

# GARSDIE FAMILIES AND GARSDIE GERMS

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ABSTRACT. Garside families have recently emerged as a relevant context for extending results involving Garside monoids and groups, which themselves extend the classical theory of (generalized) braid groups. Here we establish various characterizations of Garside families, that is, equivalently, various criteria for establishing the existence of normal decompositions of a certain type.

In 1969, F.A. Garside [21] solved the word and conjugacy problems of Artin's braid groups by using convenient monoids. This approach was pursued [1, 19, 25, 20, 8] and extended in several steps, first to Artin-Tits groups of spherical type [7, 15], then to a larger family of groups now known as Garside groups [12, 9, 10]. More recently, it was realized that going to a categorical context allows for capturing further examples [18, 3, 11, 23], and a coherent theory now emerges around a central unifying notion called a Garside family. The aim of this paper is to present the main basic results of this approach. A more comprehensive text, including examples and many further developments, will be found in [14]. Algorithmic issues are addressed in [13].

The philosophy of Garside's theory as developed in the past decades is that, in some cases, a group can be realized as a group of fractions for a monoid and that the divisibility relations of the latter provide a lot of information about the group. The key technical ingredient in the approach is a certain distinguished decomposition for the elements of the monoid and the group in terms of some fixed (finite) family, usually called the greedy normal form. Our current approach consists in analyzing the abstract mechanism underlying the greedy normal form and developing it in the general context of what we call Garside families. The leading principle is that, with Garside families, one should retrieve all results about Garside monoids and groups at no extra cost.

In the current paper, we concentrate on one fundamental question, namely characterizing Garside families. As the latter are defined to be those families that guarantee the existence of the normal form, this exactly amounts to establishing various (necessary and sufficient) criteria for this existence. Two types of characterizations will be established here: extrinsic characterizations consist in recognizing whether a subfamily of a given category is a Garside family, whereas intrinsic ones consist in recognizing whether an abstract family (more precisely, a precategory) generates a category in which it embeds as a Garside family.

Beyond the results themselves, one of our goals is to show that the new framework, which properly extends those previously considered in literature, works efficiently and provides arguments that are both simple and natural. In particular,

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1991 *Mathematics Subject Classification.* 20M05, 18B40, 20F10, 20F36.

*Key words and phrases.* Garside family, Garside monoid, Garside group, germ, greedy normal form, Coxeter group, braid monoid.

Work partially supported by the ANR grant TheoGar ANR-08-BLAN-0269-02.

appealing to a categorical framework introduces no additional complexity and helps in many places to develop a better intuition of the situation. Among the specific features in the current approach are the facts that it is compatible with the existence of nontrivial invertible elements, and requires no Noetherianity assumption.

The paper is organized as follows. In Section 1, we quickly list the basic notions about categories and the derived divisibility relations needed for further developments. In Section 2, we introduce the central notion of an  $\mathcal{S}$ -normal decomposition for an element of a category, and define a Garside family to be a family  $\mathcal{S}$  such that every element of the ambient category admits at least one  $\mathcal{S}$ -normal decomposition. Next, in Section 3, we establish what we called various extrinsic characterizations of Garside families, mainly based on closure properties and on properties of so-called head functions. Then, in Section 4, we turn to intrinsic definitions and develop the notion of a germ, which, as the name suggests, is a sort of partial structure from which a category can be constructed. In Section 5, we establish various characterizations of Garside germs, which are those germs that embed as Garside families in the category they generate. Finally, in Section 6, we give an interesting application of the previous results, namely the construction of the classical and dual braid monoids associated with a Coxeter or a complex reflection group.

## 1. CATEGORIES AND DIVISIBILITY

In this preliminary section, we fix some terminology. As mentioned in the introduction, it is convenient to work in a category framework that we describe here.

**1.1. The category framework.** The general context is that of categories, which are seen here just as monoids with a conditional, not necessarily everywhere defined product. A *precategory* is a family of elements with two objects called “source” and “target” attached to each element, and a *category* is a precategory equipped with a partial binary product such that  $fg$  exists if and only if the target of  $f$  coincides with the source of  $g$ . In addition, the product is assumed to be associative whenever defined and, for each object  $x$ , there exists an identity-element  $1_x$  attached to  $x$ , so that  $1_x g = g = g 1_y$  holds for every  $g$  with source  $x$  and target  $y$ . A monoid is the special case of a category when there is only one object, so that the product is always defined. It is convenient to represent an element  $g$  with source  $x$  and target  $y$  using an arrow as in  $x \xrightarrow{g} y$ .

**Definition 1.1.** A category  $\mathcal{C}$  is said to be *left-cancellative* (resp. *right-cancellative*) if  $fg = fg'$  (resp.  $gf = g'f$ ) implies  $f = f'$  in  $\mathcal{C}$ . It is called *cancellative* if it is both left- and right-cancellative.

(In other words, in the usual language of categories [22], a category is left-cancellative if all morphisms are epimorphisms, and it is right-cancellative if all morphisms are monomorphisms.) In the sequel, we shall always work with categories that are (at least) left-cancellative. This implies in particular that there exists a unique, well defined notion of invertible element.

**Lemma 1.2.** *If  $\mathcal{C}$  is a left-cancellative category, an element  $g$  of  $\mathcal{C}$  with source  $x$  and target  $y$  has a left-inverse, that is, there exists  $f$  satisfying  $fg = 1_y$ , if and only if it has a right-inverse, that is, there exists  $f$  satisfying  $gf = 1_x$ .*

*Proof.* Assume  $fg = 1_y$ . Right-multiplying by  $f$ , we deduce  $fgf = f$ , which, by left-cancellativity, implies  $gf = 1_x$ . Similarly,  $gf = 1_x$  implies  $gfg = g$ , whence  $fg = 1_y$ .  $\square$

If  $\mathcal{C}$  is a left-cancellative category, we shall say that an element  $g$  of  $\mathcal{C}$  is *invertible* if it admits a left- and right-inverse, naturally denoted by  $g^{-1}$ . Note that the product of two invertible elements is always invertible, that the inverse of an invertible element is invertible, and that an identity-element  $1_x$  is invertible (and equal to its inverse). Thus the invertible elements of a left-cancellative category  $\mathcal{C}$  make a subgroupoid of  $\mathcal{C}$ .

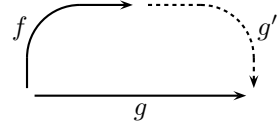
**Notation 1.3.** For  $\mathcal{C}$  a left-cancellative category, we denote by  $\mathcal{C}^\times$  the family of all invertible elements of  $\mathcal{C}$  and, for every subfamily  $\mathcal{S}$  of  $\mathcal{C}$ , we put  $\mathcal{S}^\# = \mathcal{S}\mathcal{C}^\times \cup \mathcal{C}^\times$ .

In the above notation, as everywhere in the paper, if  $\mathcal{X}_1, \mathcal{X}_2$  are subfamilies of a category  $\mathcal{C}$ , we denote by  $\mathcal{X}_1\mathcal{X}_2$  the family of all elements of  $\mathcal{C}$  that can be expressed, in at least one way, as  $g_1g_2$  with  $g_i \in \mathcal{X}_i$ ,  $i = 1, 2$ . So, for instance,  $\mathcal{S}\mathcal{C}^\times$  is the family of all elements obtained by right-multiplying an element of  $\mathcal{S}$  by an invertible element. In such a context, we naturally write  $\mathcal{X}^r$  for  $\mathcal{X}\cdots\mathcal{X}$ ,  $r$  factors.

## 1.2. The divisibility relations.

**Definition 1.4.** Assume that  $\mathcal{C}$  is a left-cancellative category. For  $f, g$  in  $\mathcal{C}$ , we say that  $f$  is a *left-divisor* of  $g$ , or, equivalently, that  $g$  is a *right-multiple* of  $f$ , written  $f \preceq g$ , if there exists  $g'$  in  $\mathcal{C}$  satisfying  $fg' = g$ . Symmetrically, we say that  $f$  is a *right-divisor* of  $g$ , or, equivalently, that  $g$  is a *left-multiple* of  $f$ , if there exists  $g'$  in  $\mathcal{C}$  satisfying  $g = g'f$ .

In terms of arrows and diagrams,  $f$  being a left-divisor of  $g$  corresponds to the existence of an arrow  $g'$  making the diagram on the right commutative.



**Notation 1.5.** For  $\mathcal{C}$  a left-cancellative category and  $f, g$  in  $\mathcal{C}$ , we write  $f =^\times g$  if there exists an invertible element  $e$  satisfying  $fe = g$ , and  $f \prec g$  for the conjunction of  $f \preceq g$  and  $g \not\preceq f$ .

**Lemma 1.6.** *If  $\mathcal{C}$  is a left-cancellative category, the relation  $\preceq$  is a partial pre-ordering on  $\mathcal{C}$ , and the conjunction of  $f \preceq g$  and  $g \preceq f$  is equivalent to  $f =^\times g$ .*

*Proof.* For every  $g$  with target  $y$ , we have  $g = g1_y$ , so  $g \preceq g$  always holds. Next, if we have both  $fg' = g$  and  $gh' = h$ , we deduce  $f(g'h') = (fg')h' = gh' = h$ , so the conjunction of  $f \preceq g$  and  $g \preceq h$  implies  $f \preceq h$ . So  $\preceq$  is reflexive and transitive.

Assume  $f \preceq g$  and  $g \preceq f$ . Then there exist  $f', g'$  satisfying  $fg' = g$  and  $gf' = f$ . We deduce  $f(g'f') = (fg')f' = gf' = f$ , whence  $g'f' = 1_y$  where  $y$  is the target of  $f$ . It follows that  $f'$  and  $g'$  must be invertible, which implies  $f =^\times g$ . Conversely,  $fe = g$  with  $e \in \mathcal{C}^\times$  implies  $f = ge^{-1}$ , so  $f =^\times g$  implies  $f \preceq g$  and  $g \preceq f$ .  $\square$

It follows that the left-divisibility relation of a category  $\mathcal{C}$  is a partial ordering on  $\mathcal{C}$  if and only if the relation  $=^\times$  is equality, that is, if and only if, the only invertible elements of  $\mathcal{C}$  are the identity-elements. In this case, we shall say that  $\mathcal{C}$  has no *nontrivial* invertible element.

**1.3. Paths and free categories.** We now fix some terminology and notation for paths, which are the natural counterparts of sequences (or words) in the context of a monoid. The notion does not depend on the product of the category but only on the sources and targets of the considered elements, and it will be useful to define it in the more general context of a precategory, which is like a category but with no product and identity-elements.

**Definition 1.7.** Assume that  $\mathcal{X}$  is a precategory. For  $q \geq 1$ , a *path in  $\mathcal{X}$* , or  *$\mathcal{X}$ -path*, of length  $q$  is a sequence  $(g_1, \dots, g_q)$  of elements of  $\mathcal{X}$  such that, for  $1 < k \leq q$ , the target of  $g_{k-1}$  coincides with the source of  $g_k$ . In this case, the source (*resp.* target) of the path is defined to be the source of  $g_1$  (*resp.* the target of  $g_q$ ). In addition, for every object  $x$ , one defines an *empty path*  $\varepsilon_x$  associated with  $x$ , whose source and target are  $x$  and whose length is zero. The family of all  $\mathcal{X}$ -paths of length  $q$  (*resp.* all  $\mathcal{X}$ -paths) is denoted by  $\mathcal{X}^{[q]}$  (*resp.* by  $\mathcal{X}^*$ ). If  $w_1, w_2$  are  $\mathcal{X}$ -paths and the target of  $w_1$  coincides with the source of  $w_2$ , the *concatenation* of  $w_1$  and  $w_2$  is denoted by  $w_1 | w_2$ .

In the case of a set, that is, a precategory with one object, the condition about source and target vanishes, and an  $\mathcal{X}$ -path is simply a finite sequence of elements of  $\mathcal{X}$ , or, equivalently, a word in the alphabet  $\mathcal{X}$ .

**Notation 1.8.** By definition, an  $\mathcal{X}$ -path  $(g_1, \dots, g_q)$  is the concatenation of the  $q$  length one paths  $(g_1), \dots, (g_q)$ . Identifying the length one path  $(g)$  with its unique entry  $g$ , we then find  $(g_1, \dots, g_q) = g_1 | \dots | g_q$ , a simplified notation used in the sequel.

**Lemma 1.9.** *For every precategory  $\mathcal{X}$ , the family  $\mathcal{X}^*$  equipped with concatenation and the empty paths  $\varepsilon_x$  for  $x$  in  $\text{Obj}(\mathcal{X})$  is a category, still denoted by  $\mathcal{X}^*$ . Every category  $\mathcal{C}$  including  $\mathcal{X}$  and generated by  $\mathcal{X}$  is a homomorphic image of  $\mathcal{X}^*$ : there exists a surjective functor of  $\mathcal{X}^*$  onto  $\mathcal{C}$  that extends the identity on  $\mathcal{X}$ .*

The proof is standard. Owing to Lemma 1.9, the category  $\mathcal{X}^*$  is called the *free category based on  $\mathcal{X}$*  (or the free monoid in the case of a set). If  $\mathcal{C}$  is a category and  $\mathcal{X}$  is a subfamily of  $\mathcal{C}$ , the unique functor of  $\mathcal{X}^*$  to  $\mathcal{C}$  that extends the identity maps on  $\mathcal{X}$  and  $\text{Obj}(\mathcal{X})$  is the evaluation map that associates to each  $\mathcal{X}$ -path  $g_1 | \dots | g_q$  the product  $g_1 \cdots g_q$  as computed using the product of  $\mathcal{C}$ . In this case, we say that the path  $g_1 | \dots | g_q$  is a *decomposition* of the element  $g_1 \cdots g_q$  of  $\mathcal{C}$ .

If  $\mathcal{X}$  is a precategory and  $\mathcal{R}$  is a family of pairs of  $\mathcal{X}$ -paths of the form  $\{w, w'\}$  with  $w, w'$  sharing the same source and the same target, one denotes by  $\langle \mathcal{X} | \mathcal{R} \rangle^+$  the category  $\mathcal{X}^* / \equiv$  where  $\equiv$  is the smallest congruence on  $\mathcal{X}^*$  that includes  $\mathcal{R}$ , that is, the smallest equivalence relation compatible with the product. If a pair  $\{w, w'\}$  lies in  $\mathcal{R}$ , the evaluations of the paths  $w$  and  $w'$  in the quotient-category  $\langle \mathcal{X} | \mathcal{R} \rangle^+$  coincide, and, therefore, it is customary to write  $w = w'$  instead of  $\{w, w'\}$  in this context, and to call such a pair a *relation* of the presented category  $\langle \mathcal{X} | \mathcal{R} \rangle^+$ .

## 2. $\mathcal{S}$ -NORMAL DECOMPOSITIONS

We now arrive at a central topic in the current approach, namely the notion of an  $\mathcal{S}$ -normal decomposition for an element of a category.

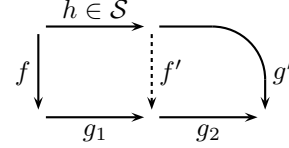
**2.1. The notion of an  $\mathcal{S}$ -greedy path.** The first step is the notion of an  $\mathcal{S}$ -greedy path, which captures the intuition that each entry in the path contains as much of  $\mathcal{S}$  as is possible.

**Definition 2.1.** Assume that  $\mathcal{S}$  is a subfamily of a left-cancellative category  $\mathcal{C}$ , that is,  $\mathcal{S}$  is included in  $\mathcal{C}$ . A  $\mathcal{C}$ -path  $g_1 | \cdots | g_q$  is called  $\mathcal{S}$ -greedy if, for every  $i < q$ , we have

$$(2.2) \quad \forall h \in \mathcal{S} \quad \forall f \in \mathcal{C} \quad (h \preceq f g_i g_{i+1} \Rightarrow h \preceq f g_i).$$

Note that  $\mathcal{S}$ -greediness is a local property in that a path  $g_1 | \cdots | g_q$  is  $\mathcal{S}$ -greedy if and only if every length two subpath  $g_i | g_{i+1}$  is  $\mathcal{S}$ -greedy.

In terms of diagrams, the fact that  $g_1 | g_2$  is  $\mathcal{S}$ -greedy means that every diagram as aside splits: whenever  $h$  left-divides  $f g_1 g_2$ , it left-divides  $f g_1$ , so there exists  $f'$  satisfying  $h f' = f g_1$ . Then the assumption  $h g' = f g_1 g_2$  implies  $h f' g_2 = h g'$ , whence  $f' g_2 = g'$  by left-cancelling  $h$ .



Before proceeding, we establish two technical results about greedy paths. The first result is that the strength of greediness is not changed by the possible existence of nontrivial invertible elements.

**Lemma 2.3.** For every subfamily  $\mathcal{S}$  of a left-cancellative category  $\mathcal{C}$ , being  $\mathcal{S}$ -greedy and  $\mathcal{S}^\sharp$ -greedy are equivalent properties.

*Proof.* By definition, a path that is  $\mathcal{S}^\sharp$ -greedy is  $\mathcal{X}$ -greedy for every  $\mathcal{X}$  included in  $\mathcal{S}^\sharp$ . As  $\mathcal{S}$  is included in  $\mathcal{S}^\sharp$ , being  $\mathcal{S}^\sharp$ -greedy implies being  $\mathcal{S}$ -greedy.

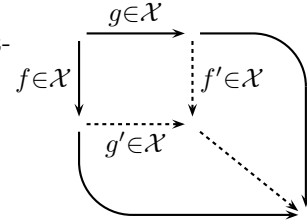
Conversely, assume that  $g_1 | g_2$  is  $\mathcal{S}$ -greedy, and we have  $h \preceq f g_1 g_2$  with  $h \in \mathcal{S}^\sharp$ . Two cases are possible. Assume first  $h \in \mathcal{S} \mathcal{C}^\times$ , say  $h = h' e$  with  $h' \in \mathcal{S}$  and  $e \in \mathcal{C}^\times$ . Then we have  $h' e \preceq f g_1 g_2$ , hence  $h' \preceq f g_1$  since  $g_1 | g_2$  is  $\mathcal{S}$ -greedy, that is, there exists  $f'$  satisfying  $h' f' = f g_1$ . We deduce  $h e^{-1} f' = f g_1$ , hence  $h \preceq f g_1$ . Assume now  $h \in \mathcal{C}^\times$ . Then we can write  $h \preceq h h^{-1}$ , whence  $h \preceq f g_1$  again. In every case,  $h \preceq f g_1$  holds, and  $g_1 | g_2$  is  $\mathcal{S}^\sharp$ -greedy.  $\square$

The second result involves the connection between the current notion of greediness and the one in the literature. If  $g_1 | g_2$  is  $\mathcal{S}$ -greedy, then, in particular, every element of  $\mathcal{S}$  that left-divides  $g_1 g_2$  left-divides  $g_1$ , which is the notion of a greediness considered for instance in [20] or [9]. The current definition is stronger as it involves the additional factor  $f$  in (2.2). However, this implication becomes an equivalence, that is, one recovers the usual notion of greediness, when the reference family  $\mathcal{S}$  satisfies some conditions.

**Definition 2.4.** A subfamily  $\mathcal{X}$  of a left-cancellative category  $\mathcal{C}$  is called *closed under right-complement* if

$$(2.5) \quad \forall f, g \in \mathcal{X} \quad \forall h \in \mathcal{C} \quad (f, g \preceq h \Rightarrow \exists f', g' \in \mathcal{X} \quad (f g' = g f' \preceq h)).$$

is satisfied.



**Lemma 2.6.** Assume that  $\mathcal{C}$  is a left-cancellative category, and  $\mathcal{S}$  is a subfamily of  $\mathcal{C}$  such that  $\mathcal{S}^\sharp$  generates  $\mathcal{C}$  and is closed under right-complement. Then  $g_1 | g_2$  is  $\mathcal{S}$ -greedy if and only if one has

$$(2.7) \quad \forall h \in \mathcal{S} \quad (h \preceq g_1 g_2 \Rightarrow h \preceq g_1).$$

*Proof.* As already observed, by definition, (2.7) holds whenever  $g_1 | g_2$  is  $\mathcal{S}$ -greedy. Conversely, assume that (2.7) holds, and we have  $h \preceq f g_1 g_2$ . By assumption,  $\mathcal{S}^\sharp$

generates  $\mathcal{C}$ , so we can write  $f = f_1 \cdots f_p$  with  $f_1, \dots, f_p$  in  $\mathcal{S}^\sharp$ . Then  $f_1$  and  $h$  belong to  $\mathcal{S}^\sharp$  and, by assumption, both left-divide  $f_1 \cdots f_p g_1 g_2$ . The assumption that  $\mathcal{S}^\sharp$  is closed under right-complement implies the existence of  $f'_1$  and  $h_1$  in  $\mathcal{S}^\sharp$  satisfying  $f_1 h_1 = h f'_1 \preceq f_1 \cdots f_p g_1 g_2$ , see Figure 1. Left-cancelling  $f_1$ , we deduce  $h_1 \preceq f_2 \cdots f_p g_1 g_2$  and, arguing similarly, we deduce the existence of  $f'_2$  and  $h_2$  in  $\mathcal{S}^\sharp$  satisfying  $f_2 h_2 = h_1 f'_2 \preceq f_2 \cdots f_p g_1 g_2$ , and so on. After  $p$  steps, we obtain  $f'_p$  and  $h_p$  in  $\mathcal{S}^\sharp$  satisfying  $f_p h_p = h_{p-1} f'_p$  and  $h_p \preceq g_1 g_2$ . Repeating the proof of Lemma 2.3, we see that (2.7) implies  $\forall h \in \mathcal{S}^\sharp (h \preceq g_1 g_2 \Rightarrow h \preceq g_1)$  and we deduce  $h_p \preceq g_1$ , whence  $h \preceq f_1 \cdots f_p g_1$  using the commutativity of the diagram. Hence  $g_1 | g_2$  is  $\mathcal{S}$ -greedy.  $\square$

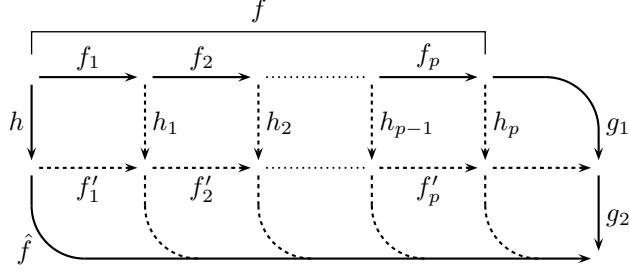


FIGURE 1. Proof of Lemma 2.6: whenever  $\mathcal{S}^\sharp$  generates the ambient category and is closed under right-complement, each relation  $h \preceq f g_1 g_2$  leads to a relation of the form  $h_p \preceq g_1 g_2$  and, therefore, the restricted form of greediness implies the general form.

**2.2. The notion of an  $\mathcal{S}$ -normal path.** Building on the notion of an  $\mathcal{S}$ -greedy path, we can now introduce the distinguished decompositions we shall be interested in.

**Definition 2.8.** Assume that  $\mathcal{S}$  is a subfamily of a left-cancellative category  $\mathcal{C}$ . A  $\mathcal{C}$ -path  $g_1 | \cdots | g_q$  is called  $\mathcal{S}$ -normal if it is  $\mathcal{S}$ -greedy and, in addition, the entries  $g_1, \dots, g_q$  lie in  $\mathcal{S}^\sharp$ .

Like greediness,  $\mathcal{S}$ -normality is a local property: a path  $g_1 | \cdots | g_q$  is  $\mathcal{S}$ -normal if and only if every length two subpath  $g_i | g_{i+1}$  is  $\mathcal{S}$ -normal. In diagrams, it will be convenient to indicate that a path  $g_1 | g_2$  is  $\mathcal{S}$ -normal by drawing a small arc connecting the ends of the arrows, as in  $\xrightarrow{g_1} \overset{\curvearrowright}{g_2} \rightarrow$ . We shall naturally say that a path  $g_1 | \cdots | g_q$  is an  $\mathcal{S}$ -normal decomposition for an element  $g$  of the ambient category if  $g_1 | \cdots | g_q$  is a decomposition of  $g$  that is  $\mathcal{S}$ -normal.

For further reference, we first observe that, in an  $\mathcal{S}$ -normal path, invertible elements can always be added at the end and that, conversely, an invertible entry cannot be followed by a non-invertible one.

**Lemma 2.9.** Assume that  $\mathcal{S}$  is a subfamily of a left-cancellative category  $\mathcal{C}$ .

- (i) If  $g_2$  is invertible, then every path of the form  $g_1 | g_2$  is  $\mathcal{S}$ -greedy.
- (ii) If  $g_1$  is invertible and  $g_2$  lies in  $\mathcal{S}^\sharp$ , then a path of the form  $g_1 | g_2$  is  $\mathcal{S}$ -greedy (if and) only if  $g_2$  is invertible.

*Proof.* (i) Assume that  $g_2$  is invertible. Then, for every  $h$  in  $\mathcal{S}$ , the relation  $h \preceq f g_1 g_2$  implies  $h \preceq f g_1 g_2 g_2^{-1}$ , whence  $h \preceq f g_1$ , and  $g_1 | g_2$  is  $\mathcal{S}$ -greedy.

(ii) Assume that  $g_1$  is invertible,  $g_2$  lies in  $\mathcal{S}^\sharp$ , and  $g_1 | g_2$  is  $\mathcal{S}$ -greedy. By Lemma 2.3,  $g_1 | g_2$  is  $\mathcal{S}^\sharp$ -greedy and we have  $g_2 = g_1^{-1} g_1 g_2$  with  $g_2 \in \mathcal{S}^\sharp$ , whence  $g_2 \preceq g_1^{-1} g_1$ . The latter relation implies that  $g_2$  is invertible.  $\square$

It follows that, in an  $\mathcal{S}$ -normal path, the non-invertible entries always occur first, and the invertible entries always occur at the end.

We now address the uniqueness of  $\mathcal{S}$ -normal decompositions. The good news is that (some form of) uniqueness comes for free.

**Definition 2.10.** (See Figure 2.) Assume that  $\mathcal{C}$  is a left-cancellative category. A  $\mathcal{C}$ -path  $f_1 | \dots | f_p$  is said to be a *deformation by invertible elements*, or  $\mathcal{C}^\times$ -deformation, of another path  $g_1 | \dots | g_q$  if there exist  $e_0, \dots, e_r$  in  $\mathcal{C}^\times$ ,  $r = \max(p, q)$ , such that  $e_0$  and  $e_r$  are identity-elements and  $e_{i-1} g_i = f_i e_i$  holds for  $1 \leq i \leq r$ , where, for  $p \neq q$ , the shorter path is expanded by identity-elements.

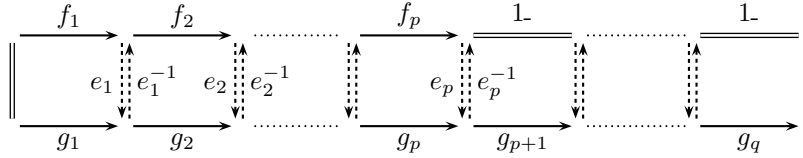


FIGURE 2. Deformation by invertible elements: invertible elements connect the corresponding entries; if one path is shorter (here we are in the case  $p < q$ ), it is extended by identity-elements.

If the ambient category  $\mathcal{C}$  contains no nontrivial invertible element, then being a  $\mathcal{C}^\times$ -deformation just means coinciding up to adding final identity-elements. The uniqueness result for  $\mathcal{S}$ -normal decompositions is as follows:

**Proposition 2.11.** *Assume that  $\mathcal{S}$  is a subfamily of a left-cancellative category  $\mathcal{C}$ . Then any two  $\mathcal{S}$ -normal decompositions of an element of  $\mathcal{C}$  (if any) are  $\mathcal{C}^\times$ -deformations of one another.*

The proof appeals to an auxiliary result.

**Lemma 2.12.** *Assume that  $\mathcal{S}$  is a subfamily of a left-cancellative category  $\mathcal{C}$ . If  $g_1 | \dots | g_q$  is an  $\mathcal{S}$ -greedy  $\mathcal{C}$ -path, then  $g_1 | g_2 \dots g_q$  is  $\mathcal{S}$ -greedy as well.*

*Proof.* Assume  $h \preceq f g_1 (g_2 \dots g_q)$ , where  $h$  lies in  $\mathcal{S}$ . Using associativity, we write  $h \preceq (f g_1 \dots g_{q-2}) g_{q-1} g_q$ , and the assumption that  $g_{q-1} | g_q$  is  $\mathcal{S}$ -greedy implies  $h \preceq (f g_1 \dots g_{q-2}) g_{q-1}$ , which is also  $h \preceq (f g_1 \dots g_{q-3}) g_{q-2} g_{q-1}$ . The assumption that  $g_{q-2} | g_{q-1}$  is  $\mathcal{S}$ -greedy implies now  $h \preceq (f g_1 \dots g_{q-3}) g_{q-2}$ , and so on. Finally, we obtain  $h \preceq (f) g_1 g_2$ , and the assumption that  $g_1 | g_2$  is  $\mathcal{S}$ -greedy implies  $h \preceq f g_1$ . So  $g_1 | g_2 \dots g_q$  is  $\mathcal{S}$ -greedy.  $\square$

*Proof of Proposition 2.11.* Assume that  $f_1 | \dots | f_p$  and  $g_1 | \dots | g_q$  are  $\mathcal{S}$ -normal decompositions of an element  $g$  of  $\mathcal{C}$ . Let  $y$  be the target of  $g$ . At the expense of adding factors  $1_y$  at the end of the shorter path, we may assume  $p = q$ : by Lemma 2.9, adding identity-elements at the end of an  $\mathcal{S}$ -greedy path yields an  $\mathcal{S}$ -greedy path.

Let  $e_0 = 1_x$ , where  $x$  is the source of  $g$ . By Lemma 2.12, the path  $g_1 | g_2 \dots g_q$  is  $\mathcal{S}$ -greedy, hence, by Lemma 2.3,  $\mathcal{S}^\sharp$ -greedy. Now  $f_1$  belongs to  $\mathcal{S}^\sharp$  and left-divides

$f_1 \cdots f_p$ , which is  $e_0 g_1 \cdots g_q$ , so it must left-divide  $g_1$ . In other words, there exists  $e_1$  satisfying  $f_1 e_1 = e_0 g_1$ . Left-cancelling  $f_1$  in  $f_1 \cdots f_p = (f_1 e_1) g_2 \cdots g_p$ , we deduce  $f_2 \cdots f_p = e_1 g_1 \cdots g_q$ . Now  $f_2$  belongs to  $\mathcal{S}^\sharp$  and, by Lemma 2.12 again,  $g_2 | g_3 \cdots g_q$  is  $\mathcal{S}$ -greedy, so we deduce the existence of  $e_2$  satisfying  $f_2 e_2 = e_1 g_1$ , and so on, giving the existence of  $e_i$  satisfying  $f_i e_i = e_{i-1} g_i$  for every  $1 \leq i \leq q$ .

Exchanging the roles of  $f_1 | \cdots | f_p$  and  $g_1 | \cdots | g_q$  and arguing symmetrically from  $e'_0 = 1_x$ , we obtain the existence of  $e'_i$  satisfying  $g_i e'_i = e'_{i-1} f_i$  for every  $1 \leq i \leq p$ . We deduce, for every  $i$ , the equalities  $(f_1 \cdots f_i) e_i = g_1 \cdots g_i$  and  $(g_1 \cdots g_i) e'_i = f_1 \cdots f_i$ , which imply that  $e'_i$  is the inverse of  $e_i$ . Hence  $f_1 | \cdots | f_p$  is a  $\mathcal{C}^\times$ -deformation of  $g_1 | \cdots | g_q$ .  $\square$

It can be checked that, conversely, every  $\mathcal{C}^\times$ -deformation of an  $\mathcal{S}$ -normal decomposition of an element  $g$  is again an  $\mathcal{S}$ -normal decomposition of  $g$  provided the relation  $\mathcal{C}^\times \mathcal{S} \subseteq \mathcal{S}^\sharp$  is satisfied, but we shall not need that result here.

A direct application of Proposition 2.11 is that the number of non-invertible entries in an  $\mathcal{S}$ -normal decomposition of an element  $g$  does not depend on the choice of that decomposition: indeed, if  $g$  and  $g'$  are connected by an equality  $g'e = e'g$  with  $e, e'$  invertible, then  $g$  is invertible if and only if  $g'$  is invertible. So the following notion is non-ambiguous.

**Definition 2.13.** Assume that  $\mathcal{S}$  is a subfamily of a left-cancellative category  $\mathcal{C}$ . The  $\mathcal{S}$ -length of an element  $g$  of  $\mathcal{S}$ , written  $\|g\|_{\mathcal{S}}$ , is the number of non-invertible entries in an  $\mathcal{S}$ -normal decomposition of  $g$  (if any).

Of course, the number  $\|g\|_{\mathcal{S}}$  is defined only if  $g$  admits at least one  $\mathcal{S}$ -normal decomposition.

**Lemma 2.14.** Assume that  $\mathcal{S}$  is a subfamily of a left-cancellative category  $\mathcal{C}$ . Then every element  $g$  of  $\mathcal{C}$  that admits an  $\mathcal{S}$ -normal decomposition admits an  $\mathcal{S}$ -normal decomposition of length  $\|g\|_{\mathcal{S}}$ .

*Proof.* Assume that  $g_1 | \cdots | g_q$  is an  $\mathcal{S}$ -normal decomposition of  $g$ . Let  $r = \|g\|_{\mathcal{S}}$ . By definition,  $r$  is the number of non-invertible entries in  $g_1 | \cdots | g_q$ , so  $r \leq q$  holds. Assume  $r < q$ . By Lemma 2.9, the entries  $g_1, \dots, g_r$  are non-invertible, whereas  $g_{r+1}, \dots, g_q$  are invertible. But, then,  $g_r \cdots g_q$  is an element of  $\mathcal{S}^\sharp$ , and therefore  $g_1 | \cdots | g_{r-1} | g_r \cdots g_q$  is an  $\mathcal{S}$ -normal decomposition of  $g$  that has length  $r$ .  $\square$

We shall prove

**Proposition 2.15.** Assume that  $\mathcal{S}$  is a subfamily of a left-cancellative category  $\mathcal{C}$ . Then  $\|g\|_{\mathcal{S}} \leq r$  holds for every element  $g$  in  $(\mathcal{S}^\sharp)^r$  that admits an  $\mathcal{S}$ -normal decomposition.

We begin with a new auxiliary result about greedy paths.

**Lemma 2.16.** Assume that  $\mathcal{S}$  is a subfamily of a left-cancellative category  $\mathcal{C}$ . If  $g_1 | \cdots | g_q$  is  $\mathcal{S}$ -greedy and  $r < q$  holds, then  $g_1 \cdots g_r | g_{r+1} \cdots g_q$  is  $\mathcal{S}^r$ -greedy.

*Proof.* Assume  $h \preceq f g_1 \cdots g_q$  with  $h \in \mathcal{S}^r$ , say  $h = h_1 \cdots h_r$  with  $h_1, \dots, h_r \in \mathcal{S}$ . By Lemma 2.12,  $g_1 | g_2 \cdots g_q$  is  $\mathcal{S}$ -greedy, hence the assumption implies  $h_1 \preceq f g_1$ , so there exists  $f'_1$  satisfying  $h_1 f'_1 = f g_1$ . Using left-cancellativity, we deduce  $h_2 \cdots h_r \preceq f'_1 g_2 \cdots g_q$ . By Lemma 2.12 again,  $g_2 | g_3 \cdots g_q$  is  $\mathcal{S}$ -greedy, and we deduce  $h_2 \preceq f'_1 g_2$ , so there exists  $f'_2$  satisfying  $h_2 f'_2 = f'_1 g_2$ . Repeating the argument, we find after  $r$  steps  $f'_r$  satisfying  $h_r f'_r = f'_{r-1} g_r$ , and we deduce  $h_1 \cdots h_r f'_r = f g_1 \cdots g_r$ , whence  $h \preceq f g_1 \cdots g_r$ . So  $g_1 \cdots g_r | g_{r+1} \cdots g_q$  is  $\mathcal{S}^r$ -greedy.  $\square$



*Proof of Proposition 2.15.* Assume that  $g_1 | \cdots | g_q$  is an  $\mathcal{S}$ -normal decomposition for an element  $g$  that belongs to  $(\mathcal{S}^\sharp)^r$ . If  $q \leq r$  holds, then  $\|g\|_{\mathcal{S}} \leq r$  is trivial. Assume now  $q > r$ . By Lemma 2.3, the path  $g_1 | \cdots | g_q$  is  $\mathcal{S}^\sharp$ -greedy so, by Lemma 2.16 (applied with  $\mathcal{S}^\sharp$ ), the sequence  $g_1 \cdots g_r | g_{r+1} \cdots g_q$  is  $(\mathcal{S}^\sharp)^r$ -greedy. As  $g$  belongs to  $(\mathcal{S}^\sharp)^r$  and it left-divides  $g_1 \cdots g_q$ , we deduce that  $g$  left-divides  $g_1 \cdots g_r$ , say  $gf = g_1 \cdots g_r$ . As we have  $g = g_1 \cdots g_q$ , by left-cancelling  $g_1 \cdots g_r$ , we deduce  $g_{r+1} \cdots g_q f = 1_y$ , where  $y$  is the target of  $f$ . It follows that  $g_{r+1}, \dots, g_q$  all must be invertible. So the non-invertible entries are among  $g_1, \dots, g_r$ , and  $\|g\|_{\mathcal{S}} \leq r$  follows.  $\square$

**2.3. Garside families.** After uniqueness, we now address the existence of  $\mathcal{S}$ -normal decompositions. If every element of the ambient category admits an  $\mathcal{S}$ -normal decomposition, then, clearly, the family  $\mathcal{S}^\sharp$  must generate the category. However this necessary condition is not sufficient in general, and we shall introduce Garside families precisely as those families that guarantee the existence of normal decompositions.

**Definition 2.17.** A *Garside family* in a left-cancellative category  $\mathcal{C}$  is a subfamily  $\mathcal{S}$  of  $\mathcal{C}$  such that every element of  $\mathcal{C}$  admits an  $\mathcal{S}$ -normal decomposition.

Garside families exist in every category: indeed, if  $\mathcal{C}$  is any left-cancellative category, then  $\mathcal{C}$  is a Garside family in itself since, for every  $g$  in  $\mathcal{C}$ , the length one path  $g$  is a  $\mathcal{C}$ -normal decomposition of  $g$ . On the other hand, it may happen that  $\mathcal{C}$  is the only Garside family in  $\mathcal{C}$ : for instance, this is the case for the monoid  $\langle \mathbf{a}, \mathbf{b} \mid \mathbf{ab} = \mathbf{ba}, \mathbf{a}^2 = \mathbf{b}^2 \rangle^+$  as, anticipating on Definition 3.5 and Proposition 3.9, one can see using induction on  $r \geq 0$  that every subset that is closed under right-comultiple and contains  $1, \mathbf{a}$ , and  $\mathbf{b}$  contains  $\mathbf{a}^{r+1}$  and  $\mathbf{a}^r \mathbf{b}$  for every  $r$ , hence coincides with the whole monoid.

The connection with the notion of a Garside monoid as developed in [12, 9] is easily described:

**Proposition 2.18.** *Assume that  $M$  is a Garside monoid with Garside element  $\Delta$ . Then the family  $\text{Div}(\Delta)$  of all divisors of  $\Delta$  is a Garside family in  $M$ .*

*Proof.* Let  $g$  be an element of  $M \setminus \{1\}$ . Let  $g_1$  be the left-gcd of  $g$  and  $\Delta$ . Then  $g_1$  belongs to  $\text{Div}(\Delta)$ , and  $g_1$  left-divides  $g$ , say  $g = g_1 g'$ . If  $g_1$  is not 1, we repeat the argument, finding a decomposition  $g' = g_2 g''$  with  $g_2$  the left-gcd of  $g'$  and  $\Delta$ , and so on. The assumption that  $M$  is atomic guarantees that the construction will stop after finitely many steps and one finds a decomposition  $g = g_1 \cdots g_q$  for  $g$  in terms of divisors of  $\Delta$ .

So it remains to see that the sequence  $g_1 | \cdots | g_q$  is  $\text{Div}(\Delta)$ -greedy. Now, we claim that the family  $\text{Div}(\Delta)$  satisfies the conditions of Lemma 2.6. Indeed, by definition of a Garside element,  $\text{Div}(\Delta)$  generates  $M$ . On the other hand, assume  $f, g \preceq \Delta$  and  $f, g \preceq h$ . Then  $f$  and  $g$  left-divide the left-gcd  $h_1$  of  $h$  and  $\Delta$ , so we have  $f g' = g f' \preceq h_1$  for some  $f', g'$ . By construction,  $h_1$  belongs to  $\text{Div}(\Delta)$ , and so do  $f'$  and  $g'$ , since all right-divisors of  $\Delta$  also are left-divisors. Hence  $\text{Div}(\Delta)$  is closed under right-complement. Now, assume  $h \preceq g_1 \cdots g_q$  with  $h \in \text{Div}(\Delta)$ . Then  $h$  left-divides the left-gcd of  $g_1 \cdots g_q$  and  $\Delta$ , which is precisely  $g_1$ . By Lemma 2.6, we deduce that  $g_1 | g_2 \cdots g_q$  is  $\text{Div}(\Delta)$ -greedy and, *a fortiori*, so is  $g_1 | g_2$ . Applying the same argument to  $g', g'' \dots$  inductively shows that  $g_i | g_{i+1}$  is  $\text{Div}(\Delta)$ -greedy as well. So the sequence  $g_1 | \cdots | g_q$  is  $\text{Div}(\Delta)$ -normal, every element of  $M$  admits a  $\text{Div}(\Delta)$ -normal decomposition, and  $\text{Div}(\Delta)$  is a Garside family in  $M$ .  $\square$

## 3. RECOGNIZING GARSIDE FAMILIES

Definition 2.17 gives no practical criterion for recognizing Garside families. Our aim for now on will be to establish various characterizations of such families, which amounts to establishing various necessary and sufficient conditions guaranteeing the existence of normal decompositions.

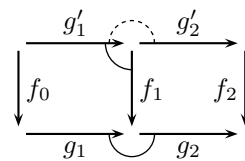
**3.1. Recognizing Garside families I: incremental approach.** The first characterization relies on the possibility of constructing  $\mathcal{S}$ -normal decompositions using an induction.

**Proposition 3.1.** *A subfamily  $\mathcal{S}$  of a left-cancellative category  $\mathcal{C}$  is a Garside family if and only if*

$$(3.2) \quad \mathcal{S}^\# \text{ generates } \mathcal{C} \text{ and every element of } (\mathcal{S}^\#)^2 \text{ admits an } \mathcal{S}\text{-normal decomposition.}$$

As already noted, if  $\mathcal{S}$  is a Garside family in  $\mathcal{C}$ , then, by definition, every element of  $\mathcal{C}$  admits a decomposition in which every entry lies in  $\mathcal{S}^\#$ , so  $\mathcal{S}^\#$  must generate  $\mathcal{C}$ , and every element admits an  $\mathcal{S}$ -normal decomposition hence, in particular, so does every element of  $(\mathcal{S}^\#)^2$ . So every Garside family necessarily satisfies (3.2), and the point is to prove that, conversely, every family satisfying (3.2) is a Garside family. We begin with preparatory results.

**Lemma 3.3** (“domino rule”). *Assume that  $\mathcal{S}$  is a subfamily of a left-cancellative category  $\mathcal{C}$ , and we have a commutative diagram with edges in  $\mathcal{C}$  as on the right. If  $g_1|g_2$  and  $g'_1|f_1$  are  $\mathcal{S}$ -greedy, then  $g'_1|g'_2$  is  $\mathcal{S}$ -greedy as well.*



*Proof.* Assume  $h \in \mathcal{S}$  and  $h \preceq fg'_1g'_2$ . As the diagram is commutative, we have  $h \preceq ff_0g_1g_2$ . As  $g_1|g_2$  is  $\mathcal{S}$ -greedy, we deduce  $h \preceq ff_0g_1$ , hence  $h \preceq fg'_1f_1$ . Now, as  $g'_1|f_1$  is  $\mathcal{S}$ -greedy, we deduce  $h \preceq fg'_1$ . Therefore  $g'_1|g'_2$  is  $\mathcal{S}$ -greedy.  $\square$

**Lemma 3.4.** *Assume that  $\mathcal{S}$  is a subfamily of a left-cancellative category  $\mathcal{C}$  that satisfies (3.2) and  $g$  is an element of  $\mathcal{C}$  that admits an  $\mathcal{S}$ -normal decomposition of length  $q$ . Then, for every  $f$  in  $\mathcal{S}^\#$ , the element  $fg$  admits an  $\mathcal{S}$ -normal decomposition of length  $q+1$  whenever it is defined. Moreover, we have  $\|g\|_{\mathcal{S}} \leq \|fg\|_{\mathcal{S}} \leq \|g\|_{\mathcal{S}} + 1$ .*

*Proof.* (See Figure 3.) Assume that  $g_1|\dots|g_q$  is an  $\mathcal{S}$ -normal decomposition of  $g$ . Put  $f_0 = f$ . By (3.2) and Proposition 2.15, the element  $f_0g_1$  of  $(\mathcal{S}^\#)^2$  admits an  $\mathcal{S}$ -normal decomposition of length two, say  $g'_1|f_1$ . Then, similarly, the element  $f_1g_2$  of  $(\mathcal{S}^\#)^2$  admits an  $\mathcal{S}$ -normal decomposition of length two, say  $g'_2|f_2$ , and so on. Finally,  $f_{q-1}g_q$  admits an  $\mathcal{S}$ -normal decomposition of length two, say  $g'_q|f_q$ . By construction,  $g'_1|\dots|g'_q|f_q$  is a decomposition of  $fg$ , and its entries lie in  $\mathcal{S}^\#$ . Moreover, for  $1 \leq i < q$ , the paths  $g_i|g_{i+1}$  and  $g'_i|f_i$  are  $\mathcal{S}$ -greedy, so the domino rule (Lemma 3.3) implies that  $g'_i|g'_{i+1}$  is  $\mathcal{S}$ -greedy as well. Thus  $g'_1|\dots|g'_q|f_q$  is  $\mathcal{S}$ -greedy, hence  $\mathcal{S}$ -normal.

As for the  $\mathcal{S}$ -length, we can assume  $q = \|g\|_{\mathcal{S}}$ , which is always possible by Lemma 2.14. Then the above argument provides an  $\mathcal{S}$ -normal decomposition of length  $q+1$  for  $fg$ , implying  $\|fg\|_{\mathcal{S}} \leq \|g\|_{\mathcal{S}} + 1$ . If  $q = 0$  holds, that is, if  $g$  is invertible, then  $\|g\|_{\mathcal{S}} \leq \|fg\|_{\mathcal{S}}$  is trivial. Otherwise, we have  $q \geq 1$  and, by assumption, the entries  $g_1, \dots, g_q$  are not invertible. This implies that  $g'_1, \dots, g'_q$

are not invertible either: indeed, by Lemma 2.9, the assumption that  $g'_i | f_i$  is  $\mathcal{S}$ -normal implies that  $f_i$  is invertible whenever  $g'_i$  is, so that  $f_{i-1}g_i$ , hence  $g_i$ , must be invertible, contrary to the assumption. Hence  $\|g\|_{\mathcal{S}} \leq \|fg\|_{\mathcal{S}}$  is always satisfied.  $\square$

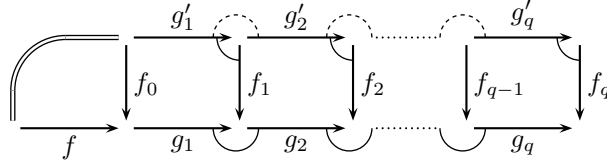


FIGURE 3. Proof of Lemma 3.4: starting from  $f$  in  $\mathcal{S}^\#$  and an  $\mathcal{S}$ -normal decomposition of  $g$ , we inductively build an  $\mathcal{S}$ -normal decomposition of  $fg$ .

We can now complete the argument.

*Proof of Proposition 3.1.* We already noted that, if  $\mathcal{S}$  is a Garside family in  $\mathcal{C}$ , then (3.2) is satisfied.

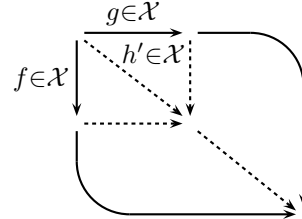
Conversely, assume that  $\mathcal{S}$  satisfies (3.2). We prove using induction on  $r$  that every element  $g$  of  $(\mathcal{S}^\#)^r$  admits an  $\mathcal{S}$ -normal decomposition. For  $r = 0$ , that is, if  $g$  has the form  $1_x$ , the empty path  $\varepsilon_x$  is an  $\mathcal{S}$ -normal decomposition of  $g$  and, for  $r = 1$ , the sequence  $(g)$  is an  $\mathcal{S}$ -normal decomposition of  $g$ . For  $r \geq 2$ , we write  $g = fg'$  with  $f$  in  $\mathcal{S}^\#$  and  $g'$  in  $(\mathcal{S}^\#)^{r-1}$ , and we apply Lemma 3.4. So every element of  $\mathcal{C}$  that belongs to  $(\mathcal{S}^\#)^r$  admits an  $\mathcal{S}$ -normal decomposition. As, by assumption,  $\mathcal{S}^\#$  generates  $\mathcal{C}$ , every element of  $\mathcal{C}$  is eligible. Hence  $\mathcal{S}$  is a Garside family in  $\mathcal{C}$ .  $\square$

**3.2. Recognizing Garside families II: closure properties.** We turn to further characterizations of Garside families involving closure properties, such as closure under right-complement (Definition 2.4). Here we introduce another similar notion.

**Definition 3.5.** A subfamily  $\mathcal{X}$  of a left-cancellative category  $\mathcal{C}$  is called *closed under right-comultiple* if

$$(3.6) \quad \forall f, g \in \mathcal{X} \quad \forall h \in \mathcal{C} \quad (f, g \preceq h \Rightarrow \exists h' \in \mathcal{X} \quad (f, g \preceq h' \preceq h)).$$

is satisfied.



So, in words, a family  $\mathcal{X}$  is closed under right-comultiple if and only if every common right-multiple of two elements of  $\mathcal{X}$  must be a right-multiple of some common right-multiple that lies in  $\mathcal{X}$ . On the other hand, we shall naturally say that a family  $\mathcal{X}$  is closed *under right-divisor* if every right-divisor of an element of  $\mathcal{X}$  belongs to  $\mathcal{X}$ . For further reference, we immediately note an easy connection between closure properties.

**Lemma 3.7.** *A subfamily  $\mathcal{X}$  of a left-cancellative category that is closed under right-comultiple and right-divisor is closed under right-complement.*

*Proof.* Assume  $f, g \preceq h$  with  $f, g \in \mathcal{X}$ . As  $\mathcal{X}$  is closed under right-comultiple, there exists  $h'$  in  $\mathcal{X}$  satisfying  $f, g \preceq h' \preceq h$ . By definition, this means that there exist  $f', g'$  satisfying  $fg' = gf' = h'$ . Now, as  $\mathcal{X}$  is closed under right-divisor, the assumption that  $h$  lies in  $\mathcal{X}$  implies that  $f'$  and  $g'$  also do. But this means that  $\mathcal{X}$  is closed under right-complement.  $\square$

We shall need one more notion.

**Definition 3.8.** Assume that  $\mathcal{S}$  is a subfamily of a left-cancellative category  $\mathcal{C}$ . For  $g$  in  $\mathcal{C}$ , we say that  $g_1$  is an  $\mathcal{S}$ -head of  $g$  if  $g_1$  belongs to  $\mathcal{S}$ , it left-divides  $g$ , and every element of  $\mathcal{S}$  that left-divides  $g$  left-divides  $g_1$ .

Here are the expected characterizations of Garside families.

**Proposition 3.9.** *A subfamily  $\mathcal{S}$  of a left-cancellative category  $\mathcal{C}$  is a Garside family if and only if it satisfies one of the following equivalent conditions:*

- (3.10)  $\mathcal{S}^\sharp$  generates  $\mathcal{C}$ , it is closed under right-divisor, and every non-invertible element of  $\mathcal{C}$  admits an  $\mathcal{S}$ -head;
- (3.11)  $\mathcal{S}^\sharp$  generates  $\mathcal{C}$ , it is closed under right-complement, and every non-invertible element of  $(\mathcal{S}^\sharp)^2$  admits an  $\mathcal{S}$ -head;
- (3.12)  $\mathcal{S}^\sharp$  generates  $\mathcal{C}$ , it is closed under right-comultiple and right-divisor, and every non-invertible element of  $(\mathcal{S}^\sharp)^2$  admits a  $\prec$ -maximal left-divisor in  $\mathcal{S}$ .

The difference between the final conditions in (3.11) and (3.12) is that, in (3.12), we do not demand that every element of  $\mathcal{S}$  that left-divides the considered element  $g$  left-divides the maximal left-divisor, but only that no proper multiple of the latter left-divides  $g$ , a weaker condition.

Contrary to Proposition 3.1, it is not obvious that the conditions of Proposition 3.9 necessarily hold for every Garside family, so a proof is needed in both directions. As usual we shall split the argument into several steps. The first one directly follows from our definitions.

**Lemma 3.13.** *Assume that  $\mathcal{S}$  is a subfamily of a left-cancellative category  $\mathcal{C}$  and  $g_1$  lies in  $\mathcal{S}$ .*

- (i) *If  $g_1 | g_2$  is  $\mathcal{S}$ -greedy, then  $g_1$  is an  $\mathcal{S}$ -head for  $g_1 g_2$ .*
- (ii) *Conversely, if  $\mathcal{S}^\sharp$  generates  $\mathcal{C}$  and is closed under right-complement, then  $g_1$  being an  $\mathcal{S}$ -head of  $g_1 g_2$  implies that  $g_1 | g_2$  is  $\mathcal{S}$ -greedy.*

*Proof.* (i) We have  $g_1 \prec g_1 g_2$ . Assume  $h \in \mathcal{S}$  and  $h \prec g_1 g_2$ . As  $g_1 | g_2$  is  $\mathcal{S}$ -greedy, we deduce  $h \prec g_1$ , which, by definition, means that  $g_1$  is an  $\mathcal{S}$ -head for  $g_1 g_2$ .

(ii) Assume that  $g_1$  is an  $\mathcal{S}$ -head of  $g_1 g_2$ . By definition, every element of  $\mathcal{S}$  that left-divides  $g_1 g_2$  left-divides  $g_1$ . Now, by Lemma 2.6, this implies that  $g_1 | g_2$  is  $\mathcal{S}$ -greedy whenever  $\mathcal{S}^\sharp$  generates  $\mathcal{C}$  and is closed under right-complement.  $\square$

**Lemma 3.14.** *Assume that  $\mathcal{S}$  is a Garside family in a left-cancellative category  $\mathcal{C}$ . Then  $\mathcal{S}^\sharp$  is closed under right-divisor, right-comultiple, and right-complement. Moreover, we have  $\mathcal{C}^\times \mathcal{S}^\sharp \subseteq \mathcal{S}^\sharp$ .*

*Proof.* Assume that  $g$  lies in  $\mathcal{S}^\sharp$  and  $f$  right-divides  $g$ . Then we have  $\|g\|_{\mathcal{S}} \leq 1$ , and Lemma 3.4 inductively implies  $\|f\|_{\mathcal{S}} \leq \|g\|_{\mathcal{S}}$ , whence  $\|f\|_{\mathcal{S}} \leq 1$ , which in turn implies  $f \in \mathcal{S}^\sharp$ . So  $\mathcal{S}^\sharp$  is closed under right-divisor.

Next, assume that  $f, g$  lie in  $\mathcal{S}^\sharp$  and  $h$  is a common right-multiple of  $f$  and  $g$ . Let  $h_1 | \dots | h_r$  be an  $\mathcal{S}$ -normal decomposition of  $h$ , which exists as  $\mathcal{S}$  is a Garside family. Then  $f$  is an element of  $\mathcal{S}^\sharp$  that left-divides  $h_1 \dots h_r$  and  $h_1 | \dots | h_r$ , hence  $h_1 | h_2 \dots h_r$ , are  $\mathcal{S}^\sharp$ -greedy. It follows that  $f$  left-divides  $h_1$ . Similarly  $g$  left-divides  $h_1$ . Hence  $h_1$  is a common right-multiple of  $f$  and  $g$  that left-divides  $h$  and lies in  $\mathcal{S}^\sharp$ . Hence  $\mathcal{S}^\sharp$  is closed under right-comultiple.

Then, Lemma 3.7 implies that  $\mathcal{S}^\sharp$  is closed under right-complement since it is closed under right-comultiple and right-divisor.

Finally, assume  $g \in \mathcal{C}^\times \mathcal{S}^\sharp$ , say  $g = eh$  with  $e \in \mathcal{C}^\times$  and  $h \in \mathcal{S}^\sharp$ . Then we have  $h = e^{-1}g$ , so  $g$  right-divides an element of  $\mathcal{S}^\sharp$ , hence it belongs to  $\mathcal{S}^\sharp$  by the first result above.  $\square$

**Lemma 3.15.** *Assume that  $\mathcal{S}$  is a subfamily of a left-cancellative category  $\mathcal{C}$  that satisfies  $\mathcal{C}^\times \mathcal{S}^\sharp \subseteq \mathcal{S}^\sharp$ . Then every element that admits an  $\mathcal{S}$ -normal decomposition admits one in which all entries except possibly the last one lie in  $\mathcal{S} \setminus \mathcal{C}^\times$ .*

*Proof.* Assume first that  $g$  is non-invertible and  $g_1 | \dots | g_q$  is an  $\mathcal{S}$ -normal decomposition of  $g$ . The assumption that  $g$  is non-invertible implies that  $\|g\|_{\mathcal{S}}$  is at least one, and, by Lemma 2.14, we can assume that  $q$  is the  $\mathcal{S}$ -length of  $g$ , implying that none of  $g_1, \dots, g_q$  is invertible.

As  $g_1$  is a non-invertible element of  $\mathcal{S}^\sharp$ , we can write  $g_1 = g'_1 e_1$  with  $g'_1$  in  $\mathcal{S}$  and  $e_1$  in  $\mathcal{C}^\times$ . Moreover, the assumption that  $g_1$  is not invertible implies that  $g'_1$  is not invertible either. Next,  $e_1 g_2$  belongs to  $\mathcal{C}^\times \mathcal{S}^\sharp$ , hence, by assumption, to  $\mathcal{S}^\sharp$ . So we can write  $e_1 g_2 = g'_2 e_2$  with  $g'_2$  in  $\mathcal{S}$  and  $e_2$  in  $\mathcal{C}^\times$  and, again, the assumption that  $g_2$  is non-invertible implies that  $g'_2$  is non-invertible. We continue in the same way until  $g_q$ , finding  $g'_q$  in  $\mathcal{S} \setminus \mathcal{C}^\times$  and  $e_q$  in  $\mathcal{C}^\times$  that satisfy  $e_{q-1} g_q = g'_q e_q$ . Then  $g'_1 | \dots | g'_{q-1} | g'_q e_q$  is a decomposition of  $g$  whose non-terminal entries are non-invertible elements of  $\mathcal{S}$ .

It remains to see that the latter path is  $\mathcal{S}$ -greedy. But this follows from the domino rule (Lemma 3.3) since, for every  $i$ , the paths  $g_i | g_{i+1}$  and  $g'_i | e_i$  are  $\mathcal{S}$ -greedy. Hence  $g'_1 | \dots | g'_{q-1} | g'_q e_q$  is an  $\mathcal{S}$ -normal decomposition of  $g$  with the expected properties.

On the other hand, if  $e$  is invertible, then, by definition, the length one path  $e$  is an  $\mathcal{S}$ -normal decomposition of  $e$  vacuously satisfying the condition of the statement.  $\square$

We can now complete one direction in the implications of Proposition 3.9.

*Proof of Proposition 3.9 ( $\Rightarrow$ ).* Assume that  $\mathcal{S}$  is a Garside family in  $\mathcal{C}$ . First, as already noted,  $\mathcal{S}^\sharp$  must generate  $\mathcal{C}$ . Next, by Lemma 3.14,  $\mathcal{S}^\sharp$  is closed under right-complement, right-comultiple, and right-divisor. Finally, let  $g$  be a non-invertible element of  $(\mathcal{S}^\sharp)^2$ . Then the  $\mathcal{S}$ -length of  $g$  is not zero.

Assume first  $\|g\|_{\mathcal{S}} = 1$ . Then  $g$  belongs to  $\mathcal{S}^\sharp \setminus \mathcal{C}^\times$ , so it can be written as  $g_1 e$  with  $g_1 \in \mathcal{S}$  and  $e \in \mathcal{C}^\times$ , in which case  $g_1 | e$  is an  $\mathcal{S}$ -normal decomposition of  $g$ , and, by Lemma 3.13,  $g_1$  is an  $\mathcal{S}$ -head of  $g$ .

Assume next  $\|g\|_{\mathcal{S}} \geq 2$ . By Lemmas 3.14 and 3.15,  $g$  admits an  $\mathcal{S}$ -normal decomposition  $g_1 | \dots | g_q$  such that  $g_1, \dots, g_{q-1}$  lie in  $\mathcal{S}$ . Then  $g_1 | g_2 \dots g_q$  is  $\mathcal{S}$ -greedy and, by Lemma 3.13,  $g_1$  is an  $\mathcal{S}$ -head of  $g$ .

Hence (3.10) and (3.11) are satisfied, and so is (3.12) as an  $\mathcal{S}$ -head is *a fortiori* a  $\prec$ -maximal left-divisor lying in  $\mathcal{S}$ .  $\square$

Before going to the second half of the proof of Proposition 3.9, we add two more observations about heads for further reference, namely a characterization of those elements that admit an  $\mathcal{S}$ -head and a connection between  $\mathcal{S}$ - and  $\mathcal{S}^\sharp$ -heads.

**Lemma 3.16.** *Assume that  $\mathcal{S}$  is a subfamily of a left-cancellative category  $\mathcal{C}$ .*

(i) *If  $\mathcal{S}$  is a Garside family, an element of  $\mathcal{C}$  admits an  $\mathcal{S}$ -head if and only if it lies in  $\mathcal{S}\mathcal{C}$ .*

(ii) *Every  $\mathcal{S}$ -head is an  $\mathcal{S}^\sharp$ -head.*

*Proof.* (i) If  $g$  admits an  $\mathcal{S}$ -head, then, by definition,  $g$  is left-divisible by an element of  $\mathcal{S}$ , so it belongs to  $\mathcal{SC}$ . Conversely, assume that  $g$  lies in  $\mathcal{SC}$ , say  $g = g_1g'$  with  $g_1 \in \mathcal{S}$ . If  $g$  is not invertible,  $g$  admits an  $\mathcal{S}$ -head by Proposition 3.9. Otherwise,  $g'$  must be invertible, so  $g = {}^\times g_1$  holds, and every element of  $\mathcal{S}$  that left-divides  $g$  also left-divides  $g_1$ . Then  $g_1$  is an  $\mathcal{S}$ -head of  $g$ .

(ii) Assume that  $g_1$  is an  $\mathcal{S}$ -head of  $g$ . First  $g_1$  lies in  $\mathcal{S}$ , hence *a fortiori* in  $\mathcal{S}^\sharp$ . Next, assume that  $h$  belongs to  $\mathcal{S}^\sharp$  and  $h \preceq g$  holds. If  $h$  is invertible, then  $h \preceq g_1$  necessarily holds. Otherwise, write  $h = h'e$  with  $h'$  in  $\mathcal{S}$  and  $e$  in  $\mathcal{C}^\times$ . Then  $h \preceq g$  implies  $h' \preceq g$ , whence  $h' \preceq g_1$  since  $g_1$  is an  $\mathcal{S}$ -head of  $g$ , hence  $h \preceq g_1$ . So  $g_1$  is an  $\mathcal{S}^\sharp$ -head of  $g$ .  $\square$

We now establish several preliminary results in view of the converse implication in Proposition 3.9.

**Lemma 3.17.** *Assume that  $\mathcal{S}$  is a subfamily of a left-cancellative category  $\mathcal{C}$  such that  $\mathcal{S}^\sharp$  generates  $\mathcal{C}$  and is closed under right-complement. Then  $\mathcal{S}^\sharp$  is closed under right-divisor.*

*Proof.* The argument is similar to that used for Lemma 2.6, see Figure 4. So assume that  $h$  belongs to  $\mathcal{S}^\sharp$  and  $g$  right-divides  $h$ , that is, we have  $h = fg$ . As  $\mathcal{S}^\sharp$  generates  $\mathcal{C}$ , we can write  $f = f_1 \cdots f_p$  with  $f_1, \dots, f_p \in \mathcal{S}^\sharp$ . The assumption that  $\mathcal{S}^\sharp$  is closed under right-complement and the fact that  $h$  and  $f_1$  left-divide  $f_1 \cdots f_p g$  imply the existence of  $f'_1, h_1$  in  $\mathcal{S}^\sharp$  satisfying  $hf'_1 = f_1h_1 \preceq f_1 \cdots f_p g$ . By left-cancelling  $f_1$ , we deduce that  $h_1$  and  $f_2$  left-divide  $f_2 \cdots f_p g$ , whence the existence of  $f'_2, h_2$  satisfying  $h_1f'_2 = f_2h_2 \preceq f_2 \cdots f_p g$ , and so on until  $h_{p-1}f'_p = f_ph_p \preceq f_p g$ , which implies  $h_p \preceq g$ , say  $h_p e = g$ . By construction,  $f'_1 \cdots f'_p e$  is an identity-element, hence all of the factors must be invertible. Now  $h_p$  belongs to  $\mathcal{S}^\sharp$ , hence  $g$ , which is  $h_p e$ , belongs to  $\mathcal{S}^\sharp \mathcal{C}^\times$ , which is  $\mathcal{S}^\sharp$ . So  $\mathcal{S}^\sharp$  is closed under right-divisor.  $\square$

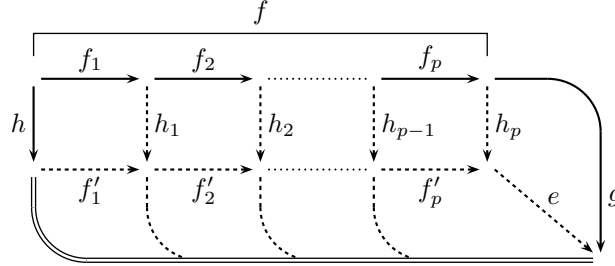


FIGURE 4. Proof of Lemma 3.17: using the closure under right-complement, one fills the diagram, and concludes that  $g$  must lie in  $\mathcal{S}^\sharp$ , since  $h_p$  lies in  $\mathcal{S}^\sharp$  and  $e$  is invertible.

**Lemma 3.18.** *Assume that  $\mathcal{S}$  is a subfamily of a left-cancellative category  $\mathcal{C}$  that generates  $\mathcal{C}$  and is closed under right-complement. Then every element of  $(\mathcal{S}^\sharp)^2$  that admits an  $\mathcal{S}$ -head admits an  $\mathcal{S}$ -normal decomposition.*

*Proof.* Assume that  $g$  is  $h_1h_2$  with  $h_1, h_2 \in \mathcal{S}^\sharp$  and  $g$  admits an  $\mathcal{S}$ -head, say  $g_1$ . By assumption, we have  $h_1 \preceq g$  with  $h_1 \in \mathcal{S}^\sharp$ . By Lemma 3.16,  $g_1$  is an  $\mathcal{S}^\sharp$ -head of  $g$ , implying  $h_1 \preceq g_1$ , say  $g_1 = h_1f$ . Then we have  $g = h_1fg_2 = h_1h_2$ , whence

$h_2 = fg_2$ . It follows that  $g_2$  right-divides  $h_2$ , an element of  $\mathcal{S}^\sharp$ . By Lemma 3.17,  $\mathcal{S}^\sharp$  must be closed under right-divisor, so  $g_2$  lies in  $\mathcal{S}^\sharp$ . Hence  $g_1|g_2$  is a decomposition of  $g$  whose entries lie in  $\mathcal{S}^\sharp$ .

Finally, by Lemma 3.13, the assumption that  $g_1$  is an  $\mathcal{S}$ -head of  $g_1g_2$  implies that  $g_1|g_2$  is  $\mathcal{S}$ -greedy, hence it is an  $\mathcal{S}$ -normal decomposition of  $g$ .  $\square$

**Lemma 3.19.** *Assume that  $\mathcal{S}$  is a subfamily of a left-cancellative category  $\mathcal{C}$  such that  $\mathcal{S}^\sharp$  is closed under right-comultiple. Then a  $\prec$ -maximal left-divisor in  $\mathcal{S}$  is an  $\mathcal{S}$ -head.*

*Proof.* Assume that  $g_1$  is a  $\prec$ -maximal left-divisor of  $g$  lying in  $\mathcal{S}$ . Write  $g = g_1g_2$ . Let  $h$  be an arbitrary left-divisor of  $g$  lying in  $\mathcal{S}$ . We wish to prove  $h \preceq g_1$ . Now,  $g$  is a common right-multiple of  $h$  and  $g_1$ , which both lie in  $\mathcal{S}$ , hence in  $\mathcal{S}^\sharp$ . As the latter is closed under right-comultiple, there must exist a common right-multiple  $h'$  of  $h$  and  $g_1$  that lies in  $\mathcal{S}^\sharp$  and left-divides  $g$ . As we have  $g_1 \preceq h'$ , the assumption that  $g_1$  is a  $\prec$ -maximal left-divisor of  $g$  implies  $g_1 = {}^\times h'$ , whence  $h \preceq h' \preceq g_1$ , and, finally,  $h \preceq g_1$ . Hence  $g_1$  is an  $\mathcal{S}$ -head of  $g$ .  $\square$

*Proof of Proposition 3.9 ( $\Leftarrow$ ).* Assume first that  $\mathcal{S}$  satisfies (3.10). We claim that  $\mathcal{S}$  is also closed under right-comultiple and right-complement. Indeed, assume that  $f, g$  belong to  $\mathcal{S}^\sharp$  and  $f, g \preceq h$  hold. Assume first  $h \notin \mathcal{C}^\times$ , and let  $h_1$  be an  $\mathcal{S}$ -head of  $h$ . As  $f$  belongs to  $\mathcal{S}^\sharp$  and it left-divides  $h$ , we must have  $f \preceq h_1$ , hence  $fg' = h_1$  for some  $g'$ , and, similarly,  $gf' = h_1$  for some  $f'$ . As  $\mathcal{S}^\sharp$  is closed under right-divisor,  $f'$  and  $g'$  must belong to  $\mathcal{S}^\sharp$ . If  $h$  is invertible, then  $h$  belongs to  $\mathcal{S}^\sharp$  and we can take  $h_1 = h$ ,  $g' = f^{-1}h_1$ , and  $f' = g^{-1}h_1$ . In all cases,  $h_1, f', g'$  witness that  $\mathcal{S}^\sharp$  is closed under right-comultiple and right-complement. Hence, (3.10) implies (3.11).

Next, assume that  $\mathcal{S}$  satisfies (3.11). Then, by assumption, every element of  $(\mathcal{S}^\sharp)^2$  admits an  $\mathcal{S}$ -head, hence, by Lemma 3.18, an  $\mathcal{S}$ -normal decomposition. Hence, by Proposition 3.1,  $\mathcal{S}$  is a Garside family in  $\mathcal{C}$ .

Finally, assume that  $\mathcal{S}$  satisfies (3.12). First, by Lemma 3.7, the assumption that  $\mathcal{S}^\sharp$  is closed under right-comultiple and right-divisor implies that it is closed under right-complement. Then, by Lemma 3.19, the existence of a  $\prec$ -maximal left-divisor in  $\mathcal{S}$  for every element  $g$  of  $(\mathcal{S}^\sharp)^2$  implies the existence of an  $\mathcal{S}$ -head whenever at least one such divisor lies in  $\mathcal{S}$ , which is guaranteed when  $g$  is non-invertible. So (3.12) implies (3.11), and  $\mathcal{S}$  must be a Garside family again.  $\square$

**3.3. Special contexts.** All results established so far are valid in an arbitrary left-cancellative category. When the ambient category happens to satisfy additional properties, some of the conditions involved in the characterizations of Garside families may be automatically satisfied or take simpler forms.

**Definition 3.20.** A category  $\mathcal{C}$  is called *right-Noetherian* if right-divisibility in  $\mathcal{C}$  is a well-founded relation, that is, every nonempty subfamily has a least element.

Above, by a least element we mean an element that right-divides every element of the considered subfamily. We recall that  $f \prec g$  stands for the conjunction of  $f \preceq g$  and  $f \not\preceq g$ , that is, by Lemma 1.6, the conjunction of  $f \preceq g$  and  $f \neq^\times g$ .

**Lemma 3.21.** *A left-cancellative category  $\mathcal{C}$  is right-Noetherian if and only if there is no bounded  $\prec$ -increasing sequence in  $\mathcal{C}$ .*

*Proof.* Assume that  $\mathcal{C}$  is right-Noetherian and we have  $g_1 \prec g_2 \prec \dots \prec g$  in  $\mathcal{C}$ . For each  $i$ , write  $g_i f_i = g_{i+1}$  and  $g_i h_i = g$ . Then we have  $g_i h_i = g_{i+1} h_{i+1} = g_i f_i h_{i+1}$ ,

whence  $h_i = f_i h_{i+1}$  by left-cancelling  $g_i$ . So the sequence  $(h_1, h_2, \dots)$  is decreasing for right-divisibility. Let  $h_N$  be a least element in  $\{h_i \mid i \geq 1\}$ . Necessarily  $f_i$  is invertible for  $i \geq N$ , which implies  $g_{i+1} = {}^\times g_i$ . So  $(g_1, g_2, \dots)$  cannot be  $\prec$ -increasing.

Conversely, assume that  $\mathcal{C}$  contains no bounded  $\prec$ -increasing sequence in  $\mathcal{C}$ , and let  $(f_1, f_2, \dots)$  be a decreasing sequence with respect to right-divisibility. Write  $f_i = g'_i f_{i+1}$ . Set  $g = f_1$ , and  $g_i = g'_1 \cdots g'_i$ . Then we have  $g_1 \preceq g_2 \preceq \cdots \preceq g$ . The assumption that  $\mathcal{C}$  contains no bounded  $\prec$ -increasing sequence implies that  $g'_i$  is invertible for  $i$  large enough. It follows that right-divisibility admits no infinite descending sequence. By the Axiom of Dependent Choices, this implies that right-divisibility is a well-founded relation, that is, that  $\mathcal{C}$  is right-Noetherian.  $\square$

**Proposition 3.22.** *A subfamily  $\mathcal{S}$  of a left-cancellative category that is right-Noetherian is a Garside family if and only if*

$$(3.23) \quad \mathcal{S}^\sharp \text{ generates } \mathcal{C} \text{ and it is closed under right-comultiple and right-divisor.}$$

*Proof.* Owing to Proposition 3.9 and (3.12), it is enough to show that every non-invertible element  $g$  in  $\mathcal{C}$  (or only in  $(\mathcal{S}^\sharp)^2$ ) admits a  $\prec$ -maximal left-divisor lying in  $\mathcal{S}$ . First, as  $\mathcal{S}^\sharp$  generates  $\mathcal{C}$  and  $g$  is non-invertible, the latter is left-divisible by some non-invertible element of  $\mathcal{S}^\sharp$ , hence by some (non-invertible) element of  $\mathcal{S}$ , say  $g_1$ . If  $g_1$  is not  $\prec$ -maximal among left-divisors of  $g$  lying in  $\mathcal{S}$ , we can find  $g_2$  satisfying  $g_1 \prec g_2 \preceq g$ . If  $g_2$  is not  $\prec$ -maximal, we find  $g_3$  satisfying  $g_2 \prec g_3 \preceq g$ , and so on. By Lemma 3.21, the construction cannot be repeated infinitely many times, which means that some  $g_N$  must be  $\prec$ -maximal among left-divisors of  $g$  lying in  $\mathcal{S}$ . Then, every non-invertible element of  $(\mathcal{S}^\sharp)^2$  admits a  $\prec$ -maximal left-divisor in  $\mathcal{S}$ . So  $\mathcal{S}$  satisfies (3.12) hence, by Proposition 3.9, it is a Garside family in  $\mathcal{C}$ .  $\square$

A further specialization leads to categories that admit least common right-multiples.

**Definition 3.24.** Assume that  $\mathcal{C}$  is a left-cancellative category. We say that  $h$  is a *least common right-multiple*, or *right-lcm*, of  $f$  and  $g$  if  $h$  is a right-multiple of  $f$  and  $g$  and every element  $h'$  that is a right-multiple of  $f$  and  $g$  is a right-multiple of  $h$ . A subfamily  $\mathcal{S}$  of  $\mathcal{C}$  is said to be *closed under right-lcm* in  $\mathcal{C}$  if every right-lcm of two elements of  $\mathcal{S}$  lies in  $\mathcal{S}$ .

In other words, a right-lcm is a least common upper bound with respect to left-divisibility.

**Proposition 3.25.** *Assume that  $\mathcal{C}$  is a left-cancellative category that is right-Noetherian and such that any two elements of  $\mathcal{C}$  that admit a common right-multiple admit a right-lcm. Then a subfamily  $\mathcal{S}$  of  $\mathcal{C}$  is a Garside family if and only if*

$$(3.26) \quad \mathcal{S}^\sharp \text{ generates } \mathcal{C} \text{ and it is closed under right-lcm and right-divisor.}$$

*Proof.* Assume that  $\mathcal{S}$  is a Garside family in  $\mathcal{C}$ . Then, by Proposition 3.9,  $\mathcal{S}^\sharp$  is closed under right-comultiple. Assume that  $f, g$  are elements of  $\mathcal{S}^\sharp$  that admit a common right-multiple. Then  $f$  and  $g$  admit a right-lcm, say  $h$ . As  $\mathcal{S}^\sharp$  is closed under right-comultiple, there exists  $h'$  in  $\mathcal{S}^\sharp$  satisfying  $f, g \preceq h' \preceq h$ . By definition of a right-lcm,  $h \preceq h'$  must hold, whence  $h' = {}^\times h$ . So there exists an invertible element  $e$  satisfying  $h = h'e$ , whence  $h \in \mathcal{S}^\sharp \mathcal{C}^\times = \mathcal{S}^\sharp$ . This shows that  $\mathcal{S}^\sharp$  is closed under right-lcm.



Conversely, assume that  $\mathcal{S}$  satisfies (3.26) and we have  $f, g \preceq h$  with  $f, g \in \mathcal{S}^\sharp$ . As  $f, g$  admit a common right-multiple, they admit a right-lcm, say  $h'$ , and, by assumption,  $h'$  belongs to  $\mathcal{S}^\sharp$ . By definition of a right-lcm,  $h' \preceq h$  holds, so  $\mathcal{S}^\sharp$  is closed under right-comultiple. Then, by Proposition 3.22,  $\mathcal{S}$  is a Garside family in  $\mathcal{C}$ .  $\square$

Specializing even more, we consider the case of categories that are both right- and left-Noetherian, the latter meaning that left-divisibility is well founded. In that case, standard arguments show that every non-invertible element is a product of *atoms*, defined to be those non-invertible elements that cannot be expressed as the product of at least two non-invertible elements. Then, if  $\mathcal{C}$  is a Noetherian category containing no nontrivial invertible element, a subfamily  $\mathcal{S}$  of  $\mathcal{C}$  generates  $\mathcal{C}$  if and only if  $\mathcal{S}$  contains all atoms of  $\mathcal{C}$ . Then Proposition 3.25 directly implies

**Proposition 3.27.** *Assume that  $\mathcal{C}$  is a left-cancellative category that is Noetherian, such that any two elements that admit a common right-multiple admit a right-lcm, and contains no nontrivial invertible element. Then a subfamily  $\mathcal{S}$  of  $\mathcal{C}$  is a Garside family if and only if*

$$(3.28) \quad \mathcal{S} \text{ contains the atoms of } \mathcal{C} \text{ and it is closed under right-lcm and right-divisor.}$$

**Corollary 3.29.** *Under the assumptions of Proposition 3.27, there exists a smallest Garside family in  $\mathcal{C}$ .*

*Proof.* Under the considered assumptions, an intersection of Garside families is a Garside family since under these assumptions a Garside family is defined by closure properties plus the fact that it contains all atoms. In particular, the intersection of all Garside families of  $\mathcal{C}$  is a smallest Garside family in  $\mathcal{C}$ .  $\square$

**3.4. Recognizing Garside families III: head functions.** We return to the general case, and establish further characterizations of Garside families, this time in terms of what is called a head function.

**Definition 3.30.** Assume that  $\mathcal{C}$  is a left-cancellative category. A partial map  $H$  of  $\mathcal{C}$  to itself is said to obey the  $\mathcal{H}$ -law if

$$(3.31) \quad H(fg) =^\times H(fH(g))$$

holds whenever both terms are defined. If (3.31) holds with equality instead of  $=^\times$ , we say that  $H$  obeys the *sharp*  $\mathcal{H}$ -law.

In the same context, in addition to the  $\mathcal{H}$ -law, we shall also consider the following conditions, supposed to hold for all  $f, g$  in the domain of the considered map  $H$ :

$$(3.32) \quad \text{(i) } H(g) \preceq g, \quad \text{(ii) } f \preceq g \Rightarrow H(f) \preceq H(g), \quad \text{(iii) } g \in \mathcal{S}^\sharp \Rightarrow H(g) =^\times g.$$

**Proposition 3.33.** *A subfamily  $\mathcal{S}$  of a left-cancellative category  $\mathcal{C}$  is a Garside family if and only if it satisfies one of the following equivalent conditions:*

$$(3.34) \quad \mathcal{S}^\sharp \text{ generates } \mathcal{C} \text{ and there exists } H : \mathcal{C} \setminus \mathcal{C}^\times \rightarrow \mathcal{S} \text{ satisfying the } \mathcal{H}\text{-law and (3.32);}$$

$$(3.35) \quad \mathcal{S}^\sharp \text{ generates } \mathcal{C}, \mathcal{C}^\times \mathcal{S}^\sharp \subseteq \mathcal{S}^\sharp \text{ holds, and there exists } H : \mathcal{S}\mathcal{C} \rightarrow \mathcal{S} \text{ satisfying the sharp } \mathcal{H}\text{-law and (3.32).}$$

*Proof.* Let us first observe that, if  $\mathcal{S}^\sharp$  generates  $\mathcal{C}$  and  $\mathcal{C}^\times \mathcal{S}^\sharp \subseteq \mathcal{S}^\sharp$  holds, then  $\mathcal{C} \setminus \mathcal{C}^\times$  is included in  $\mathcal{SC}$ : indeed, in this case, if  $g$  is non-invertible, it must be left-divisible by an element of the form  $eh$  with  $e \in \mathcal{C}^\times$  and  $h \in \mathcal{S} \setminus \mathcal{C}^\times$ , hence by an element of  $\mathcal{S} \setminus \mathcal{C}^\times$  owing to  $\mathcal{C}^\times \mathcal{S}^\sharp \subseteq \mathcal{S}^\sharp$ . Hence (3.35) implies (3.34).

Assume that  $\mathcal{S}$  is a Garside family in  $\mathcal{C}$ . For  $g$  in  $\mathcal{SC}$ , define  $H(g)$  to be an  $\mathcal{S}$ -head of  $g$ , which exists by Lemma 3.16. Then, by definition,  $H(g) \preceq g$  always holds. Next, if  $f \preceq g$  holds,  $H(f)$  is an element of  $\mathcal{S}$  that left-divides  $f$ , hence  $g$ , so  $H(f) \preceq H(g)$  must hold. Then, if  $g$  belongs to  $\mathcal{S}^\sharp \cap \mathcal{SC}$ , we have  $g ={}^\times g'$  for some  $g'$  in  $\mathcal{S}$ , and we deduce  $g' \preceq H(g) \preceq g$ , whence  $H(g) ={}^\times g$ . So  $H$  satisfies (3.32). Finally, assume that  $fg$  exists and  $g$  lies in  $\mathcal{SC}$ . If  $g$  is non-invertible, then  $H(g)$  is non-invertible as well since  $\|g\|_{\mathcal{S}} \geq 1$  holds, meaning that  $g$  is left-divisible by at least one non-invertible element of  $\mathcal{S}$ , so both  $H(fg)$  and  $H(fH(g))$  are defined in this case. On the other hand, if  $g$  is invertible, then  $H(g) ={}^\times g$  holds, and  $fg$  belongs to  $\mathcal{SC}$  if and only if  $fH(g)$  does. In all cases, by (3.32)(i), we have  $H(g) \preceq g$ , whence  $fH(g) \preceq fg$ , and, by (3.32)(ii),  $H(fH(g)) \preceq H(fg)$ . On the other hand, write  $g = H(g)g'$ . By Lemma 3.13, the path  $H(g)|g'$  is  $\mathcal{S}$ -greedy. Then the relation  $H(fg) \preceq fg$ , which holds by (3.32)(i), implies  $H(fg) \preceq fH(g)$ , whence  $H(fg) \preceq H(fH(g))$  since  $H(fg)$  lies in  $\mathcal{S}$ . So we deduce  $H(fg) ={}^\times H(fH(g))$ , and  $H$  satisfies the  $\mathcal{H}$ -law.

Next, let  $\mathcal{S}_0$  be an  $={}^\times$ -selector on  $\mathcal{S}$ , that is, a subfamily of  $\mathcal{S}$  that contains exactly one element in each  $={}^\times$ -class (which exists by the Axiom of Choice). For  $g$  in  $\mathcal{SC}$ , define  $H_0(g)$  to be the unique element of the  $={}^\times$ -class of  $H(g)$  that lies in  $\mathcal{S}_0$ . By construction,  $H_0(g) ={}^\times H(g)$  holds for every  $g$  in  $\mathcal{SC}$  and, therefore, the function  $H_0$  satisfies (3.32) and the  $\mathcal{H}$ -law: for the latter, we have  $fH(g) ={}^\times fH_0(g)$ , whence, by (3.32)(ii),  $H(fH(g)) ={}^\times H(fH_0(g))$ , and, from there,

$$H_0(fg) ={}^\times H(fg) ={}^\times H(fH(g)) ={}^\times H(fH_0(g)) ={}^\times H_0(fH_0(g)).$$

Now, by construction,  $H_0(fg)$  and  $H_0(fH_0(g))$  belong to  $\mathcal{S}_0$ , so, being  $={}^\times$ -equivalent, they must be equal. Hence  $H_0$  obeys the sharp  $\mathcal{H}$ -law, and (3.35) is satisfied. By the initial remark, (3.34) is satisfied too.

Conversely, assume that  $\mathcal{S}$  satisfies (3.34), with  $H : \mathcal{C} \setminus \mathcal{C}^\times \rightarrow \mathcal{S}$  witnessing the expected conditions. We shall prove that  $\mathcal{S}$  satisfies (3.10). The first step is to show that, for each non-invertible  $g$ , the element  $H(g)$  is an  $\mathcal{S}$ -head of  $g$ . So let  $g$  be a non-invertible element of  $\mathcal{C}$ . By assumption,  $H(g)$  belongs to  $\mathcal{S}$  and left-divides  $g$ . Assume  $h \preceq g$  with  $h \in \mathcal{S}$ . By (3.32)(ii) and (iii), we have  $h ={}^\times H(h) \preceq H(g)$ , whence  $h \preceq H(g)$ . Hence  $H(g)$  is an  $\mathcal{S}$ -head of  $g$ .

The second step is to show that  $\mathcal{S}^\sharp$  is closed under right-divisor. So assume that  $g$  belongs to  $\mathcal{S}^\sharp$  and  $f$  right-divides  $g$ , say  $g = g'f$ . If  $f$  is invertible, it belongs to  $\mathcal{S}^\sharp$  by definition. Assume now that  $f$ , and therefore  $g$ , are not invertible. Then we have  $g ={}^\times h$  for some  $h$  lying in  $\mathcal{S}$ . By (3.32)(ii) and (iii), we have  $H(g) ={}^\times H(h)$ , whence  $H(h) ={}^\times h$ . We deduce  $H(g) ={}^\times g$ , that is,  $H(g'f) ={}^\times g'f$ . Then the  $\mathcal{H}$ -law implies  $H(g'f) ={}^\times H(g'H(f))$ , whereas (3.32)(i) gives  $H(g'H(f)) \preceq g'H(f)$ , so we find

$$g'f ={}^\times H(g'f) ={}^\times H(g'H(f)) \preceq g'H(f),$$

hence  $g'f \preceq g'H(f)$ , and  $f \preceq H(f)$  by left-cancelling  $g'$ . As  $H(f) \preceq f$  always holds by (3.32)(i), we deduce  $f ={}^\times H(f)$ . As, by definition,  $H(f)$  lies in  $\mathcal{S}$ , it follows that  $f$  lies in  $\mathcal{S}^\sharp$ . So  $\mathcal{S}^\sharp$  is closed under right-divisor.

It is now easy to conclude. By assumption,  $\mathcal{S}^\#$  generates  $\mathcal{C}$ , we saw above that it is closed under right-divisor, and that every non-invertible element of  $\mathcal{C}$  admits an  $\mathcal{S}$ -head. So  $\mathcal{S}$  satisfies (3.10) and, by Proposition 3.9, it is a Garside family in  $\mathcal{C}$ .  $\square$

#### 4. GERMS

So far, we have established extrinsic characterizations of Garside families, namely conditions that describe a Garside family in a given pre-existing category. We turn now to intrinsic characterizations, that is, we do not start from a pre-existing category but consider instead an abstract family  $\mathcal{S}$  equipped with a partial product and investigate necessary and sufficient conditions for such a structure, here called a germ, to generate a category in which  $\mathcal{S}$  embeds as a Garside family.

**4.1. The notion of a germ.** If  $\mathcal{S}$  is a subfamily of a category  $\mathcal{C}$ , then, for  $f, g$  in  $\mathcal{S}$  such that  $fg$  is defined,  $fg$  may belong or not to  $\mathcal{S}$ . Restricting to the case when the product belongs to  $\mathcal{S}$  gives a partial map from  $\mathcal{S}^{[2]}$  to  $\mathcal{S}$ . The resulting structure will be called a germ, and we shall be interested in the case when the whole category can be retrieved from the germ.

In the above situation, we shall denote by  $\bullet_{\mathcal{S}}$ , or simply  $\bullet$ , the partial operation on  $\mathcal{S}$  induced by the product of the ambient category. It is easy to see that a partial operation of this type must obey some constraints.

**Lemma 4.1.** *Assume that  $\mathcal{S}$  is a subfamily of a category  $\mathcal{C}$  that contains all identity-elements of  $\mathcal{C}$ . The partial operation  $\bullet$  of  $\mathcal{S}^{[2]}$  to  $\mathcal{S}$  induced by the product of  $\mathcal{C}$  obeys the following rules:*

$$(4.2) \quad \text{If } f \bullet g \text{ is defined, the source (resp. target) of } f \bullet g \text{ is the source of } f \text{ (resp. the target of } g);$$

$$(4.3) \quad 1_x \bullet f = f = f \bullet 1_y \text{ hold for each } f \text{ in } \mathcal{S}(x, y),$$

$$(4.4) \quad \text{If } f \bullet g \text{ and } g \bullet h \text{ are defined, then } (f \bullet g) \bullet h \text{ is defined if and only if } f \bullet (g \bullet h) \text{ is, in which case they are equal.}$$

Moreover, if  $\mathcal{S}$  is closed under right-divisor in  $\mathcal{C}$ , then  $\bullet$  satisfies

$$(4.5) \quad \text{If } (f \bullet g) \bullet h \text{ is defined, then so is } g \bullet h.$$

*Proof.* Points (4.2) and (4.3) follow from the fact that  $\bullet$  is induced by the product of  $\mathcal{C}$ . Next, (4.4) follows from associativity in  $\mathcal{C}$ : saying that  $(f \bullet g) \bullet h$  exists means that  $(fg)h$  belongs to  $\mathcal{S}$ , hence so does  $f(gh)$ . As, by assumption,  $g \bullet h$  exists, this amounts to  $f \bullet (g \bullet h)$  being defined.

For (4.5), the hypotheses imply that  $(fg)h$ , hence  $f(gh)$ , belongs to  $\mathcal{S}$ . As  $\mathcal{S}$  is closed under right-divisor, this implies that  $gh$  belongs to  $\mathcal{S}$ , hence that  $g \bullet h$  is defined.  $\square$

We shall therefore start from abstract families that obey the rules of Lemma 4.1. We recall from Definition 1.7 that  $\mathcal{S}^{[r]}$  denotes the family of all length  $r$  paths in  $\mathcal{S}$ .

**Definition 4.6.** A *germ* is a triple  $(\mathcal{S}, \mathbf{1}_{\mathcal{S}}, \bullet)$  where  $\mathcal{S}$  is a precategory,  $\mathbf{1}_{\mathcal{S}}$  is a subfamily of  $\mathcal{S}$  containing an element  $1_x$  with source and target  $x$  for each object  $x$ , and  $\bullet$  is a partial map from  $\mathcal{S}^{[2]}$  to  $\mathcal{S}$  that satisfies (4.2), (4.3), and (4.4). If, moreover, (4.5) holds, the germ is said to be *left-associative*. If  $(\mathcal{S}, \mathbf{1}_{\mathcal{S}}, \bullet)$  is a germ, we denote by  $\text{Cat}(\mathcal{S}, \mathbf{1}_{\mathcal{S}}, \bullet)$  the category  $\langle \mathcal{S} \mid \mathcal{R}_{\bullet} \rangle^+$ , where  $\mathcal{R}_{\bullet}$  is the family of all relations  $f|g = f \bullet g$  with  $f, g$  in  $\mathcal{S}$  and  $f \bullet g$  defined.

In practice, we shall use the generic notation  $\underline{\mathcal{S}}$  for a germ with domain  $\mathcal{S}$ . In the sequel, an equality of the form  $g = g_1 \bullet g_2$  always means “ $g_1 \bullet g_2$  is defined and  $g$  equals it”.

For every subfamily  $\mathcal{S}$  of a category  $\mathcal{C}$ , there exists an induced germ  $\underline{\mathcal{S}}$ , and the corresponding relations  $\mathcal{R}_\bullet$  are valid in  $\mathcal{C}$  by construction. Hence  $\mathcal{C}$  is a quotient of  $\text{Cat}(\underline{\mathcal{S}})$ . In most cases, even if  $\mathcal{S}$  generates  $\mathcal{C}$ , the partial product  $\bullet$  does not determine the product of  $\mathcal{C}$ , and  $\mathcal{C}$  is a proper quotient of  $\text{Cat}(\underline{\mathcal{S}})$ . For instance, if  $\mathcal{C}$  contains no nontrivial invertible element and is generated by a family of atoms  $\mathcal{A}$ , the induced partial product on  $\mathcal{A}$  consists of the trivial instances listed in (4.3) only, and the resulting category is a free category based on  $\mathcal{A}$ . We shall now see that this cannot happen when  $\mathcal{S}$  is a Garside family: in this case, the induced structure  $\underline{\mathcal{S}}$  contains enough information to retrieve the initial category  $\mathcal{C}$ . Here one has to be careful: if  $\mathcal{S}$  is a general Garside family in  $\mathcal{C}$ , then  $\mathcal{C}$  is generated by  $\mathcal{S}^\sharp$ , hence by  $\mathcal{S} \cup \mathcal{C}^\times$ , but not necessarily by  $\mathcal{S}$  itself. To avoid problems, we shall restrict to particular Garside families.

**Definition 4.7.** A subfamily  $\mathcal{S}$  of a category  $\mathcal{C}$  is called *full* if  $\mathcal{S}$  contains all identity-elements of  $\mathcal{C}$  and it is closed under right-divisor in  $\mathcal{C}$ .

It follows from Lemma 3.14 that, for every Garside family  $\mathcal{S}$ , the family  $\mathcal{S}^\sharp$  is a full Garside family, which we know gives rise to the same greedy and normal paths. So considering full Garside families is not a proper restriction. (On the other hand, for  $\mathcal{S}$  to be full is weaker than satisfying  $\mathcal{S} = \mathcal{S}^\sharp$ : one can exhibit a full Garside family  $\mathcal{S}$  such that  $\mathcal{S}$  is properly included in  $\mathcal{S}^\sharp$ .) The precise result we shall prove is then

**Proposition 4.8.** *Assume that  $\mathcal{S}$  is a full Garside family in a left-cancellative category  $\mathcal{C}$ , and let  $\underline{\mathcal{S}}$  be the induced germ. Then  $\text{Cat}(\underline{\mathcal{S}})$  is isomorphic to  $\mathcal{C}$ .*

By definition, the category  $\text{Cat}(\underline{\mathcal{S}})$  is specified by a presentation. In order to establish Proposition 4.8, we shall show that every (full) Garside family provides a presentation of its ambient category.

**Lemma 4.9.** *Assume  $\mathcal{S}$  is a generating subfamily of a left-cancellative category  $\mathcal{C}$  that is full and closed under right-comultiple. Then  $\mathcal{C}$  admits the presentation  $\langle \mathcal{S} \mid \mathcal{R} \rangle^+$  where  $\mathcal{R}$  consists of all relations  $f|g = h$  with  $f, g, h$  in  $\mathcal{S}$  that are valid in  $\mathcal{C}$ .*

*Proof.* First, by assumption,  $\mathcal{S}$  generates  $\mathcal{C}$ . Next, by definition, all relations of  $\mathcal{R}$  are valid in  $\mathcal{C}$ . So the point is to show that every equality involving elements of  $\mathcal{S}$  that is valid in  $\mathcal{C}$  is a consequence of finitely many relations of  $\mathcal{R}$ .

So assume that  $f_1| \dots | f_p$  and  $g_1| \dots | g_q$  are two  $\mathcal{S}$ -paths with the same evaluation in  $\mathcal{C}$ , that is,  $f_1 \dots f_p = g_1 \dots g_q$  holds in  $\mathcal{C}$ . Then we inductively construct a rectangular grid as displayed in Figure 5.

Put  $h'_{i,0} = f_{i+1} \dots f_p$  for  $0 \leq i \leq p$ , and  $h'_{0,j} = g_{j+1} \dots g_q$  for  $0 \leq j \leq q$ . First, we have  $f_1 h'_{1,0} = g_1 h'_{0,1} = h'_{0,0}$  with  $f_1, g_1$  in  $\mathcal{S}$ . The assumption that  $\mathcal{S}$  is closed under right-comultiple implies the existence of  $h_{1,1}$  in  $\mathcal{S}$  and  $f_{1,1}, g_{1,1}, h'_{1,1}$  satisfying  $f_1 g_{1,1} = g_1 f_{1,1} = h_{1,1}$ ,  $h'_{0,1} = f_{1,1} h'_{1,1}$ , and  $h'_{1,0} = g_{1,1} h'_{1,1}$ . Moreover, the assumption that  $\mathcal{S}$  is closed under right-divisor implies that  $f_{1,1}$  and  $g_{1,1}$ , which right-divide  $h_{1,1}$ , belong to  $\mathcal{S}$ .

Now we repeat the same argument with  $h'_{0,1}, \dots, h'_{0,q-1}$ , then  $h'_{1,0}, \dots, h'_{1,q-1}$ , and so on until  $h'_{p-1,q-1}$ : starting from the vertex  $(i-1, j-1)$ , by induction hypothesis, we have  $f_{i,j-1} h'_{i,j-1} = g_{i-1,j} h'_{i-1,j} = h'_{i-1,j-1}$  with  $f_{i,j-1}$  and  $g_{i-1,j}$  in  $\mathcal{S}$ . As  $\mathcal{S}$

closed under right-comultiple and right-divisor, there exist  $f_{i,j}, g_{i,j}, h_{i,j}$  in  $\mathcal{S}$  and  $h'_{i,j}$  in  $\mathcal{C}$  satisfying  $f_{i,j-1}g_{i,j} = g_{i-1,j}f_{i,j} = h_{i,j}$ ,  $h'_{i-1,j} = f_{i,j}h'_{i,j}$ , and  $h'_{i,j-1} = g_{i,j}h'_{i,j}$ .

At this point, we see that the equality  $f_1 \cdots f_p = g_1 \cdots g_q$  is the consequence of  $2pq$  relations of the form  $f|g = h$  with  $f, g, h$  in  $\mathcal{S}$ , namely the relations  $f_{i,j-1}|g_{i,j} = h_{i,j}$  and  $g_{i-1,j}|f_{i,j} = h_{i,j}$ , plus the  $p + q$  relations  $f_{i,q}|h'_{i,q} = h'_{i-1,q}$  and  $g_{p,j}|h'_{p,j} = h'_{p,j-1}$ . By construction, all elements  $h'_{i,q}$  and  $h'_{p,j}$  are invertible, and so are all  $f_{i,q}$  and  $g_{p,j}$ . As  $\mathcal{S}$  generates  $\mathcal{C}$ , every element of  $\mathcal{C}^\times$  is a finite product of elements of  $\mathcal{S} \cap \mathcal{C}^\times$  and, therefore, every relation of the form  $e_1|e_2 = e$  holding in  $\mathcal{C}$  follows from finitely many relations of this form with  $e_1, e_2, e$  in  $\mathcal{S} \cap \mathcal{C}^\times$ . This completes the argument.  $\square$

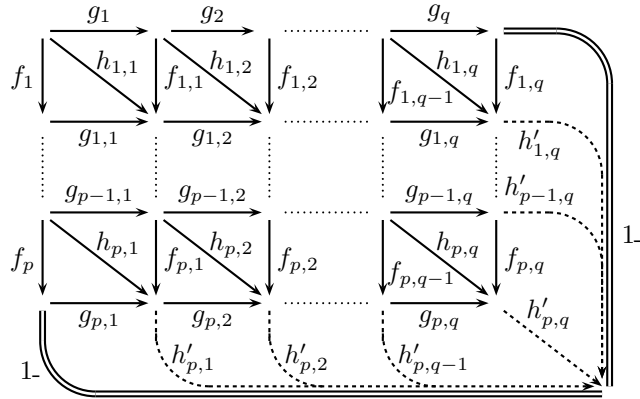


FIGURE 5. Factorization of the equality  $f_1 \cdots f_p = g_1 \cdots g_q$  in terms of relations  $fg = h$  with  $f, g, h$  in  $\mathcal{S}$ .

We can now complete the proof of Proposition 4.8.

*Proof of Proposition 4.8.* First, by Proposition 3.9, the family  $\mathcal{S}^\#$  is closed under right-comultiple. An easy direct verification shows that, in any case, a subfamily  $\mathcal{S}$  is closed under right-comultiple if and only if  $\mathcal{S}^\#$  is. So, here,  $\mathcal{S}$  is closed under right-comultiple. By assumption, it is full, so, by Lemma 4.9,  $\mathcal{C}$  is presented by the relations  $f|g = h$  with  $f, g, h$  in  $\mathcal{S}$ . This means that  $\mathcal{C}$  admits the presentation  $\langle \mathcal{S} | \mathcal{R}_\bullet \rangle^+$ . So  $\mathcal{C}$  is isomorphic to  $\text{Cat}(\underline{\mathcal{S}})$ , which, by definition, admits that presentation.  $\square$

So, in the case of a full Garside family, the induced germ contains all information needed to determine the ambient category, and it is therefore natural to investigate the germs that occur in this way.

**4.2. The embedding problem.** From now on, we start from an abstract germ  $\underline{\mathcal{S}}$ , and investigate the properties of the category  $\text{Cat}(\underline{\mathcal{S}})$  and of (the image of)  $\mathcal{S}$  in  $\text{Cat}(\underline{\mathcal{S}})$ . The first question is whether  $\underline{\mathcal{S}}$  embeds in  $\text{Cat}(\underline{\mathcal{S}})$ . This need not be the case in general, but we shall see that left-associativity is a sufficient condition.

**Notation 4.10.** For  $\underline{\mathcal{S}}$  a germ, we denote by  $\equiv$  the congruence on  $\mathcal{S}^*$  generated by the relations of  $\mathcal{R}_\bullet$ , and by  $\iota$  the prefunctor of  $\mathcal{S}$  to  $\text{Cat}(\underline{\mathcal{S}})$  that is the identity on  $\text{Obj}(\mathcal{S})$  and maps  $g$  to the  $\equiv$ -class of  $(g)$ .

So, by definition,  $\equiv$  is the equivalence relation on  $\mathcal{S}^*$  generated by all pairs

$$(4.11) \quad (g_1 | \cdots | g_i | g_{i+1} | \cdots | g_q, g_1 | \cdots | g_i \bullet g_{i+1} | \cdots | g_q),$$

that is, the pairs in which two adjacent entries are replaced with their  $\bullet$ -product, assuming that the latter exists. The category  $\text{Cat}(\underline{\mathcal{S}})$  is then  $\mathcal{S}^*/\equiv$ .

**Proposition 4.12.** *If  $\underline{\mathcal{S}}$  is a left-associative germ, the map  $\iota$  of Notation 4.10 is injective and the product of  $\text{Cat}(\underline{\mathcal{S}})$  extends the image of  $\bullet$  under  $\iota$ . Moreover,  $\iota\mathcal{S}$  is closed under right-divisor in  $\text{Cat}(\underline{\mathcal{S}})$ .*

*Proof.* We inductively define a partial map  $\Pi$  from  $\mathcal{S}^*$  to  $\mathcal{S}$  by

$$(4.13) \quad \Pi(\varepsilon_x) = 1_x \text{ and } \Pi(g|w) = g \bullet \Pi(w) \text{ if } g \text{ lies in } \mathcal{S} \text{ and } g \bullet \Pi(w) \text{ is defined.}$$

We claim that  $\Pi$  induces a well defined partial map from  $\text{Cat}(\underline{\mathcal{S}})$  to  $\mathcal{S}$ , more precisely that, if  $w, w'$  are  $\equiv$ -equivalent elements of  $\mathcal{S}^*$ , then  $\Pi(w)$  exists if and only if  $\Pi(w')$  does, and in this case they are equal. To prove this, we may assume that  $\Pi(w)$  or  $\Pi(w')$  is defined and that  $(w, w')$  is of the type (4.11). Let  $g = \Pi(g_{i+2} | \cdots | g_p)$ . The assumption that  $\Pi(w)$  or  $\Pi(w')$  is defined implies that  $g$  is defined. Then (4.13) gives  $\Pi(w) = \Pi(g_1 | \cdots | g_{i-1} | h)$  whenever  $\Pi(w)$  is defined, and  $\Pi(w') = \Pi(g_1 | \cdots | g_{i-1} | h')$  whenever  $\Pi(w')$  is defined, with  $h = \Pi(g_i | g_{i+1} | g)$  and  $h' = \Pi(g_i \bullet g_{i+1} | g)$ , that is,

$$h = g_i \bullet (g_{i+1} \bullet g) \quad \text{and} \quad h' = (g_i \bullet g_{i+1}) \bullet g.$$

So the point is to prove that  $h$  is defined if and only if  $h'$  is, in which case they are equal. Now, if  $h$  is defined, the assumption that  $g_i \bullet g_{i+1}$  is defined plus (4.4) imply that  $h'$  exists and equals  $h$ . Conversely, if  $h'$  is defined, (4.5) implies that  $g_{i+1} \bullet g$  is defined, and then (4.4) implies that  $h$  exists and equals  $h'$ .

Assume now that  $g, g'$  lie in  $\mathcal{S}$  and  $\iota g = \iota g'$  holds, that is, the length one paths  $g$  and  $g'$  are  $\equiv$ -equivalent. The above claim gives  $g = \Pi(g) = \Pi(g') = g'$ , so  $\iota$  is injective.

Next, assume that  $f, g$  belong to  $\mathcal{S}$  and  $f \bullet g$  is defined. We have  $f|g \equiv f \bullet g$ , which means that the product of  $\iota f$  and  $\iota g$  in  $\text{Cat}(\underline{\mathcal{S}})$  is  $\iota(f \bullet g)$ .

Finally, assume that  $g$  belongs to  $\mathcal{S}$  and  $\iota g'$  is a right-divisor of  $\iota g$  in  $\text{Cat}(\underline{\mathcal{S}})$ . This means that there exist elements  $f_1, \dots, f_p, g_1, \dots, g_q$  of  $\mathcal{S}$  such that  $\iota g$  is the  $\equiv$ -class of  $f_1 | \cdots | f_p | g_1 | \cdots | g_q$  and  $\iota g'$  is the  $\equiv$ -class of  $g_1 | \cdots | g_q$ . By the claim above, the first relation implies that  $\Pi(f_1 | \cdots | f_p | g_1 | \cdots | g_q)$  exists (and equals  $g$ ). By construction, this implies that  $\Pi(g_1 | \cdots | g_q)$  exists as well, hence that  $g'$  belongs to  $\mathcal{S}$ . So  $\iota\mathcal{S}$  is closed under right-divisor in  $\text{Cat}(\underline{\mathcal{S}})$ .  $\square$

In the context of Proposition 4.12, we shall from now on identify  $\mathcal{S}$  with its image in the category  $\text{Cat}(\underline{\mathcal{S}})$ , that is, drop the canonical injection  $\iota$ . Before going on, we establish a few consequences of the existence of the above function  $\Pi$ . If  $\underline{\mathcal{S}}$  is a germ, an element  $e$  of  $\mathcal{S}$  is naturally called *invertible* if there exists  $e'$  in  $\mathcal{S}$  satisfying  $e \bullet e' = 1_x$  and  $e' \bullet e = 1_y$ , with  $x$  the source of  $e$  and  $y$  its target. We denote by  $\mathcal{S}^\times$  the family of all invertible elements of  $\mathcal{S}$ . Also we introduce a local version of left-divisibility (a symmetric notion of local right-divisibility will also be considered below).

**Definition 4.14.** (i) Assume that  $\underline{\mathcal{S}}$  is a germ. For  $f, g$  in  $\mathcal{S}$ , we say that  $f \preceq_{\mathcal{S}} g$  (resp.  $f =_{\mathcal{S}}^{\times} g$ ) holds if  $g = f \bullet g'$  holds for some  $g'$  in  $\mathcal{S}$  (resp. in  $\mathcal{S}^\times$ ).

(ii) A germ  $\underline{\mathcal{S}}$  is called *left-cancellative* if there exist no triple  $f, g, g'$  in  $\mathcal{S}$  satisfying  $g \neq g'$  and  $f \bullet g = f \bullet g'$ .

**Lemma 4.15.** *Assume that  $\underline{\mathcal{S}}$  is a left-associative germ.*

- (i) *For  $f, g$  in  $\mathcal{S}$ , the relation  $f \preceq g$  holds in  $\text{Cat}(\underline{\mathcal{S}})$  if and only if  $f \preceq_{\mathcal{S}} g$  holds.*
- (ii) *The relation  $\preceq_{\mathcal{S}}$  is transitive. If  $h \bullet f$  and  $h \bullet g$  are defined, then  $f \preceq_{\mathcal{S}} g$  implies  $h \bullet f \preceq_{\mathcal{S}} h \bullet g$ , and  $f \stackrel{\times}{=}_{\mathcal{S}} g$  implies  $h \bullet f \stackrel{\times}{=} h \bullet g$ .*
- (iii) *If, in addition,  $\underline{\mathcal{S}}$  is left-cancellative, then  $f \stackrel{\times}{=}_{\mathcal{S}} g$  is equivalent to the conjunction of  $f \preceq_{\mathcal{S}} g$  and  $g \preceq_{\mathcal{S}} f$  and, if  $h \bullet f$  and  $h \bullet g$  are defined, then  $f \preceq_{\mathcal{S}} g$  is equivalent to  $h \bullet f \preceq_{\mathcal{S}} h \bullet g$ , and  $f \stackrel{\times}{=}_{\mathcal{S}} g$  is equivalent to  $h \bullet f \stackrel{\times}{=} h \bullet g$ .*

*Proof.* (i) Assume  $f, g \in \mathcal{S}$  and  $g = fg'$  in  $\text{Cat}(\underline{\mathcal{S}})$ . As  $g$  lies in  $\mathcal{S}$  and  $\mathcal{S}$  is closed under right-divisor in  $\text{Cat}(\underline{\mathcal{S}})$ , the element  $g'$  lies in  $\mathcal{S}$ . So we have  $g \equiv f|g'$ , whence, applying the function  $\Pi$  of (4.13),  $g = \Pi(g) = \Pi(f|g') = f \bullet g'$ . Therefore  $f \preceq_{\mathcal{S}} g$  is satisfied. The converse implication is straightforward.

(ii) As the relation  $\preceq$  in  $\text{Cat}(\underline{\mathcal{S}})$  is transitive, (i) implies that  $\preceq_{\mathcal{S}}$  is transitive as well. Next, assume that  $h \bullet f$  and  $h \bullet g$  are defined, and  $g = f \bullet g'$  holds. By (4.4), we deduce  $h \bullet g = h \bullet (f \bullet g') = (h \bullet f) \bullet g'$ , whence  $h \bullet f \preceq_{\mathcal{S}} h \bullet g$ . Considering the special case when  $g'$  belongs to  $\mathcal{S}^{\times}$ , we deduce that  $f \stackrel{\times}{=}_{\mathcal{S}} g$  implies  $h \bullet f \stackrel{\times}{=} h \bullet g$ .

(iii) Assume  $f = g \bullet e$  and  $g = f \bullet e'$ . We deduce  $f = (f \bullet e') \bullet e$ , whence  $f = f \bullet (e' \bullet e)$  by left-associativity. By left-cancellativity, we deduce  $e' \bullet e = 1_y$ , where  $y$  is the target of  $f$ . So  $e$  and  $e'$  are invertible, and  $f \stackrel{\times}{=}_{\mathcal{S}} g$  holds.

Assume now that  $h \bullet f$  and  $h \bullet g$  are defined and  $h \bullet f \preceq_{\mathcal{S}} h \bullet g$  holds. So we have  $h \bullet g = (h \bullet f) \bullet g'$  for some  $g'$ . By left-associativity,  $f \bullet g'$  must be defined and we obtain  $h \bullet g = h \bullet (f \bullet g')$ , whence  $g = f \bullet g'$  by left-cancellativity, and  $f \preceq_{\mathcal{S}} g$ .

Finally,  $f \stackrel{\times}{=}_{\mathcal{S}} g$  implies  $h \bullet f \stackrel{\times}{=} h \bullet g$  by (ii). Conversely,  $h \bullet f \stackrel{\times}{=} h \bullet g$  implies both  $h \bullet f \preceq_{\mathcal{S}} h \bullet g$  and  $h \bullet g \preceq_{\mathcal{S}} h \bullet f$ , hence  $f \preceq_{\mathcal{S}} g$  and  $g \preceq_{\mathcal{S}} f$  by the result above, whence in turn  $f \stackrel{\times}{=}_{\mathcal{S}} g$ .  $\square$

**4.3. Garside germs.** Our goal will be to characterize those germs that give rise to Garside families. To state the results easily, we introduce a terminology.

**Definition 4.16.** A germ  $\underline{\mathcal{S}}$  is said to be a *Garside germ* if there exists a left-cancellative category  $\mathcal{C}$  such that  $\mathcal{S}$  is a full Garside family of  $\mathcal{C}$ .

By Proposition 4.8, if  $\mathcal{S}$  is a full Garside family in some left-cancellative category  $\mathcal{C}$ , the latter must be isomorphic to  $\text{Cat}(\underline{\mathcal{S}})$ . So, a germ  $\underline{\mathcal{S}}$  is a Garside germ if and only if the category  $\text{Cat}(\underline{\mathcal{S}})$  is left-cancellative and  $\mathcal{S}$  is a full Garside family in  $\text{Cat}(\underline{\mathcal{S}})$ . In other words, in Definition 4.16, we can assume that the category  $\mathcal{C}$  is  $\text{Cat}(\underline{\mathcal{S}})$ .

We shall now state and begin to establish (the argument will be completed in Section 5 only) simple conditions that characterize Garside germs.

**Definition 4.17.** Assume that  $\underline{\mathcal{S}}$  is a germ. For  $g_1 | g_2$  in  $\mathcal{S}^{[2]}$ , we put

$$(4.18) \quad \mathcal{I}_{\mathcal{S}}(g_1, g_2) = \{h \in \mathcal{S} \mid \exists g \in \mathcal{S} (h = g_1 \bullet g \text{ and } g \preceq_{\mathcal{S}} g_2)\},$$

$$(4.19) \quad \mathcal{J}_{\mathcal{S}}(g_1, g_2) = \{g \in \mathcal{S} \mid g_1 \bullet g \text{ is defined and } g \preceq_{\mathcal{S}} g_2\}.$$

A map from  $\mathcal{S}^{[2]}$  to  $\mathcal{S}$  is called an  $\mathcal{I}$ -function (resp. a  $\mathcal{J}$ -function) for  $\underline{\mathcal{S}}$  if, for every  $g_1 | g_2$  in  $\mathcal{S}^{[2]}$ , the value at  $(g_1, g_2)$  lies in  $\mathcal{I}_{\mathcal{S}}(g_1, g_2)$  (resp. in  $\mathcal{J}_{\mathcal{S}}(g_1, g_2)$ ).

We recall that, in (4.18), writing  $h = g_1 \bullet g$  implies that  $g_1 \bullet g$  is defined. An element of  $\mathcal{J}_{\mathcal{S}}(g_1, g_2)$  is a fragment of  $g_2$  that can be added to  $g_1$  legally, that is, without going out of  $\mathcal{S}$ . Note that  $g_1$  always belongs to  $\mathcal{I}_{\mathcal{S}}(g_1, g_2)$  and that, if  $y$  is the target of  $g_1$ , then  $1_y$  always belongs to  $\mathcal{J}_{\mathcal{S}}(g_1, g_2)$ . So, in particular,  $\mathcal{I}_{\mathcal{S}}(g_1, g_2)$  and  $\mathcal{J}_{\mathcal{S}}(g_1, g_2)$  are never empty.

The connection between  $\mathcal{I}_{\mathcal{S}}(g_1, g_2)$  and  $\mathcal{J}_{\mathcal{S}}(g_1, g_2)$  is clear: with obvious notation, we have  $\mathcal{I}_{\mathcal{S}}(g_1, g_2) = g_1 \bullet \mathcal{J}_{\mathcal{S}}(g_1, g_2)$ . However, it turns out that, depending on the situation, using  $\mathcal{I}$  or  $\mathcal{J}$  is more convenient, and it is useful to introduce both notions. Note that, if  $h$  belongs to  $\mathcal{I}_{\mathcal{S}}(g_1, g_2)$  and  $\underline{\mathcal{S}}$  embeds in  $\text{Cat}(\underline{\mathcal{S}})$ , then, in  $\text{Cat}(\underline{\mathcal{S}})$ , we have  $h \in \mathcal{S}$  and  $g_1 \preceq h \preceq g_1 g_2$ . However, the latter relations need not imply  $h \in \mathcal{I}_{\mathcal{S}}(g_1, g_2)$  *a priori* since  $g_1 g \preceq g_1 g_2$  is not known to imply  $g \preceq g_2$  as long as  $\text{Cat}(\underline{\mathcal{S}})$  has not been proved to be left-cancellative.

We shall characterize Garside germs by the existence of  $\mathcal{I}$ - or  $\mathcal{J}$ -functions satisfying some algebraic laws reminiscent of the  $\mathcal{H}$ -law of (3.31).

**Definition 4.20.** If  $\underline{\mathcal{S}}$  is a germ, a map  $F$  from  $\mathcal{S}^{[2]}$  to  $\mathcal{S}$  is said to obey the  $\mathcal{I}$ -law if, for every  $g_1 | g_2 | g_3$  in  $\mathcal{S}^{[3]}$  such that  $g_1 \bullet g_2$  is defined, we have

$$(4.21) \quad F(g_1, F(g_2, g_3)) =^{\times} F(g_1 \bullet g_2, g_3).$$

The map  $F$  is said to obey the  $\mathcal{J}$ -law if, under the same hypotheses, we have

$$(4.22) \quad F(g_1, g_2 \bullet F(g_2, g_3)) =^{\times} g_2 \bullet F(g_1 \bullet g_2, g_3).$$

If, in (4.21) or (4.22),  $=$  replaces  $=^{\times}$ , we speak of the *sharp*  $\mathcal{I}$ - or  $\mathcal{J}$ -law.

The result we shall prove below is as follows.

**Proposition 4.23.** *A germ  $\underline{\mathcal{S}}$  is a Garside germ if and only if it satisfies any one of the following equivalent conditions:*

(4.24)  *$\underline{\mathcal{S}}$  is left-associative, left-cancellative, and it admits an  $\mathcal{I}$ -function that satisfies the sharp  $\mathcal{I}$ -law;*

(4.25)  *$\underline{\mathcal{S}}$  is left-associative, left-cancellative, and it admits a  $\mathcal{J}$ -function that satisfies the sharp  $\mathcal{J}$ -law.*

Note that all conditions in Proposition 4.23 are local in that they only involve the elements of  $\mathcal{S}$  and computations taking place inside  $\mathcal{S}$ . In particular, if  $\mathcal{S}$  is finite, the conditions are effectively checkable in finite time.

In this section, we prove only that the conditions are necessary. The converse is deferred to Section 5 below.

**Lemma 4.26.** *Assume that  $\underline{\mathcal{S}}$  is a germ that is left-associative and left-cancellative and  $I, J : \mathcal{S}^{[2]} \rightarrow \mathcal{S}$  are connected by  $I(g_1, g_2) = g_1 \bullet J(g_1, g_2)$  for all  $g_1, g_2$ .*

(i) *The map  $I$  is an  $\mathcal{I}$ -function for  $\underline{\mathcal{S}}$  if and only if  $J$  is a  $\mathcal{J}$ -function for  $\underline{\mathcal{S}}$ .*

(ii) *In the situation of (i), the map  $I$  obeys the  $\mathcal{I}$ -law (resp. the sharp  $\mathcal{I}$ -law) if and only if  $J$  obeys the  $\mathcal{J}$ -law (resp. the sharp  $\mathcal{J}$ -law).*

*Proof.* (i) First, the assumption that  $\underline{\mathcal{S}}$  is left-cancellative implies that, for every  $I$ , there exists at most one associated  $J$ . Then the definitions of an  $\mathcal{I}$ - and a  $\mathcal{J}$ -functions for  $\underline{\mathcal{S}}$  directly give the expected equivalence.

(ii) Assume that  $I$  is an  $\mathcal{I}$ -function obeying the  $\mathcal{I}$ -law. Assume that  $g_1 | g_2 | g_3$  lies in  $\mathcal{S}^{[3]}$  and  $g_1 \bullet g_2$  is defined. By assumption, we have  $I(g_1, I(g_2, g_3)) =^{\times} I(g_1 \bullet g_2, g_3)$ , which translates into

$$(4.27) \quad g_1 \bullet J(g_1, g_2 \bullet J(g_2, g_3)) =^{\times} (g_1 \bullet g_2) \bullet J(g_1 \bullet g_2, g_3).$$

Then the assumption that  $\underline{\mathcal{S}}$  is left-associative implies that  $g_2 \bullet J(g_1 \bullet g_2, g_3)$  is defined and, therefore, (4.27) implies

$$(4.28) \quad g_1 \bullet J(g_1, g_2 \bullet J(g_2, g_3)) =^{\times} g_1 \bullet (g_2 \bullet J(g_1 \bullet g_2, g_3)).$$



Finally, by Lemma 4.15, we may left-cancel  $g_1$  in (4.28), and what remains is the expected instance of the  $\mathcal{J}$ -law. So  $J$  obeys the  $\mathcal{J}$ -law.

The argument in the case when  $I$  obeys the sharp  $\mathcal{I}$ -law is similar: now (4.27) and (4.28) are equalities, and, applying the assumption that  $\underline{\mathcal{S}}$  is left-cancellative, we directly deduce the expected instance of the sharp  $\mathcal{J}$ -law. So  $J$  obeys the sharp  $\mathcal{J}$ -law.

Conversely, assume that  $J$  is a  $\mathcal{J}$ -function for  $\underline{\mathcal{S}}$  that satisfies the  $\mathcal{J}$ -law. Assume that  $g_1|g_2|g_3$  lies in  $\mathcal{S}^{[3]}$  and  $g_1 \bullet g_2$  is defined. By the  $\mathcal{J}$ -law, we have  $J(g_1, g_2 \bullet J(g_2, g_3)) =^{\times} g_2 \bullet J(g_1 \bullet g_2, g_3)$ . By definition of a  $\mathcal{J}$ -function, the expression  $g_1 \bullet J(g_1, g_2 \bullet J(g_2, g_3))$  is defined, hence so is  $g_1 \bullet (g_2 \bullet J(g_1 \bullet g_2, g_3))$ , and we obtain (4.28). Applying (4.4), which is legal as  $g_1 \bullet g_2$  is defined, we deduce (4.27), whence  $I(g_1, I(g_2, g_3)) =^{\times} I(g_1 \bullet g_2, g_3)$ , the expected instance of the  $\mathcal{I}$ -law. So  $I$  obeys the  $\mathcal{I}$ -law.

Finally, if  $J$  obeys the sharp  $\mathcal{J}$ -law, the argument is similar: (4.27) and (4.28) are equalities, and one obtains the expected instance of the sharp  $\mathcal{I}$ -law. So  $I$  obeys the sharp  $\mathcal{I}$ -law in this case.  $\square$

The (sharp)  $\mathcal{I}$ - and  $\mathcal{J}$ -laws are closely connected with the  $\mathcal{H}$ -law, which implies that the conditions of Proposition 4.23 are necessary.

**Lemma 4.29.** *Every Garside germ satisfies (4.24) and (4.25).*

*Proof.* Assume that  $\underline{\mathcal{S}}$  is a Garside germ. Let  $\mathcal{C} = \text{Cat}(\underline{\mathcal{S}})$ . By definition,  $\mathcal{S}$  embeds in  $\mathcal{C}$ , so, by Proposition 4.12,  $\underline{\mathcal{S}}$  must be left-associative. Next, by definition again,  $\mathcal{C}$  is left-cancellative and, therefore,  $\underline{\mathcal{S}}$  must be left-cancellative as, if  $f, g, g'$  lie in  $\mathcal{S}$  and satisfy  $f \bullet g = f \bullet g'$ , then  $fg = fg'$  holds in  $\mathcal{C}$ , implying  $g = g'$ .

Next, as  $\mathcal{S}$  is a Garside family in  $\mathcal{C}$ , it satisfies (3.35), so there exists  $H$  defined on  $\mathcal{S}\mathcal{C}$ , hence in particular on  $\mathcal{S}^2$ , satisfying the sharp  $\mathcal{H}$ -law. Then we define  $I : \mathcal{S}^{[2]} \rightarrow \mathcal{S}$  by  $I(g_1, g_2) = H(g_1g_2)$ .

First we claim that  $I$  is an  $\mathcal{I}$ -function for  $\underline{\mathcal{S}}$ . Indeed, assume  $g_1|g_2 \in \mathcal{S}^{[2]}$ . By definition, we have  $g_1 =^{\times} H(g_1) \preceq H(g_1g_2) \preceq g_1g_2$  in  $\mathcal{C}$ , hence  $H(g_1g_2) = g_1g$  for some  $g$  satisfying  $g \preceq g_2$ . As  $g$  right-divides  $H(g_1g_2)$ , which lies in  $\mathcal{S}$ , and, by Proposition 4.12,  $\mathcal{S}$  is closed under right-divisor,  $g$  must lie in  $\mathcal{S}$ . By Lemma 4.15, it follows that  $H(g_1g_2)$  lies in  $\mathcal{I}_{\mathcal{S}}(g_1, g_2)$ .

Next, assume that  $g_1|g_2|g_3$  lies in  $\mathcal{S}^{[3]}$  and  $g = g_1 \bullet g_2$  holds. Then the sharp  $\mathcal{H}$ -law gives  $H(g_1H(g_2g_3)) = H(g_1(g_2g_3)) = H(gg_3)$ . This directly translates into  $I(g_1, I(g_2, g_3)) = I(g_1 \bullet g_2, g_3)$ , the expected instance of the sharp  $\mathcal{I}$ -law. So (4.24) is satisfied.

Finally, by Lemma 4.26, (4.25) is satisfied as well.  $\square$

## 5. RECOGNIZING GARSIDE GERMS

We shall now establish two intrinsic characterizations of Garside germs, beginning with the one already stated in Proposition 4.23.

**5.1. Using the  $\mathcal{J}$ -law.** The principle for establishing that the conditions of Proposition 4.23 imply that  $\underline{\mathcal{S}}$  is a Garside germ is obvious, namely using the given  $\mathcal{I}$ - or  $\mathcal{J}$ -function to construct a head function on  $\mathcal{S}^2$  and then using Proposition 3.33. However, the argument is more delicate, because we do not know *a priori* that the category  $\text{Cat}(\underline{\mathcal{S}})$  is left-cancellative and, therefore, eligible for Proposition 3.33. So what we shall do is simultaneously constructing the head function and proving

left-cancellativity. The main point for that is to be able to control not only the head  $H(g)$  of an element  $g$ , but also its tail, defined to be the element  $g'$  satisfying  $g = H(g)g'$ . To perform the construction, using a  $\mathcal{J}$ -function is more convenient than using an  $\mathcal{I}$ -function. Here is the key technical result.

**Lemma 5.1.** *Assume that  $\underline{\mathcal{S}}$  is a left-associative, left-cancellative germ and  $J$  is a  $\mathcal{J}$ -function for  $\underline{\mathcal{S}}$  that satisfies the sharp  $\mathcal{J}$ -law. Define functions*

$$K : \mathcal{S}^{[2]} \rightarrow \mathcal{S}, \quad \underline{H} : \mathcal{S}^* \rightarrow \mathcal{S}, \quad \underline{T} : \mathcal{S}^* \rightarrow \mathcal{S}^*$$

by  $g_2 = J(g_1, g_2) \bullet K(g_1, g_2)$ ,  $\underline{H}(\varepsilon_x) = 1_x$ ,  $\underline{T}(\varepsilon_x) = \varepsilon_x$  and, for  $g$  in  $\mathcal{S}$  and  $w$  in  $\mathcal{S}^*$ ,

$$(5.2) \quad \underline{H}(g|w) = g \bullet J(g, \underline{H}(w)) \quad \text{and} \quad \underline{T}(g|w) = K(g, \underline{H}(w)) | \underline{T}(w).$$

Then, for each  $w$  in  $\mathcal{S}^*$ , we have

$$(5.3) \quad w \equiv \underline{H}(w) | \underline{T}(w),$$

and  $w \equiv w'$  implies  $\underline{H}(w) = \underline{H}(w')$  and  $\underline{T}(w) \equiv \underline{T}(w')$ .

*Proof.* First, the definition of  $K$  makes sense and is unambiguous. Indeed, by definition,  $J(g_1, g_2) \preceq_{\mathcal{S}} g_2$  holds, so there exists  $g$  in  $\mathcal{S}$  satisfying  $g_2 = J(g_1, g_2) \bullet g$ . Moreover, as  $\underline{\mathcal{S}}$

is left-cancellative, the element  $g$  is unique.

As for proving (5.3), we use induction on the length of  $w$ . For  $w = \varepsilon_x$ , we have  $\varepsilon_x \equiv 1_x | \varepsilon_x$ . Otherwise, for  $g$  in  $\mathcal{S}$  and  $w$  in  $\mathcal{S}^*$ , we find

$$\begin{aligned} g|w &\equiv g | \underline{H}(w) | \underline{T}(w) && \text{by induction hypothesis,} \\ &\equiv g | J(g, \underline{H}(w)) \bullet K(g, \underline{H}(w)) | \underline{T}(w) && \text{by definition of } K, \\ &\equiv g | J(g, \underline{H}(w)) | K(g, \underline{H}(w)) | \underline{T}(w) && \text{by definition of } \equiv, \\ &\equiv g \bullet J(g, \underline{H}(w)) | K(g, \underline{H}(w)) | \underline{T}(w) && \text{as } J(g, \underline{H}(w)) \text{ lies in } \mathcal{J}_{\mathcal{S}}(g, \underline{H}(w)), \\ &= \underline{H}(g|w) | \underline{T}(g|w) && \text{by definition of } \underline{H} \text{ and } \underline{T}. \end{aligned}$$

As for the compatibility of  $\underline{H}$  and  $\underline{T}$  with respect to  $\equiv$ , owing to the inductive definitions of  $\equiv$ ,  $\underline{H}$  and  $\underline{T}$ , it is sufficient to establish the relations

$$(5.4) \quad \underline{H}(g|w) = \underline{H}(g_1 | g_2 | w) \quad \text{and} \quad \underline{T}(g|w) \equiv \underline{T}(g_1 | g_2 | w)$$

for  $g_1 \bullet g_2 = g$  and  $w$  in  $\mathcal{S}^*$  such that  $g|w$  is a path (see Figure 6). Now, applying the sharp form of (4.22) with  $g_3 = \underline{H}(w)$  and then using the definition of  $\underline{H}(g_2 | w)$ , we obtain

$$(5.5) \quad g_2 \bullet J(g, \underline{H}(w)) = J(g_1, \underline{H}(g_2 | w)).$$

Then the first relation of (5.4) is satisfied since we can write

$$\begin{aligned} \underline{H}(g|w) &= (g_1 \bullet g_2) \bullet J(g, \underline{H}(w)) && \text{by definition of } \underline{H}, \\ &= g_1 \bullet (g_2 \bullet J(g, \underline{H}(w))) && \text{by left-associativity,} \\ &= g_1 \bullet J(g_1, \underline{H}(g_2 | w)) = \underline{H}(g_1 | g_2 | w) && \text{by (5.5) and the definition of } \underline{H}. \end{aligned}$$

We turn to the second relation in (5.4). Applying the definition of  $\underline{H}$ , we first find

$$\begin{aligned} g_2 \bullet J(g_2, \underline{H}(w)) &= \underline{H}(g_2 | w) \\ &= J(g_1, \underline{H}(g_2 | w)) \bullet K(g_1, \underline{H}(g_2 | w)) && \text{by definition of } K, \\ &= (g_2 \bullet J(g, \underline{H}(w))) \bullet K(g_1, \underline{H}(g_2 | w)) && \text{by (5.5),} \\ &= g_2 \bullet (J(g, \underline{H}(w)) \bullet K(g_1, \underline{H}(g_2 | w))) && \text{by left-associativity,} \end{aligned}$$

whence, as  $\underline{\mathcal{S}}$  is a left-cancellative germ,

$$(5.6) \quad J(g_2, \underline{H}(w)) = J(g, \underline{H}(w)) \bullet K(g_1, \underline{H}(g_2|w)).$$

We deduce

$$\begin{aligned} J(g, \underline{H}(w)) \bullet K(g, \underline{H}(w)) &= \underline{H}(w) && \text{by definition of } K, \\ &= J(g_2, \underline{H}(w)) \bullet K(g_2, \underline{H}(w)) && \text{by definition of } K, \\ &= (J(g, \underline{H}(w)) \bullet K(g_1, \underline{H}(g_2|w))) \bullet K(g_2, \underline{H}(w)) && \text{by (5.6),} \\ &= J(g, \underline{H}(w)) \bullet (K(g_1, \underline{H}(g_2|w)) \bullet K(g_2, \underline{H}(w))) && \text{by left-associativity.} \end{aligned}$$

As  $\underline{\mathcal{S}}$  is a left-cancellative germ, we may left-cancel  $J(g, \underline{H}(w))$ , and we obtain

$$K(g, \underline{H}(w)) = K(g_1, \underline{H}(g_2|w)) \bullet K(g_2, \underline{H}(w)),$$

whence  $K(g, \underline{H}(w))|T(w) \equiv K(g_1, \underline{H}(g_2|w))|K(g_2, \underline{H}(w))|T(w)$ . Owing to the definition of  $T$ , this is exactly the second relation in (5.4).  $\square$

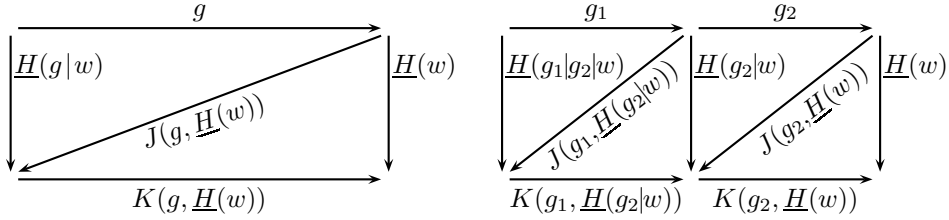


FIGURE 6. Proof of Lemma 5.1: attention! as long as the ambient category is not proved to be left-cancellative, the above diagrams should be taken with care.

We can now complete the argument easily.

*Proof of Proposition 4.23.* Owing to Lemmas 4.29 and 4.26, it suffices to prove now that (4.25) implies that  $\underline{\mathcal{S}}$  is a Garside germ.

So assume that  $\underline{\mathcal{S}}$  is a germ that is left-associative and left-cancellative, and  $J$  is a  $\mathcal{J}$ -function on  $\underline{\mathcal{S}}$  that satisfies the sharp  $\mathcal{J}$ -law. Let  $\mathcal{C} = \text{Cat}(\underline{\mathcal{S}})$ . As  $\underline{\mathcal{S}}$  is left-associative, Proposition 4.12 implies that  $\mathcal{S}$  embeds in  $\mathcal{C}$  and is a full family in  $\mathcal{C}$ .

Now we appeal to the functions  $K$ ,  $\underline{H}$ , and  $T$  of Lemma 5.1. First, for each  $w$  in  $\mathcal{S}^*$  and each  $g$  in  $\mathcal{S}$  such that the target of  $g$  is the source of  $w$ , we have

$$(5.7) \quad w \equiv J(g, \underline{H}(w))|T(g|w) :$$

indeed, we have  $g \bullet J(g, \underline{H}(w)) = \underline{H}(g|w)$  and

$$\begin{aligned} w &\equiv \underline{H}(w)|T(w) && \text{by (5.3),} \\ &= J(g, \underline{H}(w)) \bullet K(g, \underline{H}(w))|T(w) && \text{by definition of } K, \\ &\equiv J(g, \underline{H}(w))|K(g, \underline{H}(w))|T(w) && \text{by definition of } \equiv, \\ &= J(g, \underline{H}(w))|T(g|w). && \text{by definition of } T(g|w). \end{aligned}$$

Assume now  $g|w \equiv g|w'$ . First, Lemma 5.1 implies  $\underline{H}(g|w) = \underline{H}(g|w')$ , that is,  $g \bullet J(g, \underline{H}(w)) = g \bullet J(g, \underline{H}(w'))$  owing to the definition of  $\underline{H}$ . As  $\underline{\mathcal{S}}$  is a left-cancellative germ, we may left-cancel  $g$  and we deduce  $J(g, \underline{H}(w)) = J(g, \underline{H}(w'))$ .

Then, applying (5.7) twice and Lemma 5.1 again, we find

$$w \equiv J(g, \underline{H}(w)) | \underline{T}(g|w) \equiv J(g, \underline{H}(w')) | \underline{T}(g|w') \equiv w',$$

which implies that  $\mathcal{C}$  is left-cancellative.

Next, Lemma 5.1 shows that the function  $\underline{H}$  induces a well defined function of  $\mathcal{C}$  to  $\mathcal{S}$ , say  $H$ . Then (5.3) implies that  $H(g) \preceq g$  holds for every  $g$  in  $\mathcal{C}$ . On the other hand, assume that  $h$  belongs to  $\mathcal{S}$ , and that  $h \preceq g$  holds in  $\mathcal{C}$ . This means that there exists  $w$  in  $\mathcal{S}^*$  such that  $h|w$  represents  $g$ . By construction, we have  $\underline{H}(h|w) = h \bullet J(h, \underline{H}(w))$ , which implies  $h \preceq H(g)$  in  $\mathcal{C}$ . So  $H(g)$  is an  $\mathcal{S}$ -head of  $g$  and, therefore, every element of  $\mathcal{C}$  admits an  $\mathcal{S}$ -head.

Finally, by Proposition 4.12,  $\mathcal{S}$  is closed under right-divisor in  $\mathcal{C}$ , which implies that  $\mathcal{S}^\sharp$  is also closed under right-divisor: indeed, a right-divisor of an element of  $\mathcal{C}^\times$  must lie in  $\mathcal{C}^\times$ , and, if  $f$  right-divides  $ge$  with  $g \in \mathcal{S}$  and  $e \in \mathcal{C}^\times$ , then  $fe^{-1}$  right-divides  $g$ , hence it belongs to  $\mathcal{S}$ , and therefore  $f$  belongs to  $\mathcal{S}\mathcal{C}^\times$ , hence to  $\mathcal{S}^\sharp$ . Therefore,  $\mathcal{S}$  satisfies (3.10) in  $\mathcal{C}$  hence, by Proposition 3.9, it is a Garside family in  $\mathcal{C}$ , which, we recall, is  $\text{Cat}(\underline{\mathcal{S}})$ . Hence  $\underline{\mathcal{S}}$  is a Garside germ.  $\square$

**5.2. Maximum  $\mathcal{I}$ -functions.** Continuing the investigation of Garside germs, we establish alternative characterizations of the latter involving maximality conditions. A point of interest is that such characterizations are automatically satisfied in convenient Noetherian contexts, leading to simplified versions of the criteria similar to the results of Subsection 3.3.

In Proposition 4.23, we characterized Garside germs by the existence of an  $\mathcal{I}$ - or a  $\mathcal{J}$ -function that satisfies the (sharp)  $\mathcal{I}$ -law or  $\mathcal{J}$ -law. We now consider  $\mathcal{I}$ - or  $\mathcal{J}$ -functions that satisfy maximality conditions.

**Definition 5.8.** An  $\mathcal{I}$ -function (resp. a  $\mathcal{J}$ -function)  $F$  is called *maximum* if, for all  $g_1, g_2$ , every element  $h$  of  $\mathcal{I}_{\mathcal{S}}(g_1, g_2)$  (resp. of  $\mathcal{J}_{\mathcal{S}}(g_1, g_2)$ ) satisfies  $h \preceq_{\mathcal{S}} F(g_1, g_2)$ .

**Proposition 5.9.** A germ  $\underline{\mathcal{S}}$  is a Garside germ if and only if it satisfies any one of the following equivalent conditions:

(5.10)  $\underline{\mathcal{S}}$  is left-associative, left-cancellative, and, for every  $g_1 | g_2$  in  $\mathcal{S}^{[2]}$ , the family  $\mathcal{I}_{\mathcal{S}}(g_1, g_2)$  admits a  $\preceq_{\mathcal{S}}$ -greatest element;

(5.11)  $\underline{\mathcal{S}}$  is left-associative, left-cancellative, and admits a maximum  $\mathcal{I}$ -function;

(5.12)  $\underline{\mathcal{S}}$  is left-associative, left-cancellative, and for every  $g_1 | g_2$  in  $\mathcal{S}^{[2]}$ , the family  $\mathcal{J}_{\mathcal{S}}(g_1, g_2)$  admits a  $\preceq_{\mathcal{S}}$ -greatest element.

(5.13)  $\underline{\mathcal{S}}$  is left-associative, left-cancellative, and admits a maximum  $\mathcal{J}$ -function.

The next lemma establishes the equivalence of the conditions involving maximum functions and greatest elements and proves that the conditions of Proposition 5.9 are satisfied in every Garside germ (this is easy).

**Lemma 5.14.** (i) For every germ, (5.10) is equivalent to (5.11), and (5.12) is equivalent to (5.13).

(ii) Every Garside germ satisfies (5.10)–(5.13).

*Proof.* (i) By definition, an  $\mathcal{I}$ -function  $F$  on  $\mathcal{S}^{[2]}$  is maximum if and only if, for every  $g_1 | g_2$  in  $\mathcal{S}^{[2]}$ , the value  $F(g_1, g_2)$  is a  $\preceq_{\mathcal{S}}$ -greatest element in  $\mathcal{I}_{\mathcal{S}}(g_1, g_2)$ . So (5.11) directly implies (5.10). Conversely, if (5.10) is satisfied, we obtain a maximum  $\mathcal{I}$ -function by picking, possibly using the Axiom of Choice, a  $\preceq_{\mathcal{S}}$ -greatest

element in  $\mathcal{I}_{\mathcal{S}}(g_1, g_2)$  for each  $g_1 | g_2$ . So (5.10) implies (5.11). The argument is similar for (5.12) and (5.13) mutatis mutandis.

(ii) Assume that  $\underline{\mathcal{S}}$  is a Garside germ. By Lemma 4.29,  $\underline{\mathcal{S}}$  is left-associative and left-cancellative. For every  $g$  in  $\text{Cat}(\underline{\mathcal{S}})$ , let  $H(g)$  be an  $\mathcal{S}$ -head of  $g$ : by Proposition 3.9, such a head always exists as  $\mathcal{S}$  is full in  $\text{Cat}(\underline{\mathcal{S}})$  and includes  $\mathbf{1}_{\mathcal{S}}$ . Now define  $I, J : \mathcal{S}^{[2]} \rightarrow \mathcal{S}$  by  $I(g_1, g_2) = g_1 \bullet J(g_1, g_2) = H(g_1 g_2)$ . Then, as in the proof of Lemma 4.29,  $I$  is a  $\mathcal{I}$ -function and  $J$  is a  $\mathcal{J}$ -function for  $\underline{\mathcal{S}}$ .

Moreover, assume that  $g_1 | g_2$  lies in  $\mathcal{S}^{[2]}$  and we have  $h = g_1 \bullet g$  with  $g \preceq g_2$ . Then we have  $h \in \mathcal{S}$  and  $h \preceq g_1 g_2$ , whence  $h \preceq H(g_1 g_2)$ , that is,  $h \preceq I(g_1, g_2)$ , since  $H(g_1 g_2)$  is an  $\mathcal{S}$ -head of  $g_1 g_2$ . Hence  $I$  is a maximum  $\mathcal{I}$ -function for  $\underline{\mathcal{S}}$ . The argument is similar for  $J$ .  $\square$

We shall prove the converse implications by using Proposition 4.23. The main observation is that a maximum  $\mathcal{J}$ -function necessarily satisfies the  $\mathcal{J}$ -law.

**Lemma 5.15.** *Assume that  $\underline{\mathcal{S}}$  is a germ that is left-associative and left-cancellative, and  $J$  is a maximum  $\mathcal{J}$ -function for  $\underline{\mathcal{S}}$ . Then  $J$  satisfies the  $\mathcal{J}$ -law.*

*Proof.* Assume  $g_1 | g_2 | g_3 \in \mathcal{S}^{[3]}$  and  $g_1 \bullet g_2$  is defined. Set  $h = J(g_1, g_2 \bullet J(g_2, g_3))$  and  $h' = g_2 \bullet f'$  with  $f' = J(g_1 \bullet g_2, g_3)$ . Our aim is to prove  $h =^{\times} h'$ .

First,  $g_1 \bullet g_2$  is defined and  $g_2 \preceq_{\mathcal{S}} g_2 \bullet J(g_2, g_3)$  is true, so, by maximality of  $J(g_1, g_2 \bullet J(g_2, g_3))$ , we must have  $g_2 \preceq_{\mathcal{S}} J(g_1, g_2 \bullet J(g_2, g_3))$ , that is,  $g_2 \preceq_{\mathcal{S}} h$ . Write  $h = g_2 \bullet f$ . By assumption,  $h$  belongs to  $\mathcal{J}_{\mathcal{S}}(g_1, g_2 \bullet J(g_2, g_3))$ , hence we have  $h \preceq_{\mathcal{S}} g_2 \bullet J(g_2, g_3)$ , that is  $g_2 \bullet f \preceq_{\mathcal{S}} g_2 \bullet J(g_2, g_3)$ , which implies  $f \preceq_{\mathcal{S}} J(g_2, g_3)$  as  $\underline{\mathcal{S}}$  is left-cancellative, whence in turn  $f \preceq_{\mathcal{S}} g_3$  since  $J(g_2, g_3)$  belongs to  $\mathcal{J}_{\mathcal{S}}(g_2, g_3)$ . Now, by assumption,  $g_1 \bullet h$ , that is,  $g_1 \bullet (g_2 \bullet f)$ , is defined, and so is  $g_1 \bullet g_2$  by assumption. Hence  $(g_1 \bullet g_2) \bullet f$  is defined as well, and  $f \preceq_{\mathcal{S}} g_3$  holds. By maximality of  $J(g_1 \bullet g_2, g_3)$ , we deduce  $f \preceq_{\mathcal{S}} f'$ , whence  $h \preceq_{\mathcal{S}} g_2 \bullet f' = h'$ .

For the other direction, the definition of  $f'$  implies that  $(g_1 \bullet g_2) \bullet f'$  is defined and  $f' \preceq_{\mathcal{S}} g_3$  holds. By left-associativity, the first relation implies that  $g_1 \bullet (g_2 \bullet f')$ , that is,  $g_1 \bullet h'$ , is defined. On the other hand,  $g_2 \bullet f'$  is defined by assumption and  $f' \preceq_{\mathcal{S}} g_3$  holds, so the maximality of  $J(g_2, g_3)$  implies  $h' \preceq_{\mathcal{S}} g_2 \bullet J(g_2, g_3)$ . Then the maximality of  $J(g_1, g_2 \bullet J(g_2, g_3))$  implies  $h' \preceq_{\mathcal{S}} J(g_1, g_2 \bullet J(g_2, g_3))$ , that is,  $h' \preceq_{\mathcal{S}} h$ . So  $h =^{\times} h'$  is satisfied, the desired instance of the  $\mathcal{J}$ -law.  $\square$

Not surprisingly, we have a similar property for a maximum  $\mathcal{I}$ -function with respect to the  $\mathcal{I}$ -law.

**Lemma 5.16.** *Assume that  $\underline{\mathcal{S}}$  is a germ that is left-associative and left-cancellative, and  $I$  is a maximum  $\mathcal{I}$ -function for  $\underline{\mathcal{S}}$ . Then  $I$  satisfies the  $\mathcal{I}$ -law.*

*Proof.* One could mimic the argument used for Lemma 5.15, but the exposition is less convenient, and we shall instead derive the result from Lemma 5.15.

So, assume that  $I$  is a maximum  $\mathcal{I}$ -function for  $\underline{\mathcal{S}}$  and let  $J : \mathcal{S}^{[2]} \rightarrow \mathcal{S}$  be defined by  $I(g_1, g_2) = g_1 \bullet J(g_1, g_2)$ . By Lemma 4.26,  $J$  is a  $\mathcal{J}$ -function for  $\underline{\mathcal{S}}$ . Moreover, the assumption that  $I$  is maximum implies that  $J$  is maximum too. Indeed, assume  $h \in \mathcal{J}_{\mathcal{S}}(g_1, g_2)$ . Then  $g_1 \bullet h$  is defined and belongs to  $\mathcal{I}_{\mathcal{S}}(g_1, g_2)$ , hence  $g_1 \bullet h \preceq_{\mathcal{S}} I(g_1, g_2)$ , that is,  $g_1 \bullet h \preceq_{\mathcal{S}} g_1 \bullet J(g_1, g_2)$ , whence  $h \preceq_{\mathcal{S}} J(g_1, g_2)$  by Lemma 4.15. Then, by Lemma 5.15,  $J$  satisfies the  $\mathcal{J}$ -law. By Lemma 4.26 again, this in turn implies that  $I$  satisfies the  $\mathcal{I}$ -law.  $\square$

We shall now manage to go from a function obeying the  $\mathcal{I}$ -law to one obeying the sharp  $\mathcal{I}$ -law, that is, force equality instead of  $=^{\times}_{\mathcal{S}}$ -equivalence.

**Lemma 5.17.** *Assume that  $\underline{\mathcal{S}}$  is a left-associative and left-cancellative germ, and  $I$  is a maximum  $\mathcal{I}$ -function for  $\underline{\mathcal{S}}$ . Then every function  $I' : \mathcal{S}^{[2]} \rightarrow \mathcal{S}$  satisfying  $I'(g_1, g_2) =^{\times}_{\mathcal{S}} I(g_1, g_2)$  for all  $g_1, g_2$  is a maximum  $\mathcal{I}$ -function for  $\underline{\mathcal{S}}$ .*

*Proof.* Assume that  $I'(g_1, g_2) =^{\times}_{\mathcal{S}} I(g_1, g_2)$  holds for all  $g_1, g_2$ . First  $I'$  must be an  $\mathcal{I}$ -function for  $\underline{\mathcal{S}}$ . Indeed, let  $g_1 | g_2$  belong to  $\mathcal{S}^{[2]}$ . We have  $g_1 \preceq_{\mathcal{S}} I(g_1, g_2) \preceq_{\mathcal{S}} I'(g_1, g_2)$ , whence  $g_1 \preceq_{\mathcal{S}} I'(g_1, g_2)$ . Write  $I(g_1, g_2) = g_1 \bullet g$  and  $I'(g_1, g_2) = g_1 \bullet g'$ . By definition, we have  $g \preceq_{\mathcal{S}} g_2$ , that is,  $g_2 = g \bullet f$  for some  $f$ , and, by assumption,  $g' = g \bullet e$  for some invertible element  $e$  of  $\mathcal{S}$ . We find

$$g_2 = (g \bullet (e \bullet e^{-1})) \bullet f = ((g \bullet e) \bullet e^{-1}) \bullet f = (g \bullet e) \bullet (e^{-1} \bullet f),$$

whence  $g' \preceq_{\mathcal{S}} g_2$ : the second equality comes from (4.4), and the last one from the assumption that  $\underline{\mathcal{S}}$  is left-associative. Hence  $I'(g_1, g_2)$  belongs to  $\mathcal{I}_{\mathcal{S}}(g_1, g_2)$ .

Now assume  $h = g_1 \bullet g$  with  $g \preceq g_2$ . Then we have  $h \preceq_{\mathcal{S}} I(g_1, g_2)$  by assumption, hence  $h \preceq_{\mathcal{S}} I'(g_1, g_2)$  by transitivity of  $\preceq_{\mathcal{S}}$ . So  $I'$  is a maximum  $\mathcal{I}$ -function for  $\underline{\mathcal{S}}$ .  $\square$

**Lemma 5.18.** *Assume that  $\underline{\mathcal{S}}$  is a germ that is left-associative, left-cancellative, and admits a maximum  $\mathcal{I}$ -function. Then  $\underline{\mathcal{S}}$  admits a maximum  $\mathcal{I}$ -function that satisfies the sharp  $\mathcal{I}$ -law.*

*Proof.* Let  $I$  be a maximum  $\mathcal{I}$ -function for  $\underline{\mathcal{S}}$ , and let  $\mathcal{S}'$  be an  $=^{\times}_{\mathcal{S}}$ -selector on  $\mathcal{S}$ . For  $g_1 | g_2$  in  $\mathcal{S}^{[2]}$ , define  $I'(g_1, g_2)$  to be the unique element of  $\mathcal{S}'$  that is  $=^{\times}$ -equivalent to  $I(g_1, g_2)$ . Then, by construction,  $I'$  is a function from  $\mathcal{S}^{[2]}$  to  $\mathcal{S}$  satisfying  $I'(g_1, g_2) =^{\times}_{\mathcal{S}} I(g_1, g_2)$  for every  $g_1 | g_2$  in  $\mathcal{S}^{[2]}$ , hence, by Lemma 5.17,  $I'$  is a maximum  $\mathcal{I}$ -function for  $\mathcal{S}$ . By Lemma 5.16,  $I'$  satisfies the  $\mathcal{I}$ -law, that is, for every  $g_1 | g_2 | g_3$  in  $\mathcal{S}^{[3]}$  such that  $g_1 \bullet g_2$  is defined, we have

$$I'(g_1, I'(g_2, g_3)) =^{\times} I'(g_1 \bullet g_2, g_3).$$

Now, by definition of a selector, two elements in the image of the function  $I'$  must be equal whenever they are  $=^{\times}$ -equivalent. So  $I'$  satisfies the sharp  $\mathcal{I}$ -law.  $\square$

We can now complete the proof of Proposition 5.9.

*Proof of Proposition 5.9.* Owing to Lemma 5.14, it remains to prove that each of (5.11) and (5.13) implies that  $\underline{\mathcal{S}}$  is a Garside germ.

Assume that  $\underline{\mathcal{S}}$  is a germ that is left-associative, left-cancellative, and admits a maximum  $\mathcal{I}$ -function. Then, by Lemma 5.18,  $\underline{\mathcal{S}}$  admits an  $\mathcal{I}$ -function  $I$  that satisfies the sharp  $\mathcal{I}$ -law. Therefore, by Proposition 4.23,  $\underline{\mathcal{S}}$  is a Garside germ. So (5.11) implies that  $\underline{\mathcal{S}}$  is a Garside germ.

Finally, assume that  $\underline{\mathcal{S}}$  is a germ that is left-associative, left-cancellative, and admits a maximum  $\mathcal{J}$ -function  $J$ . As already seen in the proof of Lemma 5.16, the function  $I$  defined on  $\mathcal{S}^{[2]}$  by  $I(g_1, g_2) = g_1 \bullet J(g_1, g_2)$  is a maximum  $\mathcal{I}$ -function for  $\underline{\mathcal{S}}$ . So (5.13) implies (5.11) and, therefore, it implies that  $\underline{\mathcal{S}}$  is a Garside germ.  $\square$

**5.3. Noetherian germs.** Noetherianity assumptions guarantee the existence of maximum (or minimal) elements with respect to left- or right-divisibility. In the context of germs, we shall use such assumptions to guarantee the existence of a maximum  $\mathcal{J}$ -function under weak assumptions.

**Definition 5.19.** A germ  $\underline{\mathcal{S}}$  is said to be *left-Noetherian* (*resp. right-Noetherian*) if every nonempty subfamily of  $\mathcal{S}$  has a least element with respect to the local left-divisibility relation  $\preceq_{\mathcal{S}}$  (*resp.* the local right-divisibility relation). The germ is called *Noetherian* if it is both left- and right-Noetherian.

Adapting the proof of Lemma 3.21, one easily sees that a germ  $\underline{\mathcal{S}}$  that is left-associative and left-cancellative is right-Noetherian if and only if, using  $f \prec_{\mathcal{S}} g$  for “ $f \preceq_{\mathcal{S}} g$  and  $f \neq_{\mathcal{S}}^{\times} g$ ”, there exists no infinite bounded  $\prec_{\mathcal{S}}$ -increasing sequence in  $\mathcal{S}$ , that is, there is no sequence  $f_0, f_1, \dots$  satisfying  $f_0 \prec_{\mathcal{S}} f_1 \prec_{\mathcal{S}} \dots \preceq_{\mathcal{S}} g$  in  $\mathcal{S}$ .

A subfamily  $\mathcal{X}$  of a category  $\mathcal{C}$  is said to *admit common right-multiples* if any two elements of  $\mathcal{X}$  that share the same source admit a common right-multiple lying in  $\mathcal{X}$ . The principle for deducing the existence of maximal elements from Noetherianity is as follows.

**Lemma 5.20.** *Assume that  $\underline{\mathcal{S}}$  is a left-cancellative germ that is right-Noetherian,  $g$  belongs to  $\mathcal{S}$ , and  $\mathcal{X}$  is a nonempty subfamily of  $\mathcal{S}$  such that  $f \preceq_{\mathcal{S}} g$  holds for every  $f$  in  $\mathcal{X}$ . Then*

- (i) *The family  $\mathcal{X}$  admits a  $\prec_{\mathcal{S}}$ -maximal element.*
- (ii) *If  $\mathcal{X}$  admits common right-multiples,  $\mathcal{X}$  admits a  $\preceq_{\mathcal{S}}$ -greatest element.*

*Proof.* (i) Let  $f$  be an arbitrary element of  $\mathcal{X}$ . Starting from  $f_0 = f$ , we construct a  $\prec_{\mathcal{S}}$ -increasing sequence  $f_0, f_1, \dots$  in  $\mathcal{X}$ . As long as  $f_i$  is not  $\prec_{\mathcal{S}}$ -maximal in  $\mathcal{X}$ , we can find  $f_{i+1}$  in  $\mathcal{X}$  satisfying  $f_i \prec_{\mathcal{S}} f_{i+1} \preceq_{\mathcal{S}} g$ . The assumption that  $\mathcal{C}$  is right-Noetherian implies that the construction stops after a finite number  $d$  of steps. Then by construction, the element  $f_d$  is a  $\prec_{\mathcal{S}}$ -maximal element of  $\mathcal{X}$ .

(ii) By (i), there exists  $f$  in  $\mathcal{X}$  that is  $\prec_{\mathcal{S}}$ -maximal. Let  $h$  be an arbitrary element of  $\mathcal{X}$ . By assumption, there exists a common multiple  $f'$  of  $f$  and  $h$  that lies in  $\mathcal{X}$ . Now, by assumption,  $f$  is  $\prec_{\mathcal{S}}$ -maximal in  $\mathcal{X}$ , so  $f \prec_{\mathcal{S}} f'$  is impossible, and the only possibility is  $f' =_{\mathcal{S}}^{\times} f$ . But, then,  $h \preceq f'$  implies  $h \preceq_{\mathcal{S}} f$ , that is,  $f$  is a right-multiple of every element of  $\mathcal{X}$ .  $\square$

We can now characterize right-Noetherian Garside germs.

**Proposition 5.21.** *A right-Noetherian germ  $\underline{\mathcal{S}}$  is a Garside germ if and only if  $\underline{\mathcal{S}}$  is left-associative, left-cancellative, and, for every  $g_1 | g_2$  in  $\mathcal{S}^{[2]}$ , the family  $\mathcal{J}_{\mathcal{S}}(g_1, g_2)$  admits common right-multiples.*

*Proof.* Assume that  $\underline{\mathcal{S}}$  is a Garside germ. Then  $\underline{\mathcal{S}}$  is left-associative and left-cancellative by Lemma 4.29. Next, by Proposition 5.9,  $\underline{\mathcal{S}}$  admits a maximum  $\mathcal{J}$ -function  $J$ . Then, for every  $g_1 | g_2$  in  $\mathcal{S}^{[2]}$ , the element  $J(g_1, g_2)$  is a right-multiple of every element of  $\mathcal{J}_{\mathcal{S}}(g_1, g_2)$ , hence a common right-multiple of any two of them. So  $\mathcal{J}_{\mathcal{S}}(g_1, g_2)$  admits common right-multiples.

Conversely, assume that  $\underline{\mathcal{S}}$  is right-Noetherian and satisfies the conditions of the statement. Let  $g_1 | g_2$  belong to  $\mathcal{S}^{[2]}$ . By assumption, the family  $\mathcal{J}_{\mathcal{S}}(g_1, g_2)$  admits common right-multiples, and it is a subfamily of the right-Noetherian family  $\mathcal{S}$ . Hence, by Lemma 5.20,  $\mathcal{J}_{\mathcal{S}}(g_1, g_2)$  admits a  $\preceq_{\mathcal{S}}$ -greatest element. Hence, by Proposition 5.9,  $\underline{\mathcal{S}}$  is a Garside germ.  $\square$

**Remark 5.22.** In the situation of Proposition 5.21, the whole category  $Cat(\underline{\mathcal{S}})$  must be right-Noetherian. We shall not give the proof here.

When we go to the more special case of a germ that admits local right-lcms, that is, in which any two elements that admit a common right-multiple (inside the germ) admit a right-lcm (in the germ), we obtain a new sufficient condition for recognizing a Garside germ.

**Proposition 5.23.** *A germ  $\underline{\mathcal{S}}$  that is left-associative, left-cancellative, right-Noetherian, admits local right-lcms, and satisfies*

$$(5.24) \quad \text{for all } g, h, h', h'' \text{ in } \mathcal{S}, \text{ if } g \bullet h \text{ and } g \bullet h' \text{ are defined,} \\ \text{then } g \bullet h'' \text{ is defined for every right-lcm } h'' \text{ of } h \text{ and } h'.$$

*is a Garside germ.*

*Proof.* Assume that  $g_1 | g_2$  belongs to  $\mathcal{S}^{[2]}$ , and that  $h$  and  $h'$  lie in  $\mathcal{I}_{\mathcal{S}}(g_1, g_2)$ . By assumption, we have  $h \preceq_{\mathcal{S}} g_2$  and  $h' \preceq_{\mathcal{S}} g_2$ . As  $\mathcal{S}$  admits local right-lcms, there must exist a right-lcm  $h''$  of  $h$  and  $h'$  that satisfies  $h'' \preceq_{\mathcal{S}} g_2$ . If  $\underline{\mathcal{S}}$  satisfies (5.24), the assumption that  $g_1 \bullet h$  and  $g_1 \bullet h'$  are defined implies that  $g_1 \bullet h''$  is defined. But, then,  $h''$  belongs to  $\mathcal{I}_{\mathcal{S}}(g_1, g_2)$  and, therefore,  $\mathcal{I}_{\mathcal{S}}(g_1, g_2)$  admits common right-multiples. By Proposition 5.21, it follows that  $\underline{\mathcal{S}}$  is a Garside germ.  $\square$

It turns out that, when right-lcms always exist, the condition (5.24) occurring in Proposition 5.23 follows from a slight strengthening of the left-associativity assumption. We shall naturally say that a germ  $\underline{\mathcal{S}}$  is *right-associative* if the counterpart of (4.5) is satisfied, that is, if  $f \bullet g$  is defined whenever  $f \bullet (g \bullet h)$  is defined, and that  $\underline{\mathcal{S}}$  is *associative* if it is both left- and right-associative.

**Corollary 5.25.** *A germ that is associative, left-cancellative, right-Noetherian, and admits right-lcms is a Garside germ.*

*Proof.* Assume that  $\underline{\mathcal{S}}$  satisfies the hypotheses of the statement. We check that (5.24) is satisfied. So assume that  $g \bullet h$  and  $g \bullet h'$  are defined and  $h''$  is a right-lcm of  $h$  and  $h'$ . Put  $f = g \bullet h$ ,  $f' = g \bullet h'$ , and let  $\hat{f}$  be a right-lcm of  $f$  and  $f'$  (here we use the assumption that  $\mathcal{S}$  admits right-lcms, and not only local right-lcms).

First, we have  $g \preceq_{\mathcal{S}} f \preceq_{\mathcal{S}} \hat{f}$ , whence  $g \preceq_{\mathcal{S}} \hat{f}$ , so there exists  $\hat{h}$  satisfying  $\hat{f} = g \bullet \hat{h}$ . Then, by Lemma 4.15,  $f \preceq_{\mathcal{S}} \hat{f}$  implies  $h \preceq_{\mathcal{S}} \hat{h}$  and  $f' \preceq_{\mathcal{S}} \hat{f}$  implies  $h' \preceq_{\mathcal{S}} \hat{h}$ . So  $\hat{h}$  is a common right-multiple of  $h$  and  $h'$ , hence it is a right-multiple of their right-lcm  $h''$ : we have  $\hat{h} = h'' \bullet e$  for some  $e$ . By assumption,  $g \bullet \hat{h}$ , which is  $g \bullet (h'' \bullet e)$ , is defined. By right-associativity, this implies that  $g \bullet h''$  is defined, so (5.24) is true. Then,  $\underline{\mathcal{S}}$  is a Garside germ by Proposition 5.23.  $\square$

## 6. GERMS DERIVED FROM A GROUPOID

We conclude with an application of the previous constructions. Starting from a group(oid) together with a distinguished generating family, we shall derive a germ, possibly leading in turn to a new category and a new groupoid. The latter groupoid is a sort of unfolded version of the initial one. The seminal example corresponds to starting with a Coxeter group and arriving at the ordinary and dual braid monoid of the associated Artin-Tits group.

**6.1. Tight sequences.** Our aim is to associate with every groupoid equipped with a convenient family of generators a certain germ, so that this germ is a Garside germ whenever the initial groupoid has convenient properties. In order to make the construction nontrivial, we shall have to consider sequences of elements in the initial groupoid that enjoy a certain length property called tightness.

**Definition 6.1.** Assume that  $\mathcal{G}$  is a groupoid. We say that a subfamily  $\Sigma$  of  $\mathcal{G}$  *positively generates*  $\mathcal{G}$  if every element of  $\mathcal{G}$  admits an expression that is a  $\Sigma$ -path (no letter in  $\Sigma^{-1}$ ). Then, for  $g$  in  $\mathcal{G} \setminus \{1\}$ , the  $\Sigma$ -length  $\|g\|_{\Sigma}$  is defined to be the



minimal number  $\ell$  such that  $g$  admits an expression by a  $\Sigma$ -path of length  $\ell$ ; we complete with  $\|1_x\|_\Sigma = 0$  for each object  $x$ .

Note that, if  $\Sigma$  is any family of generators for a groupoid  $\mathcal{G}$ , then  $\Sigma \cup \Sigma^{-1}$  positively generates  $\mathcal{G}$ . Whenever  $\Sigma$  positively generates a groupoid  $\mathcal{G}$ , the  $\Sigma$ -length satisfies the triangular inequality  $\|fg\|_\Sigma \leq \|f\|_\Sigma + \|g\|_\Sigma$  and, more generally, for every path  $(g_1, \dots, g_r)$  in  $\mathcal{G}$

$$(6.2) \quad \|g_1 \cdots g_r\|_\Sigma \leq \|g_1\|_\Sigma + \cdots + \|g_r\|_\Sigma.$$

**Definition 6.3.** Assume that  $\mathcal{G}$  is a groupoid and  $\Sigma$  positively generates  $\mathcal{G}$ . A  $\mathcal{G}$ -path  $(g_1, \dots, g_r)$  is called  $\Sigma$ -tight if  $\|g_1 \cdots g_r\|_\Sigma = \|g_1\|_\Sigma + \cdots + \|g_r\|_\Sigma$  is satisfied.

**Lemma 6.4.** Assume that  $\mathcal{G}$  is a groupoid and  $\Sigma$  positively generates  $\mathcal{G}$ . Then  $(g_1, \dots, g_r)$  is  $\Sigma$ -tight if and only if  $(g_1, \dots, g_{r-1})$  and  $(g_1 \cdots g_{r-1}, g_r)$  are  $\Sigma$ -tight, if and only if  $(g_2, \dots, g_r)$  and  $(g_1, g_2 \cdots g_r)$  are  $\Sigma$ -tight.

*Proof.* To make reading easier, we consider the case of three entries. Assume that  $(f, g, h)$  is  $\Sigma$ -tight. By (6.2), we have  $\|fgh\|_\Sigma \leq \|fg\|_\Sigma + \|h\|_\Sigma$ , whence  $\|fg\|_\Sigma \geq \|fgh\|_\Sigma - \|h\|_\Sigma = \|f\|_\Sigma + \|g\|_\Sigma$ . On the other hand, by (6.2), we have  $\|fg\|_\Sigma \leq \|f\|_\Sigma + \|g\|_\Sigma$ . We deduce  $\|fg\|_\Sigma = \|f\|_\Sigma + \|g\|_\Sigma$ , and  $(f, g)$  is  $\Sigma$ -tight. Next we have  $\|(fg)h\|_\Sigma = \|f\|_\Sigma + \|g\|_\Sigma + \|h\|_\Sigma$ , whence  $\|(fg)h\|_\Sigma = \|fg\|_\Sigma + \|h\|_\Sigma$  since, as seen above,  $(f, g)$  is  $\Sigma$ -tight. Hence  $(fg, h)$  is  $\Sigma$ -tight.

Conversely, assume that  $(f, g)$  and  $(fg, h)$  are  $\Sigma$ -tight. Then we directly obtain  $\|fgh\|_\Sigma = \|fg\|_\Sigma + \|h\|_\Sigma = \|f\|_\Sigma + \|g\|_\Sigma + \|h\|_\Sigma$ , and  $(f, g, h)$  is  $\Sigma$ -tight.

The argument is similar when gathering final entries instead of initial ones.  $\square$

Considering the tightness condition naturally leads to introducing two partial orderings on the underlying groupoid.

**Definition 6.5.** Assume that  $\mathcal{G}$  is a groupoid and  $\Sigma$  positively generates  $\mathcal{G}$ . For  $f, g$  in  $\mathcal{G}$  with the same source, we say that  $f$  is a  $\Sigma$ -prefix of  $g$ , written  $f \leq_\Sigma g$ , if  $(f, f^{-1}g)$  is  $\Sigma$ -tight. Symmetrically, we say that  $f$  is a  $\Sigma$ -suffix of  $g$ , if  $f, g$  have the same target and  $(gf^{-1}, f)$  is  $\Sigma$ -tight.

**Lemma 6.6.** Assume that  $\mathcal{G}$  is a groupoid and  $\Sigma$  positively generates  $\mathcal{G}$ . Then being a  $\Sigma$ -prefix and being a  $\Sigma$ -suffix are partial orders on  $\mathcal{G}$  and  $1_x \leq_\Sigma g$  holds for every  $g$  with source  $x$ .

*Proof.* As  $\|1_y\|_\Sigma$  is zero, every sequence  $(g, 1_y)$  is  $\Sigma$ -tight for every  $g$  with target  $y$ , so  $g \leq_\Sigma g$  always holds, and  $\leq_\Sigma$  is reflexive. Next, as the  $\Sigma$ -length has nonnegative values,  $f \leq_\Sigma g$  always implies  $\|f\|_\Sigma \leq \|g\|_\Sigma$ . Now, assume  $f \leq_\Sigma g$  and  $g \leq_\Sigma f$ . By the previous remark, we must have  $\|f\|_\Sigma = \|g\|_\Sigma$ , whence  $\|f^{-1}g\|_\Sigma = 0$ . Hence,  $f^{-1}g$  is an identity-element, that is,  $f = g$  holds. So  $\leq_\Sigma$  is antisymmetric. Finally, assume  $f \leq_\Sigma g \leq_\Sigma h$ . Then  $(f, f^{-1}g)$  and  $(g, g^{-1}h)$ , which is  $(ff^{-1}g, g^{-1}h)$ , are  $\Sigma$ -tight. By Lemma 6.4, we deduce that  $(f, f^{-1}g, g^{-1}h)$  is  $\Sigma$ -tight, and then that  $(f, (f^{-1}g)(g^{-1}h))$ , which is  $(f, f^{-1}h)$  is  $\Sigma$ -tight. Hence  $f \leq_\Sigma h$  holds, and  $\leq_\Sigma$  is transitive. So  $\leq_\Sigma$  is a partial order on  $\mathcal{G}$ . Finally, as  $\|1_x\|_\Sigma$  is zero, every path  $(1_x, g)$  with  $x$  the source of  $g$  is  $\Sigma$ -tight, and  $1_x \leq_\Sigma g$  holds.

The verifications for  $\Sigma$ -suffixes are entirely similar.  $\square$

By definition,  $(f, g)$  is  $\Sigma$ -tight if and only if  $f$  is a  $\Sigma$ -prefix of  $fg$ . For subsequent use, we note the following weak compatibility condition of the partial order  $\leq_\Sigma$  with the product.

**Lemma 6.7.** *Assume that  $\mathcal{G}$  is a groupoid,  $\Sigma$  positively generates  $\mathcal{G}$ , and  $f, g$  are elements of  $\mathcal{G}$ . If  $(f, g)$  is  $\Sigma$ -tight and  $g'$  is a  $\Sigma$ -prefix of  $g$ , then  $f \leq_{\Sigma} fg' \leq_{\Sigma} fg$  holds.*

*Proof.* Assume  $g' \leq_{\Sigma} g$ . Then, by definition,  $(g', g'^{-1}g)$  is  $\Sigma$ -tight. On the other hand, by assumption,  $(f, g)$ , which is  $(f, g'(g'^{-1}g))$ , is  $\Sigma$ -tight. By Lemma 6.4, it follows that  $(f, g', g'^{-1}g)$  is  $\Sigma$ -tight. First we deduce that  $(f, g')$  is  $\Sigma$ -tight, that is  $f \leq_{\Sigma} fg'$  holds. Next we deduce that  $(fg', g'^{-1}g)$  is  $\Sigma$ -tight as well. As  $g'^{-1}g$  is also  $(fg')^{-1}(fg)$ , the latter relation is equivalent to  $fg' \leq_{\Sigma} fg$ .  $\square$

**6.2. Derived germ.** Here is now the basic scheme for constructing a germ. If  $\mathcal{H}$  is a subfamily of a category, we denote by  $\mathbf{1}_{\mathcal{H}}$  the family of all identity-elements  $1_x$  for  $x$  source or target of an element of  $\mathcal{H}$ .

**Definition 6.8.** Assume that  $\mathcal{G}$  is a groupoid and  $\Sigma$  positively generates  $\mathcal{G}$ . For  $\mathcal{H}$  included in  $\mathcal{G}$ , we denote by  $\mathcal{H}_{/\Sigma}$  the structure  $(\mathcal{H}, \mathbf{1}_{\mathcal{H}}, \bullet)$ , where  $\bullet$  is the partial operation on  $\mathcal{H}$  such that  $h = f \bullet g$  holds if and only if

$$(6.9) \quad h = fg \text{ holds and } (f, g) \text{ is } \Sigma\text{-tight.}$$

The structure  $\mathcal{H}_{/\Sigma}$  is called the *germ derived from  $\mathcal{H}$  and  $\Sigma$* .

So we consider the operation that is induced on  $\mathcal{H}$  by the ambient product of  $\mathcal{G}$ , but with the additional restriction that the products that are not  $\Sigma$ -tight are discarded. Speaking of germs here is legal, as we immediately see.

**Lemma 6.10.** *Assume that  $\mathcal{G}$  is a groupoid,  $\Sigma$  positively generates  $\mathcal{G}$ , and  $\mathcal{H}$  is a subfamily of  $\mathcal{G}$  that includes  $\mathbf{1}_{\mathcal{H}}$ . Then  $\mathcal{H}_{/\Sigma}$  is a cancellative germ that contains no nontrivial invertible element.*

*Proof.* The verifications are easy. First (4.2) is satisfied by definition of  $\bullet$ , and so is (4.3) since we assume that  $\mathbf{1}_{\mathcal{H}}$  is included in  $\mathcal{H}$ . Next, assume that  $f, g, h$  belongs to  $\mathcal{H}$  and  $f \bullet g, g \bullet h$  and  $(f \bullet g) \bullet h$  are defined. This means that  $fg, gh$ , and  $(fg)h$  belong to  $\mathcal{H}$  and that the pairs  $(f, g)$ ,  $(g, h)$ , and  $(fg, h)$  are  $\Sigma$ -tight in  $\mathcal{G}$ . By Lemma 6.4,  $(f, g, h)$ , and then  $(f, gh)$ , which is  $(f, g \bullet h)$ , are  $\Sigma$ -tight. As  $f(gh)$  belongs to  $\mathcal{H}$ , we deduce that  $f \bullet gh$ , that is,  $f \bullet (g \bullet h)$ , is defined, and it is equal to  $f(gh)$ . The argument is symmetric in the other direction and, therefore, (4.4) is satisfied. So  $\mathcal{H}_{/\Sigma}$  is a germ.

Assume now that  $f, g, g'$  belong to  $\mathcal{H}$  and  $f \bullet g = f \bullet g'$  holds. This implies  $fg = fg'$  in  $\mathcal{G}$ , whence  $g = g'$ . So the germ  $\mathcal{H}_{/\Sigma}$  is left-cancellative, hence cancellative by a symmetric argument.

Finally, assume  $e \bullet e' = 1_x$  with  $e, e'$  in  $\mathcal{H}$ . Then we must have  $\|e\|_{\Sigma} + \|e'\|_{\Sigma} = \|1_x\|_{\Sigma} = 0$ . The only possibility is  $\|e\|_{\Sigma} = \|e'\|_{\Sigma} = 0$ , whence  $e = e' = 1_x$ .  $\square$

We now consider Noetherianity conditions. Standard results assert that a germ  $\underline{\mathcal{S}}$  is right-Noetherian if and only if there exists a function  $\lambda : \mathcal{S} \rightarrow \mathbf{Ord}$  such that  $\lambda(f) < \lambda(g)$  holds whenever  $f$  is a proper right-divisor of  $g$  in  $\mathcal{S}$ . We also consider *left-Noetherianity*, defined as the well-foundedness of the left-divisibility relation  $\preceq_{\mathcal{S}}$ , and characterized by the existence of a function  $\lambda : \mathcal{S} \rightarrow \mathbf{Ord}$  such that  $\lambda(f) < \lambda(g)$  holds whenever  $f$  is a proper left-divisor of  $g$  in  $\mathcal{S}$ .

**Lemma 6.11.** *Assume that  $\mathcal{G}$  is a groupoid,  $\Sigma$  positively generates  $\mathcal{G}$ , and  $\mathcal{H}$  is a subfamily of  $\mathcal{G}$  that includes  $\mathbf{1}_{\mathcal{H}}$ . Then the derived germ  $\mathcal{H}_{/\Sigma}$  is both left- and right-Noetherian.*

*Proof.* Assume that  $f, g$  lie in  $\mathcal{H}$  and we have  $g = f \bullet g'$  for some non-invertible  $g'$  that lies in  $\mathcal{H}$ . By definition of  $\bullet$ , this implies  $\|g\|_\Sigma = \|f\|_\Sigma + \|g'\|_\Sigma$ , whence  $\|f\|_\Sigma < \|g\|_\Sigma$  as, by definition of the  $\Sigma$ -length, the non-invertibility of  $g'$  implies  $\|g'\|_\Sigma \geq 1$ . So the  $\Sigma$ -length witnesses both for the left- and the right-Noetherianity of  $(\mathcal{H}, \bullet)$ .  $\square$

Owing to the results of Sections 4 and 5, the only situation when a germ leads to interesting results is when it is left-associative. These properties are not automatic for a derived germ  $\mathcal{H}_\Sigma$ , but they turn out to be connected with closure under  $\Sigma$ -suffix and  $\Sigma$ -prefix, where we naturally say that  $\mathcal{H}$  is closed under  $\Sigma$ -suffix (*resp.*  $\Sigma$ -prefix) if every  $\Sigma$ -suffix (*resp.*  $\Sigma$ -prefix) of an element of  $\mathcal{H}$  lies in  $\mathcal{H}$ .

**Lemma 6.12.** *Assume that  $\mathcal{G}$  is a groupoid,  $\Sigma$  positively generates  $\mathcal{G}$ , and  $\mathcal{H}$  is a subfamily of  $\mathcal{G}$  that is closed under  $\Sigma$ -suffix (*resp.*  $\Sigma$ -prefix). Then the germ  $\mathcal{H}_\Sigma$  is left-associative (*resp.* right-associative) and an element  $f$  of  $\mathcal{H}$  is a local left-divisor (*resp.* right-divisor) of an element  $g$  in  $\mathcal{H}_\Sigma$  if and only if  $f$  is a  $\Sigma$ -prefix (*resp.*  $\Sigma$ -suffix) of  $g$ .*

*Proof.* Assume that  $\mathcal{H}$  is closed under  $\Sigma$ -suffix and  $f, g, h$  are elements of  $\mathcal{H}$  such that  $f \bullet g$  and  $(f \bullet g) \bullet h$  are defined. Then  $fg$  and  $(fg)h$  lie in  $\mathcal{H}$  and the pairs  $(f, g)$  and  $(fg, h)$  are  $\Sigma$ -tight. By Lemma 6.4,  $(f, g, h)$ , and then  $(f, gh)$  are  $\Sigma$ -tight. Hence  $gh$  is a  $\Sigma$ -suffix of  $fgh$  in  $\mathcal{G}$  and, therefore, by assumption,  $gh$  belongs to  $\mathcal{H}$ . By Lemma 6.4 again, the fact that  $(f, g, h)$  is  $\Sigma$ -tight implies that  $(g, h)$  is  $\Sigma$ -tight, and we deduce  $gh = g \bullet h$ . Thus the germ  $\mathcal{H}_\Sigma$  is left-associative.

Assume now that  $f, g$  lie in  $\mathcal{H}$  and  $f$  is a left-divisor of  $g$  in the germ  $\mathcal{H}_\Sigma$ . This means that  $f \bullet g' = g$  holds for some  $g'$  lying in  $\mathcal{H}$ . Necessarily  $g'$  is  $f^{-1}g$ , so  $(f, f^{-1}g)$  has to be  $\Sigma$ -tight, which means that  $f$  is a  $\Sigma$ -prefix of  $g$ . Conversely, assume that  $f, g$  lie in  $\mathcal{H}$  and  $f$  is a  $\Sigma$ -prefix of  $g$ . Then  $(f, f^{-1}g)$  is  $\Sigma$ -tight, so  $f^{-1}g$  is a  $\Sigma$ -suffix of  $g$ . The assumption that  $\mathcal{H}$  is closed under  $\Sigma$ -suffix implies that  $f^{-1}g$  lies in  $\mathcal{H}$ , and, then,  $f \bullet f^{-1}g = g$  holds, whence  $f \preceq_{\mathcal{H}_\Sigma} g$ .

The arguments for right-associativity and right-divisibility in  $\mathcal{H}_\Sigma$  are entirely symmetric, using now the assumption that  $\mathcal{H}$  is closed under  $\Sigma$ -prefix.  $\square$

We now wonder whether  $\mathcal{H}_\Sigma$  is a Garside germ. As  $\mathcal{H}_\Sigma$  is Noetherian, it is eligible for the criteria of Section 5.3, and we are led to looking for the satisfaction of the associated conditions. The latter involve the left-divisibility relation of the germ and, therefore, by Lemma 6.12, they can be formulated inside the base groupoid in terms of  $\Sigma$ -prefixes.

**Proposition 6.13.** *Assume that  $\mathcal{G}$  is a groupoid,  $\Sigma$  positively generates  $\mathcal{G}$ ,  $\mathcal{H}$  is a subfamily of  $\mathcal{G}$  that is closed under  $\Sigma$ -suffix, and*

$$(6.14) \quad \begin{array}{l} \text{If } g, g' \text{ lie in } \mathcal{H} \text{ and admit a common upper bound for } \leq_\Sigma \\ \text{then they admit a least common upper bound for } \leq_\Sigma \text{ in } \mathcal{H}, \end{array}$$

$$(6.15) \quad \begin{array}{l} \text{If } g, g', g'', f \text{ lie in } \mathcal{H}, f \bullet g \text{ and } f \bullet g' \text{ are defined and lie in } \mathcal{H}, \\ \text{and } g'' \text{ is a least common upper bound of } g \text{ and } g' \text{ for } \leq_\Sigma, \\ \text{then } f \bullet g'' \text{ is defined.} \end{array}$$

*Then  $\mathcal{H}_\Sigma$  is a Garside germ.*

*Proof.* By Lemmas 6.10 and 6.12, the germ  $\mathcal{H}_\Sigma$  is left-associative, cancellative, Noetherian, and it admits no nontrivial invertible element. Hence, by Proposition 5.23,  $\mathcal{H}_\Sigma$  is a Garside germ if it satisfies (5.24). Now, by Lemma 6.12, for  $g, h$

in  $\mathcal{H}$ , the relation  $g \preccurlyeq_{\mathcal{H}/\Sigma} h$  is equivalent to  $g \leq_{\Sigma} h$  and, therefore,  $h''$  is a right-lcm of  $h$  and  $h'$  in  $\mathcal{H}/\Sigma$  if and only if it is a least common upper bound of  $h$  and  $h'$  for  $\leq_{\Sigma}$ . So (6.14) means that  $\mathcal{H}/\Sigma$  admits local right-lcms, whereas (6.15) is a direct reformulation of (5.24).  $\square$

In the context of Proposition 6.13, as by assumption  $\Sigma$  generates  $\mathcal{G}$ , we can weaken (6.14) and (6.15) by restricting to the case when the elements  $g$  and  $g'$  lie in  $\Sigma$ , but one then has to assume that the germ is associative on both sides, that is,  $\mathcal{H}$  is also closed under  $\Sigma$ -prefix.

**Proposition 6.16.** *Assume that  $\mathcal{G}$  is a groupoid,  $\Sigma$  positively generates  $\mathcal{G}$ ,  $\mathcal{H}$  is a subfamily of  $\mathcal{G}$  that is closed under  $\Sigma$ -suffix and  $\Sigma$ -prefix, and*

(6.17) *If  $g, g'$  lie in  $\Sigma$  and admit a common upper bound for  $\leq_{\Sigma}$   
then they admit a least common upper bound for  $\leq_{\Sigma}$  in  $\mathcal{H}$ ,*

(6.18) *If  $g, g'$  lie in  $\Sigma$ ,  $f$  lies in  $\mathcal{H}$ ,  $f \bullet g$  and  $f \bullet g'$  are defined and lie in  $\mathcal{H}$ ,  
and  $g''$  is a least common upper bound of  $g$  and  $g'$  for  $\leq_{\Sigma}$ ,  
then  $f \bullet g''$  is defined.*

*Then  $\mathcal{H}/\Sigma$  is a Garside germ.*

*Proof.* We first establish using induction on  $\ell$  that all elements  $g, g'$  of  $\mathcal{H}$  that admit a common upper bound  $gh$  for  $\leq_{\Sigma}$  satisfying  $\|gh\|_{\Sigma} \leq \ell$  admit a least common upper bound for  $\leq_{\Sigma}$ . For  $\ell = 0$  the result is trivial and, for  $\ell \geq 1$ , we argue using induction on  $\|g\|_{\Sigma} + \|g'\|_{\Sigma}$ . First, the result is trivial if  $\|g\|_{\Sigma}$  or  $\|g'\|_{\Sigma}$  is zero. Next, if both  $g$  and  $g'$  belong to  $\Sigma$ , the result is true by (6.17). Otherwise, assuming  $g \notin \Sigma$ , we write  $g = g_1 \bullet g_2$ . Since  $g_1 | g_2$  as well as  $g_1 g_2 | h$  are  $\Sigma$ -tight by assumption,  $g_1 | g_2 | h$  and, then,  $g_1 | g_2 h$  are  $\Sigma$ -tight by Lemma 6.4. As, moreover,  $\mathcal{H}$  is closed under  $\Sigma$ -suffix, we have  $g_1 \leq_{\Sigma} gh$  and thus  $gh$  is a common upper bound of  $g_1$  and  $g'$  for  $\leq_{\Sigma}$ , as  $\mathcal{H}/\Sigma$  is cancellative by Lemma 6.10. As we have  $\|g_1\|_{\Sigma} + \|g'\|_{\Sigma} < \|g\|_{\Sigma} + \|g'\|_{\Sigma}$ , the induction hypothesis implies that  $g_1$  and  $g'$  admit a least common upper bound for  $\leq_{\Sigma}$ , say  $g_1 h_1$ . Then  $g \bullet h$  is a common upper bound of  $g_1 \bullet (g_2 \bullet h)$  and  $g_1 \bullet h_1$  for  $\leq_{\Sigma}$ , hence  $g_2 \bullet h$  is a common upper bound of  $g_2 \bullet h$  and  $h_1$  for  $\leq_{\Sigma}$ . By construction, we have  $\|g_2 h\|_{\Sigma} < \|gh\|_{\Sigma}$ , so the induction hypothesis implies that  $g_2$  and  $h_1$  admit a least common upper bound for  $\leq_{\Sigma}$ , say  $g_2 h_2$ . By Lemma 6.7, we have  $gh_2 \leq_{\Sigma} gh$ , and  $gh_2$  is a least common upper bound of  $g$  and  $g'$  for  $\leq_{\Sigma}$ . So (6.17) implies (6.14).

We now establish similarly using induction on  $\ell$  that, if  $f, g, g'$  lie in  $\mathcal{H}$ , if  $f \bullet g, f \bullet g'$  are defined and lie in  $\mathcal{H}$ , and  $g, g'$  admit a least common upper bound  $g''$  for  $\leq_{\Sigma}$  satisfying  $\|g''\|_{\Sigma} \leq \ell$ , then  $f \bullet g''$  is defined. For  $\ell = 0$ , the result is trivial and, for  $\ell \geq 1$ , we argue using induction on  $\|g\|_{\Sigma} + \|g'\|_{\Sigma}$ . As above, the result is trivial if  $\|g\|_{\Sigma}$  or  $\|g'\|_{\Sigma}$  is zero. Next, if both  $g$  and  $g'$  belong to  $\Sigma$ , the result is true by (6.18). Otherwise, assuming  $g \notin \Sigma$ , we write  $g = g_1 \bullet g_2$ . The induction hypothesis implies that, if  $g_1 h_1$  is the least common upper bound of  $g_1$  and  $g'$  for  $\leq_{\Sigma}$ , then  $f \bullet (g_1 h_1)$  is defined. Next, writing  $g'' = gh$ , the assumption that  $\mathcal{H}$  is closed under  $\Sigma$ -suffix implies that  $g_2 h$  is the least common upper bound of  $g_2$  and  $h_1$  for  $\leq_{\Sigma}$ . By construction, we have  $\|g_2 h\|_{\Sigma} < \|g''\|_{\Sigma}$  and the assumption that  $\mathcal{H}$  is closed under  $\Sigma$ -prefix implies that  $f \bullet g_1$  lies in  $\mathcal{H}$ , so the induction hypothesis implies that  $f g_1 g_2 h$ , that is,  $f g''$  lies in  $\mathcal{H}$ . So (6.18) implies (6.15), and we can apply Proposition 6.13.  $\square$

On the other hand, if the  $\Sigma$ -prefix relation  $\leq_{\Sigma}$  defines an upper-semi-lattice on the considered subfamily  $\mathcal{H}$ , that is, any two elements of  $\mathcal{H}$  admit a least common upper bound for  $\leq_{\Sigma}$ , we obtain a simpler criterion.

**Proposition 6.19.** *Assume that  $\mathcal{G}$  is a groupoid,  $\Sigma$  positively generates  $\mathcal{G}$ , and  $\mathcal{H}$  is a subfamily of  $\mathcal{G}$  that is closed under  $\Sigma$ -prefix and  $\Sigma$ -suffix and any two elements of  $\mathcal{H}$  admit a  $\leq_{\Sigma}$ -least upper bound. Then  $\mathcal{H}_{/\Sigma}$  is a Garside germ.*

*Proof.* By Lemma 6.12, the germ  $\mathcal{H}_{/\Sigma}$  is (left- and right-) associative, and the existence of least common upper bounds for  $\leq_{\Sigma}$  in  $\mathcal{G}$  implies the existence of right-lcms in  $\mathcal{H}_{/\Sigma}$ . Moreover,  $\mathcal{H}_{/\Sigma}$  is right-Noetherian by Lemma 6.11. Then the latter is a Garside germ by Corollary 5.25.  $\square$

When we consider a germ derived from the whole initial groupoid, the conditions about closure under prefix and suffix becomes trivial, so it only remains the condition about lcms.

**Corollary 6.20.** *Assume that  $\mathcal{G}$  is a groupoid,  $\Sigma$  positively generates  $\mathcal{G}$ , and any two elements of  $\mathcal{G}$  admit a  $\leq_{\Sigma}$ -least common upper bound. Then  $\mathcal{G}_{/\Sigma}$  is a Garside germ.*

So the main condition for obtaining a Garside germ along the above lines is to find a positively generating subfamily  $\Sigma$  of  $\mathcal{G}$  such that the partial order  $\leq_{\Sigma}$  admits (local) least common upper bounds.

**6.3. The ordinary Artin–Tits monoids.** A first important example of the construction described above is the construction of the Artin–Tits monoids starting from arbitrary Coxeter groups. We take for  $(G, \Sigma)$  a Coxeter system, and keep the whole of  $G$ , that is, we choose  $H = G$ . Then the monoid generated by the germ  $G_{/\Sigma}$  is the usual Artin–Tits monoid associated with  $G$ , see [24]. We will use Proposition 6.16 to show that we have a Garside germ.

First, we recall some well known consequences of the exchange lemma (see for example [5, No. 1.4 lemme 3]), which states:

**Lemma 6.21.** *If  $w$  is a  $\Sigma$ -word of minimal length representing an element  $g$  of  $G$  and  $h$  is an element of  $\Sigma$  satisfying  $\|gh\|_{\Sigma} \leq \|g\|_{\Sigma}$ , then  $gh$  is represented by some proper subword  $w'$  of  $w$ .*

The first consequence (see [5, No. 1.8 Corollaire 1]) is

**Proposition 6.22.** *Assume that  $(G, \Sigma)$  is a Coxeter system and  $I$  is included in  $\Sigma$ . Let  $G_I$  be the subgroup of  $G$  generated by  $I$ . Then all minimal  $\Sigma$ -words representing elements of  $G_I$  are  $I$ -words.*

The second is

**Proposition 6.23.** *Assume that  $(G, \Sigma)$  is a Coxeter system and  $I$  is included in  $\Sigma$ . Let  $G_I$  be the subgroup of  $G$  generated by  $I$ . Then, for  $f$  in  $G$ , the following are equivalent:*

$$(6.24) \quad f \text{ has no non-trivial } \Sigma\text{-suffix in } G_I.$$

$$(6.25) \quad \|fg\|_{\Sigma} = \|f\|_{\Sigma} + \|g\|_{\Sigma} \text{ holds for all } g \in G_I.$$

$$(6.26) \quad f \text{ has minimal } \Sigma\text{-length in its coset } fG_I.$$

Further, if  $f$  satisfies the conditions above, it is the unique element of  $fG_I$  of minimal  $\Sigma$ -length and, for every  $g$  in  $G_I$ , every  $\Sigma$ -suffix of  $fg$  in  $G_I$  is a  $\Sigma$ -suffix of  $g$ .

The analogous result (reversing left and right) applies to  $G_I f$ .

*Proof.* Equation (6.25) implies that  $f$  has minimal  $\Sigma$ -length in its coset  $fG_I$ . Conversely, assume that  $f$  satisfies (6.26) and let  $g$  be an element of minimal  $\Sigma$ -length in  $G_I$  such that (6.25) does not hold. Then, if  $w$  is a minimal word representing  $f$  and  $ua$  is a minimal word representing  $g$  with  $a$  in  $I$  and  $u$  an  $I$ -word, by minimality of  $g$  the word  $wu$  is minimal. Since the word  $wua$  is not minimal, Lemma 6.21 implies that there is a subword  $w'$  of  $wu$  representing also  $fg$ . Since  $ua$  is a minimal word, the word  $w'$  must have the form  $w''u$  with  $w''$  a subword of  $w$ . This contradicts the minimality of  $f$  in  $fG_I$ . We have shown the equivalence of (6.25) and (6.26).

Conditions (6.24) and (6.26) are equivalent. Indeed, if  $f$  has a non-trivial  $\Sigma$ -suffix  $h \in G_I$ , then we have  $f = gh$  with  $\|g\|_\Sigma = \|f\|_\Sigma - \|h\|_\Sigma$  so that  $f$  is not an element of minimal  $\Sigma$ -length in  $fG_I$ . Hence (6.26) implies (6.24). Conversely, if  $f'$  is an element of minimal  $\Sigma$ -length in  $fG_I$ , we have  $f = f'h$  with  $h \in G_I$  and  $\|f\|_\Sigma = \|f'\|_\Sigma + \|h\|_\Sigma$  by (6.25) applied to  $f'$  (we use that (6.26) implies (6.25)). Then  $h$  is a nontrivial  $\Sigma$ -suffix of  $f$  in  $G_I$ , contradicting (6.24).

Now (6.25) shows that all elements of  $fG_I$  have a  $\Sigma$ -length strictly larger than that of  $f$ , whence the unicity. Moreover, if an element  $h$  of  $G_I$  is a  $\Sigma$ -suffix of  $fg$  with  $g \in G_I$ , then  $\|fg\|_\Sigma = \|fgh^{-1}\|_\Sigma + \|h\|_\Sigma$  and by (6.25) we have  $\|fg\|_\Sigma = \|f\|_\Sigma + \|g\|_\Sigma$  and  $\|fgh^{-1}\|_\Sigma = \|f\|_\Sigma + \|gh^{-1}\|_\Sigma$  which gives  $\|g\|_\Sigma = \|gh^{-1}\|_\Sigma + \|h\|_\Sigma$ , so that  $h$  is a  $\Sigma$ -suffix of  $g$ .  $\square$

**Proposition 6.27.** *For every Coxeter system  $(G, \Sigma)$ , the germ  $G_{/\Sigma}$  is a Garside germ, and the corresponding monoid is the braid monoid associated with  $(G, \Sigma)$ .*

*Proof.* We prove that  $(G, \Sigma)$  is eligible for Proposition 6.16. We first look at (6.17); let  $a, b \in \Sigma$  which have a common upper bound for  $\leq_\Sigma$ . We let  $I = \{a, b\}$  and let  $f$  be the upper bound. Write  $f = gh$  where  $h$  is of minimal  $\Sigma$ -length in  $G_I f$  and  $g \in G_I$  with  $\|g\|_\Sigma + \|h\|_\Sigma = \|f\|_\Sigma$ . Then, since  $a$  and  $b$  are  $\Sigma$ -prefixes of  $f$ , by Proposition 6.23 they are  $\Sigma$ -prefixes of  $g$  thus  $g$  is a common upper bound of  $a$  and  $b$  for  $\leq_\Sigma$  in  $G_I$ . Since every element of  $G_I$  is equal to a product  $aba\dots$  or  $bab\dots$  with a number of factors at most the order of  $ab$  (if finite),  $ab$  has finite order and we have  $g = \Delta_{a,b}$ . Thus we found that  $\Delta_{a,b}$  is a least common upper bound of  $a$  and  $b$  for  $\leq_\Sigma$ .

Next, we show (6.18), thus we assume this time that  $a, b$  in  $\Sigma$  and  $f$  in  $G$  satisfy  $\|f\|_\Sigma + 1 = \|fa\|_\Sigma = \|fb\|_\Sigma$ , and we assume that  $a, b$  have a common upper bound for  $\leq_\Sigma$ , which we have seen is  $\Delta_{a,b}$ . We have to show  $\|f\|_\Sigma + \|\Delta_{a,b}\|_\Sigma = \|f\Delta_{a,b}\|_\Sigma$ ; but this is exactly the fact that (6.24) implies (6.25).

That the corresponding monoid is the braid monoid of  $(G, \Sigma)$  results from the presentation of the monoid associated with  $G_{/\Sigma}$ .  $\square$

**6.4. The dual monoid.** Another important example (which was part of motivating the above developments) is the dual monoid for spherical Artin groups, or, more generally, for the braid groups associated to well-generated complex reflection groups.

This time we take for  $(G, \Sigma)$  a well-generated finite complex reflection group together with the set of all its reflections. We choose a Coxeter element  $c$  in  $G$ , that is, an  $h$ -regular element where  $h$  is the highest reflection degree (which is unique since  $G$  is well-generated) and we take for  $H$  the set  $\text{Div}(c)$  of all left  $\Sigma$ -prefixes of  $c$ . Then the monoid  $H_{/\Sigma}$  is the dual braid monoid for  $G$  in the sense of David Bessis [2, 8.1]; proposition [2, 8.8] constructs this monoid according to Proposition 6.19. Bessis has shown [2, 8.2] that the group presented by this germ is the braid group of  $G$ . The lattice property for the case of the dual monoid is a deep result of which only a case-by-case proof is known in general; see [2, 8.14]. Previous to this work, Bessis had given a construction for the real case in [4], using case-by-case arguments for the lattice property. There exists a case-free proof for finite Coxeter groups due to Brady and Watt, see [6].

The same strategy can be applied to Artin groups of affine type. This time we take for  $(G, \Sigma)$  a Coxeter group of affine type with the set of all its reflections. We choose again a Coxeter element  $c$ , defined here as the product of all simple reflections in some chosen order, and we take again for  $H$  the set  $\text{Div}(c)$ . It has been proved that, if  $G$  is of type  $\tilde{G}_2$  or  $\tilde{C}_n$  or  $\tilde{A}_n$  and in the last case the order of the simple reflections is such that two consecutive elements do not commute, then this germ satisfies the assumptions of Proposition 6.19 and that the group presented by this germ is the corresponding Artin group. Moreover these are the only cases where  $\text{Div}(c)$  is a Garside germ. This last fact and the  $\tilde{G}_2$ -case are unpublished results of Crisp and McCammond; for the  $\tilde{A}$  case see [16] and for the  $\tilde{C}$  case see [17].

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