COMBINATORIAL DISTANCE BETWEEN BRAID WORDS

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Abstract. We give a simple naming argument for establishing lower bounds on the combinatorial distance between (positive) braid words.

[1] It is well-known that, for $n \geqslant 3$, Artin's braid group B_n, which is the group defined by the presentation

$$\left\langle \sigma_1, ..., \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for} \quad |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for} \quad |i-j| = 1 \end{array} \right\rangle$$

has a quadratic Dehn function, *i.e.*, there exist constants C_n, C'_n such that, if w is an n-strand braid word of length ℓ that represents the unit braid, then the number of braid relations needed to transform w into the empty word is at most $C_n\ell^2$ and, on the other hand, there exists for each ℓ at least one length ℓ word w such that the minimal number of such braid relations is at least $C'_n\ell^2$—see for instance [2].

In a recent posting [3], Hass, Kalka, and Nowik developed a knot theoretical argument for establishing lower bounds on the combinatorial distance between two equivalent positive braid words, *i.e.*, on the minimal number of braid relations needed to transform the former into the latter. Using some knot invariants introduced in [4], they prove

Proposition 1. *For each m, the combinatorial distance between the (equivalent) braid words $\sigma_1^{2m}(\sigma_2\sigma_1^2\sigma_2)^m$ and $(\sigma_2\sigma_1^2\sigma_2)^m\sigma_1^{2m}$ equals $4m^2$.*

The purpose of this note is to observe that the above result also follows from the direct combinatorial argument similar to the one developed in [1] for the reduced expressions of a permutation.

By definition, a braid word w is a finite sequence of letters σ_i and their inverses, and it naturally encodes a braid diagram $\mathcal{D}(w)$ once one decides that σ_i encodes the elementary diagram in which the $(i+1)$st strand passes over the ith strand. For instance, the diagrams associated with the words of Proposition 1 are displayed in Figure 1.

For simplicity, we restrict to positive braid words, *i.e.*, words that contain no letter σ_i^{-1}—see Remark 5 below. Each strand in a braid diagram has a well-defined initial position, hereafter called its *name*, and we can associate with each crossing of the diagram, hence with each letter in the braid word that encodes it, the names of the strands involved in the crossing. As two strands may cross more than once, we shall also include the rank of the crossing, thus using the name $\{p, q\}_a$ for the ath crossing of the strands with initial positions p and q. In this way, we associate with each positive braid word a sequence of names:

Definition 2. (See Figure 1.) For w a positive braid word, the sequence $S(w)$ is defined to be empty if w is the empty word and, for $w = w'\sigma_i$, to be the sequence

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obtained from $S(w')$ by appending $\{p,q\}_a$, where p and q are the initial positions of the strands that finish at position i and $i+1$ in $\mathcal{D}(w')$ and $a-1$ is the number of times the latter strands cross in $\mathcal{D}(w')$.

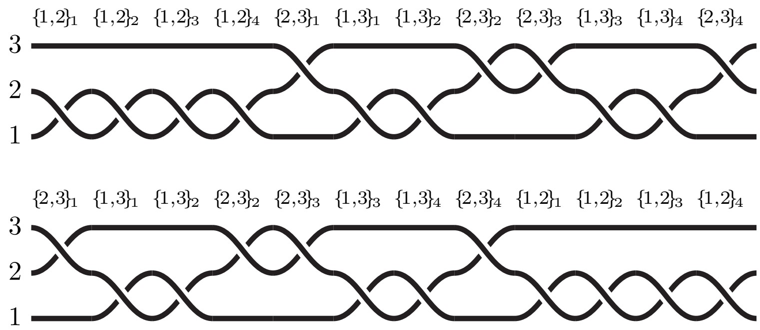

Figure 1. Braid diagrams associated with the two braid words of Proposition 1 (here with $m = 2$), together with the associated sequences of names.

The fact that three (*resp.* four) different strands are involved in a braid $\sigma_i\sigma_{i+1}\sigma_i$ (*resp.* $\sigma_i\sigma_j$ with $|i-j| \geqslant 2$) and the explicit definition of the names immediately imply

Lemma 3. *Assume that w, w' are n-strand braid words and w' is obtained from w by applying one braid relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ (resp. $\sigma_i\sigma_j = \sigma_j\sigma_i$ with $|i-j| \geqslant 2$). Then there exist pairwise distinct numbers p, q, r in $\{1, ..., n\}$ and integers a, b, c (resp. pairwise distinct p, q, r, q and integers a, b) such that $S(w')$ is obtained from $S(w)$ by reversing some subsequence $(\{p,q\}_a, \{p,r\}_b, \{q,r\}_c)$ (resp. reversing some subsequence $(\{p,q\}_a, \{r,s\}_b)$).*

Then Proposition 1 is easy.

Proof of Proposition 1. Figure 2 below makes the upper bound trivial, so we only have to prove a lower bound result. Let w_m and w'_m be the involved braid words. We consider the entries of the form $\{1,2\}_a$ and $\{2,3\}_b$ in $S(w_m)$ and $S(w'_m)$. In $S(w_m)$, they appear in the order $\{1,2\}_1, ..., \{1,2\}_{2m}, \{2,3\}_1, ..., \{2,3\}_{2m}$, whereas in $S(w'_m)$ they appear in the order $\{2,3\}_1, ..., \{2,3\}_{2m}, \{1,2\}_1, ..., \{1,2\}_{2m}$. By Lemma 3, applying one braid relation can only switch two entries in these sequences. Therefore, at least $4m^2$ braid relations (of the form $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$) are needed to exchange the $2m$ entries $\{1,2\}_a$ and the $2m$ entries $\{2,3\}_b$. ☐

We conclude with a few additional observations.

Remark 4. Each derivation from a (positive) braid word to another equivalent one can be illustrated using a van Kampen diagram, which is a planar diagram tesselated by tiles corresponding to braid relations—see for instance [2], and Figure 2 below. For each name $\{p,q\}_a$ occurring in the diagram, connecting all edges having that name yields a family of transversal curves called *separatrices* in [1]. It is easy to check that, if any two separatrices of a van Kampen diagram cross at most once, then the diagram must be optimal, in the sense that the number of faces is the minimal possible one, *i.e.*, it achieves the combinatorial distance. This criterion is clearly satisfied in the case of Figure 2. As the diagram has $4m^2$ faces, we conclude that the combinatorial distance between the bounding words is exactly $4m^2$.

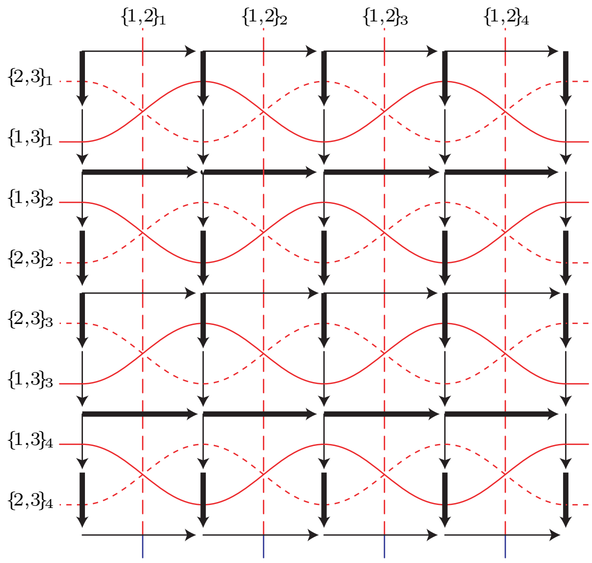

FIGURE 2. A van Kampen diagram witnessing that the two braid words of Proposition 1 are equivalent (here with $m = 2$). Thin edges represent σ_1, thick edges represent σ_2. The dotted (red) lines connect the edges bearing the same name; as any two of them cross at most once, the diagram achieves the combinatorial distance.

Remark 5. The current approach can be easily extended to arbitrary braid words. The name attributed to a negative crossing σ_i^{-1} has to be defined to be $\{p,q\}_a^{-1}$, where p, q still are the initial positions of the strands that cross, and a is the number of earlier crossings of these strands, counted algebraically, *i.e.*, it is the linking number of these two strands so far. For instance, the sequence $S(\sigma_1\sigma_2^{-3}\sigma_2\sigma_1)$ is $(\{1,2\}_1, \{1,3\}_0^{-1}, \{1,3\}_{-1}^{-1}, \{1,3\}_{-2}^{-1}, \{1,3\}_{-2}, \{1,2\}_2)$. Then Lemma 3 remains valid, as the contribution of the free group relations $\sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1$ consists in creating or deleting a subsequence of the form $(\{p,q\}_a, \{p,q\}_a^{-1})$ or $(\{p,q\}_a^{-1}, \{p,q\}_a)$.

Remark 6. A more symmetric and still more obvious example implying a number of relations that is quadratic with respect to the length of the initial words is obtained by starting with σ_1^{2m} and σ_2^{2m} and completing them into their least common right multiple. Then the sequence associated with the first braid word must begin with $\{1,2\}_1, ..., \{1,2\}_{2m}$, whereas that associated with the second one must begin with $\{2,3\}_1, ..., \{2,3\}_{2m}$, so, by Lemma 3 again, $4m^2$ braid relations are certainly needed to transform one into the other—see Figure 3 for an illustration in terms of van Kampen diagram and separatrices.

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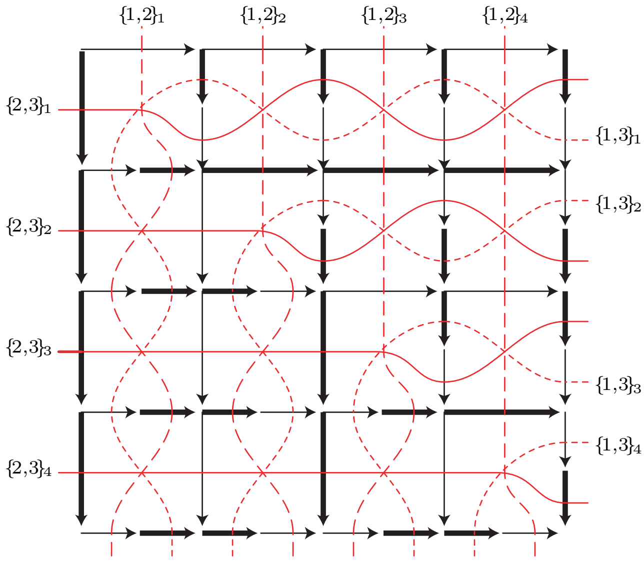

FIGURE 3. A van Kampen diagram witnessing that the braid words $\sigma_1^{2m}(\sigma_2\sigma_1^2\sigma_2)^m$ and $\sigma_2^{2m}(\sigma_1\sigma_2^2\sigma_1)^m$ are equivalent and lie at combinatorial distance $4m^2$ (here with $m = 2$). As in Figure 2, the dotted (red) lines connect the edges bearing the same name. Any two of them cross at most once, hence the diagram is optimal.

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