RECENT PROGRESS ON THE CONTINUUM HYPOTHESIS (AFTER WOODIN)

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A number of conceptually deep and technically hard results were accumulated in Set Theory since the methods of forcing and of fine structure appeared in the 1960's. This report is devoted to Woodin's recent results.Not only are these results technical breakthroughs, but they also renew the conceptual framework, making the theory more globally intelligible and emphasizing its unity. For the first time, there appear a global explanation for the hierarchy of large cardinals, and, chiefly, a realistic perspective to decide the Continuum Hypothesis—namely in the negative:

Conjecture 1 (Woodin, 1999)**.** *Every set theory that is compatible with the existence of large cardinals and makes the properties of sets with hereditary cardinality at most $\aleph_1$ invariant under forcing implies that the Continuum Hypothesis be false.*

Woodin's results come very close to this conjecture, establishing it for a substantial part of the hierarchy of large cardinals. The remaining question is whether this substantial part actually is the whole hierarchy of large cardinals. In any case—and that is what makes it legitimous to discuss these results now, without waiting for a possible solution of the above conjecture—Woodin's results contribute to show that the Continuum Problem and, more generally, the concept of uncountable infinity are not intrinsically vague and inaccessible to analysis, but that they can be the object of a genuine conceptual theory that goes far beyond simply exploring the formal consequences of more or less arbitrary axioms.

The current text owes much to the expository papers [19, 20]. It aims at describing and explaining four recent results of Woodin, here appearing as Theorems 30, 35, 38, and, chiefly, 40. It seems out of reach to give an idea of the proofs. The published part fills a significant proportion of the 900 pages of [18], and the most recent part is only alluded to in [21].

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1. A closed matter?

The Continuum Hypothesis (**CH**) is the statement: "Every infinite subset of $\mathbb{R}$ is in one-to-one correspondence either with $\mathbb{N}$, or with $\mathbb{R}$", and the Continuum Problem, raised by Cantor around 1890, is the question: "Is the Continuum Hypothesis true?". First on Hilbert's list in 1900, the Continuum Problem inspired many works throughout the XXth century. Once a general agreement was made about the Zermelo–Fraenkel system (**ZF**, or **ZFC** when the Axiom of Choice is included) as an axiomatic starting point for Set Theory, the *first* obvious step in studying the Continuum Problem is the question "Does there exist a proof of **CH**, or of its negation $\neg$**CH**, from **ZFC**?",

The answer is given in two results which have been major landmarks of Set Theory, both because of their own importance and of the methods used to establish them:

Theorem 2. (Gödel, 1938) *Provided* **ZFC** *is not contradictory, there is no proof of* $\neg$**CH** *from* **ZFC**.

Theorem 3. (Cohen, 1963) *Provided* **ZFC** *is not contradictory, there is no proof of* **CH** *from* **ZFC**.

It might be tempting to conclude that the Continuum Problem cannot be solved, and, therefore, is not a closed question, but, at least, has no interest, since every further effort to solve it is doomed to failure. This interpretation is erroneous. One may judge that studying the Continuum

Problem is inopportune if one finds little interest in the objects it involves: complicated subsets of $\mathbb{R}$, well-orderings whose existence relies on the axiom of choice[1]. But, if one does not dismiss the question *a priori*, one must see that Gödel's and Cohen's results did not close it, but, rather, opened it. As shows the huge amount of results accumulated in Set Theory in the past decades, the **ZFC** system does not exhaust our intuition of sets, and the conclusion should not be that the Continuum Hypothesis is neither true nor false[2], but simply that the **ZFC** system is incomplete, and has to be completed.

Some analogies are obvious. For instance, the proof that the axiom of parallels does not follow from the other Euclid axioms did not close geometry, but made the emergence of non-Euclidian geometries possible, and opened the question of recognizing, among all possible geometries, the most relevant for describing the physical world. Likewise, Gödel's and Cohen's results show that several universes are possible from **ZFC**, and, therefore, they open the study of the various possible universes—*i.e.*, equivalently, of the various axiomatic systems obtained by adjoining new axioms to **ZFC**—and the question of recognizing which one is the most relevant for describing the mathematical world.

Several preliminary questions arise: What can be a good axiom? What can mean "solving a problem" such as the Continuum Problem from additional axioms[3]. We shall come back to theses questions in Section 2 starting from the case of arithmetic. Various axioms possibly completing **ZFC** will be considered below. For the moment, let us simply mention the large cardinal axioms. Intuitively, they are the most natural axioms, and they play a central role. These axioms assert that higher order infinities exist, that go beyond smaller infinities in the way the infinite goes beyond the finite. They come from iterating the basic principle of Set Theory, which is precisely to postulate that infinite sets exist[4]. One reason for the success of large cardinal axioms is their efficiency in deciding a number of statements that **ZFC** cannot prove, see [9]. The important point here is that it seems reasonable to consider the large cardinal axioms as true, or, at least, to consider as plausible only those axioms **A** that are *compatible* with the existence of large cardinals in the sense that no large cardinal axiom contradicts **A**.

2. Arithmetic, incompleteness, and forcing

Let us denote by V the collection of all sets[5]. In the same way as the ultimate aim of Number Theory would be to determine all sentences satisfied in the structure $(\mathbb{N}, +, \times)$, the one of Set Theory would be to determine all sentences satisfied in the structure $(V, \in)$. This aim is inaccessible, so a possibility is to restrict to more simple structures of the type $(H, \in)$, where H is some fragment of the collection of sets. Filtrating by cardinality is then natural:

Definition 4. *For $k \geqslant 0$, we denote by H_k the set of all sets A that are* hereditarily *of cardinal strictly smaller than $\aleph_k$, i.e., are such that A, the elements of A, the elements of the elements of A, etc. all have cardinality less than $\aleph_k$* [6].

Let us first consider the structure $(H_0, \in)$, *i.e.*, the level of hereditarily finite sets. Denote by ZF^{fin} the system **ZF** without the axiom of infinity.

[1] But there are more effective versions of **CH** involving definable objects only.

[2] or is even undecidable in some mysterious sense

[3] One should suspect that just adding **CH** or $\neg$**CH** to the axioms would not be a very satisfactory solution.

[4] As the existence of a large cardinal always implies that **ZFC** is not contradictory, Gödel's Second Incompleteness Theorem forbids that this existence be provable from **ZFC**, so putting it as an hypothesis always constitutes a proper axiom.

[5] actually, the collection of all *pure* sets, defined as those obtained from the empty set by iterating the operations of going to powerset, union, and elements. One knows that such sets are sufficient to represent all mathematical objects.

[6] We recall that $\aleph_0, \aleph_1, \ldots$ is the increasing enumeration of the infinite cardinals, the latter being defined as the infinite ordinals that are equipotent to no smaller ordinal. Thus $\aleph_0$ (also denoted ω) is the smallest infinite ordinal, hence also the upper bound of all finite ordinals, and $\aleph_1$ is the smallest uncountable ordinal, hence the upper bound of all countable ordinals. Then $\aleph_0$ is the cardinality of $\mathbb{N}$, and, writing 2^κ for the cardinal of $\mathcal{P}(\kappa)$ as in the finite case, $2^{\aleph_0}$ is that of $\mathcal{P}(\mathbb{N})$, hence also of $\mathbb{R}$, so that the Continuum Hypothesis can be written as $2^{\aleph_0} = \aleph_1$.

Lemma 5. *From the axioms of* **ZF**$^{\text{fin}}$*, one can define inside* $(H_0, \in)$ *a copy of* $(\mathbb{N}, +, \times)$*. Conversely, from the Peano axioms, one can define inside* $(\mathbb{N}, +, \times)$ *a copy of* $(H_0, \in)$ [7]*.*

So, up to a coding, describing $(H_0, \in)$ is equivalent to describing $(\mathbb{N}, +, \times)$: the "hereditarily finite" level of Set Theory coincides with arithmetic. A common way of describing a structure S consists in *axiomatizing* it, *i.e.*, characterizing those sentences that are satisfied in S as those that can are *provable* from some sufficiently simple system of axioms. For arithmetic, the Peano system is well known, but Gödel's Incompleteness Theorems show that the description so obtained is not complete: there exist sentences that are satisfied in $(\mathbb{N}, +, \times)$, but are not provable from the Peano axioms, and, similarly, there are sentences that are satisfied in $(H_0, \in)$ but are not provable from **ZF**$^{\text{fin}}$.

Going to the framework of Set Theory enables one to prove more statements, hence to come closer to completeness. In the case of a sentence ϕ about H_0, this means no longer looking whether ϕ is provable from **ZF**$^{\text{fin}}$, but instead whether the sentence "$(H_0, \in)$ satisfies ϕ" [8] is provable from **ZFC**. Similarly, in the case of a sentence ϕ about $\mathbb{N}$, the question is no longer to look whether ϕ is provable from the Peano axioms, but instead whether "$(\mathbb{N}, +, \times)$ satisfies ϕ" [9] is provable from **ZFC**.

The Incompleteness Theorem still applies, and axiomatizing using **ZFC** does not provide a complete description. However, the description so obtained is satisfactory in practice, in that most of the sentences that are true but unprovable are *ad hoc* sentences steming more or less directly from logic[10]. Moreover, and chiefly, the way the incompleteness of **ZFC** appears at the level of arithmetic is fundamentally different of the way it does at higher levels, for instance in the case of the Continuum Hypothesis.

In order to explain this difference of nature, we need to introduce the notion of *forcing*, and, at first, that of a *model of* **ZFC**. As the **ZFC** axioms involve the membership relation only, it makes sense to consider abstract structures (M, E) with E a binary relation on M such that each **ZFC** axiom is satisfied when the membership relation is given the value E: such a structure is called a *model* of **ZFC**. In this framework, for the **ZFC** axioms to be true corresponds to the fact that the structure $(V, \in)$ made by *true* sets equipped with *true* membership is a model of **ZFC**[11].

It is the conceptual framework provided by the notion of a model of **ZFC** which made it possible to prove Theorems 2 and 3: in order to show that $\neg$**CH** (*resp.* **CH**) is not provable from **ZFC**, it suffices to construct one model of **ZFC** satisfying **CH** (*resp.* $\neg$**CH**). To this aim, one starts with an (arbitrary) model M of **ZFC**, and one constructs a submodel L satisfying **CH** with Gödel, and an extension $M[G]$ satisfying $\neg$**CH** with Cohen, respectively[12]. Cohen's method, called the *forcing* method, consists in adding to M a new set G whose properties are controlled ("forced") from inside M by a specific ordered set $\mathbb{P}$, called *forcing set*, which describes the elements of $M[G]$ [13]. A model of the form $M[G]$ is called a *generic extension* of M.

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[7] One obtains a copy $\underline{\mathbb{N}}$ of $\mathbb{N}$ inside H_0 by recursively defining a copy $\underline{i}$ of the natural number i by $\underline{0} = \varnothing$ and $\underline{i+1} = \underline{i} \cup \{\underline{i}\}$: this is the von Neuman representation of natural numbers by pure sets. It is then easy to construct copies $\underline{+}$ and $\underline{\times}$ of $+$ and $\times$, and to prove from **ZF**$^{\text{fin}}$ that $(\underline{\mathbb{N}}, \underline{+}, \underline{\times})$ satisfies the Peano axioms. Conversely, following Ackermann, one can define inside $(\mathbb{N}, +, \times)$ a relation $\underline{\in}$ by saying that $p \,\underline{\in}\, q$ is true if the pth digit in the binary expansion of q is 1, and then show using the Peano axioms that $(\mathbb{N}, \underline{\in})$ satisfies the axioms of **ZF**$^{\text{fin}}$ and is isomorphic to $(H_0, \in)$.

[8] *i.e.*, the sentence obtained from ϕ by adding that all variables take their values in the definable set H_0, *cf.* Note 14

[9] or, more exactly and with the notation of Note 7, "$(\underline{\mathbb{N}}, \underline{+}, \underline{\times})$ satisfies ϕ"

[10] Noticeable exceptions are the combinatorial properties studied by H. Friedman [6]; also think of the results by Matiyasevich about the existence of Diophantine equations whose solvability is unprovable [12].

[11] The description deliberately uses a platonician vocabulary referring to a true world V made of true sets; more than a philosophical choice, this is a writing option, and it consists in nothing more than fixing some reference model and giving a distinguished role to those models that share the same membership relation.

[12] Similarly, one could show that the axioms of groups do not imply commutativity, say, by starting with an arbitrary group G and constructing a commutative subgroup and a non-commutative extension of G, respectively. The construction is more delicate in the second case, because starting with $G = \{1\}$ or with $M = L$ is not excluded, so using substructures cannot work.

[13] as in the case of an algebraic field extension whose elements are described by polynomials of the ground field

The existence of forcing introduces an essential variability in Set Theory. Starting with a model M and with a sentence ϕ such that neither ϕ nor $\neg\phi$ is provable from **ZFC**, it is frequent that one can construct a generic extension $M[G_1]$ in which ϕ is satisfied by using a first forcing set $\mathbb{P}_1$, and another generic extension $M[G_2]$ in which $\neg\phi$ is satisfied by using another forcing set $\mathbb{P}_2$, so that giving any privilege to ϕ or to $\neg\phi$ seems difficult. However, this cannot happen at the level of arithmetic.

Definition 6. *Let H be a definable set*[14]*. We say that the properties of the structure $(H, \in)$ are* invariant under forcing *if, for every sentence ϕ, every model M, and every generic extension $M[G]$ of M, the sentence "$(H, \in)$ satisfies ϕ" is satisfied in M if and only if it is satisfied in $M[G]$* [15].

Proposition 7. *The properties of $(H_0, \in)$ and of $(\mathbb{N}, +, \times)$ are invariant under forcing*[16].

Thus, the manifestations of the incompletenes of **ZFC** at the level of arithmetic are not connected with the variability due to forcing, and they reduce to what could be called a residual incompleteness. We said that the latter introduces little restriction to the efficacity in practice. It is therefore natural to try to recover, for instance for the structures $(H_k, \in)$ with $k \geqslant 1$, the situation of $(H_0, \in)$ and arithmetic—if this is possible. This leads to putting the question of finding good axioms for some structure $(H, \in)$ as follows:

Problem 8. *Find an axiomatic framework,* **ZFC***, or* **ZFC** *completed with axiom(s) that are compatible with the existence of large cardinals, providing a sufficiently complete description of $(H, \in)$, and making its properties invariant under forcing.*

The approach developed below consists in giving a leading role to the criterion of invariance under forcing. Achieving invariance of the properties of $(H, \in)$ under forcing means neutralizing the action of forcing at the level of H, and, therefore, limiting the inevitable incompleteness of the description as much as possible. Invariance of the properties under forcing is a strong constraint, and it is not *a priori* clear that its can be achieved beyond H_0 [17]. But the possible satisfaction of this constraint should appear as a strong argument in favour of the axiomatic system that achieves it[18].

In the sequel, by a *solution* for a structure $(H, \in)$, we mean any axiomatic system **ZFC** completed with axioms that are compatible with the existence of large cardinals making the properties of $(H, \in)$ invariant under forcing. A solution will be said to be *complete* if, in addition, it makes the description of $(H, \in)$ empirically complete, an obviously ill-defined and imprecise notion. With such a vocabulary, we can state the previous results as "**ZFC** is a complete solution for $(H_0, \in)$", and the reader guesses that the point will be the existence of possible (complete) solutions for the structures $(H_k, \in)$ with $k \geqslant 1$.

In this framework, we can now propose an answer to the question: "What can mean establishing ϕ when neither ϕ, nor its negation $\neg\phi$ is provable from **ZFC**?", typically when one can realize both ϕ and $\neg\phi$ using forcing. Assume that ϕ involves some definable structure $(H, \in)$. If one accepts the framework of Problem 8, *i.e.*, if one gives some priority to the criterion of invariance under forcing, it should appear reasonable to consider ϕ as established when two things have been proved:

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[14] *i.e.*, H is *defined* as the set of all x's satisfying a certain formula $\psi(x)$ of the language of Set Theory. For instance, each of the sets H_k is definable.

[15] This is something subtle here: assuming that H is defined by the formula $\psi(x)$, we do not require that the sets defined by $\psi(x)$ in M and in $M[G]$, *i.e.*, "H computed in M" and "H computed in $M[G]$", coincide, we only require that they satisfy the same properties.

[16] The result here is even stronger: one obtains invariance not only with respect to generic extensions, but even with respect to arbitrary extension, (M', E') being called an *extension* of (M, E) if M is included in M', E is the restriction of E' to M, and the ordinals of (M, E) and (M', E') coincide.

[17] Woodin showed that invariance under forcing is impossible for the properties of the structure $(V_{\aleph_0+2}, \in)$ (*cf.* Note 27), so, essentially, for any fragment that contains $\mathcal{P}(\mathbb{R})$.

[18] However, there is no unanimity on this point. One objection is to consider that the variability due to forcing reflects some blurring in our perception of sets. From this point of view, requiring invariance under forcing means restricting our observation to those fragments of the universe escaping that blurring. But nothing tells us that the solution of the Continuum Problem, say, must lie in those fragments.

- (i) that there exists *at least one* solution to Problem 8 for $(H, \in)$, and,
- (ii) that *every* such solution implies that ϕ be true.

In other words, we consider a sentence ϕ as established when it is necessarily true in every coherent framework that neutralizes the action of forcing until the level of ϕ [19].

This is exactly the approach developed by Woodin in order to address the Continuum Problem, which we shall see can be expressed as a property of H_2.

3. The second level: countable sets

The simple situation of H_0 does not reproduce whenever infinite sets enter the picture: the **ZFC** system does not make the properties of H_1 invariant under forcing, and it leaves many of them open. But we shall see that, provided **ZFC** is conveniently improved, there exists an excellent solution to Problem 8 for $(H_1, \in)$. The discovery of this solution has been one of the major tasks of Set Theory in the period 1970–1985.

In the same way as one goes from $(H_0, \in)$ to $(\mathbb{N}, +, \times)$, it is easy to go from $(H_1, \in)$ to $(\mathcal{P}(\mathbb{N}), \mathbb{N}, +, \times, \in)$ [20]. Determining which sentences are satisfied in $(\mathcal{P}(\mathbb{N}), \mathbb{N}, +, \times, \in)$ amounts to studying the subsets of $\mathcal{P}(\mathbb{N})$ that are *definable* there, *i.e.*, have the form

$$A = \{x \in \mathcal{P}(\mathbb{N})\,;\, (\mathcal{P}(\mathbb{N}), \mathbb{N}, +, \times, \in) \text{ satisfies } \phi(x, \vec{a})\}, \tag{1}$$

with $\vec{a}$ a finite sequence of elements of $\mathcal{P}(\mathbb{N})$: typically, recognizing whether $\exists x \phi(x, \vec{a})$ is satisfied is equivalent to recognizing whether the set defined in (1) is nonempty.

Definition 9 (Lusin)**.** *Let X be a Polish space. A subset of X^p is said to be* projective *if it is can be obtained from a Borel set in X^{p+k} by a finite number of projections and complementations.*

The subsets of $\mathcal{P}(\mathbb{N})$ that are definable in $(\mathcal{P}(\mathbb{N}), \mathbb{N}, +, \times, \in)$ exactly are the projective subsets of $\{0,1\}^{\mathbb{N}}$ [21]. As the classes we consider always include the Borel sets, and as there exists a Borel isomorphism between the Cantor space $\{0,1\}^{\mathbb{N}}$ and the real line $\mathbb{R}$, we may replace $\mathcal{P}(\mathbb{N})$ with $\mathbb{R}$, and, finally, we conclude that studying $(H_1, \in)$ is essentially studying the projective subsets of $\mathbb{R}$.

Owing to Problem 8, the first point is to know whether the **ZFC** axiomatization provides a sufficiently complete description of $(H_1, \in)$, *i.e.*, of the projective sets. The answer is negative. Let us say that a set is *PCA* if it is the projection of the complement of a projection of a Borel set. Then, if **ZFC** is not contradictory, neither the statement “All PCA subsets of $\mathbb{R}$ are Lebesgue measurable”, nor its negation, is provable from **ZFC** [22].

According to our program, we now look for a possible new axiom providing, when added to **ZFC**, a solution to Problem 8. Gathering a number of deep results leads to the conclusion that the axiom of *Projective Determinacy* does provide such a solution.

Definition 10. *We say that a subset A of $[0,1]$ is* determined *if the infinite sentence*

$$(\exists\epsilon_1)(\forall\epsilon_2)(\exists\epsilon_3)\ldots(\textstyle\sum_i \epsilon_i 2^{-i} \in A) \text{ or } (\forall\epsilon_1)(\exists\epsilon_2)(\forall\epsilon_3)\ldots(\textstyle\sum_i \epsilon_i 2^{-i} \notin A)$$

is satisfied, where the ϵ_i's are 0 *or* 1 [23].

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[19] With still more concrete words: only ϕ remains when temperature has been lowered enough to prevent thermical agitation connected with forcing to make ϕ and $\neg\phi$ indiscernible.

[20] *i.e.*, to *second order arithmetic* where, besides natural numbers and their operations, one also considers sets of numbers and the associated membership relation

[21] If ϕ has no quantifier, (1) defines an open set; adding one existential quantifier amounts to projecting, while adding one negation amounts to taking the complement.

[22] As mentioned above, Gödel shows that $\neg$**CH** (as well as the negation of the axiom of choice) is not provable from **ZF** by constructing a certain sub-model L of V. Now L comes equipped with a canonical well-ordering (implying the axiom of choice) whose restriction to the reals is a PCA set, which cannot be Lebesgue measurable by Fubini's theorem. So L satisfies “There exists a non-measurable PCA set”, and, therefore, it is impossible that **ZFC** proves “All PCA sets are measurable”. As for the negation, one uses forcing with Martin's axiom **MA**, which will be mentioned in Section 4.

[23] Equivalently, one player has a *winning strategy* in the game G_A where two players I and II alternatively construct an infinite sequence $\epsilon_1, \epsilon_2, \ldots$ of 0's and 1's, and where I (*resp.* II) is said to win for $\sum_i \epsilon_i 2^{-i} \in A$ (*resp.* $\notin$).

All open sets are determined, and a theorem of Martin (1975) states that all Borel sets are determined. This result is the strongest that can be proved from the axioms of **ZFC** [24], so, putting as an hypothesis that all sets in a family that properly includes the Borel sets is a (proper) axiom.

Definition 11. *The axiom of* Projective Determinacy **PD** *says: "Every projective subset of* $\mathbb{R}$ *is determined"*[25].

The determinacy property is a paradigm that allows one to express a number of analytic properties, and it follows that the axiom **PD** actually provides a very complete description of projective sets. Typical results are as follows, *cf.* [14]:

Proposition 12 (Banach–Mazur, Mycielski–Swierczkowski, Moschovakis)**.** *The system* **ZFC** + **PD** *proves that every projecive set is Lebesgue measurable and has the Baire property; it also proves that projective sets have the Uniformization Property*[26].

The unprovability phenomena connected with Gödel's theorems cannot be avoided, but one can claim without cheating that the **ZFC** + **PD** axiomatization gives to the description of H_1 the same degree of empirical completeness as **ZFC** gives to that of H_0.

In view of Problem 8, the next point is to look for possible conditions making the properties of H_1 invariant under forcing. It is precisely this question that led to isolating the notion of a *Woodin cardinal*[27] and which led in 1984, building on work by Foreman, Magidor, and Shelah mentioned in Section 4 below, to the following result:

Theorem 13 (Woodin)**.** *Assume that there exists a proper class of Woodin cardinals*[28]*. Then the properties of* $(H_1, \in)$ *are invariant under forcing.*

At this point, the only missing result is a proof of the compatibility of the axiom **PD** with the existences of large cardinals—so, in some sense, a proof of **PD**. Around 1983, Woodin gave a proof from a very strong axiom, but the exact measuring of **PD** in the large cardinal hierarchy came in 1985, with the following remarkable result, *cf.* [2]:

Theorem 14 (Martin–Steel)**.** *Assume that there exist infinitely many Woodin cardinals. Then* **PD** *is true*[29].

So, the system consisting of **ZFC** enhanced with **PD** provides a good description of $(H_1, \in)$ [30], and the slightly stronger system consisting of **ZFC** enhanced with the existence of a proper class of

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[24] In the model L, some set which the projection of a Borel set is not determined.

[25] Let **AD** be the maximal determinacy axiom "Every subset of $\mathbb{R}$ is determined"; Woodin proved in 1987 that the systems **ZFC**+"there exists infinitely many Woodin cardinals" (see below) and **ZF** + **AD** are equiconsistent; one of the points is that, if there exist infinitely many Woodin cardinals in V, then **AD** holds in the minimal sub-model $L(\mathbb{R})$ of V that contains $\mathbb{R}$.

[26] *i.e.*, if A is a projective subset of $\mathbb{R}^2$, then there exists a mapping $f : \mathbb{R} \to \mathbb{R}$ with a projective graph that chooses, for each x, a distinguished y satisfying $(x, y) \in A$ whenever such a y exists; the Uniformization Property is a corollary to the Scale Property, which is crucial.

[27] As many large cardinals, Woodin cardinals are defined by the existence of *elementary embeddings*, which are homomorphisms between models of **ZFC** that preserve everything that is definable from $\in$. Let V_α denote the set of all pure sets obtained from $\varnothing$ by using the powerset operation at most α times. A cardinal κ is called *Woodin* if, for every mapping $f : \kappa \to \kappa$, there exist a class M, an elementary embedding $j : V \to M$, and an ordinal $\alpha < \kappa$ such that $\xi < \alpha$ implies $j(\xi) = \xi$ and $f(\xi) < \alpha$ and we have $j(\alpha) > \alpha$ and $V_{j(f)(\alpha)} \subseteq M$. The general idea is natural: a cardinal κ is infinite if there exists a bijection between κ and one of its proper subsets, and it will be considered "super-infinite" if the bijection is even an isomorphism, *i.e.*, an elementary embedding. The logical strength of the notion depends on how close from the ground model V the target model M is, here expressed in the condition $V_{j(f)(\alpha)} \subseteq M$. If the latter were weakened to $V_\kappa \subseteq M$, one would obtain a *measurable* cardinal, a weaker notion in terms of consistency.

[28] *i.e.*, for every cardinal κ, there exists a Woodin cardinal above κ.

[29] It turned out that the implication actually is nearly an equivalence: for each k, the system **ZFC** + **PD** proves the consistency of **ZFC**+ "there exists k Woodin cardinals"; still more connections subsequently appeared, for instance that the forcing axiom **MM** implies **PD**; all this made **PD** and Woodin cardinals central and ubiquitous in Set Theory.

[30] An additional argument is that not only gives **PD** *some* answers to all questions about H_1, but it also gives *the* heuristically satisfactory answers. For instance, the Uniformization Property dismisses any use of the Axiom of Choice when projective sets are concerned; similarly, the Lebesgue measurability forbids that any paradoxical

Woodin cardinals (somehow a delocalized version of **PD**) gives a complete solution to Problem 8 for H_1: it allows one to recover for H_1, *i.e.*, for the level of countable infinity, the same type of empirical completeness that **ZFC** guarantees for H_0, *i.e.*, for the level of finite sets and arithmetic.

4. The third level: cardinality $\aleph_1$

The next step is that of the structure $(H_2, \in)$, *i.e.*, that of cardinality $\aleph_1$. It was addressed since the beginning of the 1980's. In the same sense as above, H_2 is the level of $\mathcal{P}(\aleph_1)$ [31], and also that of the set of all length $\aleph_1$ sequences of reals. A prominent technical role is played by the *stationary* subsets of $\aleph_1$, which have no counterpart at the level of $\aleph_0$, and originate in the existence of limit points in the order topology on the ordinal $\aleph_1$.

Definition 15. *We say that a subset of* $\aleph_1$ *is* stationary *if it meets every unbounded subset of* $\aleph_1$ *that is closed for the order topology. The set of all non-stationary subsets of* $\aleph_1$ *is denoted* $\mathcal{I}_{NS}$.

The level of H_2 is the first one where the axiom of choice occurs in an essential way. Also, let us immediately observe that this is the level where the Continuum Problem arises[32]:

Lemma 16. *There exists a sentence* ϕ_{CH} *such that "*$(H_2, \in)$ *satisfies* ϕ_{CH}*" is equivalent to* **CH**.

The success obtained for H_1 leads to looking for some axiom[33] playing for $(H_2, \in)$ the role played for $(H_1, \in)$ by **PD** and by the existence of a proper class of Woodin cardinals, *i.e.*, solving Problem 8. Bad news, announced by Levy and Solovay as early as 1967, is that no large cardinal axiom can answer the question:

Proposition 17. *No large cardinal axiom can make the properties of* $(H_2, \in)$ *invariant under forcing.*

The reason is the definability of forcing by a set of the ground model. Owing to Lemma 16, it suffices to show that a large cardinal axiom **A** cannot decide the truth value of **CH**. Now, in order to break **CH**, it is sufficient to add $\aleph_2$ subsets to $\mathbb{N}$, which can be done using a forcing set of cardinality $\aleph_2$; by construction, such a small forcing preserves all large cardinals. So, starting with a model satisfying **A** + **CH**, we can always construct a generic extension satisfying **A** + ¬**CH**, and **A** cannot make **CH** invariant under forcing.

So a possible axiomatization of H_2 neutralizing the action of forcing is not to be looked at in the family of large cardinal axioms. In the past two decades, it appeared that natural candidates lie among *forcing axioms*[34]. The latter are strong versions of the well known Baire Category Theorem which states that, if X is a locally compact space, then every countable intersection of dense open subsets of X is dense. The possibility of extending the result to larger intersections obviously depends on the cardinality of X [35], and it requires that certain constraints on X or, more precisely,

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decomposition of the sphere into projective pieces. In contrary, axiomatizing with **ZFC** + **V=L** (the "fine structure" of Jensen), which means that the universe coincides with the minimal model L of Gödel, also provides a rather complete description of the sets (one that contradicts most large cardinal axioms and gives no invariance under forcing), but the answers one obtains are less satisfactory than those given by **PD**: *e.g.*, we have seen that **ZFC** + **V=L** proves the existence of non-measurable projective sets.

Another point distinguishing **PD** from **V=L** is that every model V of **ZFC** includes L as a submodel, so that a theory of L is always present as a subtheory of any theory of sets. So taking **PD** as an hypothesis does not dismiss **V=L**, while taking **V=L** as an hypothesis would restrict the scope, as developing group theory by adding the axiom that every group is commutative. It is such an argument that leads to require that every axiom for V, *i.e.*, for true sets, be compatible with large cardinals.

[31] *i.e.*, that of sets of countable ordinals, since the construction of ordinals is made in such a way that each ordinal coincides with the set of smaller ordinals: *e.g.*, $\aleph_1$ is the set of all countable ordinals.

[32] This is not obvious, as *a priori* **CH** involves the whole set $\mathcal{P}(\mathbb{R})$, which belongs to H_2 only if **CH** is true; the lemma states that one can always *encode* the Continuum Hypothesis in $(H_2, \in)$.

[33] Once for all, when we say *one* axiom, it could be a finite family, or even an infinite one provided it is sufficiently effective, typically *recursive*.

[34] Other candidiates could be the *generic large cardinal* axioms of [4].

[35] If **CH** is true, the intersection of the $\aleph_1$ open dense sets $\mathbb{R} \setminus \{a\}$ for a in $\mathbb{R}$ is empty.

on the algebra of regular open sets of X, be satisfied. These constraints are motivated by the theory of forcing and generic extensions, and that is where the name "forcing axioms" comes from.

The first forcing axiom was introduced in 1970 to solve a famous Souslin problem about characterizing the real line. It is the *Martin axiom* **MA**: "If X is a locally compact Hausdorf space where any family of pairwise disjoint open subsets is countable, then every intersection of $\aleph_1$ dense open subsets of X is dense". This axiom can be seen to express a weak form of invariance between V and the generic extensions $V[G]$ associated with a sufficiently simple forcing set.

There exists a natural linear hierarchy of extensions of **MA**, which results from a highly developed theory of "iterated forcing": see [15] (and its 1000 pages!). In [5], Foreman, Magidor, and Shelah identified the maximal class of compact spaces for which a strong form of the Baire Category Theorem is not obviously contradictory, namely those compact spaces X such that the Boolean algebra of regular open sets of X *preserves stationarity*[36]. So the strongest possible form of Martin's axiom is:

Definition 18 (Foreman–Magidor–Shelah)**.** *The* Maximum Martin *axiom* **MM** *says: "If X is a locally compact Hausdorff space such that the algebra of regular open subsets of X preserves stationarity, then every intersection of $\aleph_1$ dense open subsets of X is dense".*

As above with **PD**, the question is whether the axiom **MM** is compatible with large cardinal axioms. The answer is positive [5]; it relies on Shelah's Iterated Forcing Theorem and on some argument of Baumgartner that relates iterated forcing to supercompact cardinals[37] is:

Theorem 19 (Foreman–Magidor–Shelah)**.** *Assume that there exists a supercompact cardinal. Then the axiom* **MM** *is satisfied in some generic extension of V.*

For studying H_2, it is natural to consider a weak variant of **MM** called **MMB** (*Bounded Martin Maximum*)[38] and introduced by Goldstern and Shelah [8]. Let us say that a formula is *bounded* if the only quantifications it contains have the form $\forall y{\in}z$ and $\exists y{\in}z$. The relevance of **MMB** for $(H_2, \in)$ appears in some reformulation due to Bagaria [1], namely that **MMB** is equivalent to the statement "Every sentence $\exists x\psi(x, a)$ with ψ bounded and a in H_2 satisfied in a stationary preserving generic extension of V is already satisfied in H_2"[39]. So Axiom **MMB** proves every property of H_2 that can be expressed by a sentence of the form $\forall \ldots \exists \ldots \psi$ with ψ bounded[40] and that cannot be refuted by a stationary preserving forcing. Such properties are therefore invariant under stationary preserving forcing, and the following question is natural:

Does **MMB***—or some variation of it—solve Problem 8 for $(H_2, \in)$?*

The question is open, because there miss a completeness not restricted to $\forall\exists$ sentences, and an invariance under unconditional forcing.

This appears to be the starting point of Woodin's work. His approach consists in achieving invariance under forcing first, starting with a version of Theorem 13 asserting that, if there exists a proper class of Woodin cardinals, then the properties of the structure $(L(\mathbb{R}), \in)$ are invariant under forcing. Then Woodin essentially tries to realize Axiom **MMB** in a convenient generic extension of the model $L(\mathbb{R})$. The construction involves a certain forcing set $\mathbb{P}_{max}$ of a new, very sophisticated type: the elements of the set $\mathbb{P}_{max}$ are themselves models of **ZFC**. The argument leads to extending the invariance under forcing from $L(\mathbb{R})$ to the constructed model. The latter is $L(\mathcal{P}(\aleph_1))$, and it

[36] One says that an ordered set $\mathbb{P}$ *preserves stationarity* if every stationary subset of $\aleph_1$ in V remains stationary in the generic extension associated with $\mathbb{P}$.

[37] A cardinal κ is called *supercompact* if, for every cardinal $\lambda \geqslant \kappa$, there exists a class M and an elementary embedding $j : V \to M$ such that $\alpha < \kappa$ implies $j(\alpha) = \alpha$, one has $j(\kappa) \geqslant \lambda$, and every length λ sequence of elements of M (in V) belongs to M. The axiom "There exists a supercompact cardinal" is stronger than the axiom "There exist infinitely many Woodin cardinals" (which we have seen is nearly equivalent to Projective Determinacy in the consistency hierarchy).

[38] **MMB** is like **MM**, but restricted to intersections of dense open subsets of X that can be written as the union of at most $\aleph_1$ regular open sets.

[39] This result is directly reminiscent of a classical theorem by Levy–Shoenfied (of which Prop. 7 is a corollary), stating: "Every sentence $\exists x\psi(x, a)$ with ψ bounded and a in H_2 satisfied in V is already satisfied in H_2".

[40] In the sequel, such a sentence (with one quantifier alternation) will be called a $\forall\exists$-*sentence*.

includes H_2 by construction. It follows that the properties of H_2 are captured. The final result involves a new axiom, here called **WMM** like "Woodin's Martin Maximum" [41], which is the variant of (Bagaria's reformulation of) **MMB** in which the stationary subsets of $\aleph_1$ and a subset of $\mathbb{R}$ belonging to $L(\mathbb{R})$ can be taken as parameters:

Theorem 20 (Woodin). *Assume that there exists a proper class of Woodin cardinals. Then* **ZFC**+ **WMM** *provides an empirically complete*[42] *axiomatization of* $(H_2, \in)$*, and makes its properties invariant under forcing.*

The missing point for **ZFC** + **WMM** to be a (complete) solution for H_2 is the compatibility of Axiom **WMM** with the existence of large cardinals, *i.e.*, the counterpart of the Martin–Steel theorem. Theorem 19 guarantees this compatibility in the case of **MM**, hence of **MMB**, but the question remains open for **WMM**. However, we shall see in the sequel that Theorem 20 seems close to a solution: for the moment, there is no proof that no large cardinal axiom can contradict **WMM**, but, at least, one knows that no large cardinal admitting a canonical model can contradict **WMM**. We shall come back on this in Section 6.

5. Ω-logic

In the recent years, Woodin proposed a new conceptual framework giving a more simple formulation of the previous results, and, mainly, opening numerous perspectives. The idea is to use a specific logic that directly includes invariance under forcing and, therefore, somehow repares the blurring introduces by forcing in our perception of sets.

So far, we tried to characterize the sentences satisfied in a strcture $(H, \in)$ as those that can be proved from convenient axioms, via the usual notion of proof (in first order logic). By using a new more subtle provability notion[43], we can hope to get more simple descriptions of complicated objects, and to discover new phenomena that otherwise would remain hidden.

As other formal logics, Woodin's Ω-logic can be described using a (syntactic) notion of provability (existence of a proof, *i.e.*, of a certificate guaranteeing a certain property) and a semantic notion of validity (satisfaction is some reference structures). In the current case, the role of proofs is played not by finite sequences of sentences as in usual logic, but by specific sets of reals, namely the *universally Baire* sets of [3].

Definition 21 (Feng–Magidor–Woodin). *A subset B of $\mathbb{R}^p$ is said to be* universally Baire *if, for every continuous $f : K \to \mathbb{R}^p$ with K a compact Hausdorff space, the set $f^{-1}(B)$ has the Baire property in K* [44].

Every Borel set is universally Baire, and, if there exists a proper class of Woodin cardinals, so is every projective set.

Woodin's idea is to use the universally Baire sets as witnesses for a new provability notion. If B is a universally Baire subset of $\mathbb{R}$, it can be written as the projection of the infinite branches of some tree $\widetilde{B}$ over $\lambda_B \times \aleph_0$, where λ_B is some convenient ordinal, and $\widetilde{B}$ then plays the role of a code for B. When one goes from V to a generic extension $V[G]$, the tree $\widetilde{B}$ is preserved, but the set of its branches in $V[G]$, denoted B_G, may properly include B.

Definition 22. *Assume that M is a transitive*[45] *model of* **ZFC** *and B is a universally Baire subset of $\mathbb{R}$. We say that M is B-*closed *if, for every generic extension $V[G]$ of V, the set $B_G \cap M[G]$ belongs to $M[G]$.*

[41] The original statement, denoted $(*)$, is "**AD** is satisfied in $L(\mathbb{R})$ and $L(\mathcal{P}(\aleph_1))$ is a $\mathbb{P}_{max}$-generic extension of $L(\mathbb{R})$". A more easily understandable reformulation will be given in Section 5.

[42] in the sense that it proves every sentence that cannot be refuted by going to a generic extension

[43] Warning! We only consider using an alternative logic for *statements*, which does not change anything to *proofs*: theorems are *true* theorems...

[44] *i.e.*, there exists an open subset U of K such that both $U \setminus f^{-1}(B)$ and $f^{-1}(B) \setminus U$ are meager

[45] This means that M is included in V, has the same membership relation, and that $x \in y \in M$ implies $x \in M$.

This notion is meant as a closedness property, namely that the model M contains enough witnesses to prove the universally Baire character of B. If B happens to be a Borel set, then every transitive model of **ZFC** is B-closed. But, the more complicated the set B is, the more demanding the condition of being B-closed is.

Definition 23 (Woodin)**.** *Assume that there exists a proper class of Woodin cardinals*[46]*. We say that a universally Baire subset B of $\mathbb{R}$ is an* Ω-proof *for a sentence ϕ if ϕ is satisfied in every countable transitive model of* **ZFC** *that is B-closed. We say that ϕ is* Ω-provable *if it admits at least one Ω-proof.*

An Ω-proof is not a proof in the usual sense, but, as a proof, it will be used as certifying that the considered sentence has a certain property. Observe that an Ω-proof involves small objects only (but infinite ones, in contrast to proofs in usual logic): sets of reals, countable models of **ZFC**.

Every provable sentence is Ω-provable: if ϕ is provable (in usual logic) from **ZFC**, then ϕ is satisfied in every model of **ZFC**, so in particular in every countable transitive model, and every universally Baire set (for instance the empty set) is an Ω-proof of ϕ. Now, there exist Ω-provable sentences whose only Ω-proof are more complicated than Borel sets and which are not provable in the usual sense: Ω-logic properly extends usual logic.

Are the Ω-provable sentences true? Owing to Gödel's Completeness Theorem, if a sentence ϕ is Ω-provable but not provable, there exists at least one model (M, E) of **ZFC** in which ϕ is not satisfied. However, we shall see that ϕ must be satisfied in all models that are sufficiently close to the model V of true sets—hence, in some sense, in all models we are interested in.

For α and ordinal, we denote by V_α the set of all pure sets that can be obtaine from $\varnothing$ using the powerset operation at most α times. The structures $(V_\alpha, \in)$ can be seen as approximations of $(V, \in)$; in general, $(V_\alpha, \in)$ is not a model of **ZFC**, but this is the case whenever α is an inaccessible cardinal[47].

Proposition 24. *Assume that there exists a proper class of Woodin cardinals. Then every Ω-provable sentence is satisfied in every model of* **ZFC** *of the type "$(V_\alpha, \in)$ computed in an (arbitrary) generic extension of V".*

In other words, an Ω-provable sentence cannot be refuted using forcing from the model V of true sets. This leads us to choosing the following semantic for Ω-logic:

Definition 25. *We say that a sentence ϕ is* Ω-valid *is ϕ is satisfied in every model of* **ZFC** *of the type "$(V_\alpha, \in)$ computed in an (arbitrary) generic extension of V".*

In this way, we obtain a coherent logic: Every Ω-provable sentence is Ω-valid. Then the converse question, *i.e.*, the question of whether Ω-logic is complete, arises naturally: is every Ω-valid sentence Ω-provable?

Conjecture 26 (Ω-Conjecture, Woodin, 1999)**.** *Every Ω-valid sentence is Ω-provable*[48]*.*

Roughly speaking, the Ω-Conjecture asserts that all sentences that cannot be refuted by going to a generic extension admit a "proof" in the family of universally Baire sets. We shall come back later on equivalent reformulations.

For the moment, we shall see that the framework given by Ω-logic (with the Ω-Conjecture) enables us to simply reformulate Problem 8 and the results of Section 4.

Definition 27. *Assume that H is definable. We say that* **A** *is an* Ω-complete axiom *for the structure $(H, \in)$ is, for every ϕ, exactly one of the two sentences* **A** $\Rightarrow$ *"$(H, \in)$ satisfies ϕ,* **A** $\Rightarrow$ *"$(H, \in)$ satisfies $\neg\phi$ is Ω-provable.*

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[46]This framework of large cardinals is not needed, but it makes the formulation more simple.

[47]A cardinal κ is said to be *inaccessible* if κ is uncountable, $\lambda < \kappa$ implies $2^\lambda < \kappa$, and the conjunction of $\lambda < \kappa$ and $(\forall\alpha < \lambda)(\lambda_\alpha < \kappa)$ implies $\sup\{\lambda_\alpha; \alpha < \lambda\} < \kappa$—*i.e.*, κ cannot be reached from smaller objects by using the powerset or the limit operations. Inaccessible cardinals are the smallest large cardinals.

[48]This formulation is readily correct only for $\forall\exists$-sentences; the general case is slightly more complicated.

So, an Ω-complete axiom for $(H, \in)$ "Ω-decides" every property of H. What makes the introduction of Ω-complete axioms both interesting and natural is the following point. If A is an Ω-complete axiom for $(H, \in)$, then there can certainly exist statements ϕ such that $\mathsf{A} \Rightarrow$ "$(H, \in)$ satisfies ϕ is Ω-provable but not provable, but, in this case, we at least know by Prop. 24 that ϕ cannot be refuted by going to a generic extension from a model of $\mathsf{ZFC} + \mathsf{A}$. So, we need not have a complete description of H, but we recover the same type of forcing-free completeness as with ZFC and arithmetic, and as required in Problem 8.

Proposition 28. *If the Ω-Conjecture is true, then $\mathsf{ZFC} + \mathsf{A}$ is a solution to Problem 8 for $(H, \in)$ if and only if A is an Ω-complete axiom for $(H, \in)$.*

Proof (Sketch). Assume that A is an Ω-complete axiom for $(H, \in)$. We saw above that, by construction of Ω-logic, the properties of H necessarily are invariant under forcing. There remains to show the compatibility of A with the existence of large cardinals. In the framework of Ω-logic, this means showing that $\neg\mathsf{A}$ is not Ω-valid. Now the hypothesis that A is an Ω-complete axiom guarantees that $\neg\mathsf{A}$ is not Ω-provable. If the Ω-Conjecture is true, non-Ω-provability implies non-Ω-validity.

Conversely, assume that $\mathsf{ZFC} + \mathsf{A}$ makes the properties of $(H, \in)$ invariant under forcing. Then, for each sentence ϕ, exactly one of the two sentences $\mathsf{A} \Rightarrow$ "$(H, \in)$ satisfies ϕ", $\mathsf{A} \Rightarrow$ "$(H, \in)$ satisfies $\neg\phi$", is Ω-valid. If the Ω-Conjecture is true, this implies that at least one of them is Ω-provable. Moreover, the compatibility of A with large cardinals implies that $\neg\mathsf{A}$ is not Ω-valid, hence *a fortiori* not Ω-provable, and A is an Ω-complete axiom for $(H, \in)$. □

Let us come back to the structure $(H_2, \in)$. First, Ω-logic enables us to give a more easily understandable formulation of Axiom WMM.

Proposition 29. *Assume that there exists a proper class of Woodin cardinals. Then WMM is equivalent to: "For every A included in $\mathbb{R}$ belonging to $L(\mathbb{R})$, every $\forall\exists$-sentence about $(H_2, \mathcal{I}_{NS}, A, \in)$ whose negation is not Ω-provable is satisfied".*

This shows that WMM is a maximality principle for $(H_2, \mathcal{I}_{NS}, \in)$ analogous to algebraic closure[49]: a field K is algebraically closed if every non-contradictory system of algebraic equation with parameters in K has a solution in K, *i.e.*, precisely, if every $\forall\exists$-property of the structure $(K, +, \times)$ compatible with the axioms of fields is satisfied[50]. So, saying that Axiom WMM is true is analogous to saying that H_2 is, in some sense, algebraically closed.

From Theorem 20, Woodin proves the following result, which apprears to be its genuine content:

Theorem 30 (Woodin)**.** *Assume that there exists a proper class of Woodin cardinals. Then WMM is an Ω-complete axiom for $(H_2, \in)$.*

By applying Proposition 28, and taking into account the empirical completeness asserted in Theorem 20, we deduce:

Corollary 31. *If the Ω-Conjecture is true, then $\mathsf{ZFC} + \mathsf{WMM}$ is a complete solution to Problem 8 for $(H_2, \in)$.*

6. Ω-logic and large cardinals

Ω-logic is connected with large cardinals: in some sense, it is even the logic of large cardinals—or, at least, of those large cardinals that admit canonical models of a certain type. Describing this connection will enable us to give a precise meaning to our claim that Woodin's results come close to a proof of the Ω-Conjecture. On the other hand, we shall see that Ω-logic provides an elegant

[49]Analogy is even more relevant when the hypothesis is restricted to "A projective"; one then obtains an axiom WMM_0 that is weaker than WMM but has the same properties with respect to axiomatizing H_2. A similar formulation would be also possible for the axiom MMB of which WMM is a variation.

[50]In this case, a $\forall\exists$-sentence only involves Boolean combinations of equations since there is no relation in the considered structure.

conceptual explanation to the empirical constatation that the large cardinal axioms organize into a linear hierarchy.

It is easy to check that all so far considered large cardinal axioms enter the following syntactic framework[51]:

Definition 32. *We say that* $\exists\kappa\psi(\kappa)$ *is a* large cardinal axiom *if* ψ *is a* $\exists\forall$*-sentence such that, if* $\psi(\kappa)$ *is satisfied in* V*, then* κ *is an inaccessible cardinal and* $\psi(\kappa)$ *remains satisfied in every generic extension of* V *associated with a forcing set of cardinality smaller than* κ*. In this case, we say that* $\exists\kappa\psi(\kappa)$ *is* fulfilled *if, for every set* X*, there exists a transitive model* M *of* **ZFC** *and an ordinal* κ *of* M *such that* X *belongs to* $V_\kappa \cap M$ *and* $(M, \in)$ *satisfies* $\psi(\kappa)$*.*

For the axiom $\exists\kappa\psi(\kappa)$ to be fulfilled means that there exist many models containing cardinals with the property ψ. Observe that, if there exists a proper class of inaccessible cardinals κ satisfying $\psi(\kappa)$, then the axiom $\exists\kappa\psi(\kappa)$ is fulfilled: it suffices to take $M = V_\lambda$ with λ inaccessible and large enough.

The following result shows that Ω-provable sentences are those which are provable (in usual logic) from some large cardinal axiom for which Ω-logic is relevant:

Proposition 33. *Assume that there exists a proper class of Woodin cardinals. Then a* $\forall\exists$*-sentence* ϕ *is* Ω*-provable from* **ZFC** *if and only if there exists some large cardinal axiom* **A** *such that "***A** *is fulfilled" is* Ω*-provable and* ϕ *is provable (in usual logic) from* **ZFC**+ *"***A** *is fulfilled".*

Corollary 34. *The* Ω*-Conjecture is equivalent to the statement: «For every large cardinal axiom* **A** *that is fulfilled, the statement "***A** *is fulfilled" is* Ω*-provable».*

We can now explain how Woodin's results come close to the Ω-Conjecture. In view of Corollary 34, the problem is to determine for which large cardinals **A** the statement "**A** is fulfilled" is Ω-provable. Now, there exists a canonical model program, which roughly speaking consists in constructing for each large cardinal axiom **A** a minimal model where **A** is satisfied, on the shape of Gödel's model L [52]. This program, which relies on the so-called *comparison method*, currently reaches the level of the axiom "There exist infinitely many Woodin cardinals" [13], but not (yet?) that of "There exists a supercompact cardinal". The following result relies on a fine analysis of a general notion of canonical model:

Theorem 35. *For each large cardinal axiom* **A** *for which a canonical model based on the comparison method may exist, the statement "***A** *is fulfilled" is* Ω*-provable*[53]*.*

As Theorem 35 is essentially an equivalence, the point in the Ω-Conjecture is the possibility of extending the comparison method to every large cardinal—we therefore see that this conjecture could be refuted by showing that some *very* large cardinal is to ever remain inaccessible to any notion of canonical model.

Let us go to the second point in this section, namely the explanation for the large cardinals hierarchy. As was said above, all known large cardinal axioms organize into a linear hierarchy: for any two such axioms, there always turns out that one implies the other, or, at least, that the *consistency* of one (*i.e.*, its non-contradiction) implies the consistency of the other. One so obtains a well-founded hierarchy connected with relative consistency that calibrates the logical strength of the axioms: for instance, the consistency of (the existence of) a supercompact cardinal implies that of infinitely Woodin cardinals, which itself implies the consistency of (the existence of) one Woodin cardinal. The latter implies the consistency of (the existence of) one measurable cardinal, which itself implies that of one (even many) inaccessible cardinal.

The starting point will be the existence of some complexity scale on universally Baire sets deduced from *Wadge reducibility*.

Definition 36. *For* $B, B' \subseteq \mathbb{K}$ [54]*, we say that* B *is* reducible *(resp.* strongly reducible*) to* B' *if one has* $B = f^{-1}(B')$ *for some continuous (resp.* $1/2$*-Lipschitz) mapping* $f : \mathbb{K} \to \mathbb{K}$*.*

[51] We do not claim that every sentence of this form intuitively corresponds to a large cardinal.

[52] which corresponds to the case of **ZFC**, *i.e.*, to the case where no large cardinal axiom at all is assumed

[53] It follows that the Ω-Conjecture is true in every canonical model.

[54] here it is convenient to return to the Cantor space $\mathbb{K} = \{0,1\}^{\mathbb{N}}$ rather than staying in $\mathbb{R}$

Proposition 37. *Assume that there exists a proper class of Woodin cardinals. Then, for all universally Baire sets B, B' in $\mathbb{K}$, either B is reducible to B', or B' is strongly reducible to $\mathbb{K} \setminus B$.*

As no subset of $\mathbb{K}$ may be strongly reducible to its complement, one obtains a preordering $\prec$ on universally Baire sets by saying that $B \prec B'$ is true if both B and $\mathbb{K} \setminus B$ are strongly reducible to B'. Then one shows that the preorder $\prec$ has no infinite descending chain, which enables one to attach to each universally Baire set A an ordinal that will be called its *complexity*.

Theorem 38 (Woodin)**.** *For* **A** *a large cardinal axiom, let $\rho(\mathbf{A})$ denote the minimal complexity of an Ω-proof of the statement "***A** *is fulfilled". Then, for all levels where they are defined*[55]*, the consistency hierarchy coincides with the hierarchy associated with ρ.*

This aesthetic and deep result is a strong argument in favour of Ω-logic. If the Ω-Conjecture is true, then the hierarchy defined by ρ covers all large cardinal axioms; if it is false, this hierarchy is only the beginning of a longer hierarchy about which nothing is known so far.

7. The results on the Continuum Hypothesis

One of the most dramatic aspects in the recent developments is the new perspective they open about the Continuum Problem.

Before presenting the result—and without claiming that the brief account below is a complete story of the results about **CH** in the past decades—we shall mention a result by Woodin (1984) that stresses the critical position of the Continuum Hypothesis:

Proposition 39. *Assume that there exists a proper class of measurable Woodin cardinals. Then any two generic extensions of V satisfying* **CH** *satisfy the same existential sentences with parameter $\mathbb{R}$.*

In other words, whenever two generic extensions agree on **CH**, they also necessarily agree on all properties which have the same syntactic complexity as **CH**.

On the other hand, it has been observed for along time that there exists no symmetry between **CH** and ¬**CH**, and that at least certain variants of ¬**CH** can be proved while the corresponding counterparts for **CH** cannot. Moreover, several remarkable results (generally with difficult proofs) showed that, in various frameworks where it is not clearly determined by the hypotheses, the value of the continuum, *i.e.*, of $2^{\aleph_0}$, turns out to be $\aleph_2$. For instance, Foreman, Magidor, and Shelah showed in [5] that **MM** implies $2^{\aleph_0} = \aleph_2$, and Woodin showed in [18] that so does the hypothesis that the ideal $\mathcal{I}_{NS}$ has some combinatorial property called $\aleph_2$-saturation. Recently, using a specially elegant and direct combinatorial argument, Todorcevic showed in [17] that **MMB** implies a strong effective version of the equality $2^{\aleph_0} = \aleph_2$, namely that one can *define* a well-ordering of length $\aleph_2$ on $\mathbb{R}$ using as only parameter a single sequence of reals of length $\aleph_1$ (hence an element of H_2). These results collectively may be seen as an empirical hint against **CH**.

Let us come to the recent result. By Lemma 16, every sufficiently complete description of H_2 must include a solution of the Continuum Hypothesis. For instance, it is not difficult to see that **WMM**, as well as **MM** or **MMB**, implies that **CH** is false, and that $2^{\aleph_0} = \aleph_2$ holds. But this result does not say anything about the solution of **CH** given by *other* possible axiomatizations of $(H_2, \in)$. The last theorem of Woodin we wish to stress is the following result, established in 2000:

Theorem 40 (Woodin)**.** *Assume there exists a proper class of Woodin cardinals. Then every Ω-complete axiom for $(H_2, \in)$ whose negation is not Ω-valid implies that* **CH** *be false.*

The proof of Theorem 40—which is a technical *tour de force*—relies on analysing Ω-*recursive* sets, which play for Ω-logic the role of *recursive* sets in classical logic. A subset T of $\mathbb{N}$ is said to be *recursive* of there exists an algorithm (which can be implemented on a Turing machine) that

[55] *i.e.*, essentially, for those large cardinals accessible to the comparison method

recognizes its elements[56]. Among many equivalent characterizations, recursive sets happen to be those sets that can be defined in $(H_0, \in)$ both by an existential formula and by the negation of an existential formula. Let $L(B, \mathbb{R})$ denote the model of **ZFC** that is constructed like Gödel's model L, but starting with B and $\mathbb{R}$.

Definition 41. *A subset T of $\mathbb{N}$ is said to be* Ω-recursive *if there exists a universally Baire subset B of $\mathbb{R}$ such that T can be defined in the structure $(L(B, \mathbb{R}), \in, \{\mathbb{R}\})$ both by an existential formula and by the negation of an existential formula.*

As H_0 is definable in $L(\mathbb{R})$, every recursive set is Ω-recursive, but the existence of universally Baire sets which are (much) more complicated than Borel sets implies that there exist Ω-recursive sets which are (much) more complicated than recursive sets.

The key point in Woodin's proof consists in studying whether Ω-recursive sets can be defined in $(H_2, \in)$—or, more generally, in the set of all sets hereditarily of cardinality at most that of $\mathbb{R}$. This study requires many developments involving tools from Descriptive Set Theory (study of the subsets of the real line), from the theory of large cardinals, and from that of determinacy. A crucial part is to adapt to the framework of determinacy axioms the Mitchell–Steel construction of canonical models for Woodin cardinals [13]. To this aim, Woodin investigates a new family of canonical models denoted $\mathrm{HOD}^{L(B,\mathbb{R})}$, which are indexed by universally Baire sets. The result of this technically sophisticated analysis is

Proposition 42. *Assume that there exists a proper class of Woodin cardinals. Let T be an Ω-recursive subset of $\mathbb{N}$. Then*
(i) either T is definable in $(H_2, \in)$,
(ii) or there exists a definable surjection of $\mathbb{R}$ onto $\aleph_2$.

Theorem 40 then follows from Prop. 42 using a diagonalization argument. Indeed, if **A** is an Ω-complete axiom for $(H_2, \in)$, then the set of all numbers of those sentences ϕ such that **A** $\Rightarrow$ "$(H, \in)$ satisfies ϕ" is Ω-provable is a Ω-recursive set (this is easy). Now, a classical result by Tarski states that, for any structure S, the set of those sentences that are satisfied in S cannot be defined inside S. So, in particular, the set of all (numbers of) sentences satisfied in $(H_2, \in)$ cannot be definable in $(H_2, \in)$, and the only possibility according to Prop. 42 is case (ii), *i.e.*, the Continuum Hypothesis being (in an effective sense) false.

Corollary 43. *If the Ω-Conjecture is true, then every solution to Problem 8 for $(H_2, \in)$ implies that the Continuum Hypothesis be false*[57].

We obtain in this way the following more precise version of the conjecture stated in Introduction:

Theorem 44. *If the Ω-Conjecture is true, then every theory of sets obtained by adding to* **ZFC** *an axiom that is compatible with the existence of large cardinals and makes the properties of $(H_2, \in)$ invariant under forcing implies that the Continuum Hypothesis be false.*

We saw that, always if the Ω-Conjecture is true, axiomatizing by **ZFC** + **WMM** is a (complete) solution to Problem 8 for $(H_2, \in)$. Hence, we are, with respect to H_2 and **CH**[58], exactly in the position considered at the end of Section 2: Problem 8 has a solution, and every solution implies that **CH** is false. So we reached the following conclusion:

Corollary 45. *If the Ω-Conjecture is true, then the negation of the Continuum Hypothesis is established, in the sense described at the end of Section 2.*

[56]The effectiveness of the rules of usual logic implies that the set of all (numbers of) sentences provable from **ZFC** is the projection of a recursive set, while the set of all (numbers of) sentences that are satisfied in $(H_0, \in)$ is not, what implies that the inclusion of the former set in the latter is strict, a way of proving Gödel's First Incompleteness Theorem.

[57]Let us also mention a variant of the **CH** result. Let us enumerate all formulas with one free variable, and define $\underset{\sim}{0}^{\Omega}$ to be the set of all pairs (n, r) in $\mathbb{N} \times \mathbb{R}$ such that $\phi_n(r)$ is Ω-provable. Woodin proves that, if the set $\underset{\sim}{0}^{\Omega}$ is not universally Baire, then, essentially, every solution to Problem 8 for $(H_2, \in)$ implies that **CH** be false: in other words, the pure analysis hypothesis "$\underset{\sim}{0}^{\Omega}$ is not universally Baire" gives the same conclusion as the Ω-Conjecture.

[58]more exactly, of the sentence ϕ_{CH} that codes **CH** inside H_2

The hasty reader might only remember that the solution of Continuum Problem is just postponed to the solution of another new problem which is equally open—and whose statement is still more complicated. This would be a short view, because the nature of the Ω-Conjecture is very different from that of the Continuum Problem. The radically new feature is that one can reasonably expect a proof[59] or a refutation of the Ω-Conjecture in the future—and, from there, a solution for at least one of the aspects of the Continuum Problem. A remarkable point is that the solution of the Ω-Conjecture can come from several disjoint parts of Set Theory, which both shows how central the conjecture is and emphasizes the unity of Set Theory.

8. Conclusion

Two conclusions should appear—and be seen as evidences that Set Theory did not end up with Cohen's Theorem.

The first one is that the axiom of projective determinacy **PD** makes, when added to the **ZFC** system, the correct axiomatization of H_1, *i.e.*, of analysis, in the same way as **ZFC** is a correct axiomatization of H_0, *i.e.*, of arithmetic. This axiom leads to an empirically complete and heuristically satisfactory description for H_1, *i.e.*, for the realm of countable infinity. This success of **PD** is the strongest argument in favour of its being true. It may seem surprising to identify efficacity and truth[60]. Let us simply observe that no *a priori* intuitive evidence[61] may exist here, so the only possible truth criterion is the empirical *a posteriori* evidence given by a long familiarity. May the reader think to the widely accepted axiom that infinite sets exist: its operatory efficiency is such that no one would think to renounce to it, hence, *e.g.*, to real numbers. However, this axiom has no intrinsic justification, nor has it either any intuitive evidence but the one given by interiorizing a long familiarity. The situation with **PD** is similar, and the long familiarity with this axiom set theorists acquired now gives to this strong axiom of infinity the same *intuitive evidence* it gave to the basic axiom of infinity decades ago[62].

The second conclusion is that there exists so far no solution for H_2, *i.e.*, for the level of the subsets of $\aleph_1$, enjoying the same degree of evidence, but that there exists at least one global solution at this level, namely the one developed by Woodin from the axiom **WMM** and Ω-logic.

Even is the Ω-Conjecture is to be proved in the future, the discussion about whether invariance under forcing is the only legitimate criterion will not be closed[63], and it not clear that Ω-logic should be the only reasonable framework. That is why it would be unwise to claim that Woodin's solution to the Continuum Problem is the ony possible one.

But, even without the stronger argument that an idea of the proofs would bring, the quick overview of results we have done should make it unquestionable that there can exist a genuine conceptual theory of uncountable infinity, one that stands on its own and has its own intrinsic logic and intuition. No comparable argument can be proposed by the opponents to such a theory, in particular by those who consider the Continuum Problem as essentially impossible to solve [64].

[59] In particular, Woodin estalishes a connection between Ω-valid sentences and universally Baire, which is not yet Ω-provability, but seems to be close to. In [21], Woodin proposes a program possibly leading to a proof of the Ω-Conjecture, based on new canonical models.

[60] *A priori*, it would be strange, assuming that one is to study an unknown field, to say: "Algebraically closed fields have a good theory, hence I will assume that K is algebraically closed"; actually, the problem here is not exactly to study an unknown and possibly arbitrary field K, but rather to study "the world of fields", in which case assuming that the ambient framework is algebraically closed is a quite reasonable hypothesis.

[61] Thought experiments, which have been sometimes proposed [10], do not go very far here...

[62] cf. Gödel [7]: "*There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole discipline, and furnishing such powerful methods for solving given problems (and even solving them, as far as possible, in a constructivistic way) that quite irrespective of their intrinsic necessity they would have to be assumed at least in the same sense as any established physical theory.*"

[63] In particular, we mentioned that invariance under forcing cannot be expected much beyond H_2, *cf.* Note 17; see also Note 18.

[64] cf. Woodin [20]: ≪*There is a tendancy to claim that the Continuum Hypothesis is inherently vague and that this is simply the end of the story. But any legitimate claim that CH is inherently vague must have a mathematical basis, at the very least a theorem or a collection of theorems. My own view is that the independance of CH from*

Finally and in any case, hopefully enough has been said to let the reader suspect that Woodin's work is a remarkable piece of mathematics.

References

[1] J. Bagaria, *Bounded forcing axioms as principles of generic absoluteness*, Arch. Math. Logic **69–6** (2000) 393–401.

[2] P. Dehornoy, *La détermination projective d'après Martin, Steel et Woodin*, Séminaire Bourbaki, Astérisque **177–178** (1989) 261–276.

[3] M. Feng, M. Magidor, & H. Woodin, *Universally Baire sets of reals*, in: H. Judah, W. Just and H. Woodin (Eds), Set Theory of the Continuum, MSRI Publ. 26, 203–242, Springer, 1992.

[4] M. Foreman, *Generic large cardinals: new axioms for mathematics?*, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998).

[5] M. Foreman, M. Magidor, & S. Shelah, *Martin's maximum, saturated ideals, and nonregular ultrafilters*, Ann. of Math. **127–1** (1988) 1–47.

[6] H. Friedman, *On the necessary use of abstract set theory*, Adv. in Math. **41** (1981) 209–280.

[7] K. Gödel, *What is Cantor's Continuum Problem?*, Amer. Math. Monthly **54** (1947) 515–545.

[8] M. Goldstern & S. Shelah, *The bounded proper forcing axiom*, J. Symbolic Logic **60–1** (1995) 58–73.

[9] A. Kanamori, *The higher infinite*, Springer, Berlin, 1994.

[10] Yu. Manin, *Georg Cantor and his heritage*, arXiv:math.AG/0209244 (2002).

[11] D.A. Martin & J.R. Steel, *A proof of projective determinacy*, J. Amer. Math. Soc. **2–1** (1989) 71–125.

[12] Y. Matijasevich & J. Robinson, *Reduction of an arbitrary Diophantine equation in one in 13 unknowns*, Acta Arith. **27** (1975) 521–553.

[13] W. Mitchell & J. Steel, *Fine structure and iteration trees*, Springer, Berlin, 1994.

[14] Y. Moschovakis, *Descriptive set theory*, North-Holland, Amsterdam, 1980.

[15] S. Shelah, *Proper and improper forcing*, 2nd ed., Perspectives in Math. Logic, Springer, Berlin, 1998.

[16] J. Steel, *Mathematics need new axioms*, Bull. Symb. Logic **6–4** (2000) 422–433.

[17] S. Todorcevic, *Generic absoluteness and the continuum*, Math. Res. Letters **9** (2002) 465–471.

[18] W. Hugh Woodin, *The Axiom of Determinacy, forcing axioms, and the nonstationary ideal*, Walter de Gruyter & co., Berlin, 1999.

[19] W. Hugh Woodin, *The Continuum Hypothesis, I & II*, Notices Amer. Math. Soc. **48–6** (2001) 567–576, & **8–7** (2001) 681–690.

[20] W. Hugh Woodin, *The Continuum Hypothesis*, Proceedings Logic Colloquium 2000, Paris; to appear.

[21] W. Hugh Woodin, *The Continuum Hypothesis and the Ω-Conjecture*, Coxeter Lectures, Fields Institute, Toronto (November 2002); Notes available at http://av.fields.utoronto.ca/slides/02-03/coxeter_lectures/woodin/

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ZFC, and from ZFC togeher with large cardinal axioms, does not provide this basis. I would hope this is the minimum metamathematical assessment of the solution to CH that I have presented. Instead, for me, the independence results for CH simply show that CH is a difficult problem.»