*

Another merit of word reversing is to make it easy to recognize what we can call parabolic submonoids (and, similarly, subgroups).

Lemma 2.18. *Assume that (X, R) is a positive presentation that is complete for left reversing, and X_0 is a subset of X. Let R_0 be the set of all relations $vx = uy$ in R with $x, y \in X_0$. If all words occurring in R_0 are X_0-words, the submonoid of $\langle X\,;R\rangle^+$ generated by X_0 admits the presentation $\langle X_0\,;R_0\rangle^+$.*

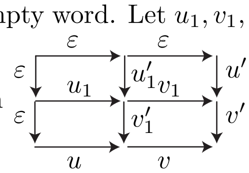
Proof. The point is to prove that, if u, v are R-equivalent X_0-words, then u and v also are R_0-equivalent, *i.e.*, no relation in $R \setminus R_0$ is neeeded to prove their equivalence. Now, by completeness, u and v being R-equivalent implies that vu^{-1} is left R-reversible to ε. The hypothesis on R_0 implies that only letters from X_0 appear during the reversing process. Therefore, the latter is an R_0-reversing, and u and v are R_0-equivalent. □

We denote by F^+ the monoid with presentation $\langle a_* \,;\, a_i a_j = a_{j-1} a_i \text{ for } j \geqslant i+2\rangle^+$, and call it *Thompson's monoid.*

Proposition 2.19. *The submonoid of $\widetilde{B}_\bullet^+$ generated by σ_* is (isomorphic to) the braid monoid B_∞^+, while the submonoid generated by a_* is (isomorphic to) Thompson's monoid F^+. Each element of $\widetilde{B}_\bullet^+$ admits a unique decomposition in $B_\infty^+ \times F^+$. The monoid $\widetilde{B}_\bullet^+$ is the Zappa-Szép product of B_∞^+ and F^+ associated with the crossed product defined for $\beta \in B_\infty^+$ and $k \geqslant 1$ by*

$$a_k \cdot \beta = \mathrm{db}_k(\beta) \cdot a_{\beta^{-1}[k]}. \tag{8}$$

Proof. An inspection of the relations in $R_\bullet$ shows that the families σ_* and a_* are eligible for the criterion of Lemma 2.18, and the first part of the proposition follows. We henceforth identify B_∞^+ and F^+ with the subgroups of $\widetilde{B}_\bullet^+$ generated by σ_* and a_*, respectively.

Formula (8) is a direct consequence of (6), and, by a straightforward induction, it implies $\widetilde{B}_\bullet^+ = B_\infty^+ \cdot F^+$. So the only point to prove is the uniqueness of the decomposition in $B_\infty^+ \times F^+$. Assume that uv and $u'v'$ are $R_\bullet$-equivalent, where u, u' are σ-words and v, v' are a-words. By completeness, this means that $uvu'^{-1}v'^{-1}$ is left reversible to the empty word. Let u_1, v_1, u_1', v_1' be the intermediate words appearing in the reversing, as shown in 

. As v and v' are positive a-words, so are v_1 and v_1'. By (6), the letters a_k^{-1} never vanish when they cross σ_i's in a left reversing. Hence the only possibility for $uv_1'^{-1}$ to reverse to a positive word u_1 is that v_1' is empty. Similarly, v_1 must be empty. As v_1 and v_1' are empty, v and v' are $R_\bullet$-equivalent. On the other hand, v_1 and v_1' being empty implies $u_1 = u$ and $u_1' = u'$, so the hypothesis that $u_1 u_1'^{-1}$ reverses to ε implies that u and u' are $R_\bullet$-equivalent. (For general Zappa-Szép products, see [7]—or [29] where the name "'crossed product" is used.) □

2.3. **The group $\widetilde{B}_\bullet$.** It is now easy to deduce results about the group $\widetilde{B}_\bullet$.

Proposition 2.20. *(i) The monoid $\widetilde{B}_\bullet^+$ embeds in the group $\widetilde{B}_\bullet$, and the latter is a group of left fractions of $\widetilde{B}_\bullet^+$, i.e., every element of $\widetilde{B}_\bullet$ can be expressed as $x^{-1}y$ with x, y in $\widetilde{B}_\bullet^+$. Moreover, every element of $\widetilde{B}_\bullet$ can be expressed as $f^{-1}\beta^{-1}\gamma g$ with β, γ in B_∞^+ and f, g in F^+.*

(ii) The group $\widetilde{B}_\bullet^+$ is torsion free.

Proof. For (i), the monoid $\widetilde{B}_\bullet^+$ satisfies Ore's conditions on the left, *i.e.*, it is cancellative and any two elements admit a left lcm. The second decomposition follows from Proposition 2.19 and the equality $\widetilde{B}_\bullet^+ = B_\infty^+ \cdot F^+$. Point (ii) follows as every torsion element in the group of fractions of a monoid admitting lcm's is a conjugate of a torsion element of the monoid [17]. As $\widetilde{B}_\bullet^+$ has no torsion element but 1, the same holds in $\widetilde{B}_\bullet$. □

Word reversing solves the word problem for the group $\widetilde{B}_\bullet$.

Lemma 2.21. *A word w represents 1 in $\widetilde{B}_\bullet$ if and only if its double left $R_\bullet$-reversing ends up with an empty word, where double left reversing consists in left reversing w into $u^{-1}v$ with u, v positive, and then left reversing vu^{-1}.*

Proof. Lemma 2.16 guarantees that, for every word w, there exist positive words u, v such that w is left $R_\bullet$-reversible to $u^{-1}v$. Then w represents 1 in $\widetilde{B}_\bullet$ if and only if u and v represent the same element of $\widetilde{B}_\bullet$, hence the same element of $\widetilde{B}_\bullet^+$, as $\widetilde{B}_\bullet^+$ embeds in $\widetilde{B}_\bullet$. Now, by definition of completeness, the latter is true if and only if the left reversing of vu^{-1} ends up with ε. □

Then we have the following group version of Lemma 2.18 for presentation of subgroups. The point is that word reversing solves the word problem without introducing any xx^{-1} or $x^{-1}x$.

Lemma 2.22. *Assume that (X, R) is a positive presentation that is complete for left reversing and such that left reversing always terminates. Let X_0 be a subset of X, and let R_0 be the set of all relations $vx = uy$ in R with $x, y \in X_0$. If all words occurring in R_0 are X_0-words, the subgroup of $\langle X\,; R\rangle$ generated by X_0 admits the presentation $\langle X_0\,; R_0\rangle$.*

Proof. The hypotheses guarantee that an $X^{\pm}$-word represents 1 in the group $\langle X\,; R\rangle$ if and only if it can be transformed to ε by double left reversing. Now, as in the proof of Lemma 2.18, the hypotheses imply that all words appearing in a (double) reversing from an $X_0^{\pm}$-word are $X_0^{\pm}$-words. So, if such a word is left R-reversible to ε, it is also left R_0-reversible to ε, and it represents 1 in $\langle X_0\,; R_0\rangle$. □

Proposition 2.23. *The subgroup of $\widetilde{B}_\bullet$ generated by σ_* is (a copy of) the braid group B_∞, and the subgroup generated by a_* is (a copy of) Thompson's group F. These subgroups generate $\widetilde{B}_\bullet$, and their intersection is $\{1\}$.*

Proof. The argument is the same as for the submonoids, replacing Lemma 2.18 with Lemma 2.22. Then, by definition, $\widetilde{B}_\bullet$ is generated by the σ_i's and the a_i's, hence by the subgroups they generate (henceforth identified with B_∞ and F). Assume $z \in B_\infty \cap F$. Every element of F is a left fraction, so we have $z = f^{-1}f'$ for some f, f' in F^+. By Garside's theory, B_∞ is both a group of left and of right fractions of B_∞^+, so we also have $z = \beta\beta'^{-1}$ for some β, β' in B_∞^+. We deduce $\beta f = \beta' f'$ in $\widetilde{B}_\bullet^+$, and the uniqueness of the decomposition in $F^+ \times B_\infty^+$ (Proposition 2.19) implies $\beta = \beta'$ and $f = f'$. □

From now on, we consider B_∞ and F as subgroups of $\widetilde{B}_\bullet$. For future use, we insist that every element of $\widetilde{B}_\bullet$ can be represented by a word in which the $a_i^{\pm 1}$ letters are gathered.

Definition 2.24. A σ,a-word is called *tidy* if it consists of letters a_i^{-1}, followed by letters $\sigma_j^{\pm 1}$, followed by letters a_k.

Propositions 2.20 implies:

Corollary 2.25. *Every element of $\widetilde{B}_\bullet$ admits a tidy representative.*

3. The self-distributive structure on $B_\bullet$

Besides their group structure, parenthesized braids are equipped with another important algebraic structure, involving the self-distributivity law.

A non-trivial property of the braid group B_∞ is the existence of a binary operation that obeys the self-distributivity law $x(yz) = (xy)(xz)$. The importance of this exotic operation originates from the fact that each element of B_∞ generates a free subsystem with respect to the self-distributive operation, a property directly connected with the existence of a canonical ordering of B_∞ [15, 19]. In this section, we show that the self-distributivity properties of B_∞ extend to $B_\bullet$, in an even stronger form as the structure involves a second related operation that has no counterpart in the case of ordinary braids.

As an application, we deduce that the groups $B_\bullet$ and $\widetilde{B}_\bullet$ are isomorphic, *i.e.*, we show that the relations $R_\bullet$ of Lemma 1.15 make a presentation of $B_\bullet$.

3.1. The self-distributive bracket on $\widetilde{B}_\bullet$.

Definition 3.1. An *LD-system* is a set equipped with a binary operation $x, y \mapsto x\lceil y\rceil$ satisfying the left self-distributivity law

$$x\lceil y\lceil z\rceil\rceil = x\lceil y\rceil\lceil x\lceil z\rceil\rceil. \tag{9}$$

An *augmented* LD-system, or ALD-system, is an LD-system equipped with a second binary operation $\circ$ satisfying the mixed laws

$$x\lceil y\lceil z\rceil\rceil = (x \circ y)\lceil z\rceil \qquad \text{and} \qquad x\lceil y \circ z\rceil = x\lceil y\rceil \circ x\lceil z\rceil. \tag{10}$$

An LD-system is said to be *left cancellative* if all left translations are injective, *i.e.*, if $x\lceil y\rceil = x\lceil z\rceil$ implies $y = z$; it is called a *rack* [20] if all left translations are bijective, which means that there exists a binary operation $x, y \mapsto x\lfloor y\rfloor$ satisfying $x\lfloor x\lceil y\rceil\rfloor = x\lceil x\lfloor y\rfloor\rceil = y$.

A group equipped with $x\lceil y\rceil = xyx^{-1}$, $x\lfloor y\rfloor = x^{-1}yx$ and $x \circ y = xy$ is an augmented rack, always satisfying the additional law $x\lceil x\rceil = x$. On the other hand, Artin's group B_∞ is an LD-system when equipped with the operation

$$\beta\lceil\gamma\rceil = \beta \cdot \partial\gamma \cdot \sigma_1 \cdot \partial\beta^{-1}, \tag{11}$$

where ∂ is the endomorphism that maps σ_i to σ_{i+1} for each i. This operation can be seen as a sort of twisted conjugacy, and there are several ways of making the definition natural [15]. The braid bracket is very different from a group conjugacy in that $\beta\lceil\beta\rceil = \beta$ never holds. Observe that there is no way to augment the LD-system B_∞, as, for instance, $1\lceil 1\lceil 1\rceil\rceil = \beta\lceil 1\rceil$ would imply $\partial\beta = \sigma_1^{-1}\sigma_2^{-1}\beta\sigma_1$, which holds for no β in B_∞.

We shall see now that the braid bracket extends to $\widetilde{B}_\bullet$, and, moreover, it can be augmented. We begin with a preparatory result.

Definition 3.2. We denote by ∂ the *shift* that maps σ_i to σ_{i+1} and a_i to a_{i+1} for each i.

Lemma 3.3. *The mapping ∂ induces an injective endomorphism of the group $\widetilde{B}_\bullet$ into itself.*

Proof. As the shift mapping on positive integers is injective, ∂ induces an isomorphism of the group $\langle a_*, \sigma_* \,;R_\bullet\rangle$ into its image $\langle \partial(a_*, \sigma_*) \,; \partial R_\bullet\rangle$. Now the explicit form of the relations in $R_\bullet$ shows that $\partial R_\bullet$ is included in $R_\bullet$, and that the criterion of Lemma 2.22 is satisfied by $\partial(a_*, \sigma_*)$ and $\partial R_\bullet$. So the subgroup of $\widetilde{B}_\bullet$ generated by $\partial(a_*, \sigma_*)$ admits the presentation $\langle \partial(a_*, \sigma_*) \,; \partial R_\bullet\rangle$, and, therefore, ∂ is an isomorphism of $\widetilde{B}_\bullet$ onto the latter subgroup. $\square$

Definition 3.4. For x, y in $\widetilde{B}_\bullet$, we set

$$x\lceil y\rceil = x \cdot \partial y \cdot \sigma_1 \cdot \partial x^{-1}, \qquad \text{and} \qquad x \circ y = x \cdot \partial y \cdot a_1. \tag{12}$$

Proposition 3.5. *The set $B_\bullet$ equipped with the operations $\lceil\ \rceil$ and $\circ$ is an ALD-system. Furthermore, the bracket is left-cancellative, i.e., $x\lceil y\rceil = x\lceil z\rceil$ implies $y = z$.*

Proof. A simple verification:

$$\begin{aligned}
x\lceil y\rceil\lceil x\lceil z\rceil\rceil &= (x \cdot \partial y \cdot \sigma_1 \cdot \partial x^{-1})\lceil x \cdot \partial z \cdot \sigma_1 \cdot \partial x^{-1}\rceil \\
&= x \cdot \partial y \cdot \sigma_1 \cdot \partial x^{-1} \cdot \partial x \cdot \partial^2 z \cdot \sigma_2 \cdot \partial^2 x^{-1} \cdot \sigma_1 \cdot \partial^2 x \cdot \sigma_2^{-1} \cdot \partial^2 y^{-1} \cdot \partial x^{-1} \\
&=^{(*)} x \cdot \partial y \cdot \partial^2 z \cdot \sigma_1\sigma_2\sigma_1\sigma_2^{-1} \cdot \partial^2 y^{-1} \cdot \partial x^{-1} \\
&= x \cdot \partial y \cdot \partial^2 z \cdot \sigma_2\sigma_1 \cdot \partial^2 y^{-1} \cdot \partial x^{-1} =^{(*)} x\lceil y \cdot \partial z \cdot \sigma_1 \cdot \partial y^{-1}\rceil = x\lceil y\lceil z\rceil\rceil.
\end{aligned}$$

The reason for $(*)$ is that $\partial^2 x$ commutes with σ_1 for every x. For left cancellativity, $x\lceil y\rceil = x\lceil z\rceil$ implies $\partial y \cdot \sigma_1 = \partial z \cdot \sigma_1$, hence $\partial y = \partial z$, and, therefore, $y = z$ by Lemma 3.3.

Then, we find similarly:

$$\begin{aligned}
x\lceil y\lceil z\rceil\rceil &= x \cdot \partial y \cdot \partial^2 z \cdot \sigma_2\sigma_1 \cdot \partial^2 y^{-1} \cdot \partial x^{-1} = x \cdot \partial y \cdot a_1 a_1^{-1} \cdot \partial^2 z \cdot \sigma_2\sigma_1 \cdot a_2 a_2^{-1} \cdot \partial^2 y^{-1} \cdot \partial x^{-1} \\
&= (x \circ y) \cdot a_1^{-1} \cdot \partial^2 z \cdot \sigma_2\sigma_1 a_2 \cdot \partial(x \circ y)^{-1} = (x \circ y) \cdot \partial z \cdot a_1^{-1}\sigma_2\sigma_1 a_2 \cdot \partial(x \circ y)^{-1}
\end{aligned}$$

(because $a_1 \cdot \partial z = \partial^2 z \cdot a_1$ always holds)

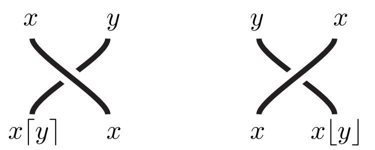
$$
\begin{aligned}
&= (x \circ y) \cdot \partial z \cdot \sigma_1 \cdot \partial(x \circ y)^{-1} = (x \circ y)\lceil z \rceil,\\
x\lceil y \circ z \rceil &= x \cdot \partial y \cdot \partial^2 z \cdot a_2\sigma_1 \cdot \partial x^{-1} = x\lceil y \rceil \cdot \partial x \cdot \sigma_1^{-1} \cdot \partial^2 z \cdot a_2\sigma_1 \cdot \partial x^{-1}\\
&= x\lceil y \rceil \cdot \partial x \cdot \partial^2 z \cdot \sigma_1^{-1} a_2\sigma_1 \cdot \partial x^{-1} = x\lceil y \rceil \cdot \partial(x\lceil z \rceil) \cdot \partial^2 x \cdot \sigma_2^{-1}\sigma_1^{-1} a_2\sigma_1 \cdot \partial x^{-1}\\
&= x\lceil y \rceil \cdot \partial(x\lceil z \rceil) \cdot \partial^2 x \cdot a_1 \cdot \partial x^{-1} = x\lceil y \rceil \cdot \partial(x\lceil z \rceil) \cdot a_1 = x\lceil y \rceil \circ x\lceil z \rceil,
\end{aligned}
$$

which completes the proof. □

The self-distributive structure so constructed will be instrumental in the sequel.

3.2. **Diagram colouring.** We now come back to proving that the relations of Lemma 1.15 make a presentation of the group $B_\bullet$. The point is to establish that the canonical morphism of $\widetilde{B}_\bullet$ to $B_\bullet$ is injective. We shall do it by showing that, for any word w, the class of w in $\widetilde{B}_\bullet$ can be recovered from the isotopy class of any diagram $\mathcal{D}_t(w)$, which depends only on the class of w in $B_\bullet$. To this end, we appeal to diagram colourings.

The principle, which can be traced back at least to Alexander, is to fix a nonempty set S (the colours), to attribute colours from S to the initial positions in a braid diagram $\mathcal{D}$, and to push the colours along the strands. If the colours never change, the output colours are a permutation of the input colours, and we do not gain much information about the diagram. Now, assume that the set of colours S is equipped with two binary operations, say $x, y \mapsto x\lceil y \rceil$ and $x, y \mapsto x\lfloor y \rfloor$—the notation is chosen to suggest that $x\lceil y \rceil$ and $x\lfloor y \rfloor$ are images of y under x. We require that, when an x-coloured strand crosses over a y-coloured strand, then the colour of the latter becomes $x\lceil y \rceil$ or $x\lfloor y \rfloor$ according to the orientation of the crossing:



In this way, for each sequence of input colours and each braid diagram $\mathcal{D}$, one obtains a sequence of output colours, and some information about $\mathcal{D}$ can be obtained by comparing the input and output colours. One of the many facets of the deep connection between braids and self-distributivity is the following observation, whose graphical verification is easy, and which appears in different forms in [4, 25, 31, 15, 19]:

Lemma 3.6. *Assume that S is a rack. Then S-colourings are invariant under Reidemeister moves II and III in the sense that, for every diagram $\mathcal{D}$ and every sequence of input colours, the corresponding output colours depend only on the isotopy class of $\mathcal{D}$.*

In order to control colourings in our current framework, it is convenient to introduce coloured trees. If $\mathcal{D}$ is an ordinary n strand braid diagram, defining an S-colouring of $\mathcal{D}$ means attributing colours from S to the n input positions $1, \dots, n$, *i.e.*, choosing a sequence in S^n. Propagating the colours along the strands of $\mathcal{D}$ gives an output sequence that lives in S^n again. Parenthesized braid diagrams are similar, but the positions belong to $\mathbf{N}_\bullet$ rather than to $\mathbf{N}$, and they form a tree rather than a sequence. Hence the objects to consider are trees of S-coloured positions, *i.e.*, *S-coloured trees*, defined to be trees (of positions) in which colours from S are attributed to the leaves. We shall use bold letters like $\boldsymbol{t}$ for coloured trees.

Definition 3.7. For x in S, we denote by $\bullet_x$ the tree with one single x-coloured node. For $\boldsymbol{t}$ an S-coloured tree, we define the *skeleton* $\boldsymbol{t}^\dagger$ of $\boldsymbol{t}$ to be the uncoloured tree t obtained by forgetting the colours in $\boldsymbol{t}$; in this case, we say that $\boldsymbol{t}$ is a colouring *of* t.

Every S-coloured tree admits a unique decomposition as a product of $\bullet_x$ with x in S. In particular, the sequence of positions $1, \dots, n$ with the colours $x_1, \dots, x_n$, as used for an ordinary S-coloured n strand braid diagram, corresponds to the S-coloured right vine $\bullet_{x_1}(\bullet_{x_2}\dots(\bullet_{x_n}\bullet)\dots)$—as the last leaf encodes no position, we give it no colour; if needed, we may assume that some distinguished colour x_0 is fixed and identify an uncoloured tree with a tree uniformly coloured x_0.

Propagating S-colours along the strands of a parenthesized braid diagram $\mathcal{D}$ amounts to defining a partial action of $\mathcal{D}$ on S-coloured trees, since, assuming that t is the initial set of positions in $\mathcal{D}$ and t' is the final one, we can associate with every S-colouring of t an S-colouring of t' (Figure 11):

Definition 3.8. For $\mathcal{D}$ a parenthesized braid diagram with initial set of positions $\mathrm{Pos}(t)$ and $\boldsymbol{t}$ an S-colouring of t, we denote by $\boldsymbol{t} \bullet \mathcal{D}$ the S-coloured tree obtained by propagating the colours of $\boldsymbol{t}$ through $\mathcal{D}$. When $\mathcal{D}$ has the form $\mathcal{D}_t(w)$ for some word w, we write $\boldsymbol{t} \bullet w$ for $\boldsymbol{t} \bullet \mathcal{D}_t(w)$.

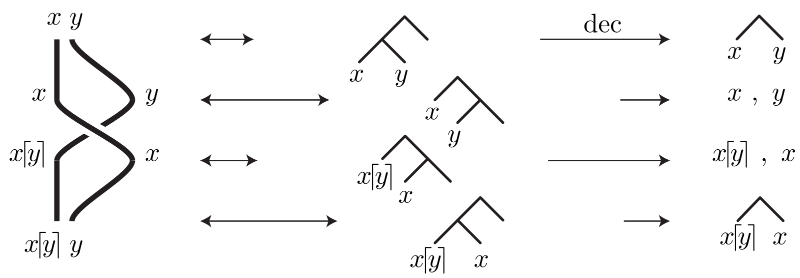


Figure 11. Correspondence between sets of S-coloured positions and S-coloured trees: here we start from (1) and (1, 1) coloured x and y, *i.e.*, from the coloured tree $(\bullet_x\bullet_y)\bullet$; then we go to (1) and (2) coloured x and y, *i.e.*, to $\bullet_x(\bullet_y\bullet)$, *etc.*; on the right, we show the decomposition of the trees, *i.e.*, the subtrees under the right branch, the last leaf excepted

It is easy to explicitly describe the action of σ_i and a_i on coloured trees.

Lemma 3.9. *Assume that* $\boldsymbol{t}$ *is a coloured tree with* $\mathrm{dec}(\boldsymbol{t}) = (\boldsymbol{t}_1, \dots, \boldsymbol{t}_n)$. *Then the coloured trees* $\boldsymbol{t} \bullet \sigma_i$ *and* $\boldsymbol{t} \bullet a_i$ *are defined for* $i < n$, *and we have then*

$$\mathrm{dec}(\boldsymbol{t} \bullet \sigma_i) = (\boldsymbol{t}_1, \dots, \boldsymbol{t}_{i-1}, \boldsymbol{t}_i\lceil\boldsymbol{t}_{i+1}\rceil, \boldsymbol{t}_i, \boldsymbol{t}_{i+2}, \dots, \boldsymbol{t}_n), \tag{13}$$

$$\mathrm{dec}(\boldsymbol{t} \bullet a_i) = (\boldsymbol{t}_1, \dots, \boldsymbol{t}_{i-1}, \boldsymbol{t}_i\boldsymbol{t}_{i+1}, \boldsymbol{t}_{i+2}, \dots, \boldsymbol{t}_n), \tag{14}$$

where $\boldsymbol{t}_i\lceil\boldsymbol{t}_{i+1}\rceil$ *denotes the tree obtained from* $\boldsymbol{t}_{i+1}$ *by replacing every colour* x *with the corresponding colour* $x_1\lceil\dots\lceil x_p\lceil x\rceil\rceil\dots\rceil$, *where* $x_1, \dots, x_p$ *form the left-to-right enumeration of the colours in* $\boldsymbol{t}_i$.

Proof. First, we observe that the rules of (13) and (14) extend those of (1) and (2): this is natural, as, when we forget the colours, we must find the previously defined action on families of positions, *i.e.*, on trees. So it only remains to look at colours. For (14), the result is clear as colours are not changed. As for (13), the result of applying σ_i is that each strand corresponding to $\boldsymbol{t}_{i+1}$ goes under all strands corresponding to $\boldsymbol{t}_i$, and it meets the latter from right to left: the first one corresponds to the rightmost position in $\boldsymbol{t}_i$, and the last one corresponds to the leftmost position in $\boldsymbol{t}_i$. Applying the rule for changes of colours at crossings, we deduce that the strand with initial colour x eventually gets the colour $x_1\lceil\dots\lceil x_p\lceil x\rceil\rceil\dots\rceil$. ☐

3.3. **Using left cancellative LD-systems.** Lemma 3.6 states that, if S is a rack, then, for each S-coloured tree $\boldsymbol{t}$, the tree $\boldsymbol{t} \bullet \mathcal{D}$ depends on the isotopy class of $\mathcal{D}$ only. It follows that, if two words w, w' are $R_\bullet$-equivalent and $\boldsymbol{t} \bullet w$ and $\boldsymbol{t} \bullet w'$ are defined, the latter are equal.

In the sequel, we shall consider a more general situation, namely when the set of colours is a left cancellative LD-system, but not necessarily a rack. In this case, all pairs of colours need not be eligible for negative crossings: we can still define $x\lfloor y\rfloor$ to be the unique element z satisfying $x\lceil z\rceil = y$ when it exists, but the operation $\lfloor\ \rfloor$ need not be everywhere defined. The following lemma gathers the results we need:

Lemma 3.10. *Let S be a left cancellative LD-system. Assume that $w_1, \dots, w_r$ are words and t is a tree such that $t \bullet w_k$ exists for each k. Then there exists at least one colouring $\boldsymbol{t}$ of t such that $\boldsymbol{t} \bullet w_k$ exists for every k.*

Proof. If S is a rack, *any* S-colouring is convenient, as the colours can always be propagated. When S is only supposed to be a left cancellative LD-system, we must be more careful. First, we observe that, if the word w is left $R_\bullet$-reversible to w', and $\boldsymbol{t} \bullet w'$ exists for some S-coloured tree $\boldsymbol{t}$, then $\boldsymbol{t} \bullet w$ exists as well, as can be checked by considering the various cases—the point is that left reversing creates no $\sigma_i^{-1}\sigma_i$. Hence, as every word is left reversible to a negative–positive word, it suffices to prove the result when each w_k is such a word. Moreover, positive words create no problem, so it is even sufficient to consider the case when each w_k is a negative word. Putting $v_k = w_k^{-1}$, our problem is to prove that, if $v_1, \dots, v_r$ are positive words, then there exist S-coloured trees $\boldsymbol{t}_1, \dots, \boldsymbol{t}_r$ such that $\boldsymbol{t}_k \bullet v_k$ exists and is equal to some tree $\boldsymbol{t'}$ independent of k. Now, by Proposition 2.17, the elements of $\widetilde{B}_\bullet^+$ represented by $v_1, \dots, v_r$ admit a left common multiple, hence there exist positive words $u_1, \dots, u_r$ such that the words $u_k v_k$ all are positively R-equivalent (*i.e.*, without introducing any negative letter) to some positive word w. Let t be a tree large enough to guarantee that $t \bullet w$ exists, and let $\boldsymbol{t}$ be any S-colouring of t. Put $\boldsymbol{t}_k = \boldsymbol{t} \bullet u_k$. Then, by construction, $\boldsymbol{t}_k \bullet v_k$ exists and is equal to $\boldsymbol{t} \bullet w$ for every k. ☐

Lemma 3.11. *Let S be a left cancellative LD-system. Assume that the parenthesized braid diagrams $\mathcal{D}_t(w)$ and $\mathcal{D}_t(w')$ are isotopic. Then there exists at least one S-colouring $\boldsymbol{t}$ of t such that $\boldsymbol{t} \bullet w$ and $\boldsymbol{t} \bullet w'$ exist and are equal.*

Proof. If S is a rack, we can take for $\boldsymbol{t}$ any S-colouring of t. Then the colours can be propagated without problem, *i.e.*, $\boldsymbol{t}\bullet w$ and $\boldsymbol{t}\bullet w'$ exist. The hypothesis that the diagrams are isotopic implies in particular that the final positions are the same, hence $t \bullet w = t \bullet w'$ holds. On the other hand, Lemma 3.6 guarantees that the sequences of output colours are the same in both diagrams, *i.e.*, the leaves of $\boldsymbol{t} \bullet w$ and $\boldsymbol{t} \bullet w'$ have the same colours. Hence $\boldsymbol{t} \bullet w$ and $\boldsymbol{t} \bullet w'$ are equal.

When S is only supposed to be a left cancellative LD-system, an arbitrary S-colouring need not be convenient. Now, the hypothesis that $\mathcal{D}_t(w)$ and $\mathcal{D}_t(w')$ are isotopic implies that there exists a finite sequence $w_1 = w$, $w_1, \dots, w_r = w'$ such that, for each k, the diagram $\mathcal{D}_t(w_{k+1})$ is obtained from $\mathcal{D}_t(w_k)$ by one Reidemeister move. By Lemma 3.10, there exists an S-colouring $\boldsymbol{t}$ of t such that $\boldsymbol{t} \bullet w_k$ is defined for each k. Now, the same argument as for Lemma 3.6 shows that the final colours in two adjacent diagrams are the same, hence in $\boldsymbol{t} \bullet w$ and $\boldsymbol{t} \bullet w'$, and we conclude as above. ☐

3.4. **Using $\widetilde{B}_\bullet$-colourings.** As $\widetilde{B}_\bullet$ equipped with its bracket is a left cancellative LD-system, we can use it to colour parenthesized braids. Here we use such colourings to answer the pending question of whether the relations $R_\bullet$ present $B_\bullet$. The key tool is a certain function that associates with every $\widetilde{B}_\bullet$-coloured tree a specific element of $B_\bullet$ constructed using the operation $\circ$.

Definition 3.12. (i) For $\boldsymbol{t}$ a $\widetilde{B}_\bullet$-coloured tree, we denote by $\mathrm{ev}(\boldsymbol{t})$ the $\circ$-*evaluation* of $\boldsymbol{t}$, *i.e.*, the image of $\boldsymbol{t}$ under the mapping inductively defined by

$$\mathrm{ev}(\bullet_x) = x \qquad \text{and} \qquad \mathrm{ev}(\boldsymbol{t}\boldsymbol{t'}) = \mathrm{ev}(\boldsymbol{t}) \circ \mathrm{ev}(\boldsymbol{t'}). \tag{15}$$

The definition is extended to uncoloured trees by identifying $\bullet$ with $\bullet_1$.

(ii) For $\boldsymbol{t}$ a $\widetilde{B}_\bullet$-coloured tree with $\mathrm{dec}(\boldsymbol{t}) = (\boldsymbol{t}_1, \dots, \boldsymbol{t}_n)$, we put

$$\mathrm{ev}^*(\boldsymbol{t}) = \mathrm{ev}(\boldsymbol{t}_1) \cdot \partial\mathrm{ev}(\boldsymbol{t}_2) \cdot \ldots \cdot \partial^{n-1}\mathrm{ev}(\boldsymbol{t}_n). \tag{16}$$

For instance, for t the right vine of size $n+1$, we have $\mathrm{ev}(t) = a_n a_{n-1} \dots a_1$, while $\mathrm{ev}((\bullet\bullet)\bullet)$ is a_1^2. We shall determine the action of the generators a_i and σ_i on the evaluation mapping ev^*. First we begin with an auxiliary result about ALD-systems.

Lemma 3.13. *Assume that S is an ALD-system. Then, for all S-coloured trees $\boldsymbol{t}, \boldsymbol{t}'$, we have*

$$\mathrm{ev}(\boldsymbol{t}\lceil \boldsymbol{t}'\rceil) = \mathrm{ev}(\boldsymbol{t})\lceil \mathrm{ev}(\boldsymbol{t}')\rceil. \tag{17}$$

Proof. We use induction on the cumuled sizes of $\boldsymbol{t}$ and $\boldsymbol{t}'$. If both $\boldsymbol{t}$ and $\boldsymbol{t}'$ have size 1, the result follows from the definition of $\boldsymbol{t}\lceil \boldsymbol{t}'\rceil$ directly. Otherwise, the definition gives

$$(\boldsymbol{t}_1\boldsymbol{t}_2)\lceil \boldsymbol{t}'\rceil = \boldsymbol{t}_1\lceil \boldsymbol{t}_2\lceil \boldsymbol{t}'\rceil\rceil \qquad \text{and} \qquad \boldsymbol{t}\lceil \boldsymbol{t}_1'\boldsymbol{t}_2'\rceil = (\boldsymbol{t}\lceil \boldsymbol{t}_1'\rceil)(\boldsymbol{t}\lceil \boldsymbol{t}_2'\rceil).$$

Applying the evaluation morphism, we deduce for $\boldsymbol{t} = \boldsymbol{t}_1\boldsymbol{t}_2$

$$\begin{aligned}\mathrm{ev}(\boldsymbol{t}\lceil \boldsymbol{t}'\rceil) &= \mathrm{ev}(\boldsymbol{t}_1\lceil \boldsymbol{t}_2\lceil \boldsymbol{t}'\rceil\rceil) = \mathrm{ev}(\boldsymbol{t}_1)\lceil \mathrm{ev}(\boldsymbol{t}_2\lceil \boldsymbol{t}'\rceil)\rceil \\ &= \mathrm{ev}(\boldsymbol{t}_1)\lceil \mathrm{ev}(\boldsymbol{t}_2)\lceil \mathrm{ev}(\boldsymbol{t}')\rceil\rceil = (\mathrm{ev}(\boldsymbol{t}_1)\circ \mathrm{ev}(\boldsymbol{t}_2))\lceil \mathrm{ev}(\boldsymbol{t}')\rceil = \mathrm{ev}(\boldsymbol{t})\lceil \mathrm{ev}(\boldsymbol{t}')\rceil\end{aligned}$$

using the induction hypothesis and the first relation in (10). Similarly, for $\boldsymbol{t}' = \boldsymbol{t}_1'\boldsymbol{t}_2'$, we find

$$\begin{aligned}\mathrm{ev}(\boldsymbol{t}\lceil \boldsymbol{t}'\rceil) &= \mathrm{ev}((\boldsymbol{t}\lceil \boldsymbol{t}_1'\rceil)(\boldsymbol{t}\lceil \boldsymbol{t}_2'\rceil)) = \mathrm{ev}(\boldsymbol{t}\lceil \boldsymbol{t}_1'\rceil)\circ \mathrm{ev}(\boldsymbol{t}\lceil \boldsymbol{t}_2'\rceil) \\ &= \mathrm{ev}(\boldsymbol{t})\lceil \mathrm{ev}(\boldsymbol{t}_1')\rceil \circ \mathrm{ev}(\boldsymbol{t})\lceil \mathrm{ev}(\boldsymbol{t}_2')\rceil = \mathrm{ev}(\boldsymbol{t})\lceil \mathrm{ev}(\boldsymbol{t}_1')\circ \mathrm{ev}(\boldsymbol{t}_2')\rceil = \mathrm{ev}(\boldsymbol{t})\lceil \mathrm{ev}(\boldsymbol{t}')\rceil\end{aligned}$$

using the induction hypothesis and the second relation in (10). ☐

Then the following technical result is crucial, as it shows that the mapping ev^* transforms the action of diagrams on trees into a multiplication in the group $\widetilde{B}_\bullet$.

Lemma 3.14. *For $\boldsymbol{t}$ a $\widetilde{B}_\bullet$-coloured tree $\boldsymbol{t}$ and w a word such that $\boldsymbol{t} \bullet w$ exists, we have*

$$\mathrm{ev}^*(\boldsymbol{t} \bullet w) = \mathrm{ev}^*(\boldsymbol{t}) \cdot \overline{w}, \tag{18}$$

where $\overline{w}$ denotes the element of $\widetilde{B}_\bullet$ represented by w.

Proof. For an induction, it is sufficient to establish (18) when w consists of one single letter σ_i or a_i. Let us assume $\mathrm{ev}(\mathrm{dec}(\boldsymbol{t})) = (x_1, \dots, x_n)$, where $\mathrm{ev}((\boldsymbol{t}_1, \dots, \boldsymbol{t}_n))$ stands for $(\mathrm{ev}(\boldsymbol{t}_1), \dots, \mathrm{ev}(\boldsymbol{t}_n))$. First, we find

$$\mathrm{ev}(\mathrm{dec}(\boldsymbol{t} \bullet \sigma_i)) = (x_1, \dots, x_{i-1}, x_i\lceil x_{i+1}\rceil, x_i, x_{i+2}, \dots, x_n), \tag{19}$$

$$\mathrm{ev}(\mathrm{dec}(\boldsymbol{t} \bullet a_i)) = (x_1, \dots, x_{i-1}, x_i \circ x_{i+1}, x_{i+2}, \dots, x_n). \tag{20}$$

Indeed, (19) follows from (13) using (17), and (20) follows from (14). Then we find

$$\begin{aligned}\mathrm{ev}^*(\boldsymbol{t} \bullet \sigma_i) &= x_1 \cdot \ldots \cdot \partial^{i-2}x_{i-1} \cdot \partial^{i-1}(x_i\lceil x_{i+1}\rceil) \cdot \partial^i x_i \cdot \partial^{i+1}x_{i+2} \cdot \ldots \cdot \partial^{n-1}x_n \\ &= x_1 \cdot \ldots \cdot \partial^{i-2}x_{i-1} \cdot \partial^{i-1}x_i \cdot \partial^i x_{i+1} \cdot \sigma_i \cdot \partial^i x_i^{-1} \cdot \partial^i x_i \cdot \partial^{i+1}x_{i+2} \cdot \ldots \cdot \partial^{n-1}x_n \\ &= x_1 \cdot \ldots \cdot \partial^{i-2}x_{i-1} \cdot \partial^{i-1}x_i \cdot \partial^i x_{i+1} \cdot \sigma_i \cdot \partial^{i+1}x_{i+2} \cdot \ldots \cdot \partial^{n-1}x_n \\ &= x_1 \cdot \ldots \cdot \partial^{n-1}x_n \cdot \sigma_i = \mathrm{ev}^*(\boldsymbol{t}) \cdot \sigma_i,\end{aligned}$$

as $\sigma_i \cdot \partial^k x = \partial^k x \cdot \sigma_i$ holds for $k \geqslant i+1$. For a_i, we find similarly

$$\begin{aligned}\mathrm{ev}^*(\boldsymbol{t} \bullet a_i) &= x_1 \cdot \ldots \cdot \partial^{i-2}x_{i-1} \cdot \partial^{i-1}(x_i \circ x_{i+1}) \cdot \partial^i x_{i+2} \cdot \ldots \cdot \partial^{n-2}x_n \\ &= x_1 \cdot \ldots \cdot \partial^{i-2}x_{i-1} \cdot \partial^{i-1}x_i \cdot \partial^i x_{i+1} \cdot a_i \cdot \partial^i x_{i+2} \cdot \ldots \cdot \partial^{n-2}x_n \\ &= x_1 \cdot \ldots \cdot \partial^{n-1}x_n \cdot a_i = \mathrm{ev}^*(\boldsymbol{t}) \cdot a_i,\end{aligned}$$

as $a_i \cdot \partial^k x = \partial^{k+1}x \cdot a_i$ holds for $k \geqslant i$. ☐

We are now able to conclude:

Proposition 3.15. *The groups $B_\bullet$ and $\widetilde{B}_\bullet$ are isomorphic, i.e., $(a_*, \sigma_*, R_\bullet)$ is a presentation for the group $B_\bullet$ of parenthesized braids.*

Proof. Assume that w and w' are words and there is a tree t such that the diagrams $\mathcal{D}_t(w)$ and $\mathcal{D}_t(w')$ are isotopic. We have to prove that w and w' are $R_\bullet$-equivalent, *i.e.*, they represent the same element of $\widetilde{B}_\bullet$. Lemma 3.11 guarantees that there exists at least one $\widetilde{B}_\bullet$-colouring $\boldsymbol{t}$ of t such that $\boldsymbol{t} \bullet w$ and $\boldsymbol{t} \bullet w'$ are defined and equal. Now—this is the point—(18) implies that both w and w' represent $\mathrm{ev}^*(\boldsymbol{t})^{-1} \cdot \mathrm{ev}^*(\boldsymbol{t} \bullet w)$. □

All algebraic results about $\widetilde{B}_\bullet$ established in Section 2 are therefore valid for $B_\bullet$. In the sequel, we shall no longer distinguish between $B_\bullet$ and $\widetilde{B}_\bullet$, and use $B_\bullet^+$ for $\widetilde{B}_\bullet^+$. In particular, we consider that B_∞ and F are included in $B_\bullet$; the elements of F are called *Thompson elements.*

3.5. **Special decompositions.** Besides its group operation, the set $B_\bullet$ is now equipped with two binary operations, namely $\lceil\ \rceil$ and $\circ$. For each parenthesized braid x, the parenthesized braids that can be constructed from β using these operations form a sub-ALD-system of $B_\bullet$. In particular, we can start from the trivial braid 1, and introduce what will be called special parenthesized braids.

Definition 3.16. A braid (*resp.* a Thompson element, *resp.* a parenthesized braid) is called *special* if it belongs to the closure of $\{1\}$ under $\lceil\ \rceil$ (*resp.* under $\circ$, *resp.* under both $\lceil\ \rceil$ and $\circ$).

For instance, 1, σ_1, a_1, and $a_1\sigma_2\sigma_1a_2^{-1}$ are special parenthesized braids , as we can write

$$\sigma_1 = 1\lceil 1\rceil, \quad a_1 = 1 \circ 1, \quad a_1\sigma_2\sigma_1a_2^{-1} = a_1\lceil \sigma_1\rceil = (1 \circ 1)\lceil 1\lceil 1\rceil\rceil.$$

We will see that every parenthesized braid admits decompositions in terms of special parenthesized braids. The following geometric characterization of special parenthesized braids is crucial for uniqueness arguments. It shows that special parenthesized braids are the ones that produce themselves starting from a right vine with trivial colours. To improve readability, we skip some parentheses in trees according to the convention that xyz stands for $x(yz)$; thus, for instance, a right vine is denoted $\bullet\bullet\dots\bullet$.

Lemma 3.17. *A parenthesized braid z is special if and only if it admits an expression w such that each sufficiently large $B_\bullet$-coloured vine $(\bullet_1\bullet_1\dots\bullet_1) \bullet w$ exists and has the form $\boldsymbol{t}\bullet_1\dots\bullet_1$. In this case, all colours in $\boldsymbol{t}$ are special braids, and we have $z = \mathrm{ev}(\boldsymbol{t})$.*

Proof. We first prove that the condition is necessary. As it is true for $z = 1$ with $w = \varepsilon$, it suffices to prove that, if the condition is true for z_1 and z_2, then it is for $z_1\lceil z_2\rceil$ and $z_1 \circ z_2$. So we assume that w_i is an expression of z_i, that $(\bullet_1\bullet_1\dots\bullet_1) \bullet w_i = \boldsymbol{t}_i\bullet_1\dots\bullet_1$ holds, and, in addition, we have $\mathrm{ev}(\boldsymbol{t}_i) = z_i$ and all colours in $\boldsymbol{t}_i$ are special braids. Then $w_1 \cdot \partial w_2 \cdot \sigma_1 \cdot \partial w_1^{-1}$ represents $z_1\lceil z_2\rceil$, and, using the induction hypothesis, we find

$$\begin{aligned}(\bullet_1\bullet_1\dots\bullet_1) \bullet (w_1 \cdot \partial w_2 \cdot \sigma_1 \cdot \partial w_1^{-1}) &= (\boldsymbol{t}_1\bullet_1\dots\bullet_1) \bullet (\partial w_2 \cdot \sigma_1 \cdot \partial w_1^{-1})\\ &= (\boldsymbol{t}_1\boldsymbol{t}_2\bullet_1\dots\bullet_1) \bullet (\sigma_1 \cdot \partial w_1^{-1}) = ((\boldsymbol{t}_1\lceil \boldsymbol{t}_2\rceil)\boldsymbol{t}_1\bullet_1\dots\bullet_1) \bullet \partial w_1^{-1} = (\boldsymbol{t}_1\lceil \boldsymbol{t}_2\rceil)\bullet_1\dots\bullet_1.\end{aligned}$$

Similarly, $w_1 \cdot \partial w_2 \cdot a_1$ represents $z_1 \circ z_2$, and we find

$$(\bullet_1\bullet_1\dots\bullet_1) \bullet (w_1 \cdot \partial w_2 \cdot a_1) = (\boldsymbol{t}_1\bullet_1\dots\bullet_1) \bullet (\partial w_2 \cdot a_1) = (\boldsymbol{t}_1\boldsymbol{t}_2\bullet_1\dots\bullet_1) \bullet a_1 = (\boldsymbol{t}_1\boldsymbol{t}_2)\bullet_1\dots\bullet_1.$$

Conversely, by (18), any equality $(\bullet_1\bullet_1\dots\bullet_1) \bullet w = \boldsymbol{t}\bullet_1\dots\bullet_1$ implies

$$\overline{w} = \mathrm{ev}^*(\bullet_1\bullet_1\dots\bullet_1) \cdot \overline{w} = \mathrm{ev}^*((\bullet_1\bullet_1\dots\bullet_1) \bullet w) = \mathrm{ev}^*(\boldsymbol{t}\bullet_1\dots\bullet_1) = \mathrm{ev}(\boldsymbol{t}).$$

By definition, if the colours in $\boldsymbol{t}$ are special braids (or, more generally, special parenthesized braids), the evaluation $\mathrm{ev}(\boldsymbol{t})$ is a special parenthesized braid. So, it only remains to show that, whenever $(\bullet_1\bullet_1\dots\bullet_1) \bullet w$ exists, then all colours in the latter tree are special braids. Now we can assume without loss of generality that w is tidy. Indeed, pushing the letters a_i^{-1} to the left and the letters a_i to the right does not change the negative crossings in the associated braid diagram, and no obstruction may appear. Now the hypothesis that $(\bullet_1\bullet_1\bullet_1\dots) \bullet w$ is defined implies that there is no initial a_i^{-1} in w, *i.e.*, that w consists of a braid word v followed by a_i's.

By [15], Propositions VI.5.8 and 5.12, if v is a σ-word and $(\bullet_1\bullet_1\dots\bullet_1)\bullet v$ is defined, then the latter has the form $\bullet_{\alpha_1}\bullet_{\alpha_2}\dots\bullet_{\alpha_n}$ where $\alpha_1,\dots,\alpha_n$ are special braids. The subsequent a_i's do not change the colours. $\square$

We give now a complete description of special Thompson elements. Note that, by definition of the operation $\circ$, such elements must be positive.

Proposition 3.18. *(i) A Thompson element not equal to* 1 *is special if and only if it has an expression* $a_{i_1}\dots a_{i_k}$ *satisfying* $i_{k+1}\geqslant i_k-1$ *for each* k *and* $i_r=1$*. This expression is unique.*

(ii) The mapping ev *establishes a one-to-one correspondence between finite binary trees of size* $n+1$ *and special Thompson elements of length* n*. So, in particular, there are* $\frac{1}{n+1}\binom{2n}{n}$ *special Thompson elements of length* n*.*

Proof. The existence of a decomposition as in (i) is true for 1, and for $f_1\circ f_2$ whenever it is for f_1 and f_2. Hence it is true for every special Thompson element. Conversely, if f admits an expression w as above, there is a unique way of expressing f as $f_1\circ f_2$, namely defining f_1 to be the element represented by the largest prefix w_1 of w that finishes with a_1 if it exists, and 1 otherwise. Then f_1 and f_2 have the same syntaxic property as f, and the parsing continues.

Then, by definition, the mapping ev establishes a surjective mapping from trees to special Thompson elements. To prove injectivity, we observe that, for every tree t, we have

$$(\bullet\bullet\bullet\dots)\bullet\mathrm{ev}(t)=(t)\bullet\bullet\dots \tag{21}$$

provided we start with a large enough vine, as shows an easy induction on the size of t. Thus $\mathrm{ev}(t)$ determines t. This proves (ii), and the uniqueness of the decomposition of (i) follows. $\square$

Lemma 3.19. *For each* $B_\bullet$*-coloured tree* $\boldsymbol{t}$*, we have*

$$\mathrm{ev}(\boldsymbol{t})=z_1\cdot\partial z_2\cdot\ldots\cdot\partial^{n-1}z_n\cdot\mathrm{ev}(\boldsymbol{t}^\dagger), \tag{22}$$

where $(z_1,\dots,z_n)$ *is the left-to-right enumeration of the colours in* $\boldsymbol{t}$*.*

Proof. First, for every special Thompson element f of length n and every parenthesized braid z, we have

$$f\cdot\partial z=\partial^{1+n}z\cdot f. \tag{23}$$

Indeed, the equality inductively follows from the relation $a_1\cdot\partial z=\partial^2 z\cdot a_1$, as the decomposition of Proposition 3.18 guarantees that, when pushing the letters a_i of f to the right, one always meets letters $a_k^{\pm1}$ or $\sigma_k^{\pm1}$ with $k\geqslant i+1$.

Now we prove (22) using induction on $\boldsymbol{t}$. The result is clear when $\boldsymbol{t}$ has size 1. For $\boldsymbol{t}=\boldsymbol{t}_1\boldsymbol{t}_2$, assuming that the colours in $\boldsymbol{t}_i$ are $z_{1,i},\dots,z_{n_i,i}$ and using the induction hypothesis, we find

$$\mathrm{ev}(\boldsymbol{t})=z_{1,1}\cdot\ldots\cdot\partial^{n_1-1}z_{n_1,1}\cdot\mathrm{ev}(\boldsymbol{t}_1^\dagger)\cdot\partial z_{1,2}\cdot\ldots\cdot\partial^{n_2}z_{n_2,2}\cdot\partial\mathrm{ev}(\boldsymbol{t}_2^\dagger)\cdot a_1.$$

By construction, $\mathrm{ev}(\boldsymbol{t}_1^\dagger)$ is a special Thompson element of length n_i-1. Applying (23) repeatedly, we push $\mathrm{ev}(\boldsymbol{t}_1^\dagger)$ to the right, and obtain

$$\mathrm{ev}(\boldsymbol{t})=z_{1,1}\cdot\ldots\cdot\partial^{n_1-1}z_{n_1,1}\cdot\partial^{n_1}z_{1,2}\cdot\ldots\cdot\partial^{n_1+n_2-1}z_{n_2,2}\cdot\mathrm{ev}(\boldsymbol{t}_1^\dagger)\cdot\partial\mathrm{ev}(\boldsymbol{t}_2^\dagger)\cdot a_1,$$

and (22) follows using $\mathrm{ev}(\boldsymbol{t}_1^\dagger)\cdot\partial\mathrm{ev}(\boldsymbol{t}_2^\dagger)\cdot a_1=\mathrm{ev}(\boldsymbol{t}^\dagger)$. $\square$

We can now express special parenthesized braids in terms of special braids and Thompson elements.

Proposition 3.20. *Every special parenthesized braid* z *admits a unique decomposition*

$$z=\beta_1\cdot\partial\beta_2\cdot\ldots\cdot\partial^{n-1}\beta_n\cdot h, \tag{24}$$

where $\beta_1,\dots,\beta_n$ *are special braids, and* h *is a special Thompson element of length* $n-1$*.*

Proof. Let z be a special parenthesized braid. By Lemma 3.17, there exists a $B_\bullet$-coloured tree $\boldsymbol{t}$, where all colours are special braids, satisfying $z = \mathrm{ev}(\boldsymbol{t})$. Then Lemma 3.19 gives a decomposition of the expected form. Next, Proposition 2.23 first implies the uniqueness of h, as $\beta \cdot h = \beta' \cdot h'$ implies $\beta^{-1}\beta' = h'h^{-1} \in B_\infty \cap F$. Then, when $\beta_1, \dots, \beta_n$ are special braids, the product $\beta_1 \cdot \partial\beta_2 \cdot \ldots \cdot \partial^{n-1}\beta_n$ determines each factor β_i as, by Lemma 3.17 again, we have $(\bullet_1 \dots \bullet_1) \bullet (\beta_1 \cdot \ldots \cdot \partial^{n-1}\beta_n) = \bullet_{\beta_1} \dots \bullet_{\beta_n}$—note that we only use the easy direction of Lemma 3.17, and not the more delicate converse that resorts to the fine study of self-distributivity. ☐

Finally, we obtain canonical decompositions for arbitrary positive parenthesized braids in terms of special parenthesized braids, hence in terms of special braids and special Thompson elements.

Proposition 3.21. *Every positive parenthesized braid x admits two unique decompositions:*

$$x = z_1 \cdot \partial z_2 \cdot \ldots \cdot \partial^{p-1} z_p, \tag{25}$$

$$x = \beta_1 \cdot \partial\beta_2 \cdot \ldots \cdot \partial^{n-1}\beta_n \cdot h_1 \cdot \partial h_2 \cdot \ldots \cdot \partial^{n-1} h_n, \tag{26}$$

where $z_1, \dots, z_p$ are special parenthesized braids, $\beta_1, \dots, \beta_n$ are special braids, and $h_1, \dots, h_n$ are special Thompson elements.

Proof. Let x be a positive parenthesized braid. By hypothesis, x admits an expression w with no σ_i^{-1} or a_i^{-1}. As w contains no σ_i^{-1}, every $B_\bullet$-colouring of a tree t such that $t \bullet w$ is defined can be propagated along the strands of the diagram $\mathcal{D}_t(w)$. Thus $\boldsymbol{t} \bullet w$ is defined for each $B_\bullet$-colouring $\boldsymbol{t}$ of t, and (18) then implies $x = \overline{w} = \mathrm{ev}^*(\boldsymbol{t})^{-1} \cdot \mathrm{ev}^*(\boldsymbol{t} \bullet w)$.

As w contains no letter a_i^{-1}, we may choose t to be a right vine $\bullet \dots \bullet$, and $\boldsymbol{t}$ to be the corresponding colouring $\bullet_1 \dots \bullet_1$. Then, by definition, we have $\mathrm{ev}^*(\boldsymbol{t}) = 1$, hence $\beta = \overline{w} = \mathrm{ev}^*(\boldsymbol{t} \bullet w)$. Moreover, by construction, each colour in $\boldsymbol{t} \bullet w$ belongs to the closure of $\{1\}$ under the bracket operation, hence it is a special braid. Then the $\circ$-evaluation of the trees occurring in the decomposition of $\boldsymbol{t} \bullet w$ are iterated $\circ$-products of special braids, hence they are special parenthesized braids. So, by definition, $\mathrm{ev}^*(\boldsymbol{t} \bullet w)$ is a shifted product of special parenthesized braids, and we obtain for x a decomposition as in (25).

Now, if $\beta_1, \dots, \beta_n$ are special parenthesized braids, Lemma 3.17 implies that, for each k, there exists an expression w_k of z_k satisfying $(\bullet_1 \dots \bullet_1) \bullet w_k = (\boldsymbol{t}_k)\bullet_1 \dots \bullet_1$, where $\boldsymbol{t}_k$ is a $B_\bullet$-coloured tree satisfying $\mathrm{ev}(\boldsymbol{t}_k) = z_k$. Provided the initial right vine is large enough, this implies

$$(\bullet_1\bullet_1 \dots \bullet_1) \bullet (w_1 \cdot \partial w_2 \cdot \ldots \cdot \partial^{n-1} w_n) = (\boldsymbol{t}_1) \dots (\boldsymbol{t}_n)\bullet_1 \dots \bullet_1.$$

This shows that the shifted product $z_1 \cdot \ldots \cdot \partial^{n-1} z_n$ determines each tree $\boldsymbol{t}_k$, hence each factor z_k, thus proving the uniqueness of the decomposition (25)—we did not prove here the (true) result that replacing w with an equivalent word w' necessarily leads to the same tree $\boldsymbol{t}$: this result is not needed here, as we only use $\mathrm{ev}(\boldsymbol{t})$, which is x in any case.

Applying Proposition 3.20 to each factor in (25) and using (23) to push the Thompson factors to the right easily gives a decomposition as in (26). For the uniqueness of the latter, the same argument as for Proposition 3.20 shows that the braid part and the Thompson part are determined, and that each special braid β_k is determined by the shifted product $\beta_1 \cdot \ldots \cdot \partial^{n-1}\beta_n$, so it only remains to verify that the uniqueness of the special Thompson factors. The latter follows from the equality

$$(\bullet_1\bullet_1 \dots \bullet_1) \bullet (h_1 \cdot \partial h_2 \cdot \ldots \cdot \partial^{n-1} h_n) = (t_1) \dots (t_n)\bullet_1 \dots \bullet_1$$

for $h_k = \mathrm{ev}(t_k)$, again a consequence of Lemma 3.17. ☐

In the case of Thompson elements we have obtained the following result, which provides a unique normal form in F^+:

Corollary 3.22. *Every positive Thompson element f admits a unique decomposition*

$$f = h_1 \cdot \partial h_2 \cdot \ldots \cdot \partial^{p-1} h_p \tag{27}$$

where $h_1, \ldots, h_p$ are special Thompson elements.

By Proposition 2.20, every parenthesized braid is a left fraction $x^{-1}y$ with x, y in $B_\bullet^+$, so another consequence of Proposition 3.21 is:

Corollary 3.23. *Every parenthesized braid x admits decompositions*

$$x = \partial^{q-1} {z'_q}^{-1} \cdot \ldots \cdot \partial {z'_2}^{-1} \cdot {z'_1}^{-1} \cdot z_1 \cdot \partial z_2 \cdot \ldots \cdot \partial^{p-1} z_p, \tag{28}$$

$$x = \partial^{n-1} {h'_n}^{-1} \cdot \ldots \cdot {h'_1}^{-1} \cdot \partial^{n-1} {\beta'_n}^{-1} \cdot \ldots \cdot {\beta'_1}^{-1} \cdot \beta_1 \cdot \ldots \cdot \partial^{n-1} \beta_n \cdot h_1 \cdot \ldots \cdot \partial^{n-1} h_n, \tag{29}$$

where $z_1, \ldots, z'_q$ are special parenthesized braids, $\beta_1, \ldots, \beta'_n$ are special braids, and $h_1, \ldots, h'_n$ are special Thompson elements.

4. A linear ordering on $B_\bullet$

Artin's braid group B_∞ admits a distinguished linear ordering that is compatible with multiplication on one side and admits a number of equivalent constructions [19]. On the other hand, it is easy to construct on Thompson's group F a linear ordering that is compatible with multiplication on both sides. Merging these orderings leads to ordering parenthesized braids.

4.1. An ordering on F^+.

One can easily order F by attaching a piecewise linear homeomorphism of $[0, 1]$ (or of the real line) to each element and comparing the derivatives. An equivalent construction involves trees. We recall that, for t a tree, $\mathrm{Dyad}(t)$ denotes the set of endpoints in the dyadic decomposition of $[0, 1]$ attached to t.

Definition 4.1. For t, t' trees, we say that $t \prec t'$ is true if $\mathrm{Dyad}(t)$ follows $\mathrm{Dyad}(t')$ in the lexicographical ordering.

For instance, the sequences attached to $\bullet(\bullet\bullet)$ and $(\bullet\bullet)\bullet$ are $(0, \frac{1}{2}, \frac{3}{4}, 1)$ and $(0, \frac{1}{4}, \frac{1}{2}, 1)$. The first entries both are 0; the second entries are $\frac{1}{2}$ and $\frac{1}{4}$, respectively: the former is larger, so we declare $\bullet(\bullet\bullet) \prec (\bullet\bullet)\bullet$.

Lemma 4.2. *The relation $\prec$ is a linear ordering on trees. An alternative definition is: $\bullet \prec t_1t_2$ is always true, and $t_1t_2 \prec t'_1t'_2$ is true if and only if $t_1 \prec t'_1$ is true, or $t_1 = t'_1$ and $t_2 \prec t'_2$ are.*

By Proposition 3.18, the evaluation mapping ev establishes a one-to-one correspondence between finite binary trees and special Thompson elements. Moreover, Corollary 3.22 shows that every positive Thompson element admits a unique decomposition in terms of special Thompson elements, hence in terms of a sequence of trees. We can therefore carry the tree ordering to F^+.

Definition 4.3. For f, f' special Thompson elements, we say that $f <_F^{sp} f'$ holds if and only if we have $\mathrm{ev}^{-1}(f) \prec \mathrm{ev}^{-1}(f')$. For f, f' in F^+, we say that $f <_F f'$ holds if the (unique) special sequence $(f_1, \ldots, f_p)$ satisfying $f = f_1 \cdot \partial f_2 \cdot \ldots \cdot \partial^{p-1} f_p$ is lexicographically $<_F^{sp}$-smaller than the special sequence $(f'_1, \ldots, f'_q)$ satisfying $f' = f'_1 \cdot \partial f'_2 \cdot \ldots \cdot \partial^{q-1} f'_q$.

For instance, we have $a_2 <_F a_1$, as the special decomposition of a_2 is $1 \cdot \partial a_1$, while a_1 is special. Now $\bullet(\bullet\bullet) \prec (\bullet\bullet)\bullet$ implies $1 = \mathrm{ev}(\bullet(\bullet\bullet)) <_F^{sp} a_1 = \mathrm{ev}((\bullet\bullet)\bullet)$, and, therefore, the sequence $(1, a_1)$ is lexicographically smaller than the sequence (a_1).

There is a canonical way of attaching to each element f of Thompson's group F a piecewise linear homeomorphism $H(f)$ of the unit inverval [10]—because of our conventions, we have $H(ff') = H(f') \circ H(f)$. The derivatives in $H(f)$ make a finite sequence of dyadic numbers, *e.g.*, $(\frac{1}{2}, 1, 2)$ in the case of a_1.

Proposition 4.4. *The relation $<_F$ is a linear ordering on F^+. It is compatible with multiplication on both sides. For f, f' in F^+, the relation $f <_F f'$ holds if and only if the first derivative not equal to* 1 *in $H(f^{-1}f')$ is smaller than* 1.

Proof. It is clear that $<_F$ is a linear ordering. The correspondence between $<_F$ and the homeomorphisms of $[0,1]$ is as follows. If w is a positive a-word representing an element f, then $(\bullet\bullet\dots)_\bullet w$ is defined provided the initial vine is large enough. Let $(\bullet\bullet\dots)_\bullet w = (t_1)\dots(t_p)\bullet\dots$. Then the special decomposition of f is the shifted product $\mathrm{ev}(t_1)\cdot\partial\mathrm{ev}(t_2)\cdot\dots$ Define $\mathrm{Dyad}(f)$ to be the union of the sets $\mathrm{Dyad}(t_i)$ contracted from $[0,1]$ to $[1-\frac{1}{2^{i-1}}, 1-\frac{1}{2^i}]$ when i varies. Then $f <_F f'$ is equivalent to $\mathrm{Dyad}(f)$ being larger than $\mathrm{Dyad}(f')$ in the lexicographical order. Now the homeomorphism $H(f^{-1}f')$ maps $\mathrm{Dyad}(f)$ to $\mathrm{Dyad}(f')$, so the first divergence between $\mathrm{Dyad}(f)$ and $\mathrm{Dyad}(f')$ results in $\mathrm{Dyad}(f)$ being declared larger if and only if the first derivative $\neq 1$ in $H(f^{-1}f')$ is less than 1.

Owing to the latter characterization, it is clear that $<_F$ is compatible with multiplication on the left. It is also compatible with multiplication on the right, as the graph of $H(ff'f^{-1})$ is obtained from the graph of $H(f')$ by using $H(f)$ to rescale the source and target intervals, which does not change the fact that the graph diverges from the diagonal downwards or upwards. □

For instance, the special decompositions of $1, a_1$, and a_2 are $(\bullet, \bullet, \dots)$, $(\bullet\bullet, \bullet, \bullet, \dots)$, and $(\bullet, \bullet\bullet, \bullet, \dots)$, respectively. So we obtain $\mathrm{Dyad}(1) = (0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots)$, $\mathrm{Dyad}(a_1) = (0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \dots)$, and $\mathrm{Dyad}(a_2) = (0, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \dots)$, hence $1 <_F a_2 <_F a_1$.

4.2. **The ordering on $B_\bullet^+$.** As every element in $B_\bullet^+$ admits a unique decomposition in terms of elements of B_∞^+ and F^+, we deduce a linear order on $B_\bullet^+$ from any linear orders on B_∞^+ and F^+. We recall that B_∞ is equipped with a distinguished linear ordering:

Proposition 4.5. [15, 19] *For β, β' in B_∞, say that $\beta <_B \beta'$ holds if and only if $\beta^{-1}\beta'$ admits an expression in which the generator σ_i with minimal index i occurs positively only, i.e., σ_i occurs but σ_i^{-1} does not. Then the relation $<_B$ is a linear ordering on B_∞, and it is compatible with multiplication on the left.*

Definition 4.6. For x, x' in $B_\bullet^+$, we say that $x <^+ x'$ holds if we have either $\beta <_B \beta'$, or $\beta = \beta'$ and $f <_F f'$, where $x = \beta f$ and $x' = \beta' f'$ are the $B_\infty^+ \times F^+$-decompositions of x and x'.

For instance, we have

$$\dots <^+ a_2 <^+ a_1 <^+ \dots <^+ \sigma_2 <^+ \sigma_1.$$

Indeed, we saw above that $a_i <_F a_j$ holds for $i > j$ (in the case $i = 1$, $j = 2$). Then, we have $1 <_B \sigma_j$, hence $a_i <^+ \sigma_j$ for all i, j—and, more generally, $f <^+ \beta$ for all f in F^+ and β in $B_\infty^+ \setminus \{1\}$. Finally, $\sigma_i <_F \sigma_j$ holds for $i > j$, as we have $\sigma_i <_B \sigma_j$ since $\sigma_i^{-1}\sigma_j$ is a braid word in which the generator with smallest index, here σ_j, occurs positively and not negatively.

Lemma 4.7. *The relation $<^+$ is a linear order on $B_\bullet^+$, compatible with left multiplication.*

Proof. As both $<_B$ and $<_F$ are linear orders and the $B_\infty^+ \times F^+$-decomposition is unique, $<^+$ is a linear order. To prove compatibility with multiplication on the left, assume $\beta f <^+ \beta' f'$. Assume first $\beta <_B \beta'$. As the braid ordering is compatible with left multiplication, we have $\sigma_k\beta <_B \sigma_k\beta'$ for every k, hence $\sigma_k \cdot \beta f <^+ \sigma_k \cdot \beta' f'$. On the other hand, (8) gives

$$a_k \cdot \beta f = \mathrm{db}_k(\beta) \cdot a_{\beta^{-1}[k]} f \quad \text{and} \quad a_k \cdot \beta' f' = \mathrm{db}_k(\beta') \cdot a_{\beta'^{-1}[k]} f'. \tag{30}$$

To compare the braids $\mathrm{db}_k(\beta)$ and $\mathrm{db}_k(\beta')$, we consider $\mathrm{db}_k(\beta)^{-1}\mathrm{db}_k(\beta')$. By construction, the latter is $\mathrm{db}_{\beta^{-1}[k]}(\beta^{-1}\beta')$. The hypothesis $\beta <_B \beta'$ means that we can represent $\beta^{-1}\beta'$ by a braid diagram in which the leftmost crossings all are positively oriented. When we double a strand, the latter property is preserved. So $\mathrm{db}_k(\beta) <_B \mathrm{db}_k(\beta')$ holds, and we deduce $a_k \cdot \beta f <^+ a_k \cdot \beta' f'$. Hence, in this case, $x \cdot \beta f <^+ x \cdot \beta' f'$ holds for every parenthesized braid x.

Assume now $\beta = \beta'$ and $f <_F f'$. Then $\sigma_k \cdot \beta f <^+ \sigma_k \cdot \beta' f'$ holds trivially for every k. As for multiplication by a_k, we use (30) again: $\beta = \beta'$ implies $\mathrm{db}_k(\beta) = \mathrm{db}_k(\beta')$, and $f <_F f'$ implies $a_{\beta^{-1}[k]} f <_F a_{\beta'^{-1}[k]} f'$, because $\beta^{-1}[k] = \beta'^{-1}[k]$ holds and $<_F$ is compatible with multiplication on the left. So, again, $x \cdot \beta f <^+ x \cdot \beta' f'$ holds for every parenthesized braid x. ☐

4.3. **The ordering on $B_\bullet$.** As every parenthesized braid is a quotient of two positive parenthesized braids, we can now easily deduce an ordering on $B_\bullet$ from the previous ordering on $B_\bullet^+$.

Definition 4.8. We denote by C the set of all elements in $B_\bullet$ that can be written as $x^{-1}x'$ with x, x' in $B_\bullet^+$ and $x <^+ x'$.

Lemma 4.9. *The set C is a positive cone, i.e., we have $C \cdot C \subseteq C$ and $C \cap C^{-1} = \emptyset$.*

Proof. Consider two elements of C, say $x_1^{-1}x_1'$ and $x_2^{-1}x_2'$ with x_i, x_i' in $B_\bullet^+$ and $x_i <^+ x_i'$ for $i = 1, 2$. The elements x_1' and x_2 admit a common left multiple in $B_\bullet^+$, say $yx_1' = y'x_2$. Then we have $(x_1^{-1}x_1') \cdot (x_2^{-1}x_2') = (yx_1)^{-1} \cdot (y'x_2')$. Using the compatibility of $<^+$ with left multiplication, we find $yx_1 <^+ yx_1' = y'x_2 <^+ y'x_2'$, hence $(x_1^{-1}x_1') \cdot (x_2^{-1}x_2') \in C$, and $C \cdot C \subseteq C$.

Assume $x \in C \cap C^{-1}$. Then we have $1 = x \cdot x^{-1} \in C \cdot C$, hence $1 \in C$ by the above result. So there must exist β, β' in B_∞^+, and f, f' in F^+ with $\beta f = \beta' f'$ and $\beta <_B \beta'$, or $\beta = \beta'$ and $f <_F f'$, contradicting the uniqueness of the $B_\infty^+ \times F^+$ decomposition in $B_\bullet^+$ in both cases. ☐

Definition 4.10. For x, x' in $B_\bullet$, we say that $x < x'$ holds if $x^{-1}x'$ belongs to C.

For instance, we have $\sigma_2 < a_1^{-1}\sigma_1 a_1 < \sigma_1$. Indeed, we find $(\sigma_2)^{-1}(a_1^{-1}\sigma_1 a_1) = a_1^{-1}\sigma_3^{-1}\sigma_1 a_1$, and $\sigma_3 <_B \sigma_1$ implies $\sigma_3 a_1 <^+ \sigma_1 a_1$. Similarly, we have $(a_1^{-1}\sigma_1 a_1)^{-1}(\sigma_1) = a_1^{-1}\sigma_1^{-1}a_1\sigma_1 = a_1^{-1}\sigma_1^{-1}\sigma_2\sigma_1 a_2$, and $\sigma_1 <_B \sigma_2\sigma_1$ implies $\sigma_1 a_1 <^+ \sigma_2\sigma_1 a_2$.

Proposition 4.11. *The relation $<$ is a linear ordering on $B_\bullet$ that is compatible with multiplication on the left, and with the shift endomorphism ∂. This linear ordering extends the orders $<^+$ on $B_\bullet^+$, $<_B$ on B_∞ and $<_F$ on F.*

Proof. Lemma 4.9 guarantees that $<$ is a partial order on $B_\bullet$. This order is linear, because $<^+$ is a linear order on $B_\bullet^+$, so, for all x, x' in $B_\bullet$, either $x^{-1}x'$ or $(x^{-1}x')^{-1}$, *i.e.*, $x'^{-1}x$, belongs to C. The order is compatible with multiplication on the left by definition. Then ∂ preserves the orders $<_B$ and $<_F$, hence the order $<^+$ on $B_\bullet^+$. This implies $\partial C \subseteq C$, hence $x < x'$ implies, and, therefore, is equivalent to, $\partial x < \partial x'$.

Assume x, x' in $B_\bullet^+$ with $x <^+ x'$. Then, by definition, $x^{-1}x'$ belongs to C, and, therefore, we have $x < x'$ in $B_\bullet$. As $<^+$ is a linear ordering, the implication is an equivalence.

Assume now β, β' in B_∞ with $\beta <_B \beta'$. Then there exists a positive braid β_0 such that $\beta_0\beta$ and $\beta_0\beta'$ belong to B_∞^+, and $\beta <_B \beta'$ implies $\beta_0\beta <_B \beta_0\beta'$, hence $\beta_0\beta <^+ \beta_0\beta'$. Then $\beta^{-1}\beta' = (\beta_0\beta)^{-1}(\beta_0\beta')$ implies $\beta^{-1}\beta \in C$, hence $\beta < \beta'$. Once again, as $<_B$ is a linear ordering, the implication is an equivalence. Finally, for f, f' in F with $f <_F f'$, the same argument shows that $f < f'$ holds in $B_\bullet$. Hence $<$ restricted to F coincides with $<_F$. ☐

Corollary 4.12. *The group $B_\bullet$ is left-orderable. The group algebra* $\mathbf{C}[B_\bullet]$ *has no zero divisor.*

4.4. **Syntaxic characterization.** We now describe the order on $B_\bullet$ in terms of words.

Definition 4.13. A σ,a-word is called *σ_i-positive* if it contains σ_i, but no σ_i^{-1} or $\sigma_j^{\pm 1}$ with $j < i$.

Proposition 4.14. *For x a parenthesized braid not in F, the following are equivalent:*
(i) We have $x > 1$, i.e., $x \in C$;
(ii) There exists i such that x admits a tidy σ_i-positive expression.

Proof. Let x be an arbitrary parenthesized braid. By Proposition 2.20, we can write $x = f^{-1}\beta f'$ with $f, f' \in F$ and $\beta \in B_\infty$. Then $x \notin F$ is equivalent to $\beta \neq 1$. In that case, $x \in C$ is equivalent to $\beta >_B 1$. By the results of [15], the latter is equivalent to β admitting at least one σ_i-positive expression. ☐

The example of the word $a_1\sigma_2 a_1^{-1}\sigma_3^{-1}$, which is σ_2-positive but represents 1 in $B_\bullet$, shows that considering tidy words is important. However, the case of σ_1 is particular, as we have:

Proposition 4.15. *If a parenthesized braid x admits a σ_1-positive expression, then $x > 1$ holds.*

Proof. Let w be a σ_1-positive word. We can transform w into an equivalent tidy word by pushing the letters a_i to the right, and the letters a_i^{-1} to the left. The point is that, in the process, the letters σ_1 cannot vanish, and no letter σ_1^{-1} can appear. Indeed, according to (8), the rules for the transformation are

$$a_k\sigma_i \mapsto \mathrm{db}_k(\sigma_i)a_{\sigma_i^{-1}[k]} \qquad \text{and} \qquad \sigma_i a_k^{-1} \mapsto a_{\sigma_i[k]}^{-1}\mathrm{db}_{\sigma_i[k]}(\sigma_i).$$

By definition of the operation of doubling a strand, the generator σ_i may be replaced with σ_{i+1} in the case $k < i$, but this cannot happen in the case $i = 1$. Thus we always obtain σ_1-positive words, and we finish with a tidy σ_1-positive word. □

A direct consequence is:

Proposition 4.16. *For all x, y in $B_\bullet$, one has $x < x\lceil y\rceil$.*

Proof. By definition, we have $(x)^{-1}\cdot(x\lceil y\rceil) = \partial x\cdot\sigma_1\cdot\partial y^{-1}$, an expression with one σ_1 and no σ_1^{-1}. □

Corollary 4.17. *Let x be an arbitrary element of $B_\bullet$. Then the closure of $\{x\}$ under the bracket operation is a free LD-system.*

Proof. According to the so-called Laver's criterion ([15], Proposition V.6.4), an LD-system S with one generator is free if and only if no equality of the form $x = x\lceil y_1\rceil\ldots\lceil y_r\rceil$ is possible in S. Now Proposition 4.16 gives

$$x < x\lceil y_1\rceil < x\lceil y_1\rceil\lceil y_2\rceil < \ldots < x\lceil y_1\rceil\ldots\lceil y_r\rceil$$

for all $x, y_1, \ldots, y_r$, hence $x \neq x\lceil y_1\rceil\ldots\lceil y_r\rceil$. □

Question 4.18. *Is the LD-system generated by $1, a_1, a_2, \ldots, a_{r-1}$ a free LD-system of rank r?*

Remark 4.19. There is no similar characterization of the order $<_F$ on F in terms of particular decompositions. However, sufficient conditions exist. Let us say that an a-word w is *a_i-positive* if w contains a_i, but no a_i^{-1} or a_j^{-1} with $j < i$. Then an a_i-positive word always represents an element larger than 1, but, conversely, $a_1^{-1}a_2a_1$ is an example of an element larger than 1 that admits no a_i-positive expression.

4.5. **The subword property.** The braid ordering is not compatible with multiplication on the right, and, more generally, there exists no linear ordering on B_∞ that is compatible with multiplication on both sides. So the same holds for $B_\bullet$, and $B_\bullet$ is not bi-orderable.

However, we shall now prove a partial compatibility result involving conjugacy. In general, a conjugate of an element x satisfying $x > 1$ need not be larger than 1: consider for instance $\sigma_1\sigma_2^{-1}$ and its conjugate $\sigma_2\sigma_1^{-1}$. We prove that this cannot happen for x in B_∞^+.

We begin with a technical result about the a_k-conjugates of σ_i or, more generally, of any braid $\mathrm{db}_i^p(\mathrm{db}_{i+1}^p(\sigma_i))$, or $\mathrm{db}_i^p\mathrm{db}_{i+1}^p(\sigma_i)$ for short, obtained from σ_i by multiplying each strand by $p+1$.

Lemma 4.20. *For all positive i, k, and $p \geqslant 0$, there exists i', k' and e in $\{0, 1\}$ satisfying*

$$a_k\cdot\mathrm{db}_i^p\mathrm{db}_{i+1}^p(\sigma_i)\cdot a_k^{-1} = a_{k'}^{-e}\cdot\mathrm{db}_{i'}^{p+e}\mathrm{db}_{i'+1}^{p+e}(\sigma_{i'})\cdot a_{k'}^{e}. \tag{31}$$

Proof. In the braid diagram $\mathrm{db}_i^p\mathrm{db}_{i+1}^p(\sigma_i)$, the strands i to $i+p$ cross over the strands $i+p+1$ to $i+2p+1$. Hence (31) is clear for $k < i$ and $k > i+2p+1$ with $e = 0$ and $i' = i$ or $i' = i-1$. For $i \leqslant k \leqslant i+p$, multplying by a_k amounts to doubling one more strand in the first block of p, so we have $a_k\cdot\mathrm{db}_i^p\mathrm{db}_{i+1}^p(\sigma_i) = \mathrm{db}_i^{p+1}\mathrm{db}_{i+1}^p(\sigma_i)\cdot a_{k+p+1}$. Then $a_{k+p+1}a_k^{-1}$ is $a_k^{-1}a_{k+p+2}$. For the same geometric reason, we have $\mathrm{db}_i^{p+1}\mathrm{db}_{i+1}^{p}(\sigma_i)\cdot a_k^{-1} = a_{k+p+2}^{-1}\cdot \mathrm{db}_i^{p+1}\mathrm{db}_{i+1}^{p+1}(\sigma_i)$, which is (31) with $e = 1$, $k' = k+p+1$ and $i' = i$. The computation is similar for $i+p+1 \leqslant k \leqslant i+2p+1$, leading now to $e = 1$, $k' = k$ and $i' = i$. □

Proposition 4.21. *For each parenthesized braid x in $B_\bullet$ and each i, we have $x\sigma_i x^{-1} > 1$.*

Proof. Write $x = f^{-1}\beta f'$ with $f, f' \in F^+$ and β in B_∞. Then we have

$$x\sigma_i x^{-1} = f^{-1}\beta f'\sigma_i f'^{-1}\beta^{-1} f.$$

By Lemma 4.20, we have $f'\sigma_i f'^{-1} = g^{-1}\mathrm{db}_{i'}^{p}\mathrm{db}_{i'+1}^{p}(\sigma_{i'})g$ for some g in F^+ and some i', p. Then we have $\beta g^{-1} = g'^{-1}\beta'$ for some g' in F^+ and β' in B_∞, hence

$$x\sigma_i x^{-1} = f^{-1}g'^{-1}\beta'\mathrm{db}_{i'}^{p}\mathrm{db}_{i'+1}^{p}(\sigma_{i'})\beta'^{-1}g'f.$$

By construction, the braid $\mathrm{db}_{i'}^{p}\mathrm{db}_{i'+1}^{p}(\sigma_{i'})$ belongs to B_∞^+. By [19], Proposition 1.2.15, every conjugate of a braid in B_∞^+ is larger than 1. Hence $\beta'\mathrm{db}_{i'}^{p}\mathrm{db}_{i'+1}^{p}(\sigma_{i'})\beta'^{-1}$ is a σ_j-positive braid for some j, and $x\sigma_i x^{-1}$ belongs to C. □

Corollary 4.22. *For each parenthesized braid x, every parenthesized braid represented by a word obtained from an expression of x by inserting letters σ_i is larger than x.*

Proof. It suffices to consider the addition of one σ_i, *i.e.*, to compare elements of the form xy and $x\sigma_i y$. Now, we have $(xy)^{-1}(x\sigma_i y) = y^{-1}\sigma_i y$. By Proposition 4.21, the latter belongs to C. □

The previous property does not extend to the letters a_i: for instance, we have $\sigma_1\sigma_1^{-1} = 1$ and $\sigma_1 a_1\sigma_1^{-1} = \sigma_1\sigma_2^{-1}\sigma_1^{-1}a_2 = \sigma_2^{-1}\sigma_1^{-1}\sigma_2 a_2$, an expression that is σ_1-negative, hence represents an element of C^{-1}. So, in this case, inserting a_1 diminishes the element.

4.6. **Order and colourings.** The order on parenthesized braids can also be characterized in terms of colourings by special braids.

Definition 4.23. For $\boldsymbol{t}$ a $B_\bullet$-coloured tree, we denote by $\mathrm{Col}(\boldsymbol{t})$ the left-to-right enumeration of the colours in $\boldsymbol{t}$. We denote by B_∞^{sp} the set of all special braids.

Proposition 4.24. *For all words w, w', the following are equivalent:*

(i) We have $\overline{w} < \overline{w'}$;

(ii) There exists a B_∞^{sp}-coloured tree $\boldsymbol{t}$ satisfying

$$\mathrm{Col}(\boldsymbol{t}\bullet w) <^{Lex} \mathrm{Col}(\boldsymbol{t}\bullet w') \quad \textit{or} \quad \mathrm{Col}(\boldsymbol{t}\bullet w) = \mathrm{Col}(\boldsymbol{t}\bullet w') \textit{ and } (\boldsymbol{t}\bullet w)^\dagger \prec (\boldsymbol{t}\bullet w')^\dagger. \tag{32}$$

(iii) For every B_∞-coloured tree $\boldsymbol{t}$ such that $\boldsymbol{t}\bullet w$ and $\boldsymbol{t}\bullet w'$ exist, (32) *holds.*

Proof. Assume (ii). Put

$$(\beta_1, \dots, \beta_n) = \mathrm{Col}(\boldsymbol{t}\bullet w), \quad (\beta'_1, \dots, \beta'_n) = \mathrm{Col}(\boldsymbol{t}\bullet w'), \quad (\beta''_1, \dots, \beta''_n) = \mathrm{Col}(\boldsymbol{t}).$$

Then Lemma 3.19 gives

$$\begin{aligned}\mathrm{ev}(\boldsymbol{t}\bullet w) &= \beta_1\cdot\ldots\cdot\partial^{n-1}\beta_n\cdot\mathrm{ev}((\boldsymbol{t}\bullet w)^\dagger),\\ \mathrm{ev}(\boldsymbol{t}\bullet w') &= \beta'_1\cdot\ldots\cdot\partial^{n-1}\beta'_n\cdot\mathrm{ev}((\boldsymbol{t}\bullet w')^\dagger),\\ \mathrm{ev}(\boldsymbol{t}) &= \beta''_1\cdot\ldots\cdot\partial^{n-1}\beta''_n\cdot\mathrm{ev}(\boldsymbol{t}^\dagger).\end{aligned}$$

Next, (3.14) gives $\overline{w} = \mathrm{ev}(\boldsymbol{t})^{-1}\cdot\mathrm{ev}(\boldsymbol{t}\bullet w)$, hence

$$\overline{w}^{-1}\cdot\overline{w'} = \mathrm{ev}((\boldsymbol{t}\bullet w')^\dagger)^{-1}\cdot\partial^{n-1}\beta_n^{-1}\cdot\ldots\cdot\partial\beta_2^{-1}\cdot\beta_1^{-1}\cdot\beta'_1\cdot\partial\beta'_2\cdot\ldots\cdot\partial^{n-1}\beta'_n\cdot\mathrm{ev}((\boldsymbol{t}\bullet w')^\dagger).$$

If $\mathrm{Col}(\boldsymbol{t}\bullet w) <^{Lex} \mathrm{Col}(\boldsymbol{t}\bullet w')$ holds, there exists k such that $\beta_i = \beta'_i$ holds for $i < k$, and $\beta_k < \beta'_k$ holds. As β_k and β'_k are special braids, this implies that $\beta_k^{-1}\cdot\beta'_k$ is σ_1-positive, hence $\overline{w}^{-1}\cdot\overline{w'}$ is σ_i-positive, and (i) is true. On the other hand, if $\mathrm{Col}(\boldsymbol{t}\bullet w)$ and $\mathrm{Col}(\boldsymbol{t}\bullet w')$ coincide, there remains $\overline{w}^{-1}\cdot\overline{w'} = \mathrm{ev}((\boldsymbol{t}\bullet w)^\dagger)^{-1}\cdot\mathrm{ev}((\boldsymbol{t}\bullet w')^\dagger)$, and, by definition, $(\boldsymbol{t}\bullet w)^\dagger \prec (\boldsymbol{t}\bullet w')^\dagger$ implies $\mathrm{ev}((\boldsymbol{t}\bullet w)^\dagger) <_F \mathrm{ev}((\boldsymbol{t}\bullet w')^\dagger$, hence $\overline{w} < \overline{w'}$. So (ii) implies (i).

Assume now (iii). For t a large enough tree, $t{\bullet}w$ and $t{\bullet}w'$ are defined, hence, by Lemma 3.10, there exists at least one $B_\bullet$-coloured tree $\boldsymbol{t}$ such that $\boldsymbol{t} \bullet w$ and $\boldsymbol{t} \bullet w'$ exist. Hence (ii) holds.

Finally, assume that (iii) fails. By the argument above, there exists $\boldsymbol{t}$ such that $\boldsymbol{t} \bullet w$ and $\boldsymbol{t} \bullet w'$ exist and (32) fails. Because $<^{Lex}$ and $\prec$ are linear orders, this implies that either w and w' are equivalent, or (32) with w and w' exchanged is true. We saw above that this implies $\overline{w} > \overline{w'}$. So, in any case, (i) fails. $\square$

5. Homeomorphisms of a punctured sphere

Artin's braid group B_n can be realized as the mapping class group of a disk with n punctures [3], and the induced action on the fundamental group gives Artin's representation of B_n in the automorphisms of a rank n free group. In this section, we prove similar results for the group $B_\bullet$. We observe that $B_\bullet$ can be mapped to the mapping class group of a sphere with a Cantor set of punctures, and deduce that $B_\bullet$ embeds in the groups of automorphisms of a free group of countable rank using the ordering of Section 4.

5.1. The mapping class group of a sphere with a Cantor set of punctures.

We aim at mapping $B_\bullet$ into the homeomorphisms of a punctured space. As $B_\bullet$ includes B_∞, disks with infinitely many punctures are to be expected. Moreover the tree-like structure of $B_\bullet$ should make it natural to meet the Cantor set. A suitable choice is to collapse the boundary of the disk, *i.e.*, to start with a 2-sphere, and to remove a Cantor set of punctures. Note that the complement of a Cantor set consists of a countable collection of open intervals naturally indexed by dyadic numbers.

Definition 5.1. (Figure 12) We fix a real number ρ in $(0,1)$—for instance $\rho = 1/3$—and we denote by $\mathbf{K}$ the Cantor subset of $[0,1]$ obtained by iteratively removing the median intervals of size ρ^k. We define $S_{\mathbf{K}}$ to be the topological space obtained from the disk of diameter $[-\rho, 1+\rho]$ in $\mathbf{R}^2$ by removing the points of $\mathbf{K}$ and collapsing the outer circle.

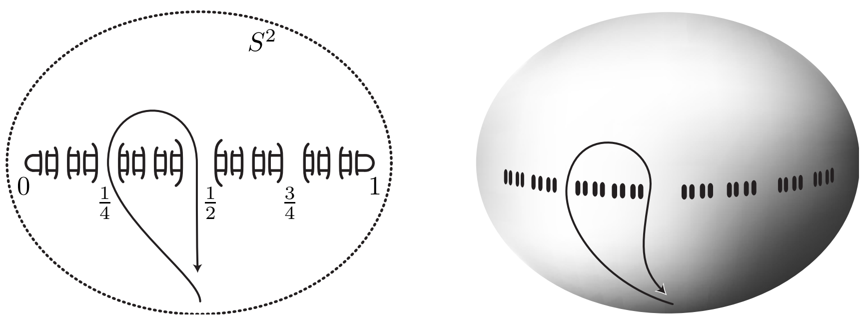


Figure 12. The space $S_{\mathbf{K}}$: a sphere with a Cantor set removed from the equator, or, equivalently, two hemispheres connected by a countable family of bridges indexed by dyadic numbers; the loop represents the element $x_{1,1}^{-1}x_1$ of the fundamental group: it starts from the South pole, crosses the bridge at $\frac{1}{4}$ to the North hemisphere, and returns to the South pole by the bridge at $\frac{1}{2}$.

We denote by $MCG(S_{\mathbf{K}})$ the mapping class group of $S_{\mathbf{K}}$, *i.e.*, the group of all homeomorphisms of $S_{\mathbf{K}}$ up to isotopy. As in the case of a finite set of punctures, a continuous motion in the disk that maps $\mathbf{K}$ to itself determines an element of $MCG(S_{\mathbf{K}})$. Imitating the standard constructions, we can define elements of $MCG(S_{\mathbf{K}})$ corresponding to Dehn's half-twists on the one hand, and to Thompson's piecewise linear homeomorphisms on the other hand.

Definition 5.2. (i) (Figure 13) Let s be a finite sequence of positive integers, say $s = (i_1, \dots, i_r)$. Put $\rho_s := 2^{-i_1-\dots-i_r}\rho$. Then D_s is defined to be the (image in $S_{\mathbf{K}}$ of the) disk with diameter

$$[0.1^{i_1-1}01^{i_2-1}0\dots01^{i_r-1} - \rho_s\ ,\ 0.1^{i_1-1}01^{i_2-1}0\dots01^{i_{r-1}-1}01^{i_r} - \rho_s/2]$$

(referring to the dyadic expansion of rationals; ρ is the constant used in the realization of the Cantor set $\mathbf{K}$, *e.g.*, 1/3).

(ii) (Figure 14) For $i \geqslant 1$, we define $\phi(\sigma_i)$ to be the class in $MCG(S_{\mathbf{K}})$ of a clockwise half-turn (with rescaling) that exchanges D_i and D_{i+1} and is the identity on all other D_j's. We define $\phi(a_i)$ to be the class in $MCG(S_{\mathbf{K}})$ of a motion that fixes D_j for $j < i$, dilates $D_{i,1}$ to D_i, translates $D_{i,j+1}$ to $D_{i+1,j}$ for every j, and contracts D_j to D_{j+1} for $j > i$.

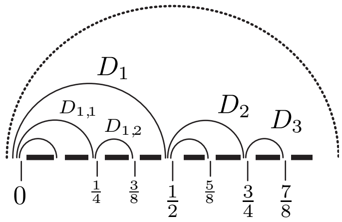


FIGURE 13. The disks D_s: essentially, D_s is the disk based on s and its immediate successor in the lexicographical ordering: for instance, D_1 is essentially the disk with diameter $[0, \frac{1}{2}]$, and $D_{1,1}$ is essentially the disk with diameter $[0, \frac{1}{4}]$; the adjustments guarantee that the disks $D_{s,i}$ are disjoint and nested in D_s

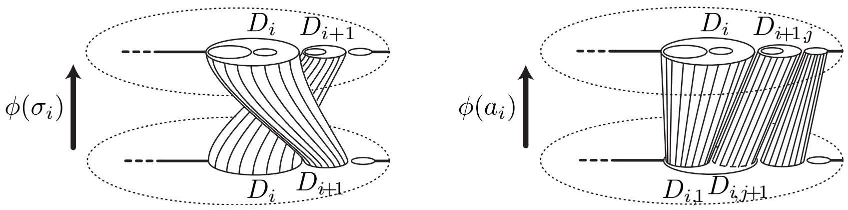


FIGURE 14. Homeomorphisms of $S_{\mathbf{K}}$ associated with σ_i and a_i: a Dehn half-twist, and a dilatation–contraction

An immediate verification shows that all relations in $R_\bullet$ induce isotopies, so we have:

Lemma 5.3. *The mapping ϕ induces a morphism of $B_\bullet$ into $MCG(S_{\mathbf{K}})$.*

5.2. **Action on the fundamental group.** The homeomorphisms of $S_{\mathbf{K}}$ induce automorphisms of its fundamental group, and those coming from the elements of $B_\bullet$ can be described explicitly. We first identify $\pi_1(S_{\mathbf{K}})$.

Definition 5.4. (Figures 12 and 15) For s a finite nonempty sequence of positive integers, we define x_s to be the class in $\pi_1(S_{\mathbf{K}})$ of a loop that starts from the South pole of $S_{\mathbf{K}}$, reaches the South pole of D_s, turns around D_s clockwise, and returns to the South pole of $S_{\mathbf{K}}$. We define x_s to be 1 for s the empty sequence.

Lemma 5.5. *The fundamental group of $S_{\mathbf{K}}$ is the free group $F_\bullet$ based on the x_s's.*

Proof. As $S_{\mathbf{K}}$ is open in S^2, a loop, which is compact, may cross the equator only finitely many times. So, in order to prove that $\pi_1(S_{\mathbf{K}})$ is generated by the x_s's, it is sufficient to show that, for every sequence s, the loop γ_s that starts from the South pole, crosses the equator at the left of 0 and returns to the South hemisphere by the bridge immediately at the right of D_s can be expressed as a product of x_s's. Indeed, as S^2 has no boundary, the loop crossing near 0 and returning near 1 is trivial, and, if we can obtain γ_s, then, by using loops of the form $\gamma_s^{-1}\gamma_{s'}$, we obtain every loop crossing the equator twice, and, from there, every loop crossing the equator finitely many times. Now, one easily checks that, for $s = (i_1, \dots , i_r)$, one can take for γ_s any loop representing

$$(x_1x_2 \dots x_{i_1-1})(x_{i_1,1}x_{i_1,2} \dots x_{i_1,i_2-1}) \dots (x_{i_1,\dots,i_{r-1},1}x_{i_1,\dots,i_{r-1},2} \dots x_{i_1,\dots,i_{r-1},i_r-1}).$$

It remains to show that the x_s's form a free family. Assume that we have a relation in $\pi_1(S_{\mathbf{K}})$, say $w(x_{s_1}, \dots , x_{s_n}) = 1$ with w a freely reduced word. If the disks $D_{s_1}, \dots , D_{s_n}$ are pairwise disjoint, collapsing each of them to a point induces a surjective homomorphism of the subgroup of $\pi_1(S_{\mathbf{K}})$ generated by $x_{s_1}, \dots , x_{s_n}$ onto the fundamental group of a disk with n punctures. The latter is a free group of rank n, so w must be trivial.

Assume now that some disk D_{s_i} includes another disk D_{s_j}. This means that s_i is a prefix of s_j. For each such i, we define $y_i = x_{s_i,1}x_{s_i,2} \dots x_{s_i,p_i}$, where p_i is the minimal p such that (s_i, p) is a prefix of no other index s_j. Note that the process creates no new inclusion. Let φ be the result of collapsing all $x_{s_i,p}$'s with $p > p_i$. By construction, we have $\varphi(x_{s_i}) = y_i$, and, therefore, $w(x_{s_1}, \dots , x_{s_n}) = 1$ implies $w(y_1, \dots , y_n) = 1$. Now, for each i, the variable x_{s_i,p_i} occurs in y_i only, and the disks D_{s_i,p_i} are disjoint. Then the same argument as above shows that w must be trivial. □

The homeomorphisms of $S_{\mathbf{K}}$ induce automorphisms of its fundamental group $F_\bullet$, and we obtain a morphism of $MCG(S_{\mathbf{K}})$ into $\mathrm{Aut}(\pi_1(S_{\mathbf{K}}))$, *i.e.*, into $\mathrm{Aut}(F_\bullet)$.

Proposition 5.6. *Let ψ denote the composition of the above morphism of $MCG(S_{\mathbf{K}})$ to $\mathrm{Aut}(F_\bullet)$ with the morphism ϕ of $B_\bullet$ to $MCG(S_{\mathbf{K}})$. Then ψ maps $B_\bullet$ into $\mathrm{Aut}(F_\bullet)$, and we have*

$$\psi(\sigma_i) : \quad x_{j,s} \mapsto x_{j,s} \quad \text{for } j \neq i, i+1, \qquad x_{i,s} \mapsto x_i x_{i+1,s} x_i^{-1}, \qquad x_{i+1,s} \mapsto x_{i,s}, \tag{33}$$

$$\psi(a_i) : \begin{cases} x_{j,s} \mapsto x_{j,s} \text{ for } j < i, \quad x_{j,s} \mapsto x_{j+1,s} \text{ for } j > i, \\ x_i \mapsto x_i x_{i+1}, \qquad x_{i,1,s} \mapsto x_{i,s}, \qquad x_{i,j+1,s} \mapsto x_{i+1,j,s} \text{ for } j \geqslant 2. \end{cases} \tag{34}$$

Proof. That ψ is a morphism follows from the construction—or from a direct verification, once the explicit formulas for $\psi(\sigma_i)$ and $\psi(a_i)$ are known. The latter can be read in Figure 15. □

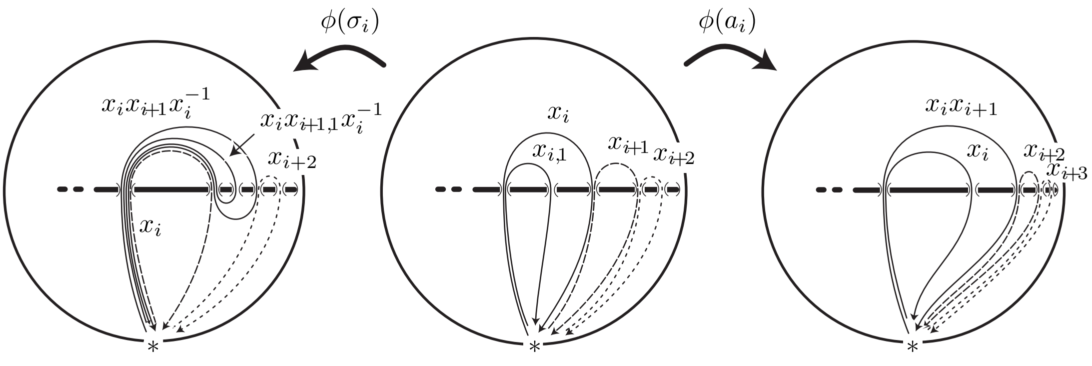


FIGURE 15. Generators of $\pi_1(S_{\mathbf{K}})$, and action of $\phi(\sigma_i)$ and $\phi(a_i)$ on these generators

5.3. **Determining the automorphism.** Once the automorphisms attached with σ_i and a_i are known, we can determine the automorphism of $F_\bullet$ associated with any x in $B_\bullet$ by composing the automorphisms associated with the successive letters of any word representing x. Here we give an alternative description involving $F_\bullet$-coloured trees, *i.e.*, finite binary trees in which the leaves wear colours from $F_\bullet$.

Definition 5.7. We use finite sequences of positive integers as addresses for the nodes in binary trees, as described in Figure 16. Moreover, we define for each node its *natural $F_\bullet$-colour* to be $x_{s,k-1}^{-1}x_{k-2}^{-1}\dots x_{s,1}^{-1}x_s$ for the node with address (s,k).

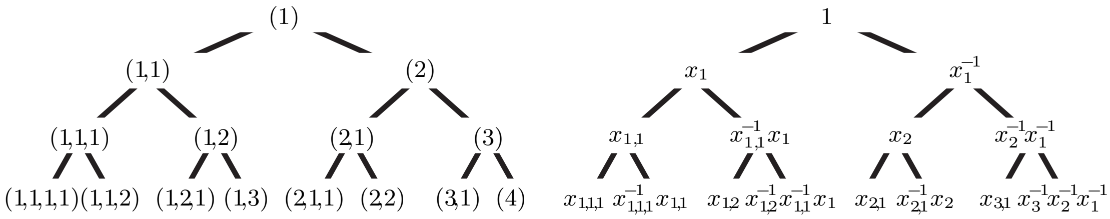


FIGURE 16. Addresses for the nodes in trees, and the associated natural $F_\bullet$-colours; for each s, the variable x_s is the natural colour of the node with address $(s,1)$; we recall that x_s is 1 for s the empty sequence, whence the colours on the right branch

In the sequel, it will be convenient to consider trees in which not only the leaves, but also the inner nodes are given $F_\bullet$-colours.

Definition 5.8. An $F_\bullet$-coloured tree will be called *coherent* if the colour at each inner node is the product of the colours of the left and right sons of the node (in this order).

By construction, when we give to each node in a tree t its natural $F_\bullet$-colour, we obtain a coherent $F_\bullet$-coloured tree that will be called the *natural $F_\bullet$-colouring* of t.

We now introduce a partial action of words on $F_\bullet$-coloured trees extending the action on uncoloured trees. As in the case of $B_\bullet$-coloured trees, the point is to specify how colours behave.

Definition 5.9. For $\boldsymbol{t}$ a coherent $F_\bullet$-coloured tree with $\mathrm{dec}(\boldsymbol{t}) = (\boldsymbol{t}_1, \dots, \boldsymbol{t}_n)$ and $n > i$, the trees $\boldsymbol{t} \bullet \sigma_i$ and $\boldsymbol{t} \bullet a_i$ are determined by:

$$\mathrm{dec}(\boldsymbol{t} \bullet \sigma_i) = (\boldsymbol{t}_1, \dots, \boldsymbol{t}_{i-1}, \boldsymbol{t}', \boldsymbol{t}_i, \boldsymbol{t}_{i+2}, \dots, \boldsymbol{t}_n), \tag{35}$$

$$\mathrm{dec}(\boldsymbol{t} \bullet a_i) = (\boldsymbol{t}_1, \dots, \boldsymbol{t}_{i-1}, \boldsymbol{t}_i\boldsymbol{t}_{i+1}, \boldsymbol{t}_{i+2}, \dots, \boldsymbol{t}_n), \tag{36}$$

where $\boldsymbol{t}'$ is the tree obtained from $\boldsymbol{t}_{i+1}$ by replacing each colour y with xyx^{-1}, where x is the colour of the root in $\boldsymbol{t}_i$. Then, for w a word, $\boldsymbol{t} \bullet w$ is defined so that $\boldsymbol{t} \bullet w^{-1} = \boldsymbol{t}'$ is equivalent to $\boldsymbol{t}' \bullet w = \boldsymbol{t}$, and $\boldsymbol{t} \bullet (w_1w_2) = (\boldsymbol{t} \bullet w_1) \bullet w_2$ holds.

It is easy to check that the previous action preserves coherence. Then we have the following effective method for determining the automorphism of $F_\bullet$ associated with a word w.

Proposition 5.10. *For w a parenthesized braid word, put $\widehat{w} = \psi(\overline{w})$* [1]. *Then $\widehat{w}$ can be determined as follows:*

(i) Choose a tree t that is large enough to ensure that $t \bullet w$ exists;

(ii) Compute $\boldsymbol{t} \bullet w$, where $\boldsymbol{t}$ is the natural $F_\bullet$-colouring of t;

(iii) Then $\widehat{w}$ maps the natural colour of every node in $\boldsymbol{t} \bullet w$ to its actual colour in $\boldsymbol{t} \bullet w$.

Proof. (See Figure 17 for an example). For $\boldsymbol{t}$ an $F_\bullet$-coloured tree and θ a mapping of $F_\bullet$ into itself, we denote by $\boldsymbol{t}^\theta$ the tree obtained from $\boldsymbol{t}$ by replacing each colour x with $\theta(x)$. What we want to prove is the equality $\boldsymbol{t} \bullet w = \boldsymbol{t}'^{\widehat{w}}$ where $\boldsymbol{t}'$ is the natural $F_\bullet$-colouring of $(\boldsymbol{t} \bullet w)^\dagger$.

A direct inspection shows that the result is true when w is a single letter $\sigma_i^{\pm1}$ or $a_i^{\pm1}$. So the point is to show that the result is true for $w = w_1w_2$ when it is for w_1 and w_2. Assume that $t \bullet w$ exists. Denote by $\boldsymbol{t}_1$ the natural $F_\bullet$-colouring of $t \bullet w_1$. By induction hypothesis, we have $\boldsymbol{t} \bullet w_1 = \boldsymbol{t}_1^{\widehat{w_1}}$, hence $\boldsymbol{t} \bullet w = \boldsymbol{t}_1^{\widehat{w_1}} \bullet w_2$. By induction hypothesis again, we have $\boldsymbol{t}_1 \bullet w_2 = \boldsymbol{t}'^{\widehat{w_2}}$, which means that each node with colour x in $\boldsymbol{t}'$, has colour $\widehat{w_2}(x)$ in $\boldsymbol{t}_1 \bullet w_2$. By

[1] where we recall $\overline{w}$ denotes the element of $B_\bullet$ represented by w

construction, this colour is an expression $E(x_{s_1}, \dots, x_{s_p})$ involving some variables $x_{s_1}, \dots, x_{s_p}$ with products and inverses. When we substitute $\boldsymbol{t}_1$ with $\boldsymbol{t}_1^{\widehat{w_1}}$ and let w_2 act, the result is the corresponding expression $E(\widehat{w_1}(x_{s_1}), \dots, \widehat{w_1}(x_{s_p}))$, which is also $\widehat{w_1}(E(x_{s_1}, \dots, x_{s_p}))$ as $\widehat{w_1}$ is a group automorphism. This means that $\boldsymbol{t}_1^{\widehat{w_1}} \bullet w_2$, which is $\boldsymbol{t} \bullet w$, is $\boldsymbol{t}'^{\widehat{w_1} \circ \widehat{w_2}}$, *i.e.*, $\boldsymbol{t}'^{\widehat{w}}$, as expected. ☐

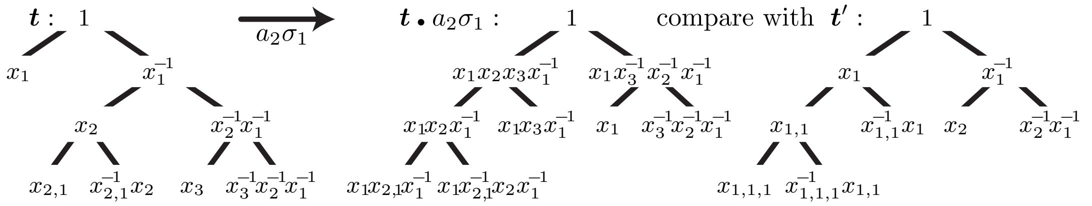


FIGURE 17. Computing the automorphism of $F_\bullet$ associated with $a_2\sigma_1$: we let $a_2\sigma_1$ act on a tree $\boldsymbol{t}$ with natural $F_\bullet$-colours, and compare the colours in $\boldsymbol{t} \bullet a_2\sigma_1$ with the natural ones: the node with natural colour x has colour $\psi(a_2\sigma_1)(x)$ in $\boldsymbol{t} \bullet a_2\sigma_1$. For instance, x_1 is mapped to $x_1x_2x_3x_1^{-1}$ and that $x_{1,1}^{-1}x_1$ is mapped to $x_1x_3x_1^{-1}$.

Remark 5.11. The (partial) actions of $B_\bullet$ on $F_\bullet$- and $B_\bullet$-coloured trees extends to all S-coloured trees where S is a left cancellative ALD-system.

5.4. **The injectivity result.** Artin's representation of B_∞ is an embedding [3]. We extend the result to $B_\bullet$, so obtaining a realization of $B_\bullet$ as a group of automorphisms of a free group.

Proposition 5.12. *The representation ψ of $B_\bullet$ in* $\mathrm{Aut}(F_\bullet)$ *is an embedding.*

Corollary 5.13. *The morphism ϕ of $B_\bullet$ into $MCG(S_{\mathbf{K}})$ is injective.*

The method for proving Proposition 5.12 relies on the possibility of considering words w of a specific form, in connection with the linear ordering of $B_\bullet$ constructed in Section 4. In the case of braids, the method was first used by D. Larue in [26], and it gives a powerful method for proving the possible injectivity of a representation [30, 12].

Definition 5.14. For u a word in the letters $x_s^{\pm 1}$, we denote $\mathrm{red}(u)$ for the freely reduced word obtained from u by removing all pairs xx^{-1} and $x^{-1}x$.

Thus $F_\bullet$ identifies with the set of all freely reduced words. We recall that $\widehat{w}$ denotes the automorphism $\psi(\overline{w})$ of $F_\bullet$ associated with w.

We begin with two auxiliary results. The first one is similar to Proposition 5.1.6 of [19] for braids. The only change is that variables x_s with s of length more than 1 may occur, but this does not change the argument.

Lemma 5.15. *The image of a word ending with x_i^{-1} under $\widehat{\sigma}_i$ or $\widehat{\sigma}_j^{\pm 1}$ with $j > i$ ends with x_i^{-1}.*

Proof. Assume that u ends with x_i^{-1}, say $u = u'x_i^{-1}$. Then we have

$$\widehat{\sigma}_i(u) = \mathrm{red}(\widehat{\sigma}_i(u')x_ix_{i+1}^{-1}x_i^{-1}). \tag{37}$$

In order to prove that the word above ends with x_i^{-1}, it is sufficient to check that the final x_i^{-1} cannot be cancelled during the reduction by some x_i coming from $\widehat{\sigma}_i(u')$. By (33), an x_i in $\widehat{\sigma}_i(u')$ must come from some x_i, x_i^{-1}, or x_{i+1} in u'. We consider the three cases, displaying the supposed involved letter in u'. For $u' = u''x_iu'''$, (37) becomes

$$\widehat{\sigma}_i(u) = \mathrm{red}(\widehat{\sigma}_i(u'')x_ix_{i+1}x_i^{-1}\widehat{\sigma}_1(u''')x_ix_{i+1}^{-1}x_i^{-1}).$$

The assumption that the first x_i cancels the final x_i^{-1} implies $\widehat{\sigma}_i(u''') = \varepsilon$, hence $u''' = \varepsilon$, contradicting the hypothesis that $u''x_iu'''x_i^{-1}$ is reduced. For $u' = u''x_i^{-1}u'''$, (37) is

$$\widehat{\sigma}_i(u) = \mathrm{red}(\widehat{\sigma}_i(u'')x_ix_{i+1}^{-1}x_i^{-1}\widehat{\sigma}_1(u''')x_ix_{i+1}^{-1}x_i^{-1}).$$

The assumption that the first x_i cancels the final x_i^{-1} implies now that $x_{i+1}^{-1}x_i^{-1}\widehat{\sigma}_i(u''')x_ix_{i+1}^{-1}$ reduces to ε, hence $\widehat{\sigma}_i(u''') = x_ix_{i+1}^2x_i^{-1}$, and, therefore, $u''' = x_i^2$, again contradicting the hypothesis that $u''x_i^{-1}u'''$ is reduced. Finally, for $u' = u''x_{i+1}u'''$, (37) says

$$\widehat{\sigma}_i(u) = \mathrm{red}(\widehat{\sigma}_i(u'')x_i\widehat{\sigma}_1(u''')x_ix_{i+1}^{-1}x_i^{-1}).$$

The assumption that the first x_i cancels the final x_i^{-1} implies that $\widehat{\sigma}_i(u''')x_ix_{i+1}^{-1}$ reduces to ε, hence $\widehat{\sigma}_i(u''') = x_{i+1}x_i^{-1}$, and, then, $u''' = x_{i+1}^{-1}x_i$, contradicting the hypothesis that $u''x_{i+1}u'''$ is reduced. We similarly consider the action of $\widehat{\sigma}_j^e$ with $j > i$ and $e = \pm 1$. We find

$$\widehat{\sigma}_j(u) = \mathrm{red}(\widehat{\sigma}_j^e(u')x_i^{-1}), \tag{38}$$

and aim at proving that the final x_i^{-1} cannot vanish in reduction. Now it could do it only with some x_i in $\widehat{\sigma}_j^e(u')$, itself coming from some x_i in u'. For a contradiction, we display the latter as $u' = u''x_iu'''$. Then (38) becomes $\widehat{\sigma}_j(u) = \mathrm{red}(\widehat{\sigma}_j^e(u'')x_i\widehat{\sigma}_j^e(u''')x_i^{-1})$. As above, we must have $\widehat{\sigma}_j^e(u''') = \varepsilon$, hence $u''' = \varepsilon$, contradicting the hypothesis that $u''x_iu'''x_i^{-1}$ is reduced. □

The second preliminary result is specific to our current situation.

Definition 5.16. A word in the letters $x_s^{\pm 1}$ is said to be *special* if it is freely reduced and it admits a suffix of the form $x_s^{-1}x_{s,j_1,s_1}\dots x_{s,j_r,s_r}$ with $r \geqslant 0$, where $s, s_1, \dots, s_r$ are sequences, and $j_1, \dots, j_r$ are positive integers.

Thus x_1^{-1} and $x_1x_2^{-1}x_{2,1}$ are special words.

Lemma 5.17. *For each i, the image of a special word under $\widehat{a}_i^{-1}$ is a special word.*

Proof. Let $u = u'x_{j,s}^{-1}x_{j,s,j_1,s_1}\dots x_{j,s,j_r,s_r}$ be a special word. We consider the image of u under $\widehat{a}_i^{-1}$, according to the mutual positions of i and j. Assume first $j < i$. Then we have $\widehat{a}_i^{-1}(x_{j,s}) = x_{j,s}$, and, similarly, $\widehat{a}_i^{-1}(x_{j,s,j_k,s_k}) = x_{j,s,j_k,s_k}$ for each k, hence

$$\widehat{a}_i^{-1}(u) = \mathrm{red}(\widehat{a}_i^{-1}(u')x_{j,s}^{-1}x_{j,s,j_1,s_1}\dots x_{j,s,j_r,s_r}). \tag{39}$$

In order to conclude that this word is special, it suffices to prove that the displayed letter $x_{j,s}^{-1}$ cannot vanish during reduction. Now assume it does. The letter $x_{j,s}^{-1}$ is cancelled by some letter $x_{j,s}$ coming from $\widehat{a}_i^{-1}(u')$. The explicit formulas for $\widehat{a}_i^{-1}$ are

$$\widehat{a}_i^{-1} : \begin{cases} x_{j,s} \mapsto x_{j,s} \text{ for } j < i, \quad x_{j+1,s} \mapsto x_{j,s} \text{ for } j > i, \\ x_{i,s} \mapsto x_{i,1,s}, \qquad x_{i+1} \mapsto x_{i,1}^{-1}x_i, \qquad x_{i+1,j,s} \mapsto x_{i,j+1,s} \text{ for } j \geqslant 2. \end{cases} \tag{40}$$

So a letter $x_{j,s}$ in $\widehat{a}_i^{-1}(u')$ must come from a letter $x_{j,s}$ of u'. Let us display the considered letter and write $u' = u''x_{j,s}u'''$. Then (39) becomes

$$\widehat{a}_i^{-1}(u') = \mathrm{red}(\widehat{a}_i^{-1}(u'')x_{j,s}\widehat{a}_i^{-1}(u''')).$$

The assumption that the final $x_{j,s}^{-1}$ in $\widehat{a}_i^{-1}(u')x_{j,s}^{-1}$ is cancelled by the displayed $x_{j,s}$ implies $\widehat{a}_i^{-1}(u''') = \varepsilon$, hence $u''' = \varepsilon$ as $\widehat{a}_i$ is an automorphism. This means that u' finishes with $x_{j,s}$, contradicting the hypothesis that $u'x_{j,s}^{-1}$ is reduced.

The argument is similar for x_j with $j > i + 1$, and, more generally, it works for all x_t's except x_i and x_{i+1}. Indeed, in these cases, $\widehat{a}_i^{-1}$ maps $x_{j,s}$ to a (possibly different) letter $x_{j',s'}$ so that a letter $x_{j',s'}$ in $\widehat{a}_i^{-1}(v)$ must come from a $x_{j,s}$ in v. Then, the previous argument shows that the letter $x_{j,s}^{-1}$ witnessing for specialness becomes a letter $x_{j',s'}^{-1}$ that cannot be cancelled.

On the other hand, (40) shows that, in all considered cases, the final letters x_{j,s,j_k,s_k} become letters x_{j',s',j_k,s_k}, so the word $\widehat{a}_i^{-1}(u)$ is special.

There remain the cases of x_i and x_{i+1}. To simplify reading, we assume $i = 1$. Let us first consider x_1, *i.e.*, $u = u' x_1^{-1} x_{1,j_1,s_1} \dots x_{1,j_r,s_r}$, which gives

$$\widehat{a}_1^{-1}(u) = \mathrm{red}(\widehat{a}_1^{-1}(u') x_{1,1}^{-1} x_{1,1,j_1,s_1} \dots x_{1,1,j_r,s_r}). \tag{41}$$

If the displayed $x_{1,1}^{-1}$ does not vanish during reduction, the above word is special. We shall see now that $x_{1,1}^{-1}$ may vanish, but one nevertheless obtains a special word. Indeed, (40) shows that an $x_{1,1}$ in $\widehat{a}_1^{-1}(u')$ comes either from an x_1 or from an x_2^{-1} in u'. By the same argument as above, x_1 is excluded. So assume $u' = u'' x_2^{-1} u'''$. Then (41) becomes

$$\widehat{a}_1^{-1}(u) = \mathrm{red}(\widehat{a}_1^{-1}(u'') x_1^{-1} x_{1,1} \widehat{a}_1^{-1}(u''') x_{1,1}^{-1} x_{1,1,j_1,s_1} \dots x_{1,1,j_r,s_r},$$

and the assumption is $\widehat{a}_1^{-1}(u''') = \varepsilon$. As above, we deduce $u''' = \varepsilon$, hence $u' = u'' x_2^{-1}$—which is not forbidden. In this case, we find

$$\widehat{a}_1^{-1}(u) = \mathrm{red}(\widehat{a}_1^{-1}(u'') x_1^{-1} x_{1,1,s_1} \dots x_{1,1,s_r}). \tag{42}$$

To show that this word is special, it is sufficient to prove that the x_1^{-1} cannot disappear. Now the only way x_1^{-1} could vanish is with some x_1 in $\widehat{a}_1^{-1}(u'')$, necessarily coming from some x_2 in u''. Write $u'' = u''' x_2 u''''$. As above, we obtain $\widehat{a}_1^{-1}(u'''') = \varepsilon$, hence $u'''' = \varepsilon$, implying that u'' finishes with x_2, and contradicting the hypothesis that $u'' x_2^{-1}$ is reduced. So the study for x_1 is complete.

Finally, let us consider the case of x_2. The problem here is that $\widehat{a}_1^{-1}$ maps x_2 to $x_{1,1}^{-1} x_1$, which is not a single letter. So assume $u = u' x_2^{-1} x_{2,j_1,s_1} \dots x_{2,j_r,s_r}$. We obtain

$$\widehat{a}_1^{-1}(u) = \mathrm{red}(\widehat{a}_1^{-1}(u') x_1^{-1} x_{1,1} x_{1,j_1+1,s_1} \dots x_{1,j_r+1,s_r}). \tag{43}$$

In order to show that this word is special, it suffices to prove that the letter x_1^{-1} cannot vanish. Now a letter x_1 in $\widehat{a}_1^{-1}(u')$ must come from a letter x_2 in u', and we argue as above. ☐

We can now prove the injectivity of the homomorphism ψ of $B_\bullet$ into $\mathrm{Aut}(F_\bullet)$.

Proof of Proposition 5.12. Our aim is to show that, if w is a word that represents a non-trivial element of $B_\bullet$, then the automorphism $\widehat{w}$ (*i.e.*, $\psi(\overline{w})$) is not the identity mapping, *i.e.*, there exists at least one letter x_s such that $\widehat{w}(x_s)$ is not x_s. By Proposition 4.14, at the expense of replacing w by an equivalent word and possibly exchanging w and w^{-1}, we may assume that w is either σ_i-positive or is a non-trivial a-word.

Case 1: w is σ_i-positive. By definition, we can write $w = w_1^{-1} w_2 w_3$, where w_1 and w_3 are positive a-words, and w_2 is a σ_i-positive σ-word. First, because w_3 contains positive letters a_k only, there exists a vine t such that $t \bullet w_3$ is defined and we may assume in addition that the right height of t is at least $i+1$. Let $\boldsymbol{t}$ be the natural $F_\bullet$-colouring of t. By construction, x_i is a colour in $\boldsymbol{t}$, hence in $\boldsymbol{t} \bullet w_3$, and Proposition 5.10 implies that there must exist x in $F_\bullet$ such that $\widehat{w_3}$ maps x to x_i. All colours in a natural $F_\bullet$-colouring are not single variables, but this is always the case for nodes with addresses ending with 1. So, in any case, the left son of the node where x_i occurs has colour $x_{i,1}$ in $\boldsymbol{t} \bullet w_3$, and colour x_s for some s in the natural colouring of $\boldsymbol{t} \bullet w_3$. In other words, there exists s satisfying $\widehat{w_3}(x_s) = x_{i,1}$.

We now consider $\widehat{w_2}(\widehat{w_3}(x_s))$, *i.e.*, $\widehat{w_2}(x_{i,1})$. Write $w_2 = w'_0 \sigma_i w'_1 \sigma_i \dots \sigma_i w'_r$, where w'_k contains no $\sigma_j^{\pm 1}$ with $j \leqslant i$. Then $\widehat{w'_r}$ fixes $x_{i,1}$, while σ_i maps it to $x_i x_{i+1,1} x_i^{-1}$, a reduced word ending with x_i^{-1}. Applying Lemma 5.15 repeatedly, we deduce that the final x_i^{-1} cannot disappear, and, so, $\widehat{w_2}(\widehat{w_3}(x_s))$ is a reduced word ending with x_i^{-1}.

Consider now the action of $\widehat{w_1}^{-1}$ on the latter word. Every reduced word ending with x_i^{-1} is a special word, hence, by Lemma 5.17, its image under $\widehat{w_1}^{-1}$ is a special word. Hence $\widehat{w}(x_s)$ is a special word. As x_s is not a special word, $\widehat{w}$ cannot be the identity mapping.

Case 2: w is a non-trivial a-word. Let t, t' be trees satisfying $t' = t \bullet w$. The hypothesis that w is non-trivial implies $t' \neq t$. Then there must exist an address s such that $(s, 1)$ is an address in t' and not in t. Then x_s occurs in the natural $F_\bullet$-colouring $\boldsymbol{t'}$ of t', and not in the natural $F_\bullet$-colouring $\boldsymbol{t}$ of t. Proposition 5.10 implies that $\widehat{w}(x_s)$ is a combination of colours occurring in $\boldsymbol{t'}$, so it cannot be x_s, and $\widehat{w}$ is not the identity mapping. □

An application of Proposition 5.12 is an alternative proof of the fact that the relations $R_\bullet$ make a presentation of the group $B_\bullet$. Indeed, ignoring the injectivity of $\pi : \widetilde{B}_\bullet \to B_\bullet$, we can construct a morphism $\widetilde{\psi}$ of $\widetilde{B}_\bullet$ to $\mathrm{Aut}(F_\bullet)$ using the explicit formulas of Proposition 5.6. Then Proposition 5.10 shows that, for each word w, the automorphism $\widetilde{\psi}(\overline{w})$ can be recovered from the action of w on $F_\bullet$-coloured trees. Now the latter can in turn be deduced from the diagram $\mathcal{D}(w)$ using $F_\bullet$-colourings, hence from the isotopy class of $\mathcal{D}(w)$ as isotopy preserves colours. So $\widetilde{\psi}(\overline{w})$ depends on the image of w in $B_\bullet$ only, *i.e.*, $\widetilde{\psi}$ factors through $B_\bullet$:

$$\begin{array}{ccc} \widetilde{B}_\bullet = \langle a_*, \sigma_* \,;\, R_\bullet \rangle & \xrightarrow{\ \widetilde{\psi}\ } & \mathrm{Aut}(F_\bullet) \\ \pi \downarrow & \overset{\psi}{\nearrow} & \\ B_\bullet = \{\text{parenthesized diagrams}\}/\text{isotopy} & & \end{array}$$

What Proposition 5.12 shows is that $\widetilde{\psi}$ is injective, which implies that both π and ψ are injective.

6. Miscellani

We conclude with a few additional remarks about $B_\bullet$.

6.1. **Pure parenthesized braids.** Each braid induces a permutation of positive integers, which leads to a surjective homomorphism of B_∞ onto the group S_∞ of eventually trivial permutations. The group S_∞ is the quotient of B_∞ under the relations $\sigma_i^2 = 1$, and the kernel is the pure braid group PB_∞. The situation is similar with $B_\bullet$. The quotient of $B_\bullet$ obtained by adding the relations $\sigma_i^2 = 1$ is the subgroup $S_\bullet$ of Thompson's group V made of the elements that, in the action of V on the Cantor set $\mathbf{K}$, preserve the right endpoint; see [18], and [5, 6] where this group is called $\widehat{V}$. Then the kernel of the projection $B_\bullet \to S_\bullet$ is a non-trivial normal subgroup $PB_\bullet$ of $B_\bullet$, whose elements can be called *pure* parenthesized braids.

Proposition 6.1. *We have* $PB_\bullet = (F^+)^{-1} \cdot PB_\infty \cdot F^+$.

One inclusion is trivial, and the other follows from the equality $B_\bullet = (F^+)^{-1} \cdot B_\infty \cdot F^+$.

6.2. **Alternative presentations.** Alternative presentations of $B_\bullet$ have been considered. On the one hand, exactly as Thompson's group F is generated by the two elements here denoted a_1 and a_2, the group $B_\bullet$ is generated by $\sigma_1, \sigma_2, a_1, a_2$, and it is a finitely presented group [6].

On the other hand, large presentations may also of interest. The presentation $(a_*, \sigma_*, R_\bullet)$ gives different roles to the left and right sides. This in particular implies that $B_\bullet$ is a group of left fractions of $B_\bullet^+$ only, and that right common multiples need not exist in $B_\bullet^+$. As shown in [18], $B_\bullet$, as well as Thompson's groups F and V, can be given a balanced presentation. The principle is to consider new generators similar to σ_i and a_i but acting at any possible address in a tree, and not only at addresses on the rightmost branch. In the current framework, it is natural to denote by σ_s and a_s such generators, with s a finite sequence of positive integers. For instance, $\sigma_{1,1}$ corresponds to applying σ_1 at the address $(1,1)$ (in the sense of Figure 16) instead of at (1), which amounts to defining $\sigma_{1,1} = a_1^{-1} a_2^{-1} \sigma_1 a_2 a_1$. We obtain in this way an extended double family of generators σ_s, a_s, and, using the techniques of [18], one can show:

Proposition 6.2. *In terms of the generators σ_s and a_s, a presentation of $B_\bullet$ is:*

$$x_{s,i,s'}y_{s,j,s''} = y_{s,j,s''}x_{s,i,s'} \qquad \text{for } j \neq i, \tag{44}$$

$$\sigma_{s,i}x_{s,j,s'} = x_{s,j,s'}\sigma_{s,i} \qquad a_{s,i}x_{s,j,s'} = x_{s,j-1,s'}a_{s,i} \qquad \text{for } j \geqslant i+2, \tag{45}$$

$$x_{s,i,j,s'}\sigma_{s,i} = \sigma_{s,i}x_{s,i+1,j,s'}, \qquad x_{s,i+1,j,s'}\sigma_{s,i} = \sigma_{s,i}x_{s,i,j,s'}, \tag{46}$$

$$x_{s,i,1,s'}a_{s,i} = a_{s,i}x_{s,i,1,1,s'}, \qquad x_{s,i+1,j,s'}a_{s,i} = a_{s,i}x_{s,i,j+1,s'}, \tag{47}$$

$$\sigma_{s,i}\sigma_{s,i+1}\sigma_{s,i} = \sigma_{s,i+1}\sigma_{s,i}\sigma_{s,i+1}, \ \sigma_{s,i+1}\sigma_{s,i}a_{s,i+1} = a_{s,i}\sigma_{s,i}, \ \sigma_{s,i}\sigma_{s,i+1}a_{s,i}, = a_{s,i+1}\sigma_{s,i}, \tag{48}$$

$$\sigma_{s,i}a_{s,i+1}a_{s,i} = a_{s,i+1}a_{s,i}a_{s,i,1}, \qquad a_{s,i}a_{s,i} = a_{s,i+1}a_{s,i}a_{s,i,1}, \tag{49}$$

with i, j positive integers, s, s', s'' sequences of positive integers, and x, y denoting any of σ or a.

Despite its apparent complexity, the above presentation is simple: in addition to the relations of $R_\bullet$, it only contains more or less trivial commutation relations, plus the last relation in (49), which is MacLane's pentagon relation [27]. The advantage of this presentation is that it restores the symmetry between left and right—this becomes more evident when sequences of 0's and 1's are used as addresses [18]. In particular, the presentation leads to a new monoid, larger than $B_\bullet^+$, in which both left and right lcm's exist, and $B_\bullet$ is both a group of left and right fractions of this monoid.

6.3. **Artin's representation of the group BV.** In [5, 6], M. Brin introduces two groups denoted BV and $\widehat{BV}$, for which he establishes presentations. The presentation of $\widehat{BV}$ shows that this group is isomorphic to $B_\bullet$. The group BV, which is an extension of Thompson's group V, includes $\widehat{BV}$, hence $B_\bullet$, as a subgroup, but, at the same time, it identifies with the subgroup $B_\bullet^{(1)}$ of $B_\bullet$ consisting of the parenthesized braids in which only the strands starting at a positions $(1, s)$—*i.e.*, 1 or infinitely close—may be braided. For instance, $a_1^{-1}\sigma_1 a_1$ is a typical element of $B_\bullet^{(1)}$. By using the Artin representation of $B_\bullet^{(1)}$, we obtain a representation of the group BV into $\mathrm{Aut}(F_\bullet)$. From the point of view of an action on trees, BV can be obtained from $B_\bullet$ by adding new generators c_i, $i \geqslant 1$, whose effect is to switch the subtrees t_i and $t_{i+1}\dots t_n\bullet$ of the right decomposition.

Proposition 6.3. *Defining*

$$\psi(c_i) : x_{j,s} \mapsto x_{j,s} \text{ for } j < i, \quad x_i \mapsto x_i^{-1}, \quad x_{i,j,s} \mapsto x_i x_{i+j,s} x_i^{-1}, \quad x_{i+j,s} \mapsto x_{i,j,s}$$

extends the embedding ψ of Proposition 5.6 to the group BV.

6.4. **Further questions.** Owing to the many results about B_∞ and F, in particular in terms of (co)-homology and geometry of the Cayley graph, investigating $B_\bullet$ in these directions seems a promising project.

7. Appendix: The cube condition for the presentation $(a_*, \sigma_*, R_\bullet)$

The algebraic results of Section 2 rely on the fact that the presentation $(a_*, \sigma_*, R_\bullet)$ satisfies the so-called left and right cube conditions. Verifying these combinatorial properties requires that we consider all possible triples of letters. There are infinitely many of them, but only finitely many different patterns may appear, and the needed verifications are finite in number. Here we give some details.

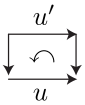
The left cube condition. The left cube condition for a triple of letters (x, y, z) claims that, whenever the word $xy^{-1}yz^{-1}$ is left reversible to some word $v^{-1}u$ with u, v containing no negative letter, then $vxz^{-1}u^{-1}$ is left reversible to the empty word ε.

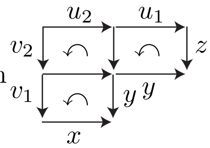
In the presentation $(a_*, \sigma_*, R_\bullet)$, there exists exactly one relation $ux = vy$ for each pair of letters x, y, hence there exists at most one way to reverse a word w to a word of the form $v^{-1}u$ with u, v positive. We shall denote by u/v the unique positive u' such that uv^{-1} is left reversible to $v'^{-1}u'$ for some positive v', if such words exist. If w is left reversible to w', then w^{-1} is left reversible to w'^{-1}, and therefore, if uv^{-1} is left reversible to $v'^{-1}u'$, the latter is $(v/u)^{-1}(u/v)$. So, for instance, we have $\sigma_1/\sigma_2 = \sigma_2\sigma_1$ and $\sigma_2/\sigma_1 = \sigma_1\sigma_2$, and (7) rewrites as

(50) $$\sigma_i/a_j = \mathrm{db}_j(\sigma_i), \qquad a_j/\sigma_i = a_{\sigma_i[j]}.$$

In the case of two a_i's, the formula for / always takes the form $a_i/a_j = a_{i'}$. The index i' will be denoted i/j. For instance, one has $1/2 = 1$ and $2/1 = 3$. It is then easy to verify the equalities

(51) $$\mathrm{db}_k(\sigma_i)/\mathrm{db}_k(\sigma_j) \equiv \mathrm{db}_{\sigma_j[k]}(\sigma_i/\sigma_j), \qquad \sigma_k[i]/\sigma_k[j] = \mathrm{db}_j(\sigma_k)[i/j],$$

where $\equiv$ denotes $R_\bullet$-equivalence. Let us write v'  v when uv^{-1} is left reversible to $v'^{-1}u'$.

The left cube condition for (x, y, z) means that, when we fill the diagram , then

the word $v_1v_2xz^{-1}u_1^{-1}u_2^{-1}$ must be left reversible to ε, *i.e.*, filling the corresponding diagram leads to ε edges on the left and the top side.

We are ready to consider all possible triples of letters. We sort them according to the numbers of σ's and a's. In the case of three σ's or of three a's, the condition is already known. So, we have only to consider the four cases corresponding to one a and two σ's, or two σ's and one a. The values follow from the formulas of (50) and (51). Figure 18 gives the details for the (σ, σ, a) case; the other three cases are similar.

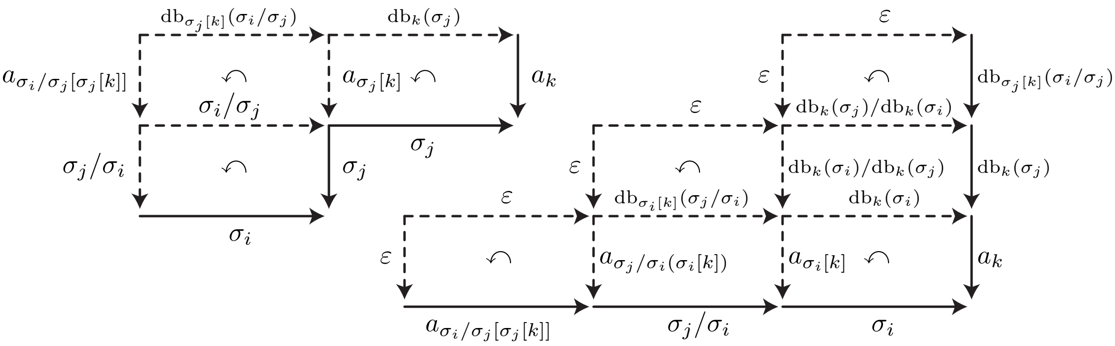


FIGURE 18. Left cube condition for the triples (σ, σ, a): one first reverses $\sigma_i\sigma_j^{-1}\sigma_j a_k^{-1}$ to $(\sigma_j/\sigma_i)^{-1}(a_{\sigma_i/\sigma_j[\sigma_j[k]]})^{-1}(\mathrm{db}_{\sigma_j[k]}(\sigma_i/\sigma_j))(\mathrm{db}_k(\sigma_j))$, then restart from $(a_{\sigma_i/\sigma_j[\sigma_j[k]]})\,(\sigma_j/\sigma_i)\,(\sigma_i)\,(\sigma_k)^{-1}\,(\mathrm{db}_k(\sigma_j))^{-1}\,(\mathrm{db}_{\sigma_j[k]}(\sigma_i/\sigma_j))^{-1}$ and check that the latter is left reversible to ε; the values follow from (51) and the fact that the permutations associated with $(\sigma_i/\sigma_j)\sigma_j$ and $(\sigma_j/\sigma_i)\sigma_i$ coincide, as both come from the left lcm of the involved braid.

The right cube condition. The verifications for the right cube condition are similar, except that we use right reversing, *i.e.*, we push the negative letters to the right. Again, right reversing leads to at most one final word of the from uv^{-1} with u, v positive, but, in contrast to left reversing, right reversing need not converge: $R_\bullet$ contains no relation of the form $a_iu = a_{i+1}v$ or $\sigma_iu = a_iv$, hence $a_i^{-1}a_{i+1}$ and $\sigma_i^{-1}a_i$ are not right reversible.

It is possible to establish general formulas similar to (50) and (51). Denote by $u\backslash v$ and $v\backslash u$ the unique positive words such that $u^{-1}v$ is right reversible to $(u\backslash v)(v\backslash u)^{-1}$, if such words exist. Then, if u, v are σ-words, $u\backslash(va_j)$, when it exists, is obtained from $u\backslash(v\sigma_j)$ by replacing the final σ_k with the corresponding a_k, and $a_j\backslash u$, when it exists, is obtained from u by erasing the j-th strand (in the braid diagram coded by u). However, such formulas are not very convenient as they do not guarantee that the considered words exist, and it is actually easier to systematically consider all possible cases, which are not so many owing to symmetries and trivial cases. Because of the above mentioned formula, all words appearing have length 6 at most, and the less trivial cases are when the indices are neighbours. A typical example is given in Figure 19; all other cases are similar or more simple.

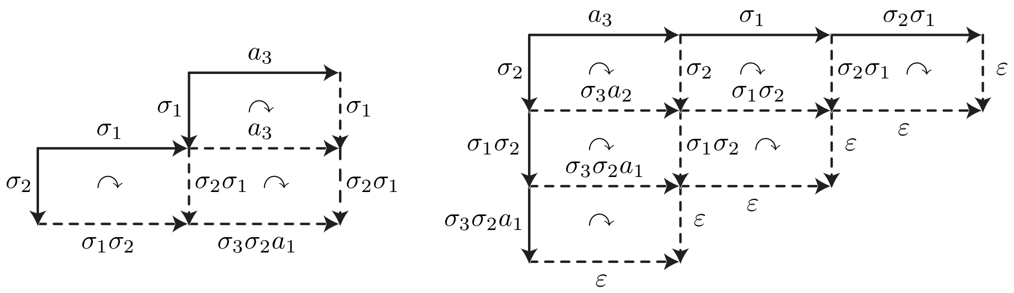


FIGURE 19. Right cube condition for the triple $(\sigma_2, \sigma_1, a_3)$: one first reverses $\sigma_2^{-1}\sigma_1\sigma_1^{-1}a_3$ to a positive–negative word, here $\sigma_1\sigma_2\sigma_3\sigma_2a_1\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}$, and, then, one checks that $a_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}a_3\sigma_1\sigma_2\sigma_1$ is right reversible to ε.

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Index of terms and notation

$\curvearrowright$ (word reversing): Def. 2.1
$x\lceil y\rceil$ (operation on $B_\bullet$): Def. 3.4
$x \circ y$ (operation on $B_\bullet$): Def. 3.4
$\bullet_x$ (coloured tree): Def. 3.7
$\prec$ (tree ordering): Def. 4.1
$<^{sp}_F$ (special order on F): Def. 4.3
$<^+$ (positive ordering): Def. 4.6
$<$ (ordering): Def. 4.10
a_*: (family of all a_i's): Def. 1.16
address (node of a tree): Def. 5.7
ALD-system (augmented LD-system): Def. 3.1
a-word (parenthesized braid word): Sec. 1.3
$B_\bullet$ (group of parenthesized braids): Def. 1.14
$\widetilde{B}_\bullet$ (group presented by $R_\bullet$): Def. 1.16
$\widetilde{B}^+_\bullet$ (monoid presented by $R_\bullet$): Def. 2.9
B^{sp}_∞ (special braids): Def. 4.23
c_n (right vine): Example 1.3
$c_{t,t'}$ (diagram completion): Def. 1.11
Col($\boldsymbol{t}$) (colours in a tree): Def. 4.23
C (positive cone): Def. 4.8
coherent ($F_\bullet$-coloured tree): Def. 5.8
complete (presentation): Def. 2.3
completion (of a braid diagram): Def. 1.11
∂ (shift mapping): Def. 3.2
$\mathcal{D}_t(w)$ (braid diagram): Def. 1.6
$\mathrm{db}_k(w)$ (strand doubling): Def. 2.15
dec(t) (tree decomposition): Def. 1.7
D_s (disk): Def. 5.2
Dyad(t) (rationals associated with t): Def. 1.2
dyadic realization (sequence): Def. 1.1
equivalent (braid diagrams): Def. 1.10
ev($\boldsymbol{t}$) (evaluation of a coloured tree): Def. 3.12
ev*($\boldsymbol{t}$) (evaluation of a coloured tree): Def. 3.12
ϕ (morphism of $B_\bullet$ to $MCG(S_{\mathbf{K}})$): Def. 5.2
$H(f)$ (homeomorphism): Prop. 4.4
homogeneous (presentation): Def. 2.4
$\mathbf{K}$ (Cantor set): Def. 5.1
LD-system: Def. 3.1
$\mathbf{N}_\bullet$ (set of all positions): Def. 1.1
naturel (colouring): Def. 5.7
parenthesized braid: Def. 1.14
Pos(t) (positions associated with t): Def. 1.2
position: Def. 1.1
ψ (morphism of $B_\bullet$ to Aut($F_\bullet$)): Prop. 5.6
ρ (construction of a Cantor set): Def. 5.1
$R_\bullet$ (relations): Def. 1.16
rack: Def. 3.1
red(u) (free reduced word): Def. 5.14
reversing: Def. 2.1
$s^\#$ (dyadic realization): Def. 1.1
special (parenthesized braid): Def. 3.16
special (word): Def. 5.16
σ_* (family of all σ_i's): Def. 1.16
$S_{\mathbf{K}}$ (sphere with a Cantor removed): Def. 5.1
σ-word, σ,a-word (parenthesized braid word): Sec. 1.3
σ_i-positive (word): Def. 4.13
skeleton (tree): Def. 3.7
$t^\dagger$ (skeleton of t): Def. 3.7
$\boldsymbol{t} \bullet w$ (action on $B_\bullet$-coloured tree): Def. 3.8
$\boldsymbol{t} \bullet w$ (action on $F_\bullet$-coloured tree): Def. 5.9
tidy (word): Def. 2.24
$w[k]$ (initial position): Def. 2.15
$\overline{w}$ (element represented by w): Lemma 3.14
$\widehat{w}$ (automorphism of $F_\bullet$): Prop. 5.10
x_s (loop class): Def. 5.4

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