GAUSSIAN GROUPS

Patrick DEHORNOY
Université de Caen

• The program:
- Artin's braid groups B_n have nice properties, which can be established from their standard presentation (Artin, Garside, Thurston, etc.);

$\longrightarrow$ Look for other groups for which
(i) the results,
(ii) the methods
apply; hopefully: new results + better proofs.

• Successive steps:
- 0. Braid groups (+ free groups);
- 1. Artin groups (Brieskorn, Deligne, Charney);
- 2. (Thin) Gaussian groups (OK);
- 3. (Thin) groups of fractions (in progress).

- Braid groups: $B_n =$

$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{cases} \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geq 2 \end{cases} \right\rangle.$$

Then, for each braid group B_n:

- the word problem is decidable (Artin);
- the conjugacy problem is decidable (Garside);
- there exists a bi-automatic structure (Thurston);
- the center is monogenic (Chow);
- the group is linear (Bigelow, Krammer), etc.

• Extension #1: Spherical Artin groups

$\longrightarrow$ Similar presentations but longer relations:

$$\sigma_i \sigma_j \sigma_i \cdots = \sigma_i \sigma_j \sigma_i \cdots$$

$\longrightarrow$ Said to be spherical if the associated Coxeter group is finite.

• Then, for every spherical Artin group:

- the word problem is decidable (Brieskorn);
- the conjugacy problem is decidable (Deligne);
- there exists a bi-automatic structure (Charney);
- the group decomposes into a product of groups with monogenic centers (Brieskorn-Saito);
- the group is linear (Cohen, Digne), etc.

• Extension #2: Thin Gaussian groups

Two properties of braid groups (Garside, Thurston), and, more generally, of every spherical Artin group G:

- G is the group of fractions of some monoid M;
- M is cancellative and it admits lcm's.

$\longrightarrow$ Question 1: Do the previous results extend to all groups of fractions of monoids with lcm's?
$\longrightarrow$ Question 2: What corresponds to sphericality?

$\longrightarrow$ Answer to Question 1: Yes, provided we add the additional hypothesis that M is Noetherian (= there is no infinite descending chain for division: trivial for Artin groups);
$\longrightarrow$ Answer to Question 2: Thinness (see below).

Definition: A monoid M is Gaussian if it is cancellative, Noetherian, and it admits (left and right) lcm's. A group G is Gaussian if it can be expressed, in at least one way, as the group of fractions of a Gaussian monoid.

Assume that M is a Gaussian monoid. For x, y in M, there exists a unique z s.t.

$$xz = \operatorname{lcm}(x, y) :$$

denote this z by $x \backslash y$ (the complement of x in y.

Definition: A Gaussian monoid M is thin if M admits a finite set of generators closed under $\backslash$.

Fact: *An Artin monoid is Gaussian; it is thin Gaussian iff it is spherical.*

$\longrightarrow$ Question 3: Other examples?

$\longrightarrow$ Question 3': How to recognize (thin) Gaussian groups (or monoids)?

$\longrightarrow$ Question 4: Do the properties of spherical Artin groups extend to all thin Gaussian groups?

Answer to Question 3': Complemented presentations.

Definition: A monoid presentation (Σ, R) is complemented if, for all a, b in Σ, there exists in R exactly one relation

$$a \cdots = b \cdots .$$

Coxeter presentations are complemented.

$\longrightarrow$ Word reversing:

Definition: Assume that (Σ, R) is a complemented presentation. For w, w' words on $\Sigma \cup \Sigma^{-1}$, we say that $w \curvearrowright w'$ holds—" w **reverses** to w' "—if w' is obtained from w by iteratively

- deleting some subword $a^{-1}a$,
- replacing some subword $a^{-1}b$ with vu^{-1} s.t. $av = bu$ is a relation of R.

Fact: *(i) If $w \curvearrowright w'$ holds, then w and w' represent the same element of the group $\langle \Sigma; R \rangle$;*
(ii) If u and v are words on Σ and $u^{-1}v \curvearrowright \varepsilon$ holds, then u and v represent the same element of the monoid $\langle \Sigma; R \rangle^{+}$.

Definition: We say that a complemented presentation (Σ, R) satisfies the cube property if, for all a, b, c in Σ, we have

Proposition: *(i) Every thin Gaussian monoid admits a complemented presentation (Σ, R) that satisfies the cube property, admits a [pseudolength], and is such that the closure of Σ under $\curvearrowright$ is finite;*
(ii) Conversely, every presentation as above defines a thin Gaussian monoid.

$\longrightarrow$ Examples: A thin Gaussian monoid M admits a Garside element Δ (the lcm of the closure of the atoms under $\backslash$, and it is completeley determined by the finite lattice made by the divisors of Δ
$\longrightarrow$ Specify M by this finite lattice.

• Answer to Question 4 (Do all properties of spherical Artin groups extend to all thin Gaussian groups?): Yes—but not obvious, as the Coxeter relations are very special: they preserve the length, no square divides Δ, Δ is the lcm of the atoms, etc.

• Then, for every thin Gaussian group:

- the word problem is decidable by a double reversing process: w represents 1 in the group iff we have $w \curvearrowright uv^{-1}$ for some positive words u, v, and then $v^{-1}u \curvearrowright \varepsilon$ (D.-Paris) ;
- the group is torsion-free (D.);
- the conjugacy problem is decidable (Picantin);
- there exists a bi-automatic structure associated with an explicit transducer whose states are the divisors of Δ (D.);
- the group decomposes into a crossed product of groups with monogenic centers (Picantin);
- existence of n-th roots is decidable (Sibert);
- the homology of the group can be computed explicitly (D.-Lafont).

• Extension #3: Thin groups of fractions

$\longrightarrow$ Question: Can we go further?
$\longrightarrow$ Answer: Yes: renounce to lcm's, i.e., keep the existence of common multiples, but skip uniqueness.

Definition: A subset Σ of a monoid M spans M if Σ generates M and, for all x, y in Σ, if z is a common right multiple of x and y, then there exist x', y' in Σ such that $xy' = yx'$ holds and z is a right muliple of xy'.

A monoid M is thin if it admits a finite spanning subset.

$\longrightarrow$ Every thin Gaussian monoid is thin as above,
$\longrightarrow$ New examples:

$$\langle a, b; ab = ba, a^2 = b^2 \rangle$$
$$\langle a, b, c; ac = ca = b^2, ab = bc, cb = ba \rangle$$

What remains valid:

- existence of finite presentations with an effective criterion for recognizing thinness involving nondeterministic word reversing;

- word reversing solution of the word problem, and quadratic isoperimetric inequality;

- under additional hypotheses (existence of a “Garside” element), automatic structure.

	braid groups	(spherical) Artin groups	(thin) Gaussian groups	(thin) groups of fractions
ex.				
word pb. (by word reversing)	Artin47 D.97	Brieskorn71 D.-Paris99	D.[00]	D.[01]
conjugacy pb.	Garside69	Briesk-Saito73	Picantin00	??
quadratic isop. ineq.	Thurston88	Tatsuoka92	D.-Paris99	D.[01]
automatic struct.	(bi-) Thurston88	(bi-) Charney93	(bi-) D.[00]	(one-) D.[01]
center + decomp.	Chow68	Briesk-Saito73	Picantin[00]	??
torsion freeness		Squier80	D.99	(false)
homology (by algeb.)	Arnol'd70	Goriunov78 Squier80,95	D.-Lafont[01]	
extract. of roots	Stychnev78		Sibert[00]	

References

P. Dehornoy; Petits groupes gaussiens; Ann. Sci. Ec. Norm. Sup. Paris, to appear.

P. Dehornoy; Braids and Self-Distributivity; Progress in Math. vol. 192, Birkhäuser (2000).

P. Dehornoy; preprints
http://www.math.unicaen.fr/∼dehornoy/

P. Dehornoy & L. Paris; Gaussian groups and Garside groups, two generalizations of Artin groups; Proc. London Math. Soc.; 79-3; 1999; 569–604.

M. Picantin; The center of small Gaussian groups; J. of Algebra, to appear.

→ http://www.math.unicaen.fr/gdrtresses/