

MULTIFRACTION REDUCTION IV: PADDING AND ARTIN–TITS MONOIDS OF SUFFICIENTLY LARGE TYPE

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ABSTRACT. We investigate the padded version of reduction, an extension of multifraction reduction as defined in [arXiv:1606.08991](#), and connect it both with ordinary reduction and with the so-called Property H. As an application, we show that all Artin–Tits groups of sufficiently large type satisfy some weakened version Conjecture $\mathbf{A}^{\text{padded}}$ of Conjecture \mathbf{A} , thus showing that the reduction approach is relevant for these groups.

Reduction of multifractions, which was introduced in [2] and [3], is a new approach to the word problem for Artin–Tits groups and, more generally, for groups that are enveloping groups of monoids in which the divisibility relations have weak lattice properties (“gcd-monoids”). It is based on a rewrite system (“ \mathcal{R} -reduction”) that extends the usual free reduction for free groups, as well as the rewrite systems known for Artin–Tits groups of spherical type, and more generally Garside groups. It was proved in [2] that \mathcal{R} -reduction is convergent for all Artin–Tits groups of type FC, and in [3] that a certain condition called semi-convergence, weaker than convergence, is sufficient to obtain the decidability of the word problem, leading to the main conjecture (“Conjecture \mathbf{A} ”) that \mathcal{R} -reduction is semi-convergent for every Artin–Tits monoid.

The aim of the current paper is to exploit the observation that semi-convergence up to Turing-computable padding, a weakening of semi-convergence, is again sufficient to solve the word problem. By *padding*, we mean the insertion of an even number of trivial components at the beginning of a multifraction.

The main results we prove are as follows. First, we have a simple criterion for the word problem:

Proposition 1.6. *If M is a strongly noetherian gcd-monoid with finitely many basic elements, for which \mathcal{R} -reduction is semi-convergent up to f -padding for some Turing-computable map f , then the word problem of $\mathcal{U}(M)$ is decidable.*

Next, we establish a simple connection between the padded version of semi-convergence, the semi-convergence of a variant of \mathcal{R} -reduction (“split reduction” or “ \mathcal{S} -reduction”) and Property H of [1, 4, 8]. When we say below that (S, R) is an lcm presentation for M , we mean that every relation $s\dots = t\dots$ in R specifies the least common multiple of s and t , and similarly on the right; see Definition 1.14.

Proposition 1.14. *If M is a gcd-monoid and (S, R) is an lcm presentation for M , then the following are equivalent:*

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- (i) \mathcal{R} -reduction is semi-convergent for M up to padding;
- (ii) \mathcal{S} -reduction is semi-convergent for M ;
- (iii) Property H is true for (S, R) .

Finally, we consider the specific case of Artin–Tits groups. In view of Proposition 1.6, we propose

Conjecture $\mathbf{A}^{\text{padded}}$. *For every Artin–Tits monoid, \mathcal{R} -reduction is semi-convergent up to f -padding for some Turing-computable map f .*

By the results of [2], Conjecture $\mathbf{A}^{\text{padded}}$ is true for every Artin–Tits monoid of type FC. Here we prove:

Theorem 1. *Conjecture $\mathbf{A}^{\text{padded}}$ is true for all Artin–Tits monoids of sufficiently large type.*

We recall from [10] that an Artin–Tits group is said to be of *sufficiently large type* if, in any triangle in the associated Coxeter diagram, either no edge has label 2, or all three edges have label 2, or at least one edge has label ∞ . The result follows directly from the more precise result stated as Proposition 2.1 below, which gives an explicit quadratic upper bound on the padding that is needed. The proof relies on a careful analysis of the techniques of [10] and [8]. With this result, the family of Artin–Tits for which multifraction reduction is relevant is greatly enlarged.

The paper is organised in two sections. The first one is devoted to padded reduction and its variants in a general context of gcd-monoids, and contains a proof of Propositions 1.6 and 1.15. The second section is devoted to the specific case of Artin–Tits monoids of sufficiently large type, with a proof of Theorem 1.

1. PADDED MULTIFRACTION REDUCTION

After recalling in Subsection 1.1 the definitions that we need for multifraction reduction and the rewrite system \mathcal{R} , we introduce in Subsection 1.2 padded versions of semi-convergence and use them to solve the word problem of the enveloping group. Next, we introduce in Subsection 1.3 a new rewrite system \mathcal{S}_M , a variant of \mathcal{R} called split reduction, and we connect its semi-convergence with the padded semi-convergence of \mathcal{R} . Finally, we establish the connection with subword reversing and Property H in Subsection 1.4.

1.1. Multifraction reduction. If M is a monoid, we denote by $\mathcal{U}(M)$ the enveloping group of M , and by ι the canonical (not necessarily injective) morphism from M to $\mathcal{U}(M)$. We say that a finite sequence $\underline{a} = (a_1, \dots, a_n)$ of elements of M , also called a *multifraction* on M and denoted by $a_1/\dots/a_n$, *represents* an element g of $\mathcal{U}(M)$ if

$$(1.1) \quad g = \iota(a_1)\iota(a_2)^{-1}\iota(a_3)\dots\iota(a_n)^{(-1)^{n-1}}$$

holds in $\mathcal{U}(M)$. In this context, the parameter n is called the *depth* of \underline{a} , denoted by $\|\underline{a}\|$, and the right hand side of (1.1) is denoted by $\iota(\underline{a})$. We use \mathcal{F}_M for the family of all multifractions on M , and $\underline{1}_n$ for the depth n multifraction with all entries equal to 1, skipping n when no ambiguity is possible. Our aim is to recognise which multifractions represent 1 in $\mathcal{U}(M)$.

We collect together the definitions that we need concerning monoids.

Definition 1.2. A monoid M is called a *gcd-monoid* if it is cancellative, 1 is its only invertible element, and any two elements of M admit a left- and a right-gcd, where, for a, b in M , we say that a *left-divides* b , written $a \leq b$, if we have $b = ax$ for some x in M , and that d is a *left-gcd* of a and b if d left-divides a and b and every common left-divisor of a and b left-divides d . Right-division and right-gcd are defined symmetrically, with $b = xa$ replacing $b = ax$.

An *atom* in M is an element that is not expressible as ab with $a, b \neq 1$. A right- (*resp.* left-) *basic element* is one that is obtained from atoms using the operation \setminus (*resp.* $/$) defined by $a(a \setminus b) = a \vee b$ (*resp.* $(a/b)b = a \tilde{\vee} b$), where \vee (*resp.* $\tilde{\vee}$) stands for the right-lcm (*resp.* left-lcm).

The monoid M is called *noetherian* if there is no infinite descending sequence with respect to proper left- or right-division, and *strongly noetherian* if there exists a map λ from $M \setminus \{1\}$ to the positive integers satisfying $\lambda(ab) \geq \lambda(a) + \lambda(b)$.

Throughout the paper, we restrict our attention to gcd-monoids: Artin–Tits monoids are typical examples. In a gcd-monoid, any two elements a, b that admit a common right-multiple (*resp.* left-multiple) admit a least one, the *right-lcm* (*resp.* *left-lcm*) of a and b , denoted by $a \vee b$ (*resp.* $a \tilde{\vee} b$).

If M is a gcd-monoid, a family \mathcal{R} of rewrite rules on \mathcal{F}_M is defined in [2]: for $\underline{a}, \underline{b}$ in \mathcal{F}_M , and for $i \geq 1$ and $x \in M$, we write $\underline{a} \bullet R_{i,x} = \underline{b}$ if we have $\|\underline{b}\| = \|\underline{a}\|$, $b_k = a_k$ for $k \neq i-1, i, i+1$, and there exists x' satisfying

$$\begin{array}{lll} \text{for } i \text{ even:} & b_{i-1} = a_{i-1}x', & xb_i = a_i x' = x \vee a_i, & xb_{i+1} = a_{i+1}, \\ \text{for } i \geq 3 \text{ odd:} & b_{i-1} = x'a_{i-1}, & b_i x = x'a_i = x \tilde{\vee} a_i, & b_{i+1}x = a_{i+1}, \\ \text{for } i = 1: & & b_i x = a_i, & b_{i+1}x = a_{i+1}. \end{array}$$

We say that \underline{a} *reduces* to \underline{b} in one step, and write $\underline{a} \Rightarrow \underline{b}$, if $\underline{a} \bullet R_{i,x} = \underline{b}$ holds for some i and some $x \neq 1$. We use \Rightarrow^* for the reflexive–transitive closure of \Rightarrow .

By [2, Lemma 3.8 and Cor. 3.20], $\underline{a} \Rightarrow^* \underline{b}$ implies that \underline{a} and \underline{b} represent the same element in $\mathcal{U}(M)$, and, conversely, the relation $\iota(\underline{a}) = \iota(\underline{b})$ is essentially the equivalence relation generated by \Rightarrow (up to deleting trivial final entries). Furthermore, whenever the monoid M is noetherian, \mathcal{R} -reduction is terminating for M .

By [2, Prop. 4.16] and [3, Prop. 3.2], \mathcal{R} -reduction is locally confluent for M if and only if M satisfies the *3-Ore condition*, meaning that any three elements of M that pairwise admit a common right- (*resp.* left-) multiple admit a global one. In this case, \mathcal{R} -reduction is convergent, implying that a multifraction \underline{a} represents 1 in $\mathcal{U}(M)$ if and only if $\underline{a} \Rightarrow^* \underline{1}$ is satisfied. In the case of Artin–Tits monoids, this exactly corresponds to type FC.

When M does not satisfy the 3-Ore condition, \mathcal{R} -reduction is not confluent for M , and no convergence result can be expected. However, a condition weaker than convergence can be used to solve the word problem:

Definition 1.3 ([3]). We say that \mathcal{R} -reduction is *semi-convergent* for M if $\underline{a} \Rightarrow^* \underline{1}$ holds for every multifraction \underline{a} that represents 1 in $\mathcal{U}(M)$.

Convergence implies semi-convergence, but explicit examples of gcd-monoids M for which \mathcal{R} -reduction is semi-convergent but not convergent are known [6]. However:

Proposition 1.4 ([3, Prop. 3.16]). *If M is a strongly noetherian gcd-monoid with finitely many basic elements for which \mathcal{R} -reduction is semi-convergent, then the word problem of $\mathcal{U}(M)$ is decidable.*

Conjecture A ([3]). *\mathcal{R} -reduction is semi-convergent for every Artin–Tits monoid.*

Conjecture A is true for every Artin–Tits monoid of FC type, but other cases remain open. Every Artin–Tits monoid satisfies the finiteness assumptions of Proposition 1.4 and, therefore, the word problem of the enveloping group of every Artin–Tits monoid satisfying Conjecture A is decidable.

1.2. Padded semi-convergence. We now introduce a weakening of semi-convergence, in which trivial initial entries may be added to multifractions. For p a non-negative integer and \underline{a} a multifraction, we write $\underline{1}_{2p}/\underline{a}$ for the multifraction obtained from \underline{a} by adding $2p$ trivial entries on the left of \underline{a} . Note that the number of 1s that are added must be even in order to preserve the image of the multifraction in the group.

Definition 1.5. If f is a map from \mathcal{F}_M to \mathbb{N} , we say that \mathcal{R} -reduction is semi-convergent for M up to f -padding if $\underline{1}_{2f(\underline{a})}/\underline{a} \Rightarrow^* \underline{1}$ holds for every multifraction \underline{a} that represents 1 in $\mathcal{U}(M)$.

By definition, semi-convergence is semi-convergence up to 0-padding. Because $\underline{a} \Rightarrow^* \underline{b}$ trivially implies $\underline{1}_{2p}/\underline{a} \Rightarrow^* \underline{1}_{2p}/\underline{b}$ for every p , semi-convergence of \mathcal{R} -reduction up to f -padding implies its semi-convergence up to f' -padding for $f' \geq f$ (meaning $\forall \underline{a} \in \mathcal{F}_M (f'(\underline{a}) \geq f(\underline{a}))$): appending trivial initial entries preserves the existing reductions and possibly adds new ones.

Proposition 1.6. *If M is a strongly noetherian gcd-monoid with finitely many basic elements for which \mathcal{R} -reduction is semi-convergent up to f -padding for some Turing-computable map f on \mathcal{F}_M , then the word problem of $\mathcal{U}(M)$ is decidable.*

Speaking of a Turing-computable map defined on \mathcal{F}_M makes sense because, under the assumptions of Proposition 1.6, M admits a finite presentation in terms of the finite atom family S of M , and every element of the monoid M is represented by finitely many words in S only, which implies the decidability of the word problem of M with respect to S . Then a Turing-computable map on M means one that is induced by a Turing-computable map on words in S that is compatible with the relations defining M .

If (S, R) is a monoid presentation, that is, R is a family of pairs of words in S , we denote by \equiv_R^+ , or simply \equiv^+ , the congruence on S^* generated by R , implying $\langle S \mid R \rangle^+ = S^*/\equiv^+$. For w a word in S , we denote by $\overline{[w]^+}$ the element of $\langle S \mid R \rangle^+$ represented by w , that is, the \equiv^+ -class of w . Let \overline{S} be a disjoint copy of S that consists of a new element \overline{s} for each element s of S . For w a word in $S \cup \overline{S}$, we denote by \overline{w} the word obtained from w by exchanging s and \overline{s} everywhere and reversing the order of letters. In this context, the element of S (*resp.* \overline{S}) are called positive (*resp.* negative), and we denote by $[w]$ the element of the group $\langle S \mid R \rangle$ represented by w when \overline{s} represents s^{-1} .

Proof of Proposition 1.6. The argument is the same as the one in [3] for Proposition 1.4. Let S be the atom set of M , and let w be a word in $S \cup \overline{S}$. To decide whether w represents 1 in $\mathcal{U}(M)$, we first express it as $w_1 w_2^{-1} w_3 \dots$ with w_1, w_2, \dots in S^* (no negative letters). Then, writing $\underline{w} = (w_1, w_2, \dots)$ and $\underline{[w]^+}$ for $[w_1]^+/[w_2]^+/\dots$, we append $2f(\underline{[w]^+})$ trivial entries on the left, and exhaustively enumerate all sequences \underline{w}' satisfying $\underline{1}_{2f(\underline{[w]^+})}/\underline{[w]^+} \Rightarrow^* \underline{[w]}'^+$: this is possible because, under the assumptions of the statement, the existence of common multiples and, from there,

the relation \Rightarrow^* are decidable. Indeed, call a word *basic* if it represents a basic element of M . Then, if a word w (*resp.* w') can be expressed as the concatenation of ℓ (*resp.* ℓ') basic words, which is the case if w (*resp.* w') has length ℓ (*resp.* ℓ') since all letters of S are basic, and $[w]^+, [w']^+$ admit a common right multiple, hence a right lcm, the latter can be expressed as the concatenation of $\ell + \ell'$ basic words, namely any diagonal path in a rectangular $\ell \times \ell'$ -grid all edges of which correspond to basic words: this directly follows from the definition of basic elements and the rules for an iterated lcm (equivalently, the element $[w]^+ \setminus [w']^+$ is a product of at most ℓ' basic elements). Thus, if $[w]^+$ and $[w']^+$ admit a common right multiple, they admit one represented by a word of length at most $(\ell + \ell')L$, where L is an upper bound for the lengths of basic words. Then w represents 1 in $\mathcal{U}(M)$ if and only if a trivial sequence $\varepsilon / \dots / \varepsilon$ appears in the list, where ε is the empty word. \square

We are thus led to the new conjecture, Conjecture $\mathbf{A}^{\text{padded}}$, stated in the introduction. Conjecture $\mathbf{A}^{\text{padded}}$ is *a priori* weaker than Conjecture \mathbf{A} but, by Proposition 1.6, it nevertheless implies the decidability of the word problem of every Artin–Tits group for which it is true.

1.3. Split reduction. We now connect the padded version of semi-convergence of \mathcal{R} -reduction with the ordinary semi-convergence of a new rewrite system, a variant of \mathcal{R} -reduction that can naturally be called *split reduction*.

In the definition of multifraction reduction used in [2, 3, 6] and repeated in Section 1.2, in order for $\underline{a} \bullet R_{i,x}$ to be defined, x and a_i have to admit a common multiple, a left- or right-multiple depending on the parity of i . In order to define a variant of this reduction, we now relax the assumption and only require that x and some divisor of a_i admit a common multiple; then, we again remove x from a_{i+1} and push it to the left or right side using the lcm operation, this time through the divisor of a_i that is involved and not necessarily through the whole of a_i .

Definition 1.7. (See Figure 1 right.) Assume that M is a gcd-monoid. For $\underline{a}, \underline{b}$ in \mathcal{F}_M , and for $i \geq 1$ even (*resp.* odd) and x, y in M such that $x \vee y$ (*resp.* $x \tilde{\vee} y$) exists, we write $\underline{a} \bullet R_{i,x,y}^{\text{break}} = \underline{b}$ if we have $\|\underline{b}\| = \|\underline{a}\| + 2$, $b_k = a_k$ for $k < i$, $b_k = a_{k-2}$ for $k > i + 3$, and

$$\begin{array}{ll} \text{for } i \text{ even:} & yb_i = a_i, \quad yb_{i+1} = xb_{i+2} = x \vee y, \quad xb_{i+3} = a_{i+1}, \\ \text{for } i \text{ odd:} & b_i y = a_i, \quad b_{i+1} y = b_{i+2} x = x \tilde{\vee} y, \quad b_{i+3} x = a_{i+1}. \end{array}$$

In addition we define “trimming” rules R_i^{trim} , such that $\underline{a} \bullet R_i^{\text{trim}} = \underline{b}$ holds if we have $a_{i+1} = 1$, $\|\underline{b}\| = \|\underline{a}\| - 2$, and $b_k = a_k$ for $k < i$, $b_k = a_{k+2}$ for $k > i + 1$, and $b_i = a_i a_{i+2}$ (*resp.* $a_{i+2} a_i$) if i is odd (*resp.* even). We write $\underline{a} \Rightarrow_{\text{split}}^* \underline{b}$ if either $\underline{a} \bullet R_{i,x,y}^{\text{break}} = \underline{b}$ holds for some i, x, y with $xy \neq 1$ or $\underline{a} \bullet R_i^{\text{trim}} = \underline{b}$ holds for some i , and denote by $\Rightarrow_{\text{split}}^*$ the reflexive–transitive closure of $\Rightarrow_{\text{split}}$. The rewrite system \mathcal{S}_M so obtained is called *split reduction* on M .

Adding the trimming rules R_i^{trim} is a technical artefact that will make statements simpler. It directly follows from the definition that, if $\underline{a} \Rightarrow_{\text{split}}^* \underline{b}$ is true, then \underline{a} and \underline{b} represent the same element in the group $\mathcal{U}(M)$. A major difference between \mathcal{R} - and \mathcal{S} -reductions is that, even if the monoid M is noetherian, no termination can be expected for the latter in general.

Example 1.8. Let M be the Artin–Tits monoid of type \tilde{A}_2 , that is,

$$\langle a, b, c \mid aba = bab, bcb = cbc, cac = aca \rangle^{\dagger},$$

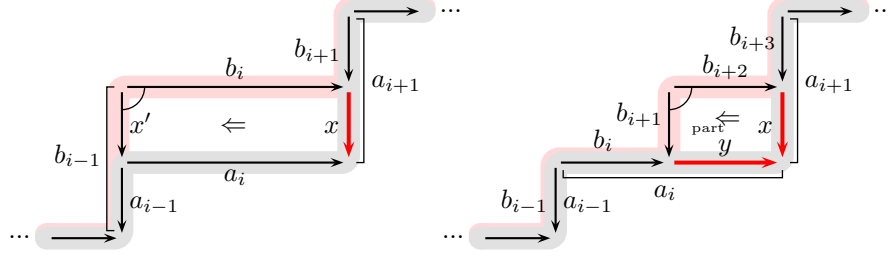


FIGURE 1. Comparing the reduction $R_{i,x}$ (left) and the split reduction $R_{i,x,y}^{\text{break}}$ (right), here for i odd: in both cases, we divide a_{i+1} by x and push x to the left using an lcm, but, for $R_{i,x}$, we demand that x crosses the whole of a_i , whereas, for $R_{i,x,y}^{\text{break}}$, we only require that x crosses the possibly proper divisor y of a_i ; as a consequence, we cannot gather the elements at level $i-1$ and the depth increases by 2.

and let $\underline{a} = ab/c$. Then we find $\underline{a} \cdot R_{1,c,b}^{\text{break}} R_{1,c,a}^{\text{break}} = 1/ac/ca/b/cb/1$, whence

$$\underline{a} \cdot R_{1,c,b}^{\text{break}} R_{1,c,a}^{\text{break}} R_{3,b,a}^{\text{break}} R_{3,b,c}^{\text{break}} R_{5,a,c}^{\text{break}} R_{5,a,b}^{\text{break}} = 1/ac/1/cb/1/ba/ab/c/ac/1/ba/1/cb/1,$$

that is, with obvious notation, $\underline{a} \xrightarrow{\text{split}}^6 1/ac/1/cb/1/ba/\underline{a}/ac/1/ba/1/cb/1$, leading to an infinite sequence of split reductions from \underline{a} .

The following result directly follows from the definition; we state it both and as a useful example for practice and for future reference.

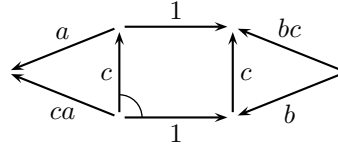
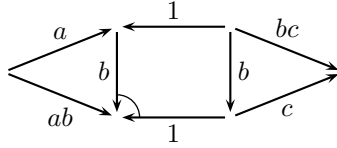
Lemma 1.9. *If M is a gcd-monoid, and $\underline{a} = \dots/a/1/bc/\dots$ is a multifraction on M with elements a , 1 , and bc in positions $i-1$, i , and $i+1$, then we have*

$$\begin{aligned} \underline{a} \cdot R_{i,b} &= \dots/a/1/bc/\dots \cdot R_{i,b} = \dots/ab/1/c/\dots \quad \text{for } i \text{ even,} \\ \underline{a} \cdot R_{i,c} &= \dots/a/1/bc/\dots \cdot R_{i,c} = \dots/ca/1/b/\dots \quad \text{for } i \text{ odd.} \end{aligned}$$

In particular, for $a = 1$, we find

$$\begin{aligned} \underline{a} \cdot R_{i,b} &= \dots/1/1/bc/\dots \cdot R_{i,b} = \dots/b/1/c/\dots \quad \text{for } i \text{ even,} \\ \underline{a} \cdot R_{i,c} &= \dots/1/1/bc/\dots \cdot R_{i,c} = \dots/c/1/b/\dots \quad \text{for } i \text{ odd.} \end{aligned}$$

Proof. The diagrams provide a proof.



□

Lemma 1.10. *Assume that M is a gcd-monoid and $\underline{a}, \underline{b}$ belong to \mathcal{F}_M .*

- (i) *If $\underline{a} \xrightarrow{*} \underline{b}$ holds, then so does $\underline{a} \xrightarrow{\text{split}}^* \underline{b}$.*
- (ii) *If $\underline{a} \xrightarrow{\text{split}}^* \underline{b}$ holds, then $\underline{1}_{2p}/\underline{a} \xrightarrow{*} \underline{b}/\underline{1}_{2q}$ holds for some p and q .*

Proof. (i) For an induction it is sufficient to prove that $\underline{a} \Rightarrow \underline{b}$ implies $\underline{a} \xrightarrow{\text{split}}^* \underline{b}$. So assume that $\underline{a} \cdot R_{i,x} = \underline{b}$ with, say, i even. Writing $x \vee a_i = xb_i = a_i x'$ and applying the definition of $R_{i,x,y}^{\text{break}}$, we find

$$\underline{a} \cdot R_{i,x,a_i}^{\text{break}} = a_1/\dots/a_{i-1}/1/x'/b_i/\dots/b_n = b_1/\dots/b_{i-2}/a_{i-1}/1/x'/b_i/\dots/b_n.$$

The definition of $R_{i,x}$ gives $b_{i-1} = a_{i-1}x'$, so we deduce

$$\underline{a} \cdot R_{i,x,a_i}^{\text{break}} R_i^{\text{trim}} = b_1/\dots/b_{i-2}/a_{i-1}x'/b_i/\dots/b_n = \underline{b},$$

whence $\underline{a} \Rightarrow_{\text{split}}^2 \underline{b}$. The argument is similar for i odd.

(ii) We prove using induction on m that $\underline{a} \Rightarrow_{\text{split}}^m \underline{b}$ implies the existence of p and q satisfying $\underline{1}_{2p}/\underline{a} \Rightarrow^* \underline{b}/\underline{1}_{2q}$. Put $n := \|\underline{a}\|$. The result is straightforward for $m = 0$ with $p = q = 0$. Suppose that $m = 1$. Suppose first $\underline{a} \bullet R_{i,x,y}^{\text{break}} = \underline{b}$ with, say, i even. By hypothesis, y left-divides a_i , and we have $a_i = yb_i$. Starting from $\underline{1}_2/\underline{a}$, we can first use i reduction steps to split a_i into b_i and y by pushing a_1, \dots, a_{i-1} , and b_i to the left using Lemma 1.9 repeatedly: precisely, we find

$$\underline{1}_2/\underline{a} \bullet R_{2,a_1} R_{3,a_2} \cdots R_{i,a_{i-1}} R_{i+1,b_i} = a_1/\cdots/a_{i-1}/b_i/1/y/a_{i+1}/\cdots/a_n,$$

whence $\underline{1}_2/\underline{a} \Rightarrow^i b_1/\cdots/b_i/1/y/a_{i+1}/b_{i+4}/\cdots/b_{n+2}$. Let \underline{a}' be the latter multifraction. By hypothesis, x divides a_{i+1} , which is a'_{i+3} , and the lcm of x and y , that is, of x and a'_{i+2} , exists, so \underline{a}' is eligible for $R_{i+2,x}$. Then we obtain

$$\underline{a}' \bullet R_{i+2,x} = b_1/\cdots/b_i/b_{i+1}/b_{i+2}/b_{i+3}/b_{i+4}/\cdots/b_{n+2} = \underline{b},$$

whence $\underline{1}_2/\underline{a} \Rightarrow^* \underline{a}' \Rightarrow \underline{b}$, the expected result with $p = 1$ and $q = 0$.

Suppose now $\underline{a} \bullet R_i^{\text{trim}} = \underline{b}$, still with i even. By hypothesis, we have $a_{i+1} = 1$, and we find

$$\underline{a} \bullet R_{i+1,a_{i+2}} = a_1/\cdots/a_{i-1}/a_i a_{i+2}/1/1/a_{i+3}/\cdots/a_n$$

and from there, by the same argument as above with $\underline{1}_2/\underline{a}$, we can push a_{i+3}, \dots, a_n to the left, obtaining

$$\underline{a} \bullet R_{i+1,a_{i+2}} R_{i+2,a_{i+3}} \cdots R_{n-1,a_n} = a_1/\cdots/a_{i-1}/a_i a_{i+2}/a_{i+3}/\cdots/a_n/1/1,$$

which implies $\underline{a} \Rightarrow^* \underline{b}/\underline{1}_2$, the expected result with $p = 0$ and $q = 1$. The verifications for i odd are similar.

Suppose finally $m \geq 2$. Write $\underline{a} \Rightarrow_{\text{split}}^{m-1} \underline{c} \Rightarrow_{\text{split}} \underline{b}$. The induction hypothesis implies the existence of p and q satisfying $\underline{1}_{2p}/\underline{a} \Rightarrow^* \underline{c}/\underline{1}_{2q}$, and that of p' and q' satisfying $\underline{1}_{2p'}/\underline{c} \Rightarrow^* \underline{b}/\underline{1}_{2q'}$. As \mathcal{R} -reduction is compatible with left and right multiplication, $\underline{1}_{2p}/\underline{a} \Rightarrow^* \underline{c}/\underline{1}_{2q}$ implies $\underline{1}_{2p+2p'}/\underline{a} \Rightarrow^* \underline{1}_{2p'}/\underline{c}/\underline{1}_{2q}$, and $\underline{1}_{2p'}/\underline{c} \Rightarrow^* \underline{b}/\underline{1}_{2q'}$ implies $\underline{1}_{2p'}/\underline{c}/\underline{1}_{2q} \Rightarrow^* \underline{b}/\underline{1}_{2q+2q'}$. By transitivity of \Rightarrow^* , we deduce $\underline{1}_{2p+2p'}/\underline{a} \Rightarrow^* \underline{b}/\underline{1}_{2q+2q'}$ and this is the expected result. \square

We deduce a simple connection between \mathcal{R} - and \mathcal{S} -reductions to $\underline{1}$:

Proposition 1.11. *If M is a gcd-monoid then, for every multifraction \underline{a} on M ,*

$$(1.12) \quad \underline{a} \Rightarrow_{\text{split}}^* \underline{1} \text{ is equivalent to } \exists p (\underline{1}_{2p}/\underline{a} \Rightarrow^* \underline{1}).$$

Proof. Assume $\underline{a} \Rightarrow_{\text{split}}^* \underline{1}$, say $\underline{a} \Rightarrow_{\text{split}}^* \underline{1}_r$. By Lemma 1.10, there exist p, q satisfying $\underline{1}_{2p}/\underline{a} \Rightarrow^* \underline{1}_{r+2q}$, whence $\underline{1}_{2p}/\underline{a} \Rightarrow^* \underline{1}$.

Conversely, assume $\underline{1}_{2p}/\underline{a} \Rightarrow^* \underline{1}$. Suppose first $\|\underline{a}\| \leq 1$. Then $\underline{1}_{2p}/\underline{a} \Rightarrow^* \underline{1}$ is possible only for $\underline{a} = 1$, in which case $\underline{a} \Rightarrow_{\text{split}}^* \underline{1}$ is trivial. Now suppose $\|\underline{a}\| \geq 2$ with \underline{a} not trivial, that is, at least one entry of \underline{a} is not 1. Let i be minimal satisfying $a_i \neq 1$. Then \underline{a} is eligible for $R_{i,1,a_1}^{\text{break}}$, and we find $\underline{a} \bullet R_{i,1,a_1}^{\text{break}} = \underline{1}_2/\underline{a}$, whence $\underline{a} \Rightarrow_{\text{split}}^* \underline{1}_{2p}/\underline{a}$ in p steps. Next, by Lemma 1.10 again, $\underline{1}_{2p}/\underline{a} \Rightarrow^* \underline{1}$ implies $\underline{1}_{2p}/\underline{a} \Rightarrow_{\text{split}}^* \underline{1}$. We deduce $\underline{a} \Rightarrow_{\text{split}}^* \underline{1}$ by transitivity of $\Rightarrow_{\text{split}}^*$. \square

We naturally say that \mathcal{S} -reduction is semi-convergent for M if $\underline{a} \Rightarrow_{\text{split}}^* \underline{1}$ holds for every multifraction \underline{a} that represents 1 in $\mathcal{U}(M)$. Proposition 1.11 implies:

Corollary 1.13. *If M is a gcd-monoid, then \mathcal{S} -reduction is semi-convergent for M if and only if \mathcal{R} -reduction is semi-convergent for M up to padding.*

1.4. Connection with subword reversing and Property H. In the absence of a solution of the word problem for general Artin–Tits groups, Property H (see Definition 1.16 below) was proposed [1] as a weaker statement in the direction of such a solution along the line of Dehn’s algorithm for hyperbolic groups, and it was proved that Property H is indeed satisfied for the standard presentation of all Artin–Tits groups of type FC [4] and of sufficiently large type [8]. We now establish a simple connection between Property H and multifraction reduction. To state the result, we first need the notion of an lcm presentation:

Definition 1.14. A presentation (S, R) of a monoid M is a *right-lcm presentation* if R contains one relation for each pair (s, t) in $S \times S$ such that s and t admit a common right-multiple and this relation has the form $sv = tu$ where both sv and tu represent the right-lcm $s \vee t$. A left-lcm presentation is defined symmetrically.

By [5, Thm. 4.1], every strongly noetherian gcd-monoid admits a right-lcm presentation and a left-lcm presentation. For instance, the standard presentation of an Artin–Tits monoid is an lcm presentation on both sides.

Proposition 1.15. *If M is a gcd-monoid and (S, R) is an lcm presentation for M (on both sides), then the following are equivalent:*

- (i) \mathcal{R} -reduction is semi-convergent for M up to padding;
- (ii) \mathcal{S} -reduction is semi-convergent for M ;
- (iii) Property H is true for (S, R) .

By Corollary 1.13, (i) and (ii) in Proposition 1.15 are equivalent, and we are left with establishing the equivalence of (ii) and (iii), which will occupy the rest of the section.

We first recall a few definitions. If (S, R) is a monoid presentation, and w, w' are words in $S \cup \overline{S}$, then we say that w' is obtained from w by one step of *right R -reversing* if w' is obtained by deleting some length 2 factor \overline{ss} of w or replacing some length 2 factor \overline{st} of w with $\overline{v\overline{u}}$ such that $sv = tu$ is a relation of R . Symmetrically, we say that w' is obtained from w by one step of *left R -reversing* if w' is obtained by deleting some length 2 factor $s\overline{s}$ of w or replacing some length 2 factor $s\overline{t}$ of w with $\overline{v}u$ such that $vs = ut$ is a relation of R .

Definition 1.16. Suppose that (S, R) is a monoid presentation. If w, w' are words in $S \cup \overline{S}$, we say that $w \rightsquigarrow w'$ is true if one can go from w to w' using finitely many *special transformations* of the following types: (i) replacing a positive factor u of w by u' with $u' \equiv^+ u$ (“positive equivalence”), (ii) replacing a negative factor \overline{u} of w by $\overline{u'}$ with $u' \equiv^+ u$ (“negative equivalence”), (iii) right R -reversing, (iv) left R -reversing. We say that (S, R) satisfies *Property H* if $[w] = 1$ implies $w \rightsquigarrow \varepsilon$, where ε is the empty word.

All special transformations map a word in $S \cup \overline{S}$ to one that represents the same element in $\mathcal{U}(M)$, so $w \rightsquigarrow \varepsilon$ always implies $[w] = 1$: Property H says that this implication is an equivalence. Note that special transformations add no new trivial factor $s\overline{s}$ or \overline{ss} , a situation reminiscent of Dehn’s algorithm for hyperbolic groups [4, Sec. 1.2].

The point here is that, if (S, R) is a right-lcm presentation, then right-reversing computes the right-lcm in $\langle S \mid R \rangle^+$ in the following sense:

Lemma 1.17 ([1, Prop. 3.6]). *Assume that M is a strongly noetherian gcd-monoid and (S, R) is a right-lcm presentation for M . Then, for all words u, v in S^* , the*

elements $[u]^+$ and $[v]^+$ of M admit a common right-multiple if and only if the right R -reversing of \overline{uv} leads to word $v'\overline{u'}$ with u', v' in S^* and, in this case, one has $[uv']^+ = [vu']^+ = [u]^+ \vee [v]^+$.

Of course, if (S, R) is a left-lcm presentation of M , left R -reversing computes the left-lcm in M in a similar sense.

Our aim from now on is to connect the relation \rightsquigarrow on words in $S \cup \overline{S}$ with partial reduction of multifractions. The convenient bridge between words and multifractions is as follows:

Definition 1.18. Suppose that $M = \langle S \mid R \rangle^+$. A word w in $S \cup \overline{S}$ is said to represent a depth n multifraction \underline{a} on M if there exist words w_1, \dots, w_n in S^* satisfying $w = w_1 \overline{w_2} w_3 \overline{w_4} \dots$ and $[w_i]^+ = a_i$ for every i . We say that w sharply represents \underline{a} if, in addition, w_i is nonempty for $1 < i < n$.

A word w in $S \cup \overline{S}$ may represent several multifractions, since the expression of w as $w_1 \overline{w_2} w_3 \overline{w_4} \dots$ need not be unique if no maximality condition is required, but it sharply represents a unique multifraction not finishing with a trivial entry.

Lemma 1.19. Assume that M is a gcd-monoid with lcm presentation (S, R) and w is a word in $S \cup \overline{S}$ representing a multifraction \underline{a} . Then $\underline{a} \Rightarrow_{\text{split}} \underline{b}$ implies $w \rightsquigarrow w'$ for some word w' in $S \cup \overline{S}$ representing \underline{b} .

Proof. Put $n = \|\underline{a}\|$, and let w_1, \dots, w_n satisfy $w = w_1 \overline{w_2} w_3 \overline{w_4} \dots$ and $[w_i]^+ = a_i$ for every i . Suppose first $\underline{a} \bullet R_i^{\text{trim}} = \underline{b}$ with, say, i odd. By hypothesis, we have $a_{i+1} = 1$, which requires $w_{i+1} = \varepsilon$, because there is no nontrivial invertible element in M , so the empty word must be the only element of its \equiv^+ -class. Then we have $b_i = a_i a_{i+2}$, and w also represents \underline{b} , as witnessed by the decomposition $w_1, \dots, w_{i-1}, w_i w_{i+2}, w_{i+3}, \dots, w_n$. The argument is similar for i even, replacing $w_i w_{i+2}$ with $w_{i+2} w_i$.

Suppose now $\underline{a} \bullet R_{i,x,y}^{\text{break}} = \underline{b}$ with, say, i even. The assumption that $\underline{a} \bullet R_{i,x,y}^{\text{break}}$ is defined implies that y left-divides a_i , and x left-divides a_{i+1} . It follows that there exist words u, v, u'', v'' in S satisfying

$$w_i \equiv^+ u u'' \quad \text{and} \quad w_{i+1} \equiv^+ v v'' \quad \text{with} \quad [u]^+ = y \quad \text{and} \quad [v]^+ = x.$$

By Lemma 1.17, the assumption that $x \vee y$ exists implies that the right- R -reversing of \overline{uv} leads to some $-$ word $v'\overline{u'}$ with u', v' in S^* such that uv' and vu' both represent $x \vee y$. Put $w'_k := w_k$ for $k < i$, $w'_i := u''$, $w'_{i+1} := v'$, $w'_{i+2} := u'$, $w'_{i+3} := v''$, and $w'_{k+2} := w_k$ for $k > i + 1$, and $w' := w'_1 \overline{w'_2} w'_3 \overline{w'_4} \dots$. By construction, w' is obtained from w using special transformations (namely positive and negative equivalences and right-reversing) and, by the definition of $R_{i,x,y}^{\text{break}}$, it represents \underline{b} . The argument is similar for i odd, exchanging right- and left-reversings. \square

Some care is needed for the other direction, because positive and negative equivalences might cause trouble. This is where sharp representations are useful.

Lemma 1.20. Assume that M is a gcd-monoid with lcm presentation (S, R) and w is a word in $S \cup \overline{S}$ sharply representing a multifraction \underline{a} . Then $w \rightsquigarrow w'$ implies $\underline{a} \Rightarrow_{\text{split}}^* \underline{b}$ for some multifraction \underline{b} sharply represented by w' .

Proof. Put $n = \|\underline{a}\|$, and let w_1, \dots, w_n satisfy $w = w_1 \overline{w_2} w_3 \overline{w_4} \dots$ and $[w_i]^+ = a_i$ for every i . Suppose first that w' is obtained from w by positive equivalence. Because the representation is sharp, the transformation involves one word w_i , with odd i

only. Then we have $w'_i \equiv^+ w_i$, and $w'_k = w_k$ for $k \neq i$, so w' sharply represents \underline{a} , and the result is true with $\underline{b} = \underline{a}$. The case when w' is obtained from w by negative equivalence is similar.

Suppose now that w' is obtained from w by one step of right-reversing. This means that there exists an even integer i , generators s, t in S , words u', v' in S^* , and a relation $sv = tu$ in R satisfying $w_i = tv'$, $w_{i+1} = su'$, and such that w' is obtained from w by replacing the factor $\overline{w_i}w_{i+1}$, which is $\overline{v'}\overline{t}su'$, with $\overline{v'}\overline{u}\overline{v}u'$. Now, if we define $w'_k := w_k$ for $k < i$, $w'_i := v'$, $w'_{i+1} := u$, $w'_{i+2} := v$, $w'_{i+3} := u'$, and $w'_{k+2} := w_k$ for $k > i + 1$, then we have $w' = w'_1\overline{w'_2}w'_3\overline{w'_4}\dots$, whereas, by very definition, the words w'_1, \dots, w'_{n+2} witness that w' represents the multifraction $\underline{a} \bullet R_{i,s,t}^{\text{break}}$. If all words u, v, u', v' are nonempty, then the representation is sharp, and taking $\underline{b} := \underline{a} \bullet R_{i,s,t}^{\text{break}}$ gives the result. Otherwise, we restore sharpness by inserting trimming steps. Assume for instance that u is empty, but u' and vv' are not. Then $\underline{a} \bullet R_{i,s,t}^{\text{break}}$ is eligible for R_i^{trim} , and w' sharply represents $\underline{b} := \underline{a} \bullet R_{i,s,t}^{\text{break}} R_i^{\text{trim}}$. The other cases are treated similarly, possibly using several trimmings. Note that free reduction, which corresponds to $s = t$ and $u = v = \varepsilon$, is not special. Finally, the argument for left-reversing is similar, with i now necessarily odd. \square

We can now complete the argument.

Proof of Proposition 1.15. As said above, it only remains to establish the equivalence of (ii) and (iii). Suppose first that \mathcal{S}_M is semi-convergent, and let w be a word in $S \cup \overline{S}$ representing 1 in $\mathcal{U}(M)$. Let w_1, \dots, w_n be words in S^* satisfying $w = w_1\overline{w_2}w_3\dots$, and let $\underline{a} := [w_1]^+ / \dots / [w_n]^+$. Then we find

$$(1.21) \quad \iota(\underline{a}) = \iota([w_1]^+) \iota([w_2]^+)^{-1} \iota([w_3]^+) \dots = [w_1][w_2]^{-1}[w_3]\dots = [w],$$

so \underline{a} represents 1 in $\mathcal{U}(M)$. By hypothesis, this implies $\underline{a} \Rightarrow_{\text{split}} \underline{1}$. By Lemma 1.19, we must have $w \rightsquigarrow w'$ for some word w' in $S \cup \overline{S}$ representing $\underline{1}$. The word w' must be the empty word, so we have $w \rightsquigarrow \varepsilon$. Thus (ii) implies (iii).

Suppose now that Property H is satisfied for (S, R) , and let \underline{a} be a multifraction satisfying $\iota(\underline{a}) = 1$. First, by applying as many trim steps R_i^{trim} as possible to \underline{a} , we obtain $\underline{a} \Rightarrow_{\text{split}}^* \underline{a}'$ for some multifraction \underline{a}' satisfying $a'_i \neq 1$ for $1 < i < \|\underline{a}'\|$. Now let w be a word in $S \cup \overline{S}$ representing \underline{a}' . The conditions $a'_i \neq 1$ imply that w sharply represents \underline{a}' . By construction, $\iota(\underline{a}') = 1$ holds in $\mathcal{U}(M)$, and, therefore, by the computation of (1.21), w represents 1 in $\mathcal{U}(M)$. By Property H, we must have $w \rightsquigarrow \varepsilon$. By Lemma 1.20, one has $\underline{a}' \Rightarrow_{\text{split}} \underline{b}$ for a multifraction \underline{b} sharply represented by ε . The latter is necessarily of the form $\underline{1}$, whence $\underline{a} \Rightarrow_{\text{split}} \underline{1}$. Thus (iii) implies (ii). \square

It was noted in [1] that Property H need not imply the decidability of the word problem in general; Proposition 1.15 precisely measures the gap between the two properties: by Proposition 1.6, the decidability of the word problem follows from the semi-convergence of \mathcal{R} -reduction up to a Turing-computable padding, whereas Property H entails no effective upper bound on the padding that is needed.

Remark 1.22. As stated above, Property H makes sense only for those gcd-monoids that admit a presentation that is both a right-lcm and a left-lcm presentation—as is the case for the standard presentation of Artin–Tits groups. If M is a strongly noetherian gcd-monoid with atom set S , then M admits a right-lcm presentation (S, R) and a left-lcm presentation (S, \widetilde{R}) which need not coincide.

Then Property H can be stated for the double presentation (S, R, \widetilde{R}) , using R for right-reversing and \widetilde{R} for left-reversing, and all results stated above extend without change to this general version.

2. ARTIN–TITS GROUPS OF SUFFICIENTLY LARGE TYPE

We now investigate a special family of gcd-monoids, namely Artin–Tits monoids of sufficiently large type, for which we establish the padded semi-convergence result stated as Theorem 1.

A presentation (S, R) of a monoid M is an *Artin–Tits presentation* if, for each pair (s, t) in $S \times S$ with $s \neq t$, the set R contains at most one relation, and that relation has the form $st\cdots = ts\cdots$, relating two alternating products of length m_{st} , for some integer $m_{st} \geq 2$. If $m_{st} = 2$, then (s, t) is called a *commuting pair*, and if there is no such relation between s and t , then (s, t) is called a *free pair*, and we define $m_{st} = \infty$. The associated *Coxeter diagram* is a complete graph with vertex set S , and a label m_{st} on the edge joining s and t . A monoid M is called an Artin–Tits monoid if it possesses an Artin–Tits presentation (S, R) . We say that M and its enveloping group $\mathcal{U}(M)$ are of *sufficiently large type* if, in any triangle in the associated Coxeter diagram, either no edge has label 2, or all three edges have label 2, or at least one edge has label ∞ .

Since all relations in R have the form $u = v$, where u and v are positive words of the same length, we see that the word-length map λ induces a well-defined map on the presented monoid, verifying strong noetherianity. We use λ also to denote the induced map on the monoid and, for $a \in M$, we call $\lambda(a)$ the *word-length* of a . We extend λ to multifractions by $\lambda(\underline{a}) := \sum_i \lambda(a_i)$, and call $\lambda(\underline{a})$ the *word-length* of \underline{a} .

2.1. The result. The main result that we shall prove is:

Proposition 2.1. *If M is an Artin–Tits monoid of sufficiently large type and \underline{a} is a multifraction of word-length ℓ that represents 1 in $\mathcal{U}(M)$, then we have $\underline{1}_{2p}/\underline{a} \Rightarrow^* \underline{1}$ with $p = 3\ell(\ell + 2)/4$.*

Note that, since all of the defining relators of the Artin–Tits groups have even length, a word in $S \cup \overline{S}$ that represents the identity must have even length, so ℓ must be even in the above proposition.

So, \mathcal{R} -reduction is semi-convergent for every Artin–Tits monoid of sufficiently large type up to a quadratic padding. As the square of word-length is obviously Turing-computable, Proposition 2.1 verifies that M satisfies Conjecture $\mathbf{A}^{\text{padded}}$, and therefore that the word problem of the associated Artin–Tits group is decidable (as was known already). By Proposition 1.15, we also obtain an alternative proof to the main result of [8], namely that the standard presentation of Artin–Tits groups of sufficiently large type satisfies Property H.

Proposition 2.1 follows from directly the following technical result. We say that a multifraction \underline{a} is *geodesic* if its word-length is minimal over all multifractions that represent the same group element.

Proposition 2.2. *If M is an Artin–Tits monoid of sufficiently large type and \underline{a} is a multifraction of word-length ℓ that is not geodesic, then $\underline{1}_{6\ell}/\underline{a}$ is \mathcal{R} -reducible to a multifraction of word-length $\ell - 2$.*

The proof makes use of the existence of an effective rewrite system for Artin–Tits groups of sufficiently large type that is defined and explored in [9, 10], most

of whose rules involve two generators only. We show that appropriate sequences of rules from that system can be simulated within multifraction reduction to reduce $\underline{1}_{6\ell}/\underline{a}$ as required.

The following subsection, Subsection 2.2, explains our strategy. Further technical results are established in Subsection 2.3, and then the proof given in Subsection 2.4.

2.2. Strategy for the proof of Proposition 2.2. For the rest of this section, we assume that M is an Artin–Tits monoid of sufficiently large type, and that (S, R) is the associated presentation.

We shall refer to the elements of S as *generators* and to those of $S \cup \overline{S}$ as *letters*. For $s \in S$, we call s the *name* of both of the letters s and \overline{s} . We use $f[w]$ and $l[w]$ to denote the first and last letter of a word w in $S \cup \overline{S}$. Furthermore, in situations where an element a of M is represented by a unique word w in S , we use $f[a]$ and $l[a]$ also to denote the first and last letters of w .

The rewrite system described in [9, 10] is based on two kinds of length preserving rules relating words, τ -moves and κ -moves, together with free reduction. By definition, a τ -move rewrites a certain type of 2-generator subword w on non-commuting non-free generators to another word $\tau(w)$ on the same two generators, where w and $\tau(w)$ have the same length and represent the same group element. A κ -move applies commutation relations to subwords on two or more generators. In this article, in order to simplify the exposition, we choose to break κ -moves down as sequences of κ -moves on 2-generator subwords, and we introduce the term τ^+ -move to mean either a τ -move or a κ -move applied to a 2-generator subword.

So τ^+ -moves can be applied only to certain geodesic 2-generator words, which we call *critical* words. When v is critical, then $\tau^+(v)$ is also critical, on the same set of generators. In all cases, one can check from the list of rewrite rules that $f[v]$ and $f[\tau^+(v)]$ have distinct names, as do $l[v]$ and $l[\tau^+(v)]$. Furthermore, if v is neither a positive nor a negative word, then exactly one of $f[v]$ and $f[\tau^+(v)]$ lies in S , and the same is true for $l[v]$ and $l[\tau^+(v)]$.

Suppose that w is a freely reduced non-geodesic word in $S \cup \overline{S}$. Then, by [10, Proposition 3.2], w admits a so-called *leftward length reducing critical sequence*. This consists of a certain type of factorisation (see [10] for a precise definition) $w = \alpha w_1 w_2 \cdots w_k \beta$ for some $k \geq 1$, where w_1, \dots, w_k are geodesic 2-generator subwords of w , and w_k is critical. We can locate such a decomposition, if it exists, by a finite search.

To perform the length reduction, as described in [10], we first replace w_k by $\tau^+(w_k)$. Then, if $k > 1$, we replace w'_{k-1} by $\tau^+(w'_{k-1})$, where $w'_{k-1} := w_{k-1} f[\tau^+(w_k)]$. We continue in this manner, finally replacing w'_1 by $\tau^+(w'_1)$, where $w'_1 := w_1 f[\tau^+(w_2)]$. At that stage, we have $l[\alpha] = \overline{f[\tau^+(w'_1)]}$, and we can freely cancel these two adjacent letters.

To mimic the length reduction summarised above, we need to simulate the replacement of w_k by $\tau^+(w_k)$. If w_k is a positive or a negative word then, by definition, $\tau^+(w_k)$ is another such word representing the same group element, with $f[\tau^+(w_k)] \neq f[w_k]$. In that case, the only multifraction operations that we may need to carry out are moves that ensure that the part of the multifraction that is represented by w_k occupies a single component.

When w_k is neither a positive nor a negative word, the simulation of the replacement of w_k by $\tau^+(w_k)$ is generally carried out in several stages. Initially, a certain

part of the multifraction, which may be spread across several components, is represented by the word w_k . We use \mathcal{R} -reduction (if necessary) to adjust that part of the multifraction so that it occupies exactly two adjacent components, numbered i and $i + 1$ for some i , and is represented by a word v_k with $[w_k] = [v_k]$. Furthermore, if $f[w_k] \in S$ (i.e. w_k starts with a positive letter), then i is odd and v_k is a concatenation of nonempty positive and negative words, whereas if $f[w_k] \in \overline{S}$, then i is even and v_k is a concatenation of nonempty negative and positive words. We then carry out a single \mathcal{R} -reduction, after which the part of the multifraction in question occupies components i and $i - 1$, and is represented by a word u'_k with $[u'_k] = [v_k]$, which is a concatenation of negative and positive words in the first case, and of positive and negative words in the second.

Let u_k be the maximal proper suffix of u'_k . We shall show later in Lemma 2.6 that $f[u'_k] = f[\tau^+(w_k)]$, so u_k represents the same group element as $f[\tau^+(w_k)]^{-1}\tau^+(w_k)$. Then, for $k > 1$, we adjoin $f[u'_k]$ to w_{k-1} to give the word w'_{k-1} defined above.

We repeat the above procedure on w'_{k-1} to define words u'_{k-1} , u_{k-1} and then, provided that $k > 2$, to define w'_{k-2} , u'_{k-2} , u_{k-2} , ..., w'_1 , u'_1 , u_1 . Then the first letter $f[u'_1]$ of u'_1 freely reduces with the final letter $l[\alpha]$ of α , and this cancellation can be effected by a single \mathcal{R} -reduction.

So the word has effectively been transformed to $\alpha' u_1 \cdots u_k \beta$, where α' is the maximal prefix of α . Now we use \mathcal{R} -reduction to simulate the replacement of each u_i by a 2-generator geodesic v_i . The resulting word $v = \alpha' v_1 \cdots v_k \beta$, where $|\alpha'| = |\alpha| - 1$, $|v_k| = |w_k| - 1$ and, for $1 \leq i < k$, $|v_i| = |w_i|$, is the same as the word resulting from the application of sequence of τ^+ -moves to w followed by the free cancellation. So we have successfully used \mathcal{R} -reduction to simulate the reduction resulting from the leftward critical sequence, and reduce the length of the word by 2.

2.3. Technical results for two-generator Artin–Tits monoids. In order to simulate the τ^+ transformations, we need a few preliminary results. By definition, the elements involved in the transformations belong to a non-free two-generator submonoid of the ambient monoid, and we establish here various technical statements involving the elements of such submonoids and their enveloping groups.

The first point is that every element in the group admits distinguished fractional decompositions.

Proposition 2.3. *If M_0 is a non-free two-generator Artin–Tits monoid, then every element of $\mathcal{U}(M_0)$ admits a unique expression ab^{-1} with a, b in M_0 and $a \wedge b = 1$, and a unique expression $c^{-1}d$ with c, d in M_0 and $c \wedge d = 1$.*

Proof. The lcm Δ of the two generators is a Garside element in M_0 and, as a consequence, any two elements of M_0 admit a common right-multiple and a common left-multiple. By Ore’s classical theorem, this implies that every element of $\mathcal{U}(M_0)$ can be expressed as a right fraction ab^{-1} and as a left fraction $c^{-1}d$. As any two elements of M_0 admit a left gcd and a right gcd, dividing the numerator and the denominator of a right (*resp.* left) fraction by their right gcd (*resp.* left gcd) yields irreducible fractions. Uniqueness follows from the fact that, if the right gcd of a and b is trivial, then ca is the right lcm of c and d whenever we have $ca = db$ (see for instance [7, Lemma 9.3.5]). \square

In the above situation, we shall call ab^{-1} and $c^{-1}d$ the *right* and *left fractional normal forms* of the associated element.

The next point is that, under mild assumptions, the numerator and the denominator of these fractional normal forms are represented by unique words in the generators:

Lemma 2.4. *If M_0 is a non-free two-generator Artin–Tits monoid, and ab^{-1} and $c^{-1}d$ are the fractional normal forms of an element g of $\mathcal{U}(M_0)$ and, moreover, a and b (or c and d) are both $\neq 1$, then none of a , b , c or d is divisible by the Garside element Δ , and there are unique positive words in M_0 representing each of them.*

Proof. If, for example, a were divisible by Δ then it would have a representative word with the same last letter as a representative word for b , which would violate the condition that a and b have no common right divisor. The uniqueness of the word representing a is proved in [11, Proposition 4.3]. \square

In this situation, we shall unambiguously write $f[a]$ and $l[a]$ for the first and last letters of the unique 2-generator word that represents a , and similarly for b , c , and d .

Next, the generators occurring at the ends of the numerators and denominators obey some constraints:

Lemma 2.5. *If M_0 is a non-free two-generator Artin–Tits monoid, and ab^{-1} and $c^{-1}d$ are the fractional normal forms of an element g of $\mathcal{U}(M_0)$ and, moreover, a and b (or c and d) are both $\neq 1$, then we have $f[a] \neq l[c]$ and $f[b] \neq l[d]$.*

Proof. We proved in Lemma 2.4 that none of a, b, c, d is divisible by the Garside element Δ of M_0 . Now, from the equality $ca = db$ and the fact that c and d have no nontrivial common left divisor, we have $f[c] \neq f[d]$, and so the element ca of M_0 is represented by more than one word in the generators. By [11, Proposition 4.3] again (or just because ca must be divisible by the two atoms of M_0), any such word must admit Δ as a subword. Since neither c nor a is divisible by Δ , we have $\Delta = c'a'$, where c' and a' are nontrivial right and left divisors of c and a . But the two words representing Δ consist of alternating letters, and so we have $f[a] \neq l[c]$ and similarly $f[b] \neq l[d]$. \square

Using the previous result, we obtain some control about the first (or the last) generator occurring in the fractional normal form of an arbitrary geodesic word.

Lemma 2.6. *Assume that M_0 is a non-free two-generator Artin–Tits monoid, and ab^{-1} and $c^{-1}d$ are the fractional normal forms of an element g of $\mathcal{U}(M_0)$ lying neither in M_0 nor in M_0^{-1} . Then, if w is a geodesic word representing g , we have $f[a] = f[w]$ if $f[w]$ is positive, and $l[c] = \overline{f[w]}$ if $f[w]$ is negative.*

Proof. The proof is by induction on the length n of w . In this proof, for a generator $x \in S$, we shall write x^\pm to mean either x or \bar{x} . Suppose that $w = x_{i_1}^\pm \cdots x_{i_n}^\pm$. There is nothing to prove for $n \leq 1$. Let us assume that $f[w] = x_{i_1}$, that is, $f[w]$ is positive; the other case, when $f[w]$ is negative, is similar. Let v be the maximal prefix $x_{i_1} x_{i_2}^\pm \cdots x_{i_{n-1}}^\pm$ of w . If v is a positive word, then ab^{-1} with $a = [v]$ and $b = [x_{i_n}]$ is a fractional normal form of $[w]$, so $f[a] = x_{i_1}$. Otherwise, let $a'b'^{-1}$ be the right fractional normal form of $[v]$. Then by induction we have $f[a'] = f[v] = f[w]$.

Suppose first $l[w] = \bar{x}_{i_n}$. Now $[a']$ and $[x_{i_n} b']$ may have a nontrivial greatest common right divisor a'' but, since $[w]$ is not a negative element, this cannot be the whole of a' . By Lemma 2.4, a' is represented by a unique positive word. Hence $a' = aa''$ and so $f[a] = f[a'] = f[w]$.

Otherwise we have $l[w] = x_{i_n}$. If $x_{i_n} = f[b']$ then x_{i_n} simply cancels with $l[b'^{-1}]$ and we are done. Otherwise the left fractional normal form of $[b'^{-1}x_{i_n}]$ is $[b']^{-1}[x_{i_n}]$ and it follows from Lemma 2.5 that the right fractional normal form $a''b''^{-1}$ of $[b'^{-1}x_{i_n}]$ has $f[a''] \neq l[b']$ and hence, since $l[a'] \neq l[b']$, that $f[a''] = l[a']$. So $a'a''$ is not divisible by Δ . Hence $a'a''$ has a unique geodesic representative and so, if $a'a''$ and b'' had a nontrivial common right divisor, then so would a'' and b'' , which is not the case. So $a'a''b''^{-1}$ is the right fractional normal form of $[w]$, with $f[a'] = f[w]$, and we are done. \square

We now establish a series of results involving reduction of multifractions, still in the non-free two-generator case. Although very easy, the following result will be crucial.

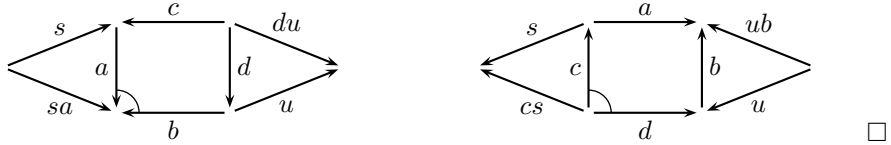
Lemma 2.7. *If M_0 is a non-free two-generator Artin–Tits monoid, and ab^{-1} and $c^{-1}d$ are the fractional normal forms of an element of $\mathcal{U}(M_0)$, then we have*

$$\begin{aligned} \dots/s/c/dt/\dots &\Rightarrow^{R_{i,d}} \dots/sa/b/t/\dots && \text{when } i \text{ is even} \\ \dots/s/a/tb/\dots &\Rightarrow^{R_{i,b}} \dots/cs/d/t/\dots && \text{when } i \text{ is odd,} \end{aligned}$$

where the multifractions are of depth at least 3, and the entries shown are in positions $i - 1, i, i + 1$. In particular, for $s = 1$, we have

$$\begin{aligned} \dots/1/c/dt/\dots &\Rightarrow^{R_{i,d}} \dots/a/b/t/\dots && \text{when } i \text{ is even} \\ \dots/1/a/tb/\dots &\Rightarrow^{R_{i,b}} \dots/c/d/t/\dots && \text{when } i \text{ is odd,} \end{aligned}$$

Proof. The diagrams provide a proof.



Using Lemma 2.7 repeatedly, we deduce that \mathcal{R} -reduction can be used to go from a fractional normal form to a geodesic expression.

In the sequel, $\underline{1}_m$ stands for $1/\dots/1$, m terms. Note that, for a, b in M , the multifraction $\underline{1}_m/a/b$ represents the element ab^{-1} of $\mathcal{U}(M)$ when m is even, and the element $a^{-1}b$ when m is odd.

Lemma 2.8. *Assume that M_0 is a non-free two-generator Artin–Tits monoid, and ab^{-1} and $c^{-1}d$ are the fractional normal forms of an element g of $\mathcal{U}(M_0)$ lying neither in M_0 nor in M_0^{-1} . Let v be a geodesic word representing g . Then, using \mathcal{R} -reduction, we can transform each of the multifractions $\underline{1}_{4|v|}/a/b$ and $\underline{1}_{4|v|-1}/c/d$ to a multifraction that is represented by v .*

Proof. The proof is by induction on $|v|$. Since we are assuming that v is not positive or negative, there is nothing to prove when $|v| \leq 1$.

Let $x = f[v]$ and let v' be the maximal proper suffix of v (of geodesic length $|v| - 1$). By Lemma 2.6, x is equal to either $f[a]$ or to $\overline{l[c]}$. If $x = f[a]$ then we use Lemma 2.7 to transform $\underline{1}_{4|v|-1}/c/d$ to $\underline{1}_{4|v|-2}/a/b/1$, and if $x = \overline{l[c]}$, then we transform $\underline{1}_{4|v|}/a/b$ to $\underline{1}_{4|v|-1}/c/d/1$. In either case, from the fractional normal form for g whose representative word does not start with x , we derive a multifraction represented by a word beginning with x , and this is equal to the multifraction representing the other fractional normal form of g (or that minus two initial entries

containing 1). We apply Lemma 1.9 to move the letter x left to the first or second entry of the multifraction.

In the first case we have identified reductions

$$\underline{1}_{4|v|}/a/b \Rightarrow^* [x]/\underline{1}_{4|v|-1}/a'/b, \quad \underline{1}_{4|v|-1}/c/d \Rightarrow^* [x]/\underline{1}_{4|v|-3}/a'/b/1,$$

and in the second case we have identified reductions

$$\underline{1}_{4|v|}/a/b \Rightarrow^* 1/[x]/\underline{1}_{4|v|-3}/c'/d/1, \quad \underline{1}_{4|v|-1}/c/d \Rightarrow^* 1/[x]/\underline{1}_{4|v|-3}/c'/d.$$

If a' or c' is equal to the identity, then b or d is a negative or positive element of G' and hence by Lemma 2.4 is represented by the unique geodesic word v' , and we are done. Otherwise, by induction, we can apply sequences of \mathcal{R} -reductions to transform each of $\underline{1}_{4|v|-4}/a'/b$ and $\underline{1}_{4|v|-5}/c'/d$ to multifractions represented by v' . Hence there are sequences of \mathcal{R} -reductions that transform each of $\underline{1}_{4|v|}/a/b$ and $\underline{1}_{4|v|-1}/c/d$ to multifractions represented by v . \square

2.4. Proof of Proposition 2.2. Using the results of Section 2.3, we now describe how the steps outlined in Subsection 2.2 can be executed using \mathcal{R} -reduction. As in Subsection 2.2 and for the rest of this section, M is a fixed Artin–Tits monoid of sufficiently large type.

Let w be a non-geodesic word representing the multifraction $\underline{a} = a_1/a_2/\cdots/a_n$ on M . Then w has a leftward length reducing critical factorisation $w = \alpha w_1 \cdots w_k \beta$. Suppose that the subword w_i starts within the entry a_{j_i} of the multifraction \underline{a} , that is, that we have

$$a_{j_i} = \begin{cases} [y_i z_i] & \text{for } j_i \text{ odd} \\ [z_i y_i] & \text{for } j_i \text{ even} \end{cases}$$

for positive words z_i, y_i , where w_i shares a prefix with z_i or \bar{z}_i . Specifically, unless w_i is a negative or positive word, one of z_i or \bar{z}_i is a proper prefix of w_i , but if w_i is positive or negative, then w_i might instead be a prefix of z_i or \bar{z}_i .

It follows from Lemma 1.9 that, via an application of the sequence of moves $R_{2k, a_1} R_{2k+1, a_2} \cdots R_{2k+j_1-2, a_{j_1-1}}$, we have

$$\underline{1}_{2k}/a_1/\cdots/a_n \Rightarrow^* \underline{1}_{2k-2}/a_1/a_2/\cdots/a_{j_1-1}/1/1/a_{j_1}/a_{j_1+1}/\cdots/a_n$$

and then application of $R_{2k+j_1, [y_1]}$ transforms this to

$$\underline{1}_{2k-2}/a_1/a_2/\cdots/a_{j_1-1}/[y_1]/1/[z_1]/a_{j_1+1}/\cdots/a_n.$$

Hence we see that a sequence of such reductions transforms the multifraction $\underline{1}_{2k}/a_1/\cdots/a_n$ to a multifraction

$$\underline{a}' := a_1/a_2/\cdots/a_{j_1-1}/[y_1]/1/[z_1]/a_{j_1+1}/\cdots/1/[z_2]/\cdots/1/[z_k]/\cdots/a_n$$

for which $[z_i]$ is an entry of \underline{a}' for each of the subwords z_i . Let us redefine j_i to denote the index of the entry of \underline{a}' in which $[z_i]$ now lies.

Let us assume that j_k is odd, and hence that w_k has a positive prefix - the other case is similar.

Suppose first that w_k is a positive word. Then w_k might be distributed over more than one entry of \underline{a}' as a prefix of a multifraction $\underline{b} := [z_k]/1/a_{j_k+2}/1/\cdots/a_{j_k+2s}$. But then a sequence of reductions $R_{j_k+2s-1, a_{j_k+2s}} \cdots R_{j_k+1, a_{j_k+2}} \cdots a_{j_k+2s}$ would transform \underline{b} to $\underline{b}' := [z_k]a_{j_k+2} \cdots a_{j_k+2s}/\underline{1}_{2s}$. So we can assume that w_k lies within a single multifraction entry, as a prefix of z_k . The required τ^+ -move on w_k can then be executed just by choosing a different representative word for w_k , which does not change \underline{b}' .

Otherwise, w_k is distributed over more than one entry of \underline{a}' , and has a positive prefix. Suppose that w_k occupies $s > 2$ such entries. So it has a suffix x_k (or \bar{x}_k) which defines a left (resp. right) divisor of the $(j_k + s - 1)$ -th entry of \underline{a}' . Then, by Lemma 2.7, we can apply a reduction $R_{j_k+s-2, [x_k]}$, which effectively replaces w_k by a word representing the same group element that occupies at most $s - 1$ multifraction entries. A sequence of at most $s - 2$ such reductions transforms w_k into a word occupying two entries, which represents the fractional normal form ab^{-1} of $[w_k]$, with a the j_k -th entry and b a right divisor of the $(j_k + 1)$ -th entry of the new multifraction \underline{a}'' . (Since w_k is not a positive or negative element of the group, it could not be transformed to a word that occupies fewer than two multifraction entries.) By Lemma 2.6 we have $f[a] = f[z_k]$. Now we apply Lemma 2.7, with $i = j_k$ and $t = 1$, to transform \underline{a}'' into a multifraction in which ab^{-1} is represented by its other fractional normal form, $c^{-1}d$, where now c is the $(j_k - 1)$ -th entry and d the j_k -th entry.

Let u'_k be a product of geodesic words representing c^{-1} and d . Then, from Lemma 2.5, we have $l[c] \neq f[a] = f[w_k]$, and so c^{-1} and hence also u'_k begins with the first letter that we wish to append to w_{k-1} to give w'_{k-1} .

We can repeat this process on $w'_{k-1}, w'_{k-2}, \dots, w'_1$ in turn. Then we can carry out the free reduction between the first letter of u'_1 and the final letter of α using a single \mathcal{R} -reduction. Denote the multifraction after the free reduction by \underline{b} . Then a word representing \underline{b} contains each of the words u_1, \dots, u_k as a subword, where each u_i is either a positive or negative word within a single entry of \underline{b} , or else it is a word in fractional normal form ab^{-1} or $c^{-1}d$, where the entries of \underline{b} in position $j_i - 1$ and j_i are a and b , or c and d , respectively.

We complete the proof by describing \mathcal{R} -reductions that transform the multifraction $\underline{1}_N/\underline{b}$ into one that is represented by the word $\alpha'v_1 \dots v_k\beta$, for an appropriately chosen integer N . In the case where u_i is a positive or a negative word, u_i and v_i are geodesic words with $[u_i]^+ = [v_i]^+$, so no change is needed to the multifraction.

We can apply Lemma 2.8 to deal with those u_i that are neither positive nor negative words. We deduce that we can find a sequence of \mathcal{R} -reductions that transform $\underline{1}_{N+2k}/\underline{a}$ via $\underline{1}_N/\underline{b}$ to a multifraction that is represented by the word $\alpha'v_1 \dots v_k\beta$ provided that $N \geq 4 \sum_{i=1}^k |v_i|$. Since clearly both k and $\sum_{i=1}^k |v_i|$ are bounded above by $|w|$, the proof of the Proposition 2.2 is completed using Lemma 2.8. This completes also the proof of Proposition 2.1. \square

The results of the current paper leave the question of whether Artin–Tits monoids of sufficiently large type satisfy Conjecture **A** open, but, at least, we know now that \mathcal{R} -reduction is relevant for them. In view of this result, and those of [2], which settle the case of type FC, the “first” case for which nothing is known in terms of reduction (nor of the word problem) is the Artin–Tits monoid with exponents 3, 3, 3, 3, 3, 2, that is, the monoid

$$\langle \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \mid \mathbf{aba} = \mathbf{bab}, \mathbf{aca} = \mathbf{cac}, \mathbf{bcb} = \mathbf{cbc}, \mathbf{ada} = \mathbf{dad}, \mathbf{bdb} = \mathbf{bdb}, \mathbf{cd} = \mathbf{dc} \rangle^+$$

We hope that further progress will arise soon

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