DYNNIKOV'S FORMULAS FOR THE BRAID ORDERING

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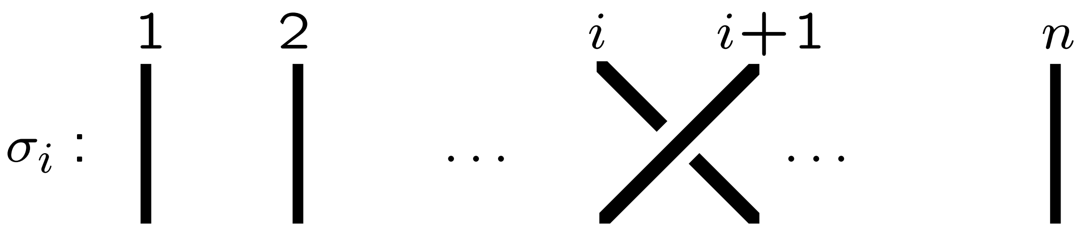
The linear ordering of braids, which was first discovered using results of self-distributive algebra, has now received several alternative constructions. Here, we mention some of them, in particular the recent approach developped by Ivan Dynnikov using laminations.

The linear ordering of braids

• Standard presentation of Artin's braid group B_n:

$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{cases} \sigma_i\sigma_j\sigma_i = \sigma_i\sigma_j\sigma_i & \text{for } |i-j| = 1 \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geq 2 \end{cases} \right\rangle$$

with geometric interpretation:



Proposition (A) (acyclicity): *A σ_1-positive braid is not trivial.*

• σ_1-positive = admits at least one expression where σ_1 appears but σ_1^{-1} does not

Proposition (C) (comparison): *Every braid is σ_1-positive, σ_1-negative, or σ_1-neutral.*

• σ_1-negative = ... σ_1^{-1} but no σ_1 ...
• σ_1-neutral = ... no σ_1 and no σ_1^{-1} ...

• **Corollary**: Let P_1 be the set of all σ_1-positive braids. Let

$$P = P_1 \cup \mathsf{sh}(P_1) \cup \mathsf{sh}^2(P_1) \cup \cdots .$$

where

$$\mathsf{sh} : \sigma_i \mapsto \sigma_{i+1} \text{ (the shift endomorphism).}$$

Then P is a positive cone:

• **Proposition:** Define

$$b_1 <_L b_2 \text{ iff } b_1^{-1} b_2 \in P;$$

then $<_L$ is a **left invariant linear ordering** on B_∞.

• **Remark 1:** If we define

$$b_1 <_R b_2 \text{ iff } b_2 b_1^{-1} \in P,$$

then $<_R$ is a right invariant linear ordering on B_∞ and

$$b_1 <_R b_2 \text{ is equivalent to } b_1^{-1} <_L b_2^{-1}.$$

• **Remark 2** (D. Rolfsen): There can exist **no** bi-invariant ordering on B_∞:

$$\Delta_3 \sigma_1 \Delta_3^{-1} = \sigma_2 \quad \text{and} \quad \Delta_3 \sigma_2 \Delta_3^{-1} = \sigma_1.$$

• Some **properties** of the linear ordering:

(R. Laver) **For each n, the restriction of the linear ordering $<_L$ to the monoid B_n^+ of Garside positive braids is a well-order.**

$\longrightarrow$ assigns a unique, well-defined ordinal to each Garside positive braid;

$\longrightarrow$ assigns a pair of ordinals to each braid;

but also

$\longrightarrow$ assigns a pair of ordinals to each conjugacy class of braids;

$\longrightarrow$ assigns a pair of ordinals to each Markov class of braids, etc.

• Some **applications** of the linear ordering:

• (The group B_∞ is torsion-free.)

• The algebra $\mathbb{C}B_\infty$ has no zero divisor.

• (E. Formanek) Each group B_n is isolated in B_∞, *i.e.*, $b^k \in B_n$ implies $b \in B_n$.

• Convergence of handle reduction (a practically very efficient solution for the braid isotopy problem).

• Proving that some representation of braids is faithful: it suffices to show that the image of a σ-positive brid is not trivial (used by V. Shpilrain for some Wada's representation).

• What is important in the braid ordering?

- its existence
$\rightarrow$ Acyclicity and Comparison Properties,

- its characterization
$\rightarrow$ expressions by words with σ_1 and no σ_1^{-1}.

• Natural task: to find other approaches (corresponding to other ways of introducing braids), and, for each of them,
- characterize the order, and
- reprove Properties A and C
(if possible).

• **Example 0**: Braid colorings and self-distributive systems.

[the original approach, relying on the study of the identity $x(yz) = (xy)(xz)$;]

- characterization: the one above.
- proofs of Properties A and C.

• **Example 1**: Automorphisms of a free group. (D. Larue)

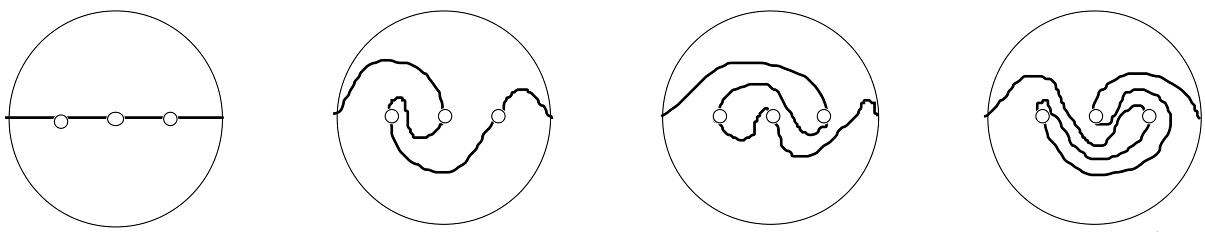
Embed B_n into Aut(F_n), where F_n free group generated by x_1, ... , x_n by

$$\sigma_i(x_k) = \begin{cases} x_i x_{i+1} x_i^{-1} & \text{for } k = i, \\ x_i & \text{for } k = i+1, \\ x_k & \text{otherwise.} \end{cases}$$

- characterization: b is σ_1-positive iff $b(x_1)$ ends with x_1^{-1}.
- *simple* proof of Property A;
- consequence of Property C: faithfulness.

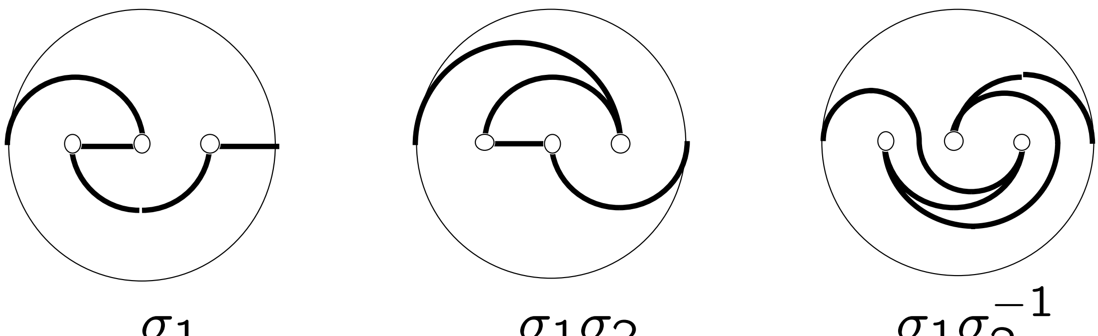
• **Example 2**: Mapping class group and curve diagrams. (R. Fenn, M. Greene, D. Rolfsen, C. Rourke, B. Wiest)

Introduce B_n as the mapping class group of a disk with n punctures; for each braid b (= diffeomorphism of the punctured disk that leaves the boudary fixed), look at the image of the main diameter (called the **curve diagram** of the braid):



1 $\quad$ σ_1 $\quad$ $\sigma_1\sigma_2$ $\quad$ $\sigma_1\sigma_2^{-1}$

• Main result: existence and uniqueness of some **standardized** curve diagram



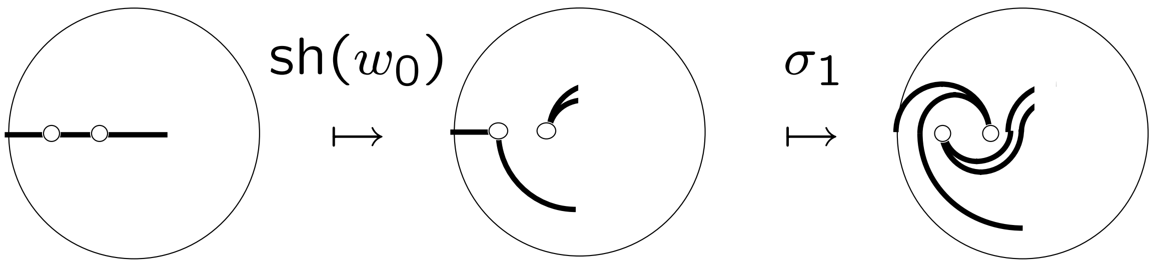
σ_1 $\quad$ $\sigma_1\sigma_2$ $\quad$ $\sigma_1\sigma_2^{-1}$

• Then

- characterization:

Proposition: *A braid b is σ-positive if and only if the standardized curve diagram associated with b diverges from the main diameter to the left.*

- (easy) proof of Property A:



- (difficult) proof of Property C ["useful arcs"].

• **Example 3**: Hyperbolic geometry. (H. Short, B. Wiest, after Nielsen and W. Thurston)

Identify B_n with the mapping class group of a punctured disk D_n, consider the universal covering of D_n in the hyperbolic plane; for φ in B_n, look at the action of the lifting of φ on the boundary of $\widetilde{D}_n$:

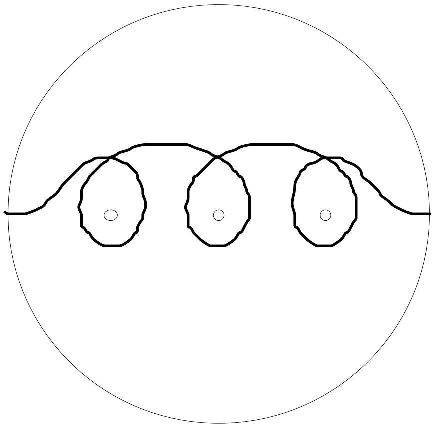
Proposition: There exists an action of B_n on the real line consisting of order preserving homeomorphisms, and the action is faithful on an uncountable Borel set U (actually a dense G_δ), *i.e.*, for every x in U, $b_1 \neq b_2$ implies $b_1(x) \neq b_2(x)$.

• Then define:

$$b_1 <_x b_2 \iff b_1(x) < b_2(x).$$

$\to$ a linear order $<_x$ on B_n for every x in U.

• The order $<_L$ corresponds to the geodesic:

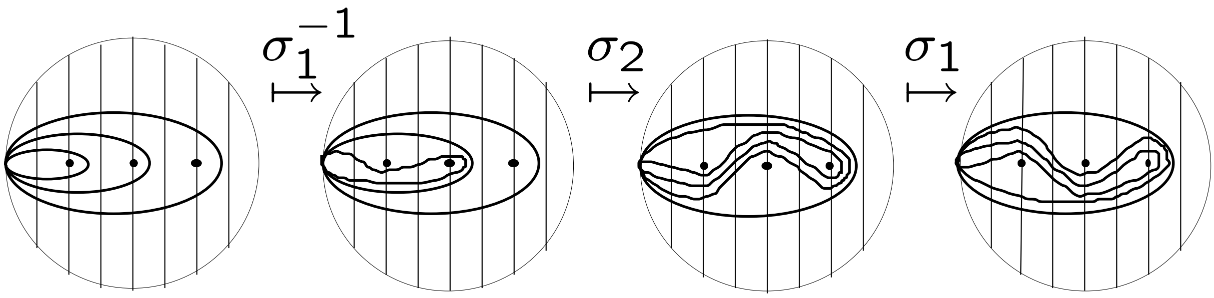


• **Example 4**: Laminations. (I. Dynnikov)

Instead of considering the image of the main diameter, look at the image of a family of circles L_0 starting from the origin and encircling 1, 2, ... , n punctures respectively.

$\longrightarrow$ a *lamination* consisting of n non-intersecting curves in the disk.

Examples:



- Encoding laminations:

- count the intersections with fixed vertical (half)-lines:

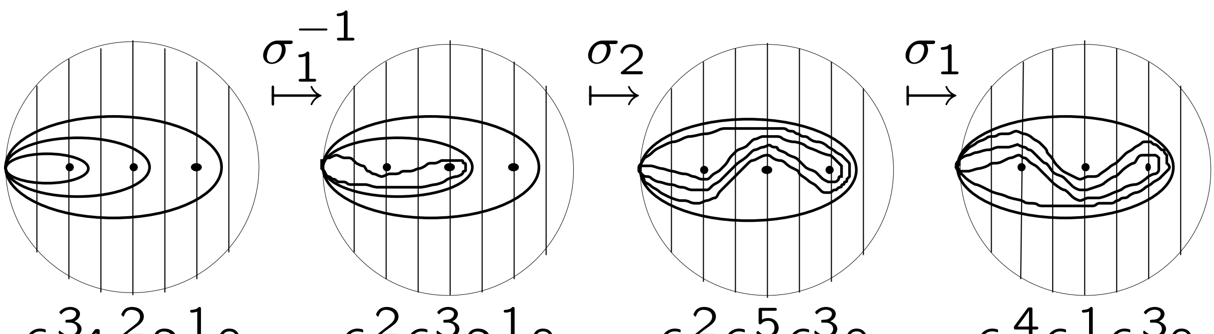
$$\begin{cases} x_i = \#\ \text{int. with half-line } x = i, y > 0 \\ y_i = \#\ \text{int. with half-line } x = i, y < 0 \\ z_i = \#\ \text{int. with line } x = i + 1/2 \end{cases}$$

(require x_i, y_i, and z_i to be minimal in the isotopy class);

- then encode by $(a_1, b_1, a_2, b_2, \dots, a_n, b_n)$ with

$$a_i = (x_i - y_i)/2 \quad \text{and} \quad b_i = (z_{i-1} - z_i)/2$$

Examples:



$6\frac{3}{3}4\frac{2}{2}2\frac{1}{1}0$ $\quad$ $6\frac{2}{4}6\frac{3}{3}2\frac{1}{1}0$ $\quad$ $6\frac{2}{4}6\frac{5}{1}6\frac{3}{3}0$ $\quad$ $6\frac{4}{2}6\frac{1}{5}6\frac{3}{3}0$

$(0, 1, 0, 1, 0, 1)$ $\quad$ $(-1, 0, 0, 2, 0, 1)$ $\quad$ $(-1, 0, 2, 0, 0, 3)$ $\quad$ $(1, 0, -2, 0, 0, 3)$

Notation: $x^+ = \sup(x, 0)$, $x^- = \inf(x, 0)$.

Proposition: (Dynnikov's formulas) *If L is coded by $(a_1, b_1, \ldots, a_n, b_n)$, then $L\sigma_i$ is coded by $(a'_1, b'_1, \ldots, a'_n, b'_n)$, with $a'_k = a_k$, $b'_k = b_k$ for $k \neq i, i+1$, and*

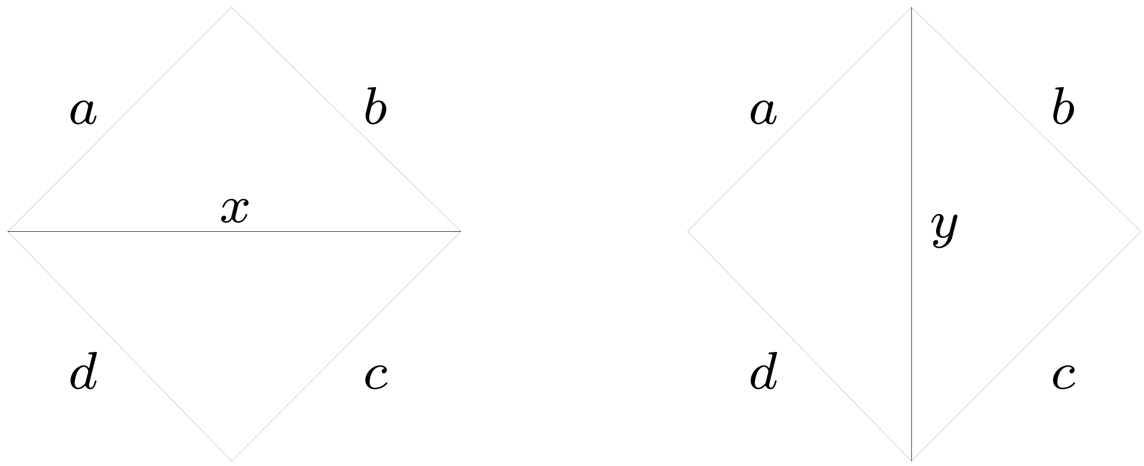
$$a'_i = a_i + b_i^+ + (b_{i+1}^+ - c)^+$$
$$b'_i = b_{i+1} - c^+$$
$$a'_{i+1} = a_{i+1} + b_{i+1}^- + (b_i^- + c)^-,$$
$$b'_{i+1} = b_i + c^+$$

with $c = a_i - b_i^- - a_{i+1} + b_{i+1}^+$,
(resp., $L\sigma_i^{-1}$ is coded by ... and

$$a'_i = a_i - b_i^+ - (b_{i+1}^+ + d)^+$$
$$b'_i = b_{i+1} + d^-$$
$$a'_{i+1} = a_{i+1} - b_{i+1}^- - (b_i^- - d)^-$$
$$b'_{i+1} = b_i - d^-$$

with $d = a_i + b_i^- - a_{i+1} - b_{i+1}^+$).

• Proof: Count the intersections with triangulations; decompose the action of $\sigma_i^{\pm 1}$ into elementary flips; the basic formula is:



$$x + y = \sup(a + c, b + d)$$

• Dynnikov's formulas define a right action of B_n on $\mathbb{Z}^{2n}$
→ can check the compatibility with the braid relations directly (tedious, but easy).

Lemma: Assume that b is a σ_1-positive braid. Then the first entry in the sequence

$$(0, 1, 0, 1, \dots) * b$$

is positive.

Proof: (straightforward) The action of σ_1 is associated with

$$a_1' = a_1 + b_1^+ + (b_2^+ - c)^+$$

for some c, so $a_1 \leq a_1'$ always, and $a_1 < a_1'$ for $b_1 > 0$, which is the initial case. $\square$

- characterization:

Proposition: *The braid b is σ-positive (resp. σ_1-positive) iff the first non-zero entry of odd index (resp.the first entry) in $(0, 1, \dots, 0, 1) * b$ is positive.*

- proof of Property A: straightforward;

- consequence of Property C: Dynnikov's action of B_n on $\mathbb{Z}^{2n}$ is faithful.

• **Application**: New solution to the isotopy problem of braids (from the standard presentation):
- starting from a braid word w, compute the sequence of integers $(0, 1, 0, 1, \dots) * w$ using Dynnikov's formulas.

Proposition: *The word problem of B_∞ and the linear order $<_L$ on B_∞ are decidable in* **quadratic** *time: deciding for w requires at most $C \cdot lg(w)^2$ steps, where C is some constant not depending on the number of strands.*

$\rightarrow$ adding one letter $\sigma_i^{\pm 1}$ increases each integer by at most one digit (only additions, **no** multiplication).

References

P. Dehornoy; Braids and Self-Distributivity; Progress in Math. vol. 192, Birkhäuser, (2000)

R. Fenn, M.T. Greene, D. Rolfsen, C. Rourke & B. Wiest; Ordering the braid groups; Pacific J. of Math.; 191; 1999; 49–74.

H. Short & B. Wiest; Ordering the mapping class groups after Thurston; preprint (1999).