FROM SETS TO BRAIDS

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Patrick Dehornoy

Laboratoire de Mathématiques
Nicolas Oresme, Caen

- A 4-strand braid diagram

- A 4-strand braid diagram = 2D-projection of a 3D-figure

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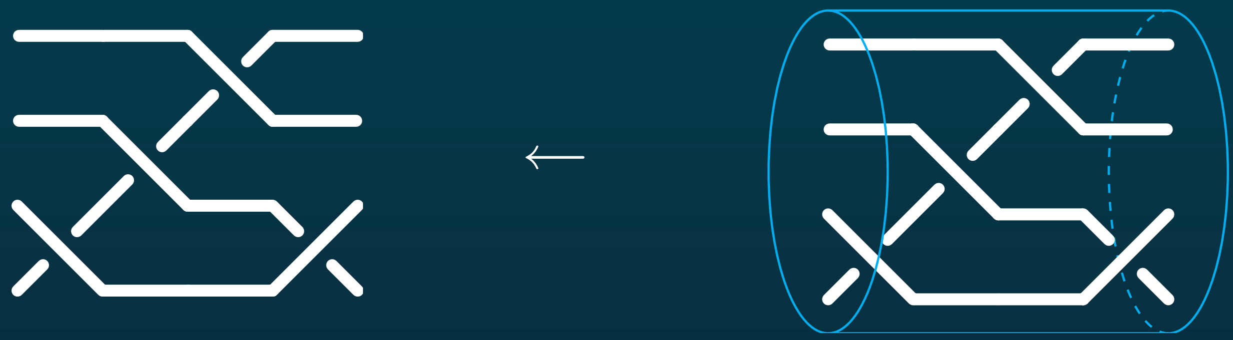

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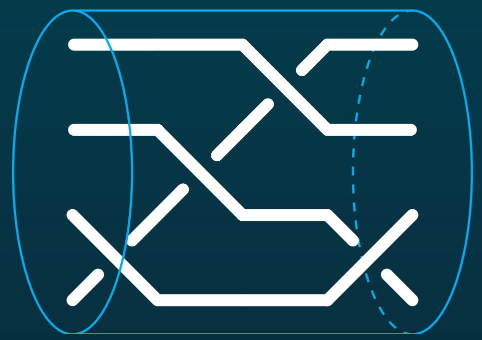

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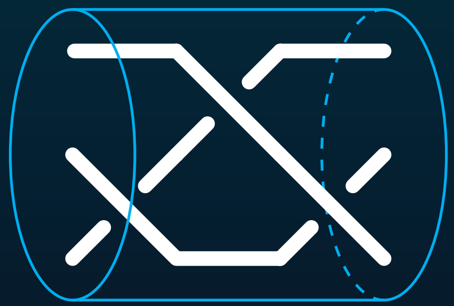

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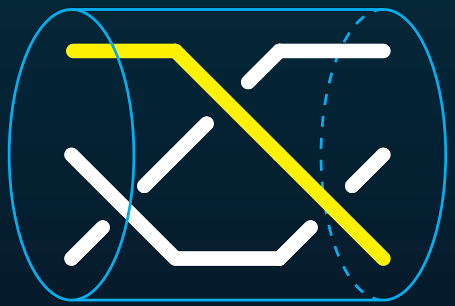

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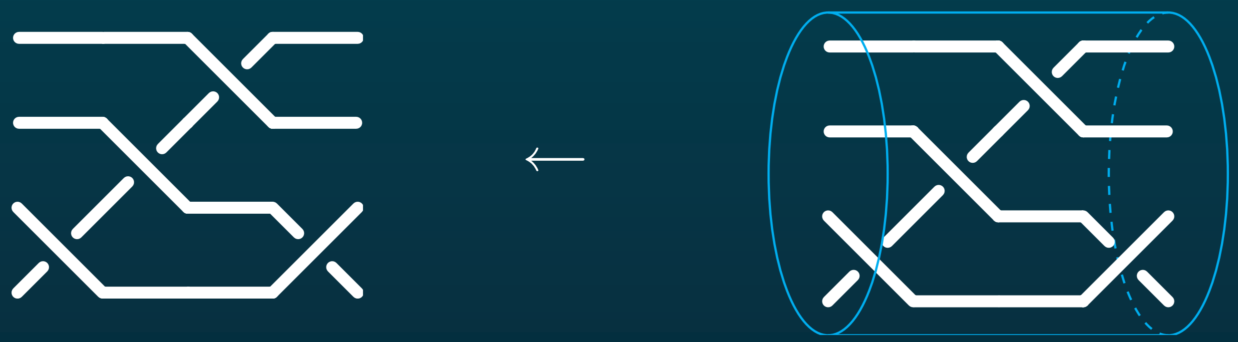

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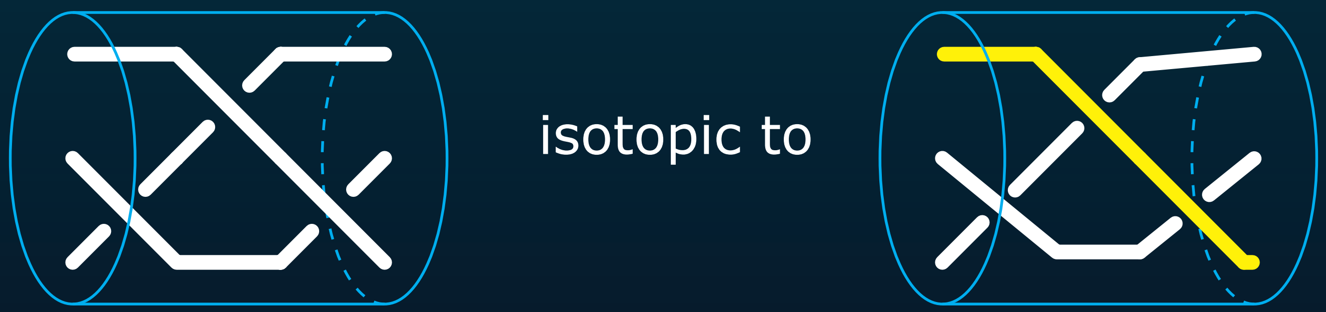

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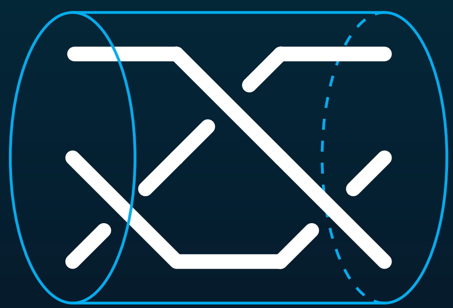

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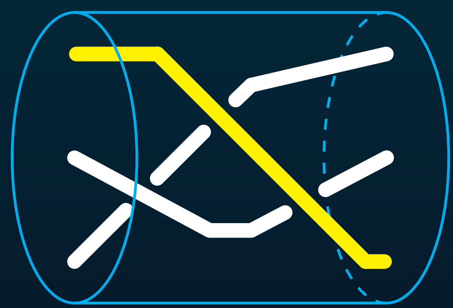

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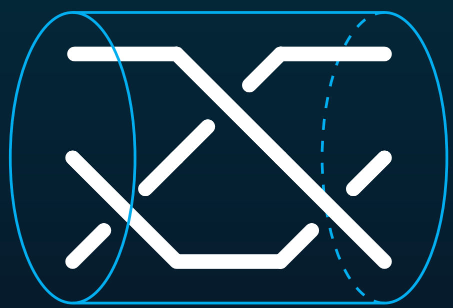

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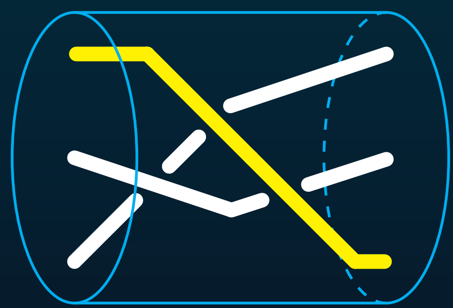

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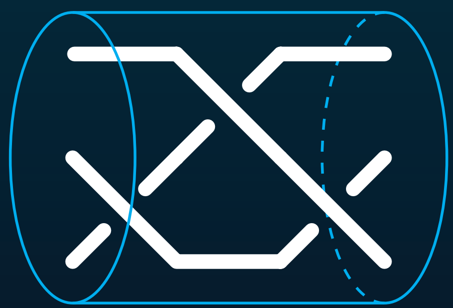

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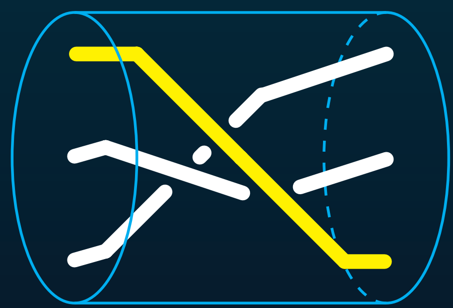

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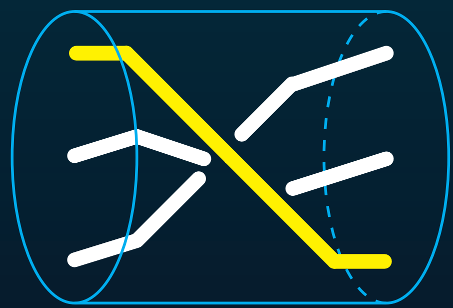

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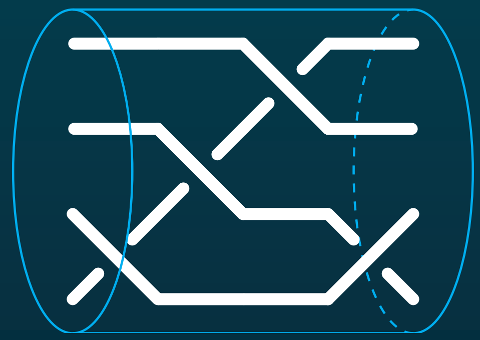

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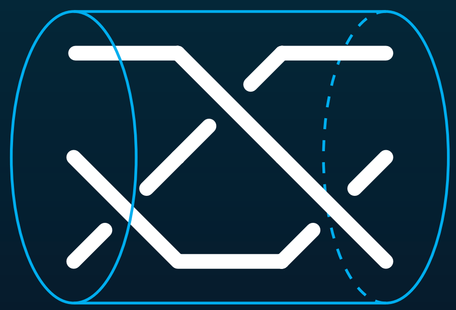

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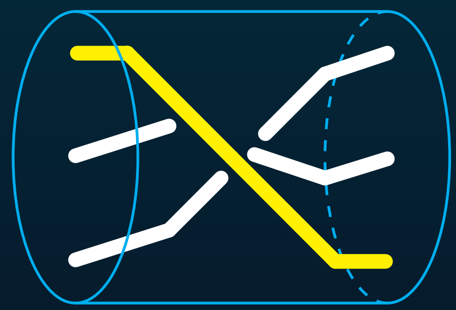

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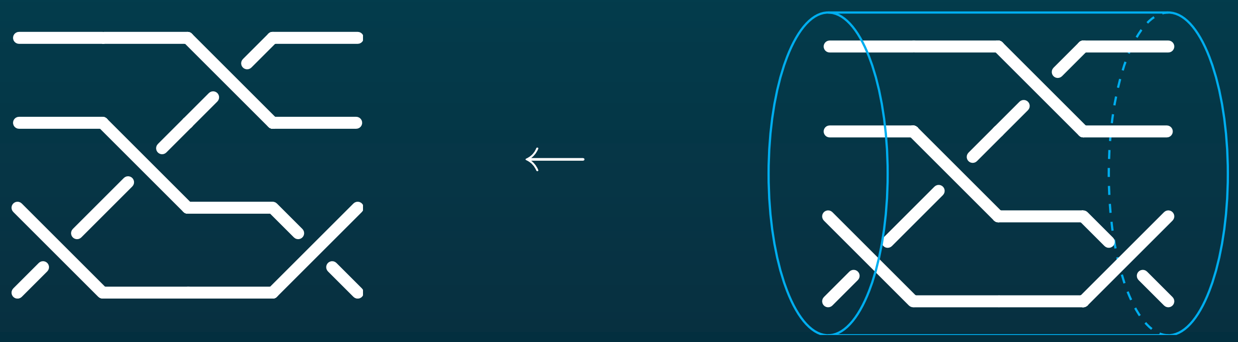

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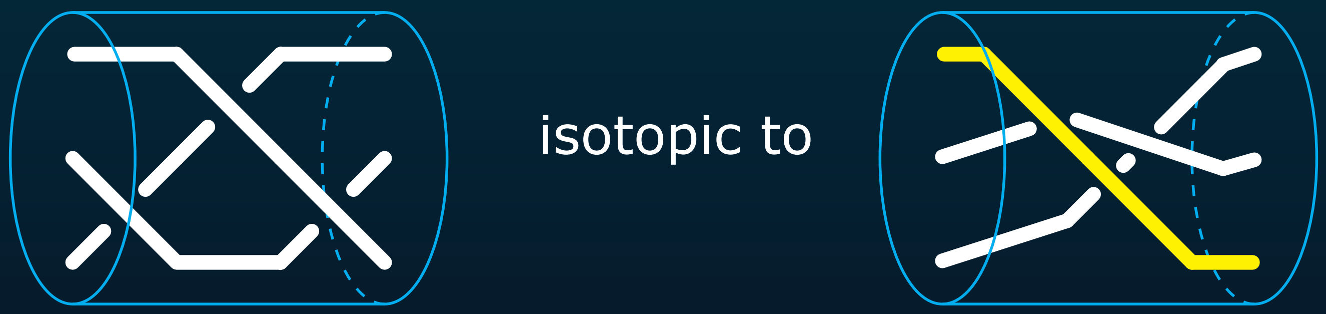

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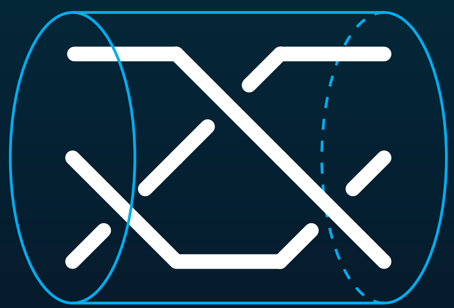
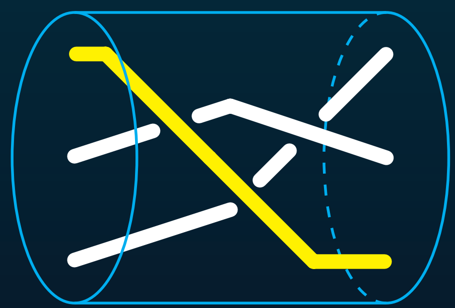
 isotopic to 

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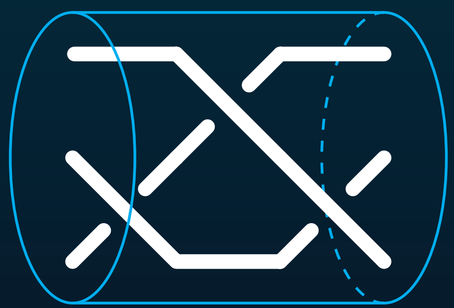

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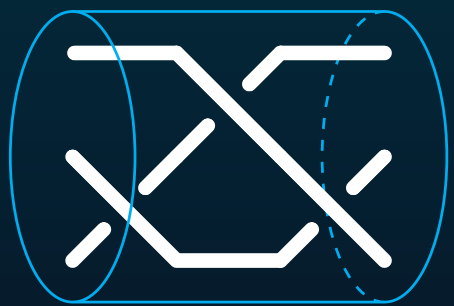

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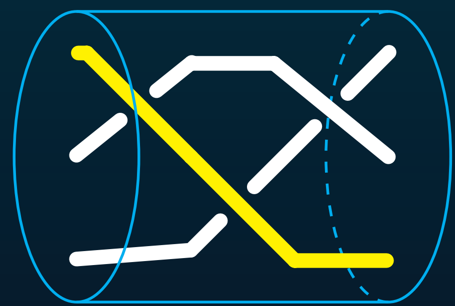

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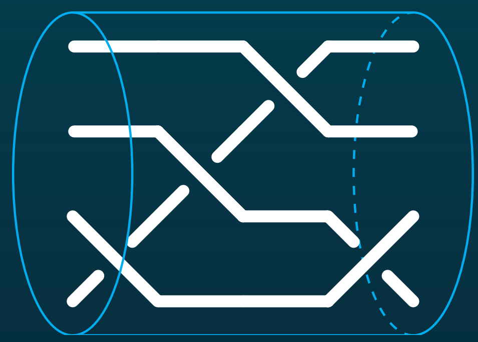

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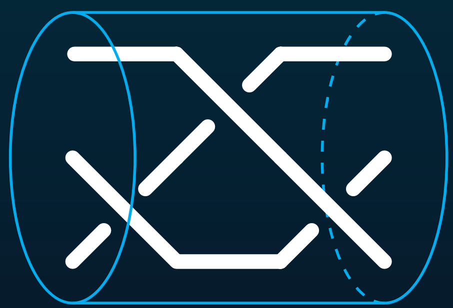

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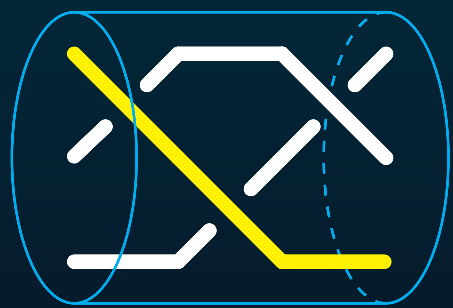

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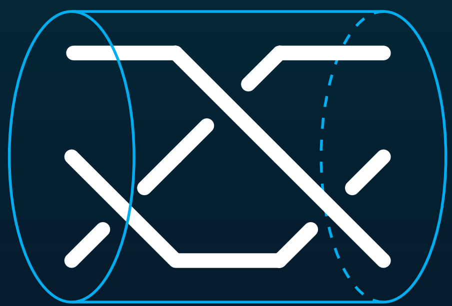

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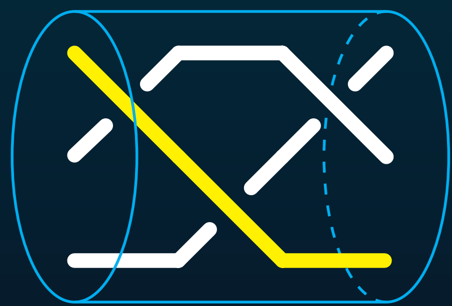

- a braid = an isotopy class $\rightsquigarrow$ represented by 2D-diagram,

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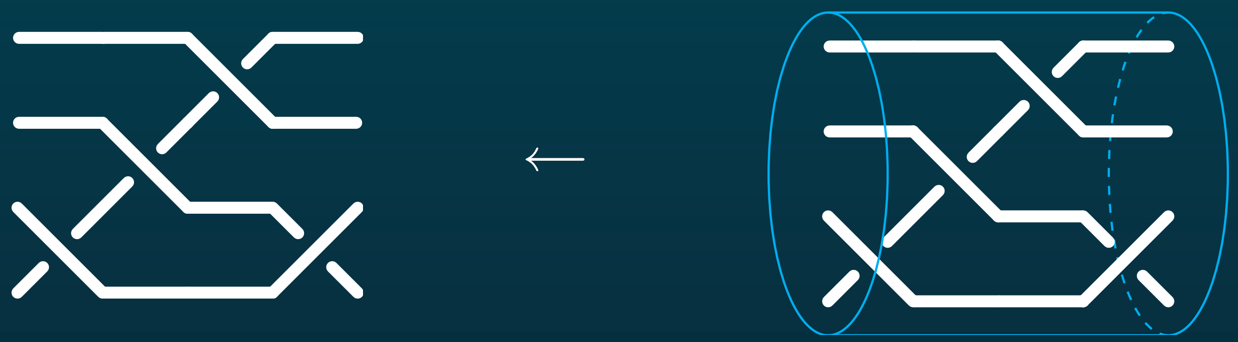

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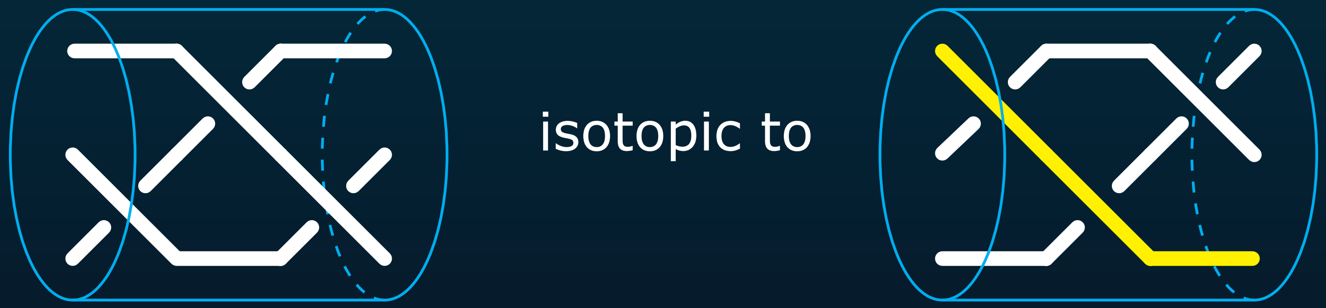

- a braid = an isotopy class $\rightsquigarrow$ represented by 2D-diagram,

  but different 2D-diagrams may give rise to the same braid.

- Product of two braids:

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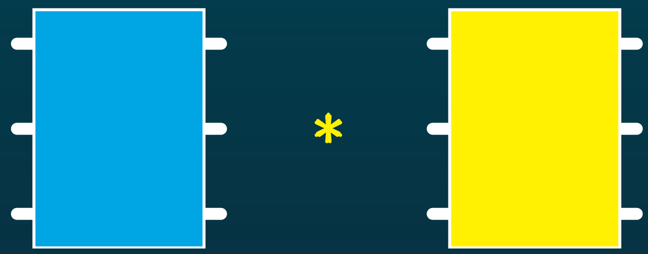

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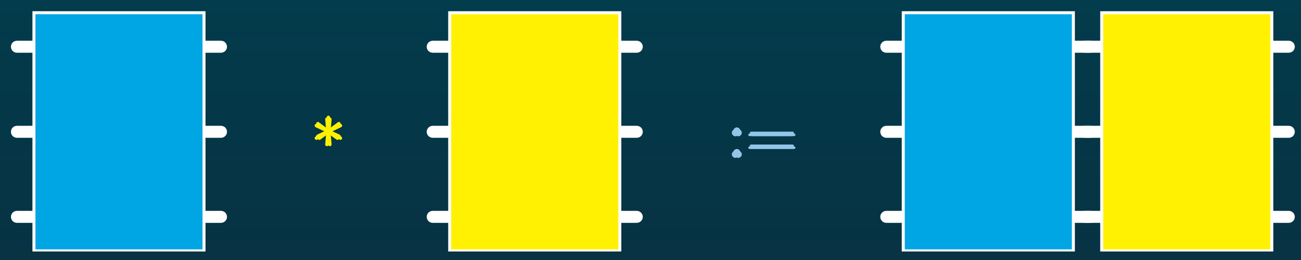

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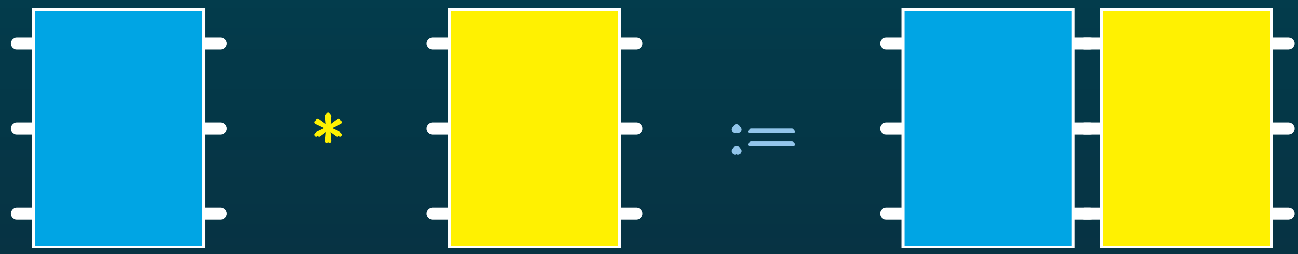

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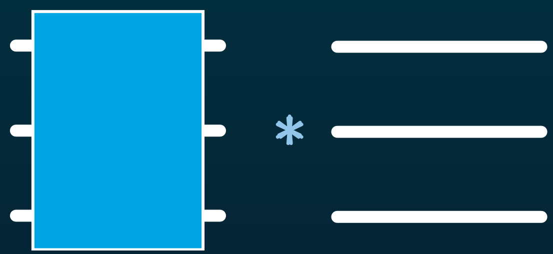

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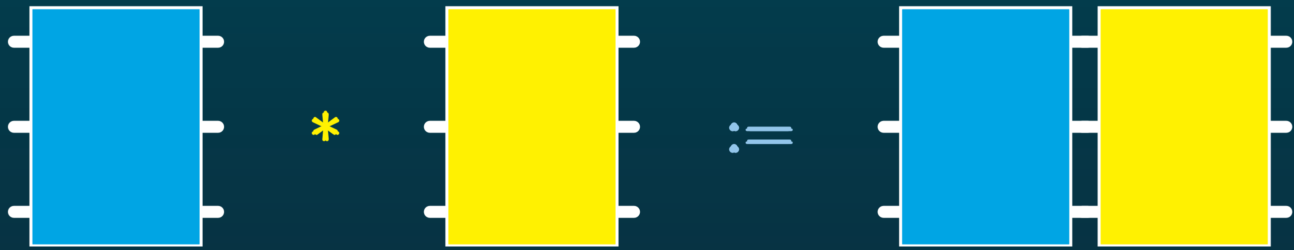

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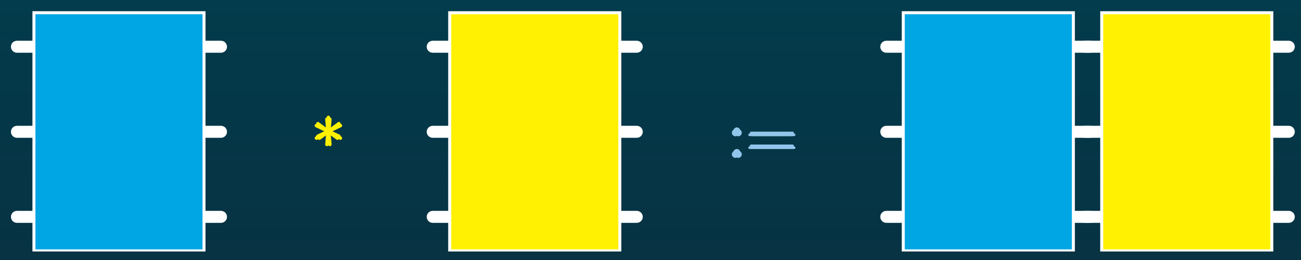

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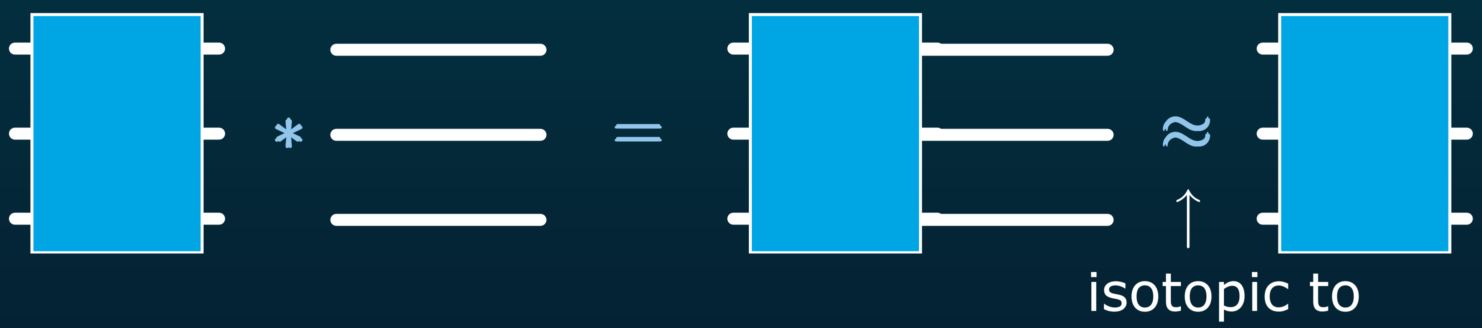

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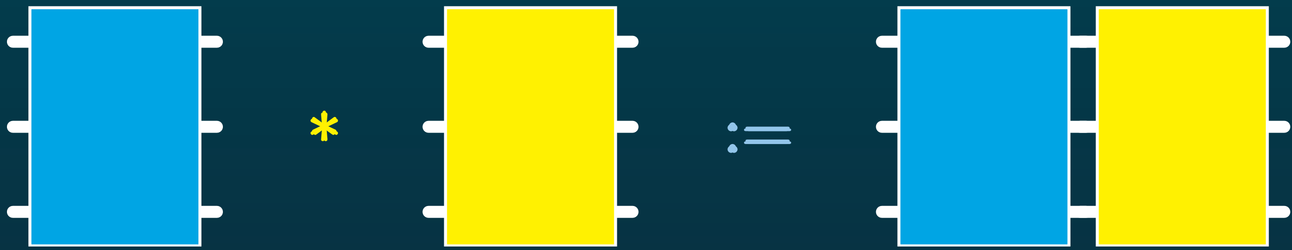

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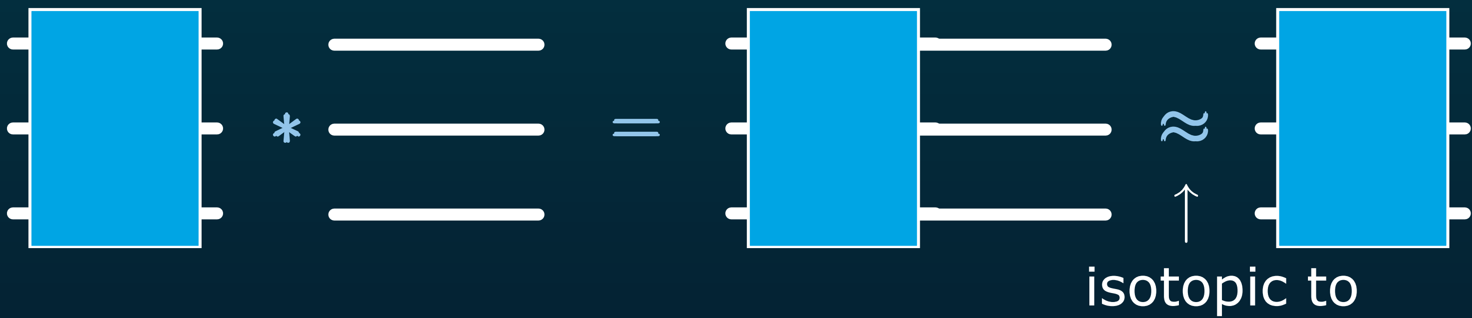

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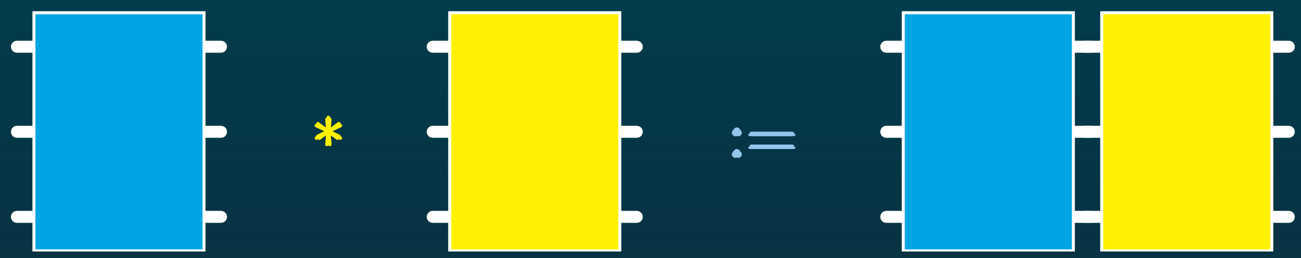

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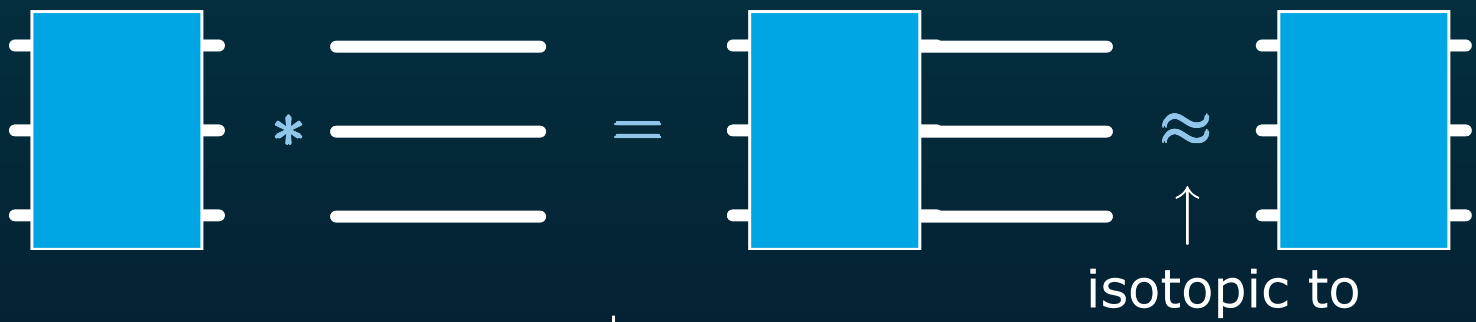

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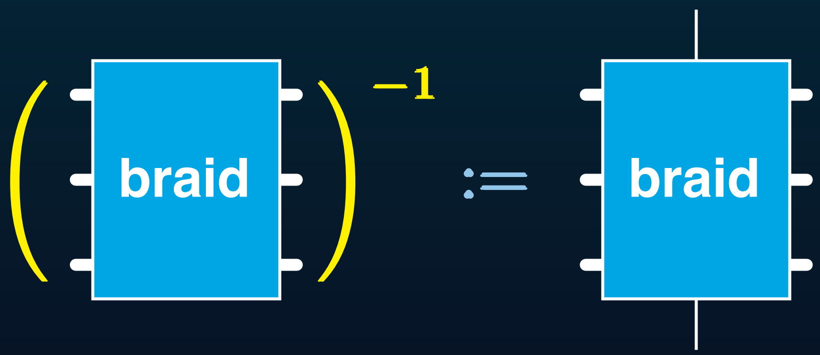

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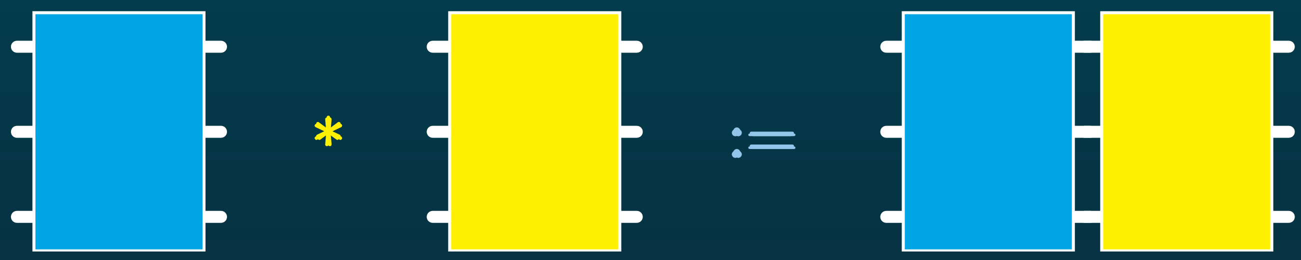

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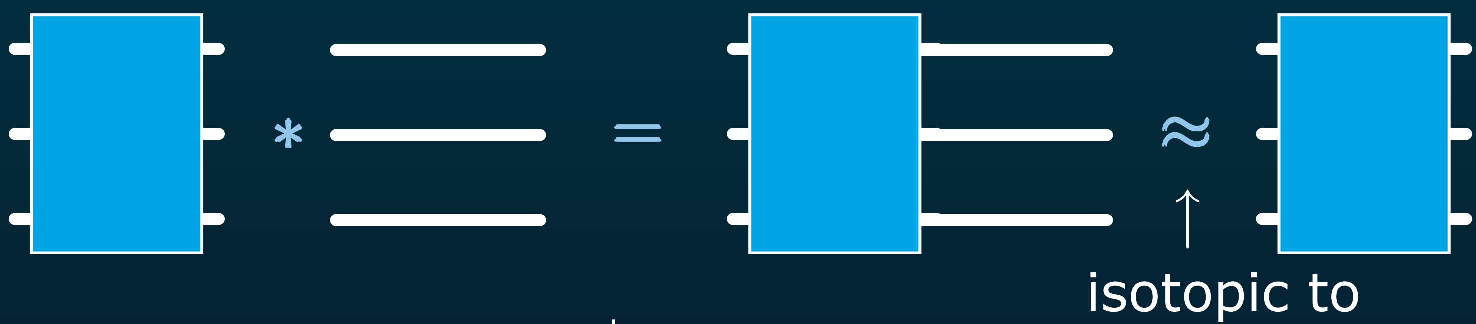

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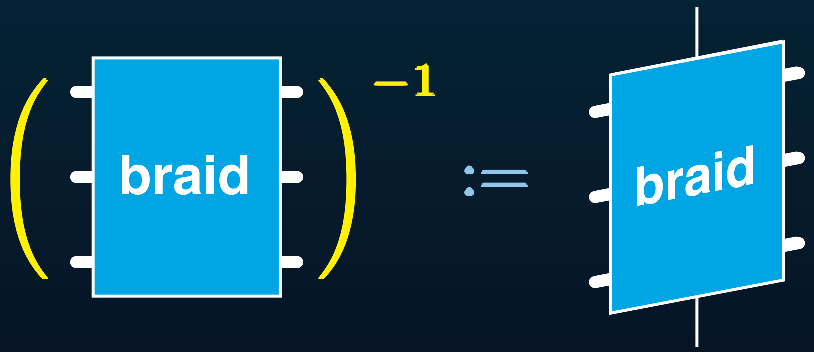

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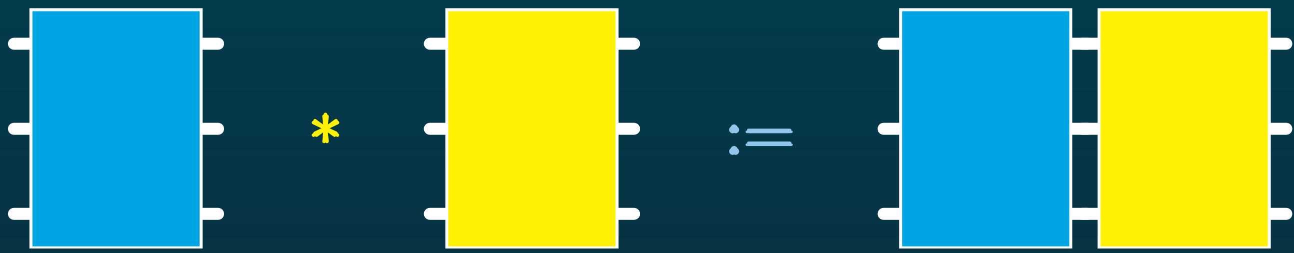

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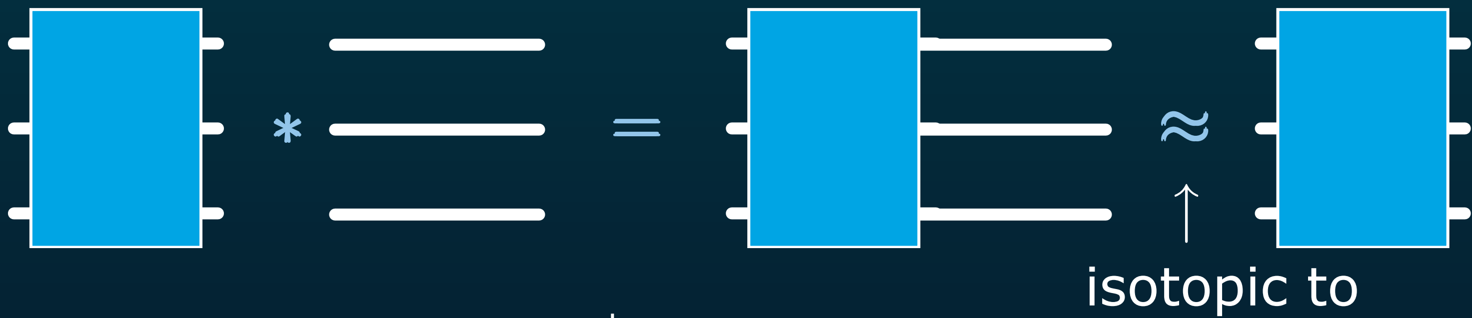

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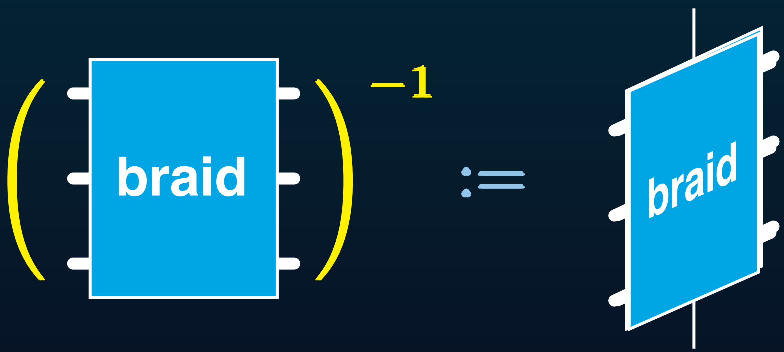

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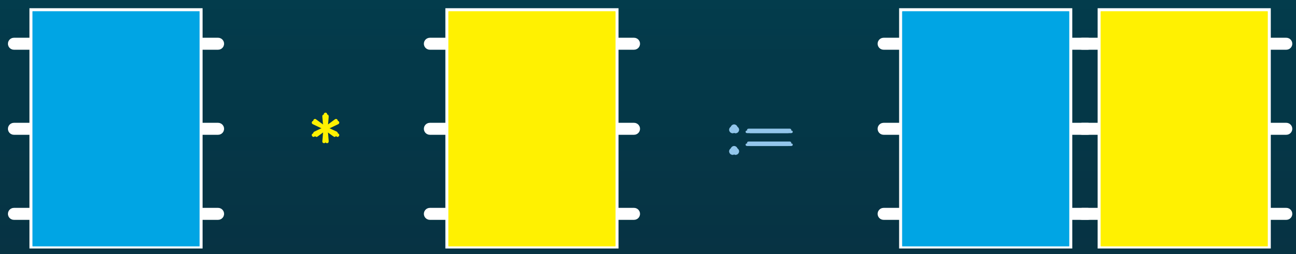

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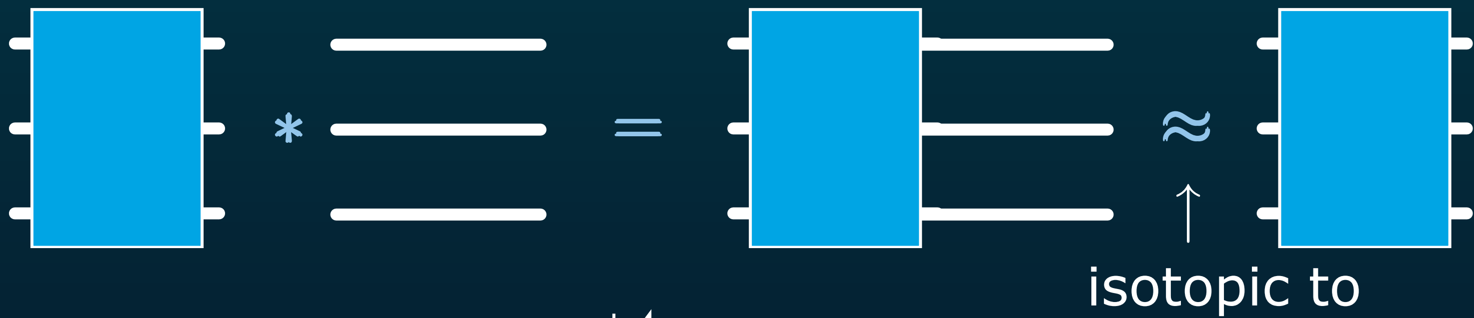

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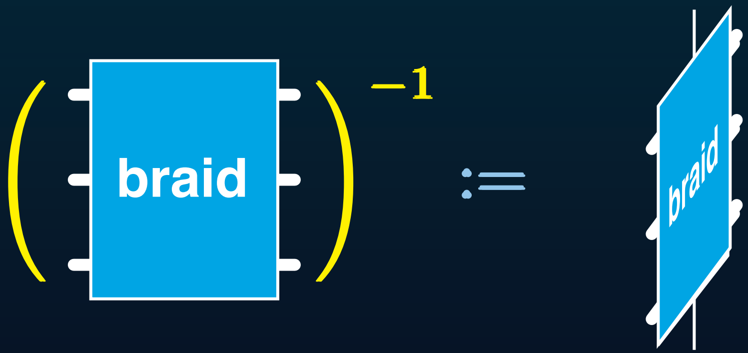

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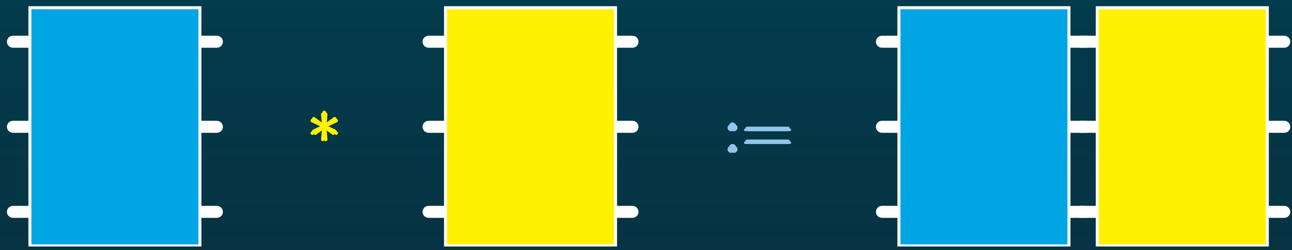

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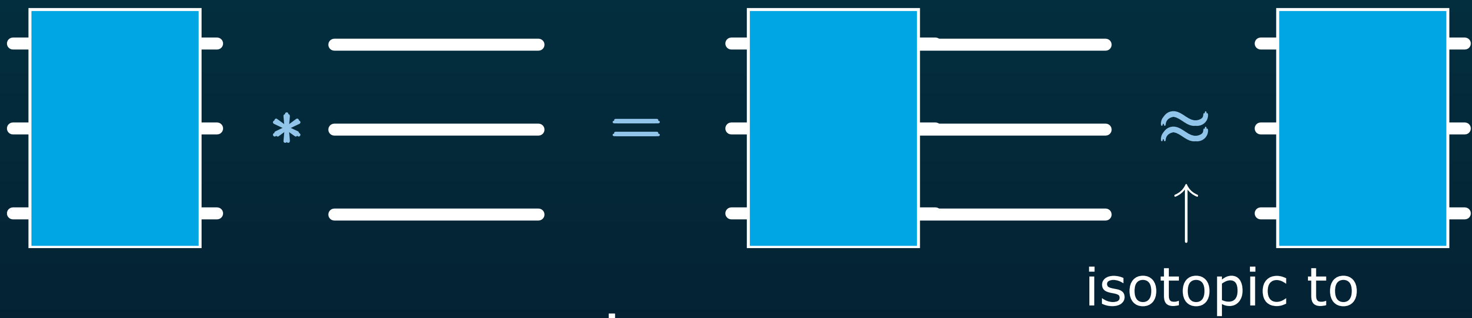

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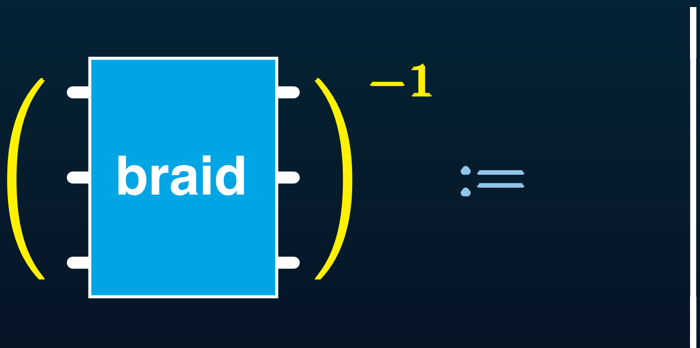

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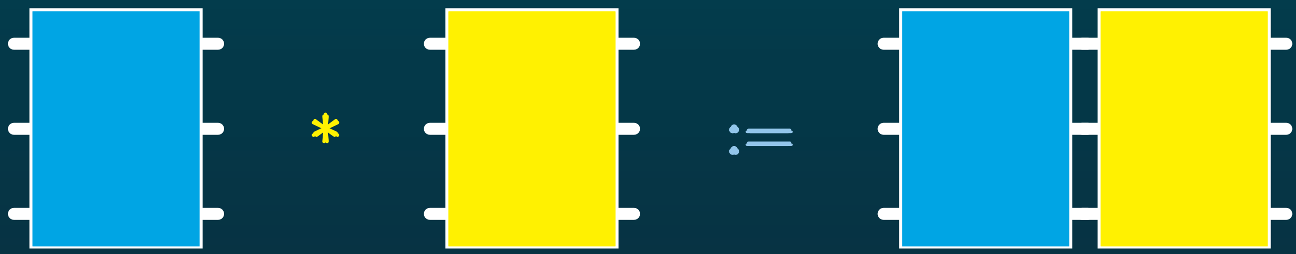

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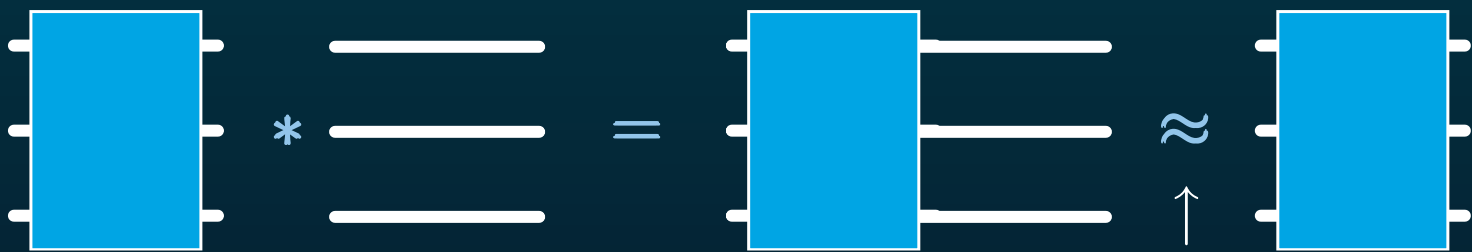

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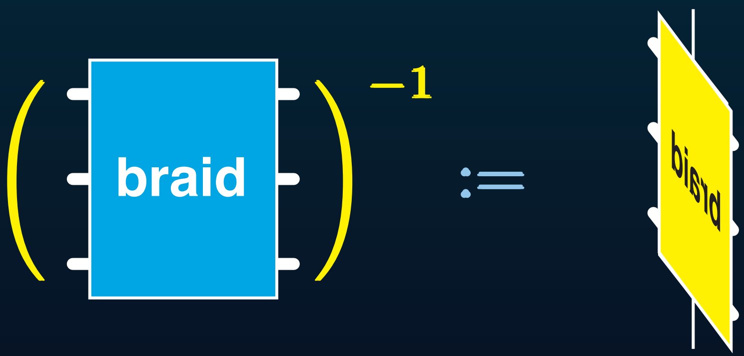

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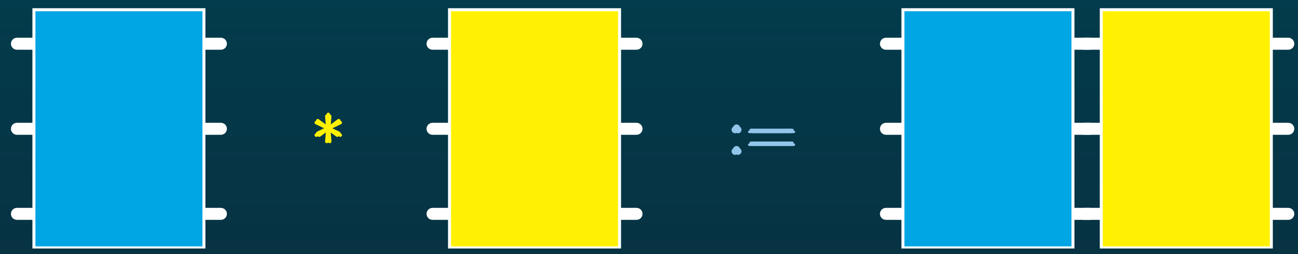

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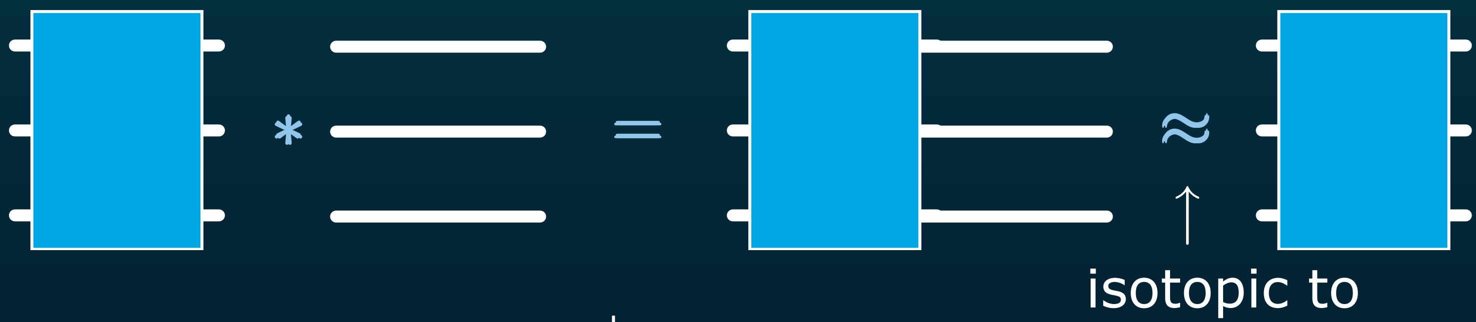

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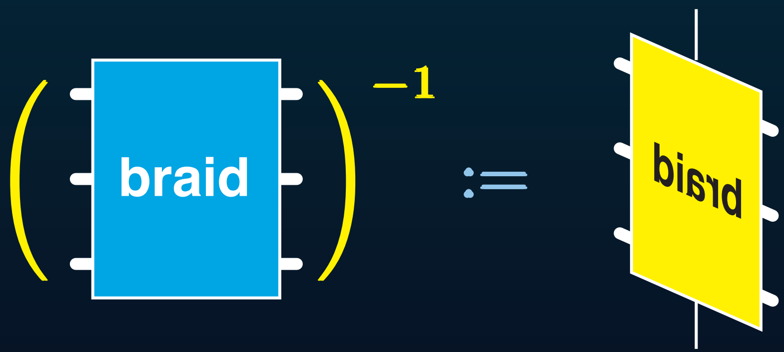

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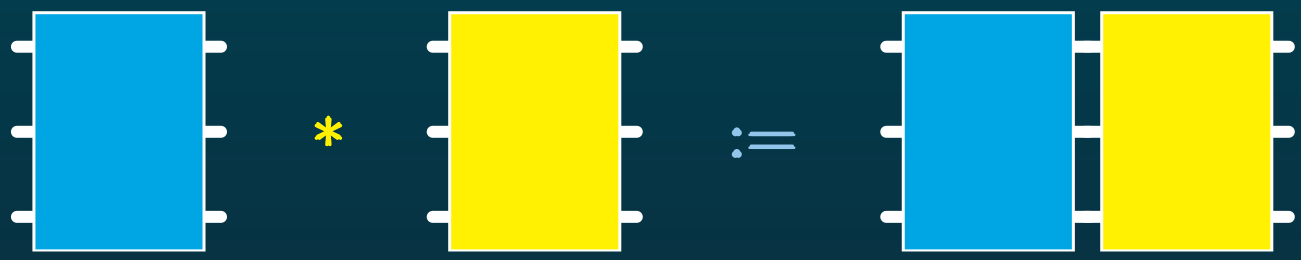

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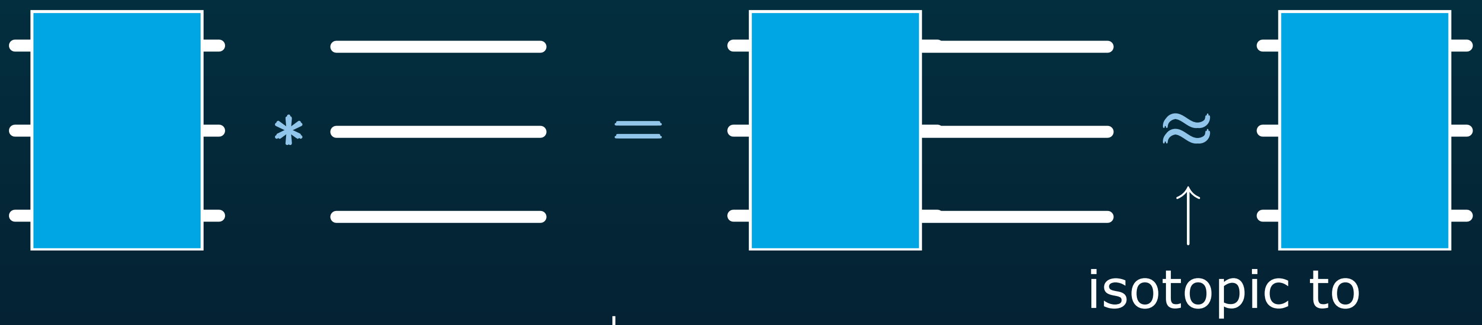

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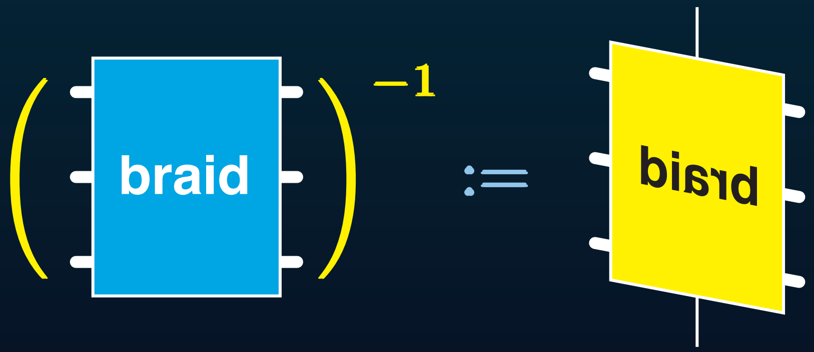

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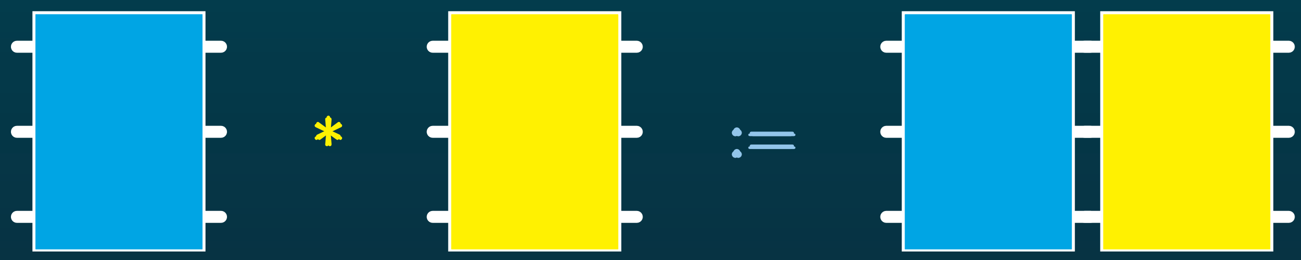

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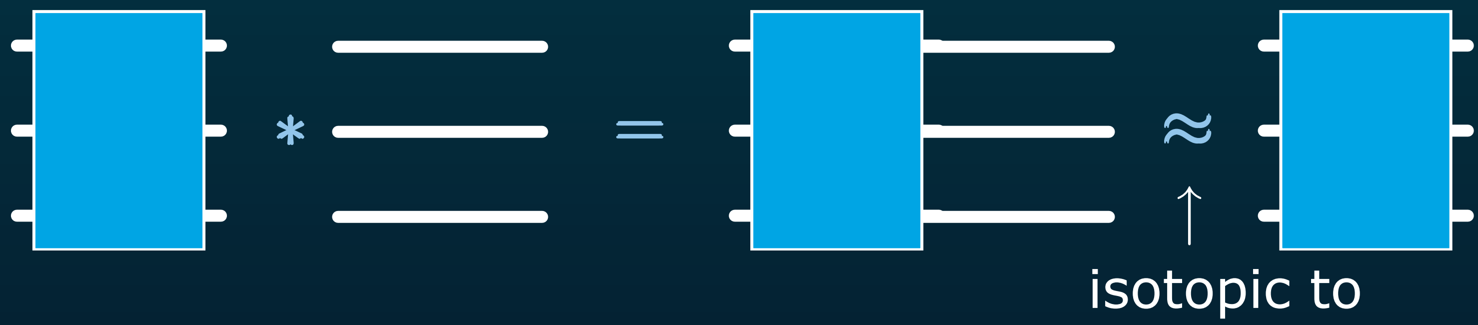

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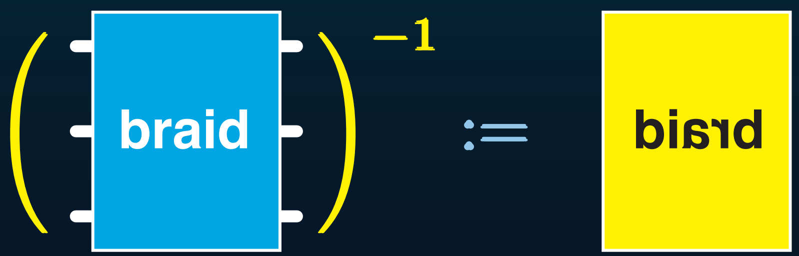

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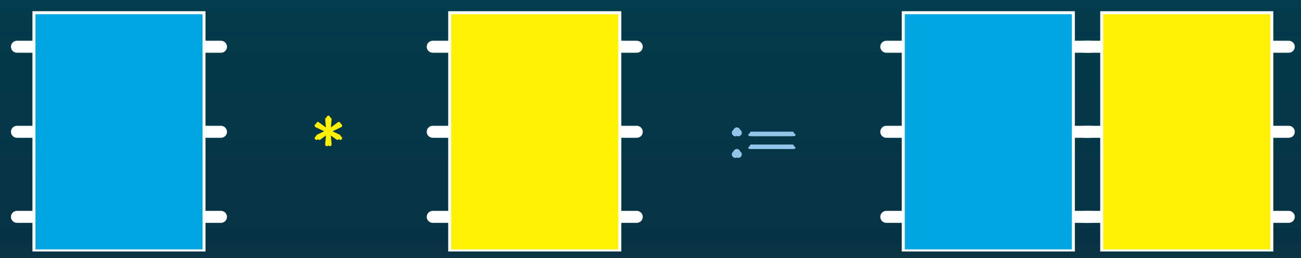

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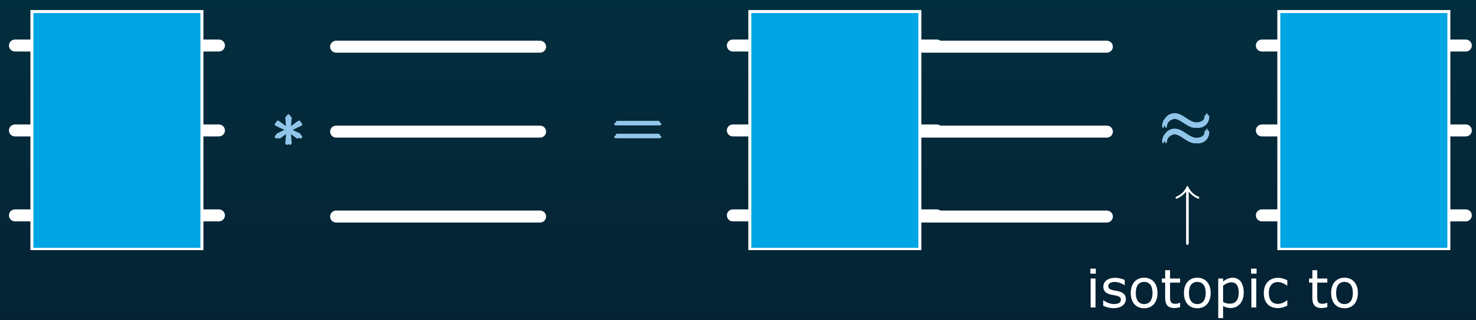
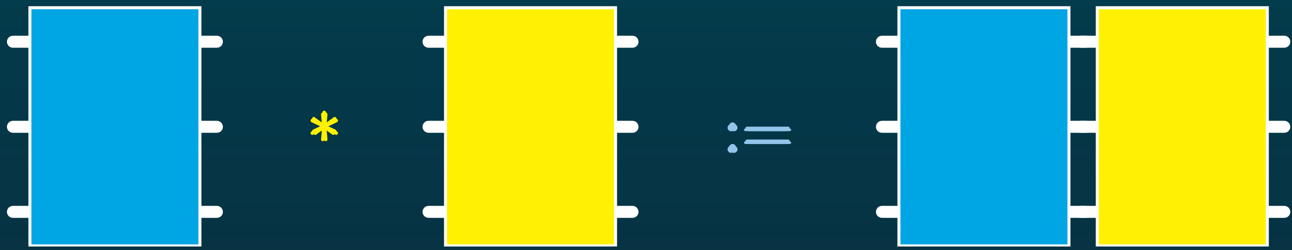


- and

- Product of two braids:



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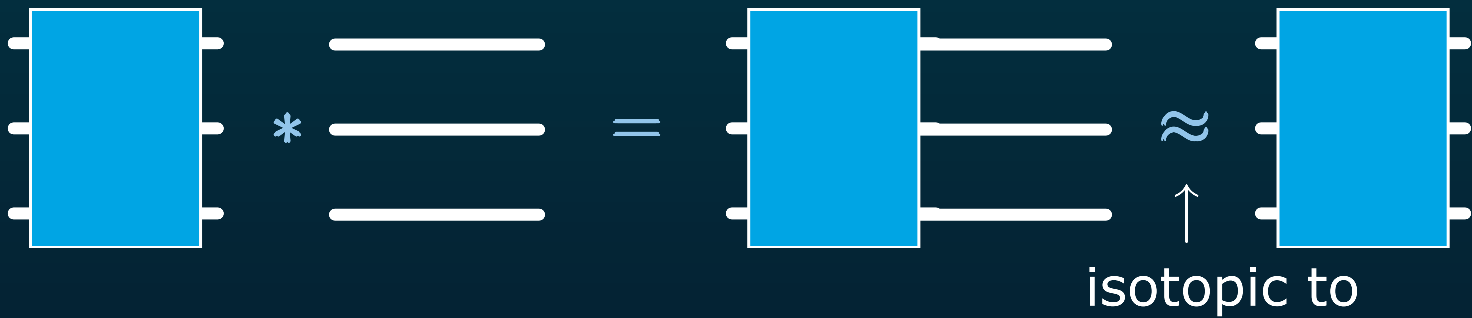

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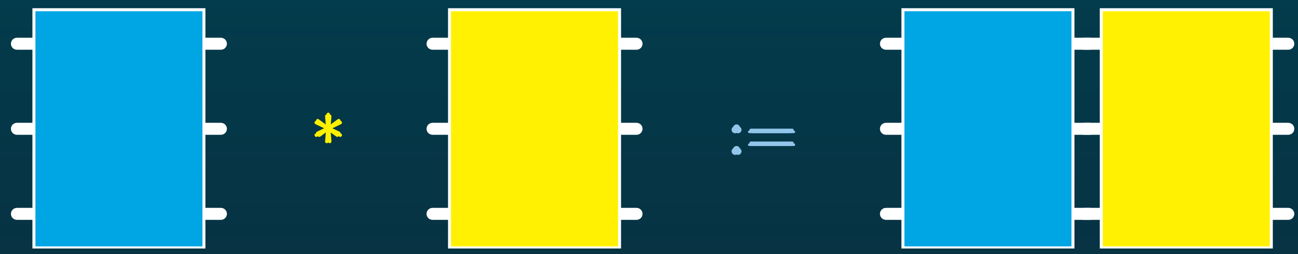

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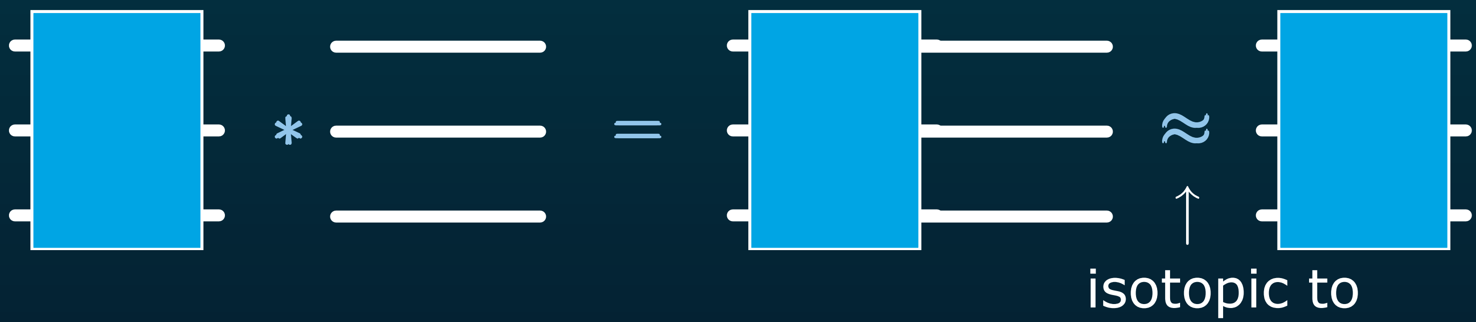

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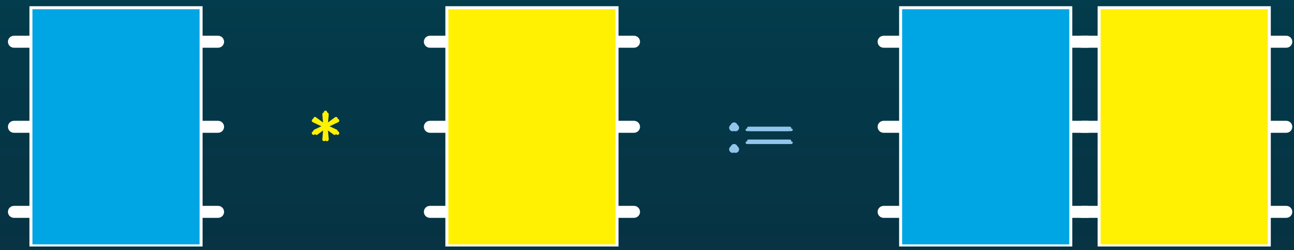

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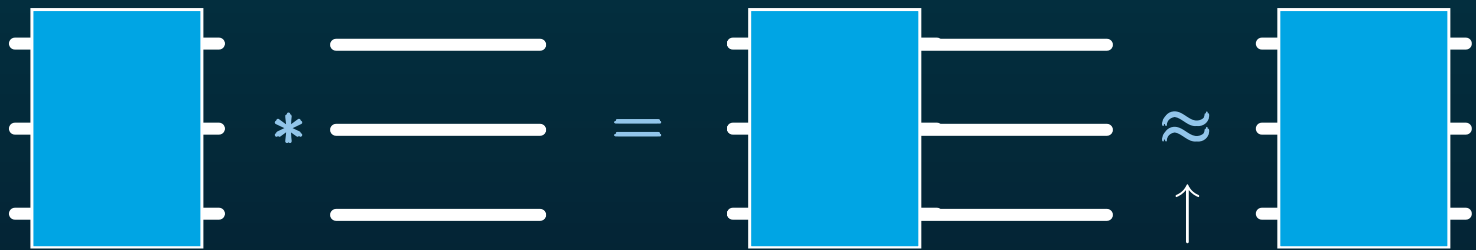
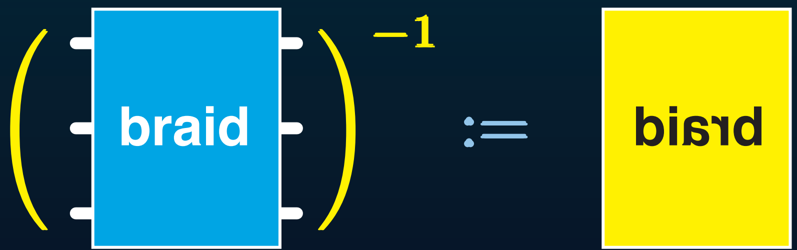


isotopic to

- and



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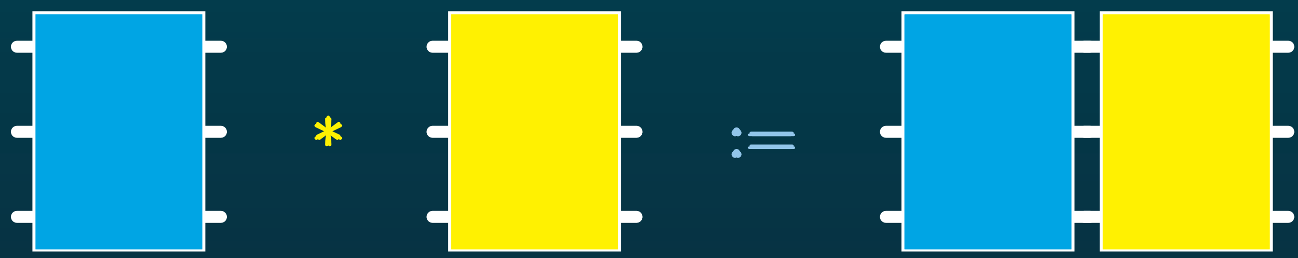

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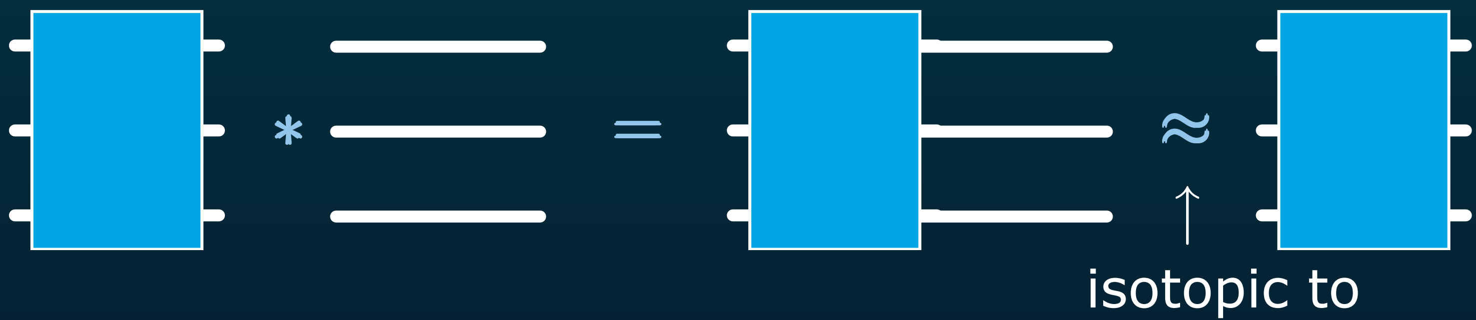

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THE BRAID GROUPS

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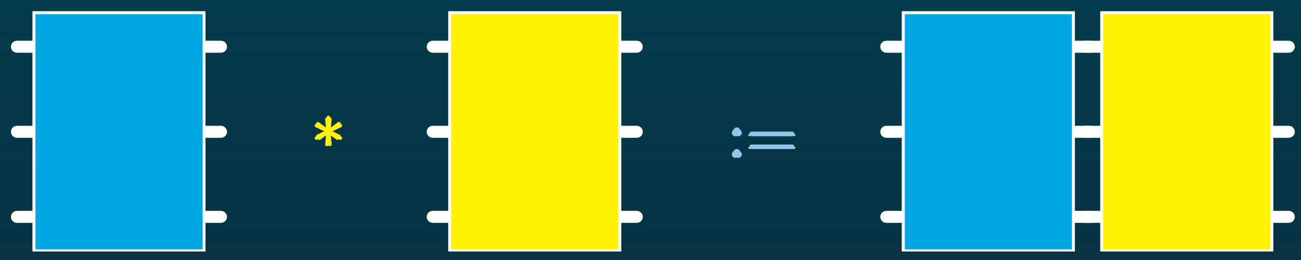

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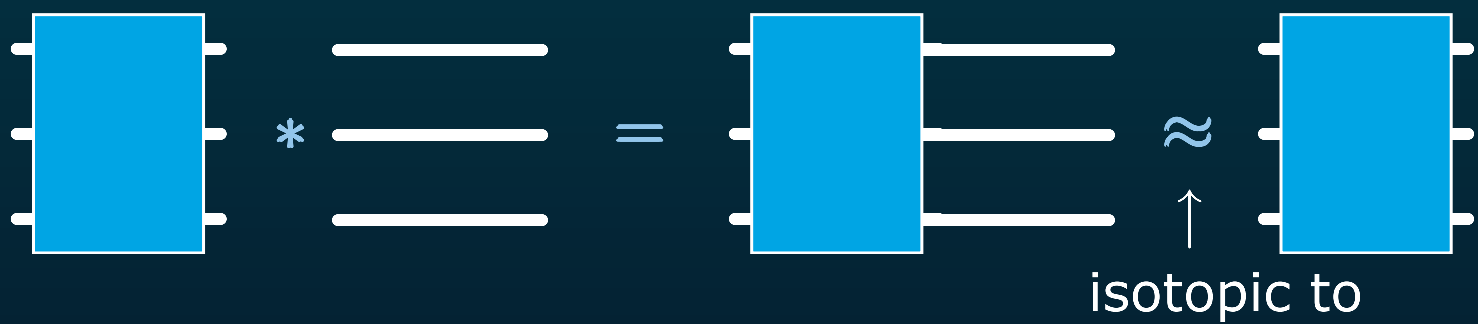

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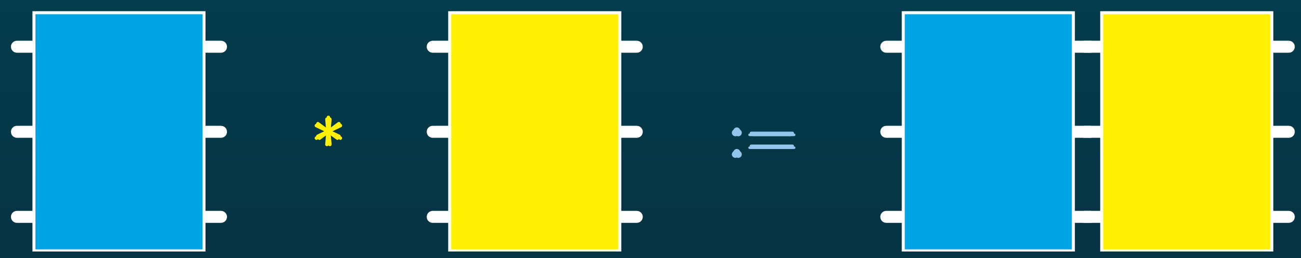

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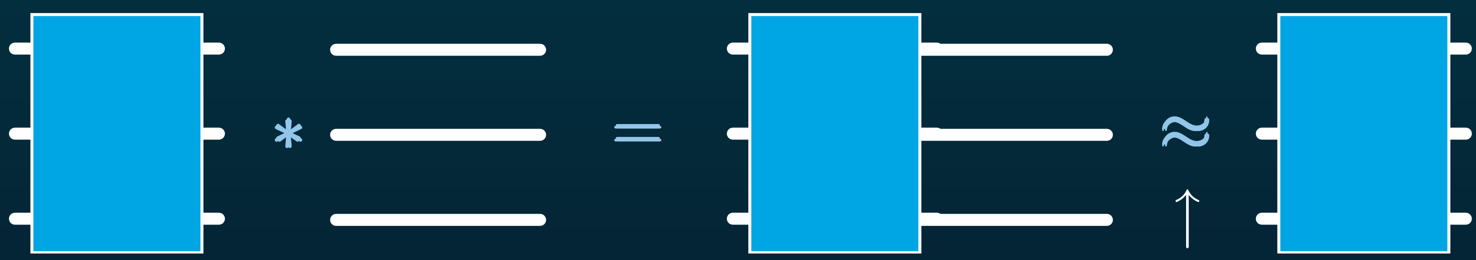

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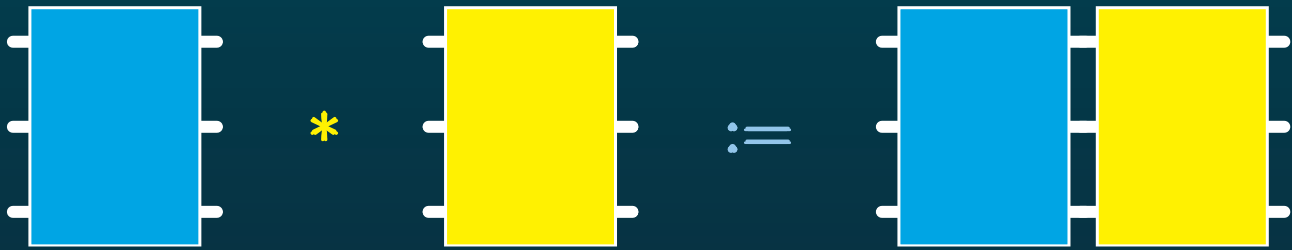

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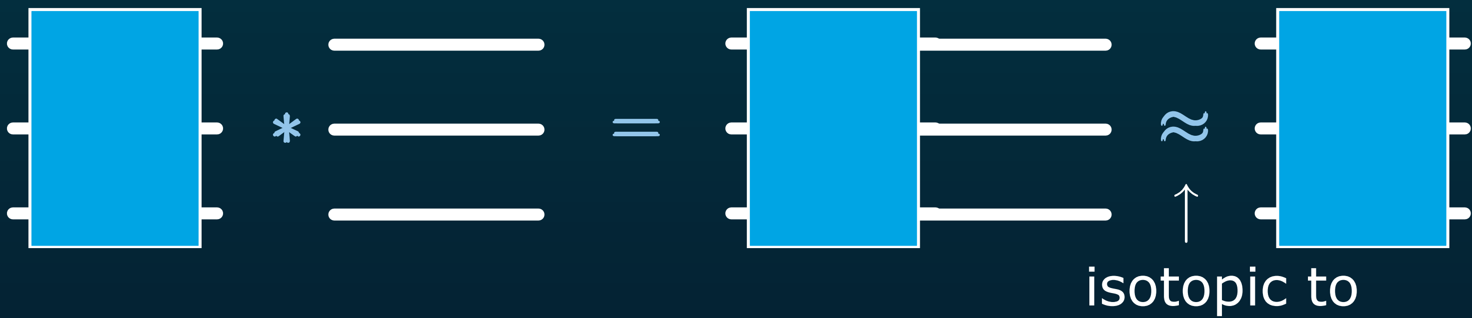

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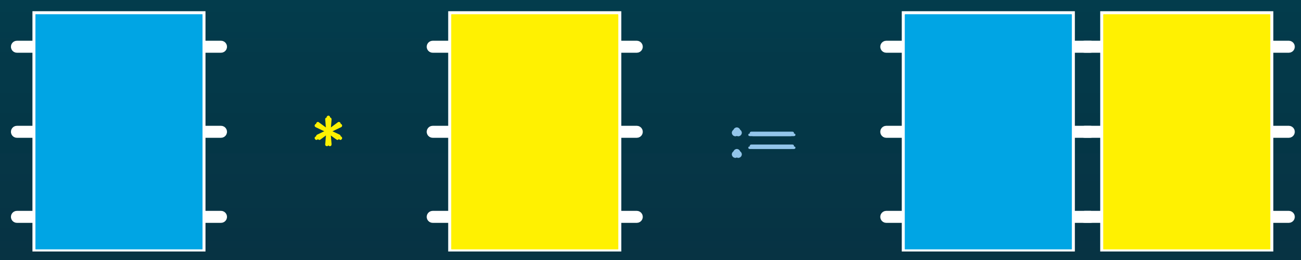

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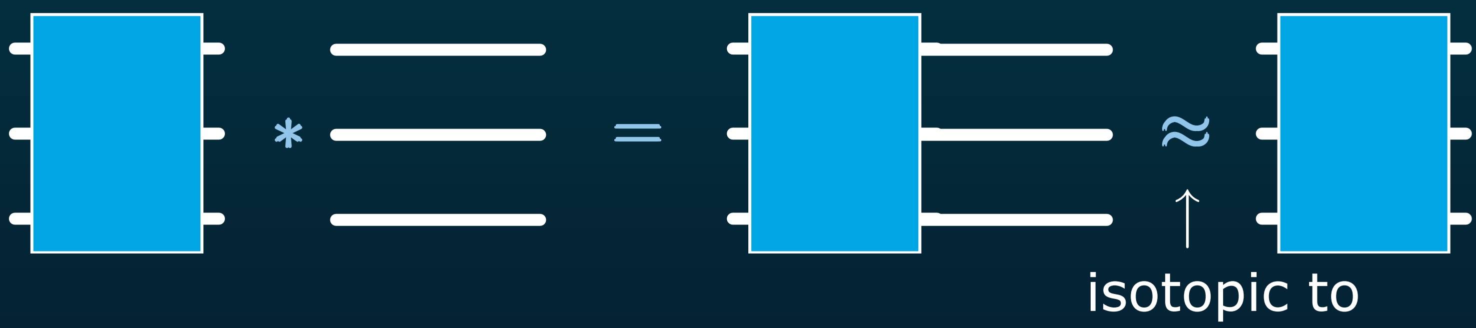

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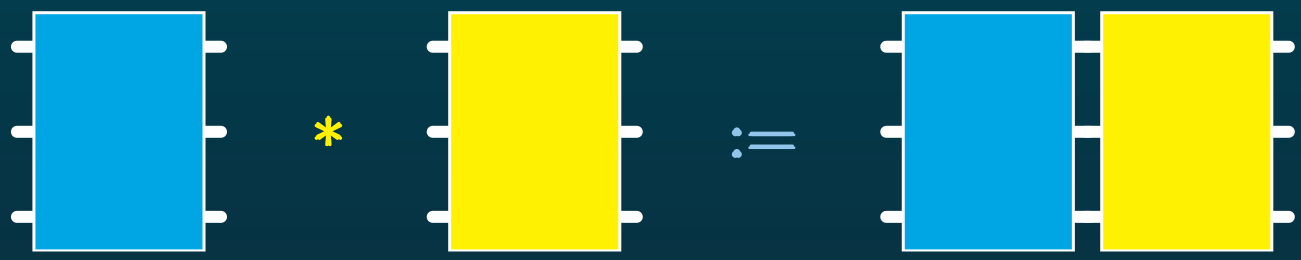

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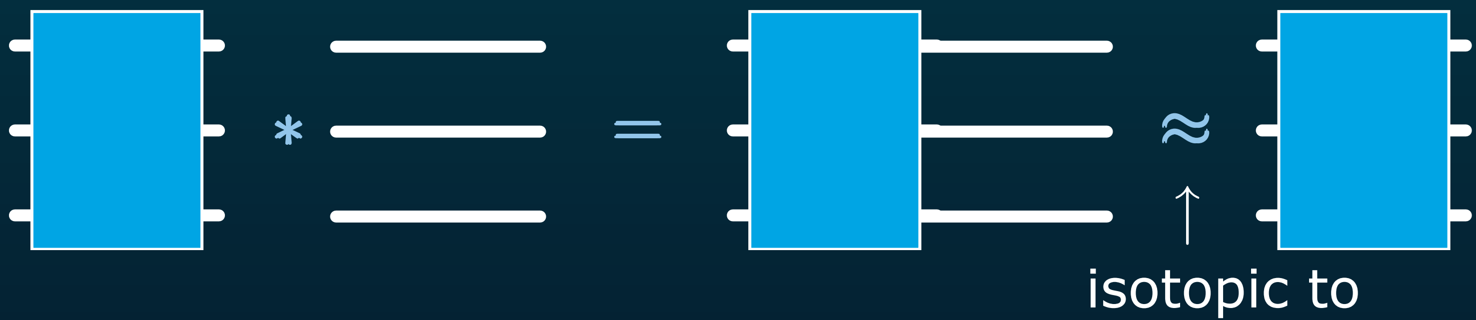

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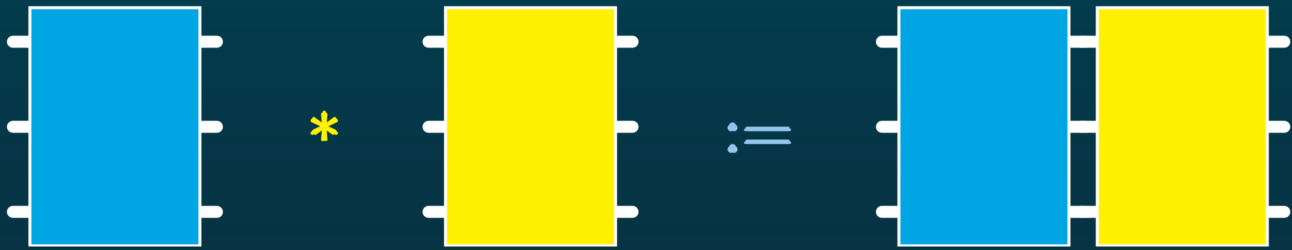

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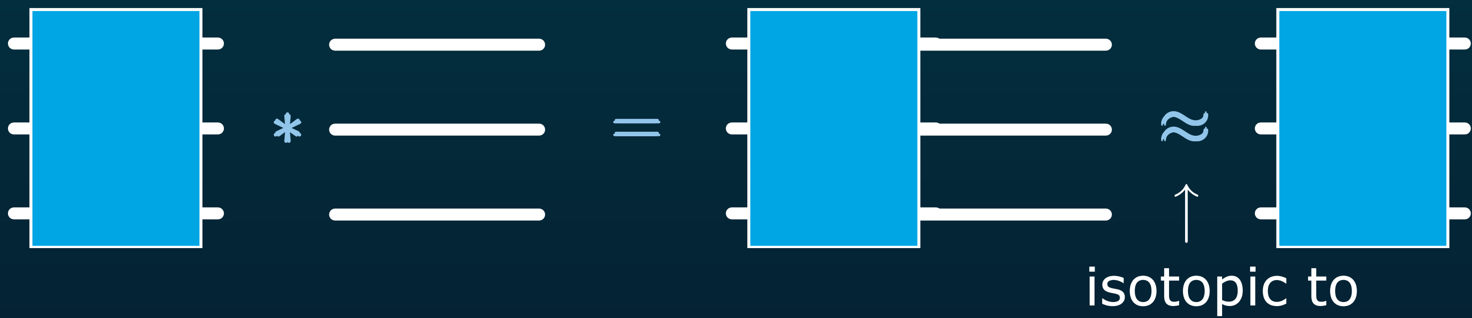

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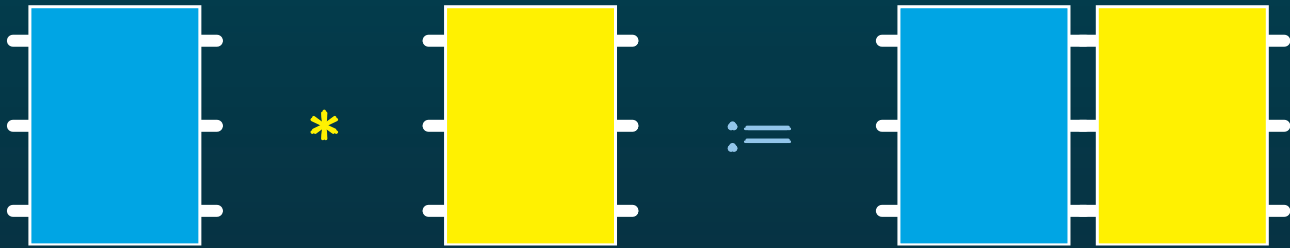

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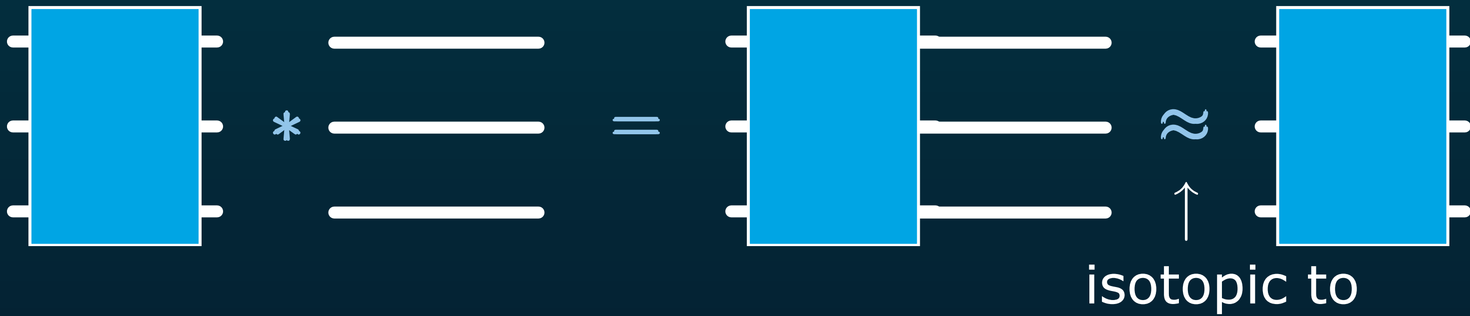
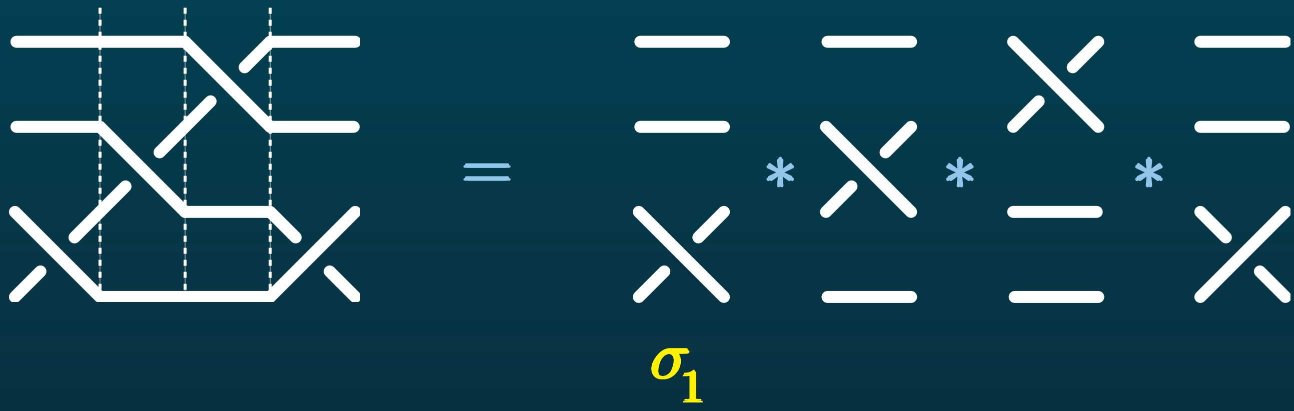
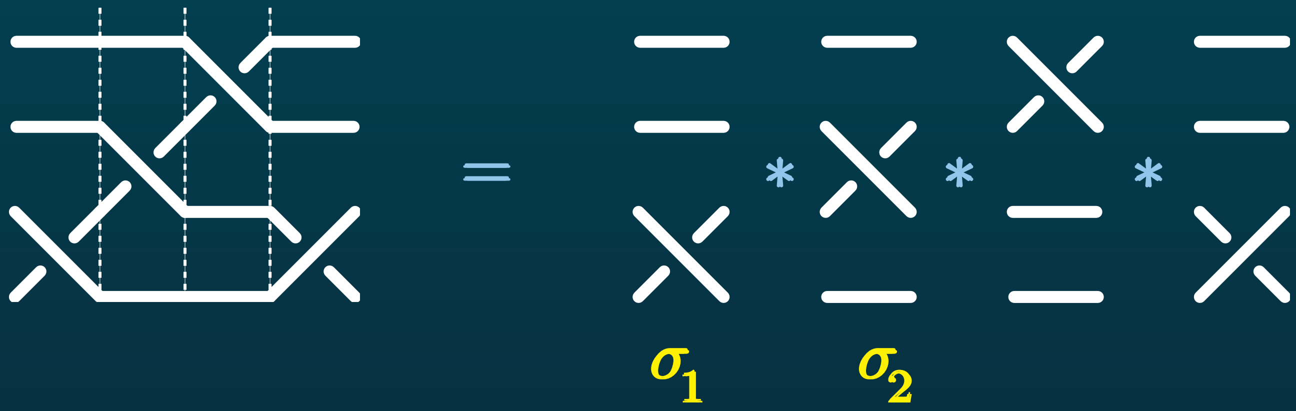


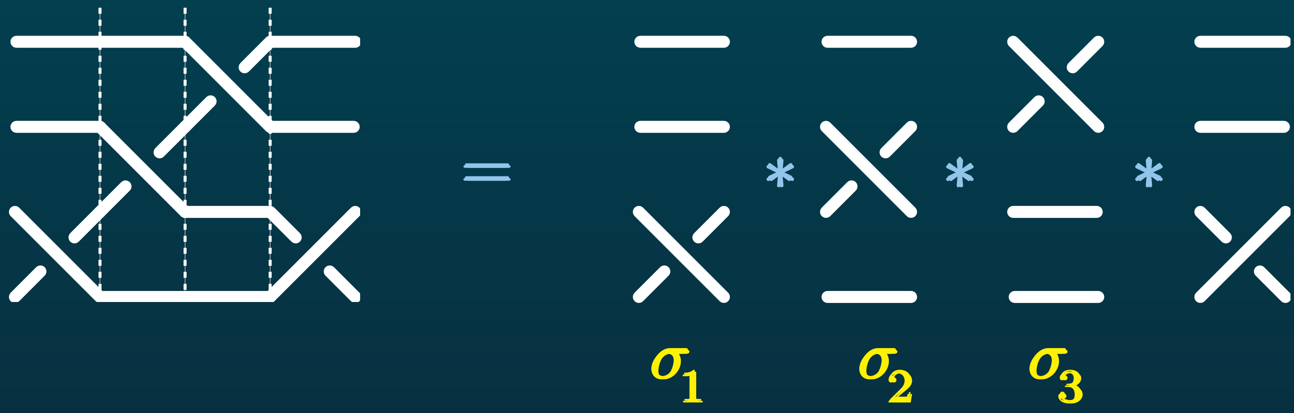
- and

$\rightsquigarrow$ For each n, the group B_n of n strand braids (E. Artin, $\sim$1925).

A PRESENTATION OF B_n




σ_1
σ_2


=
*
*
*
σ_1
σ_2
σ_3

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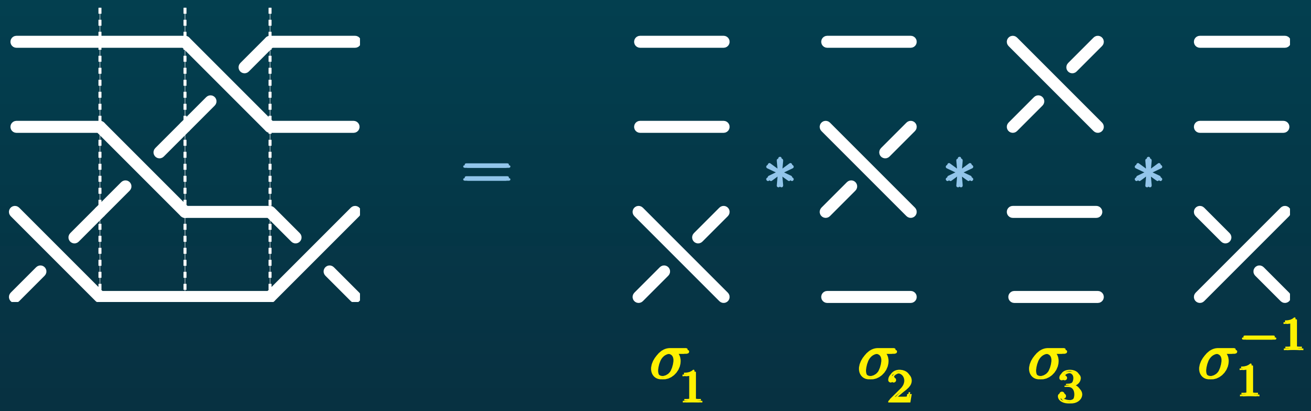

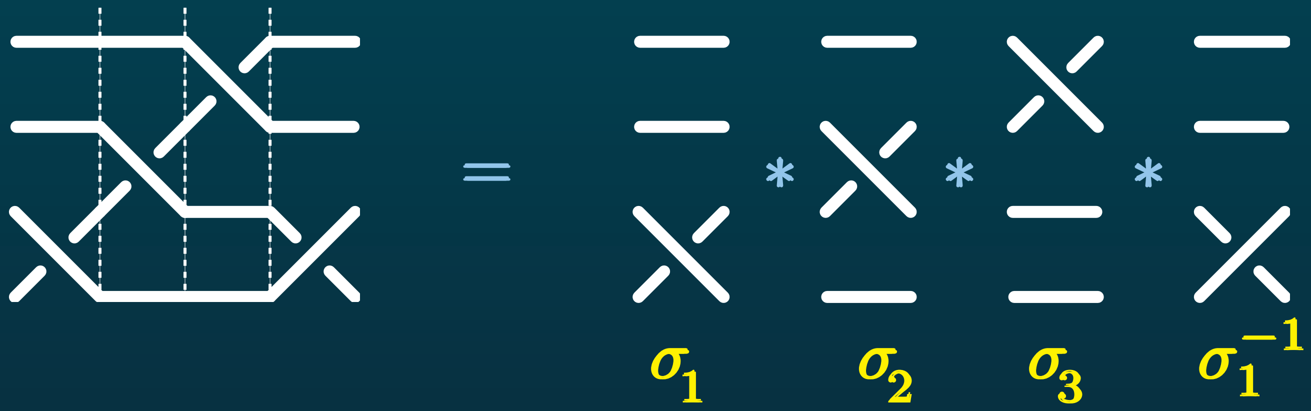

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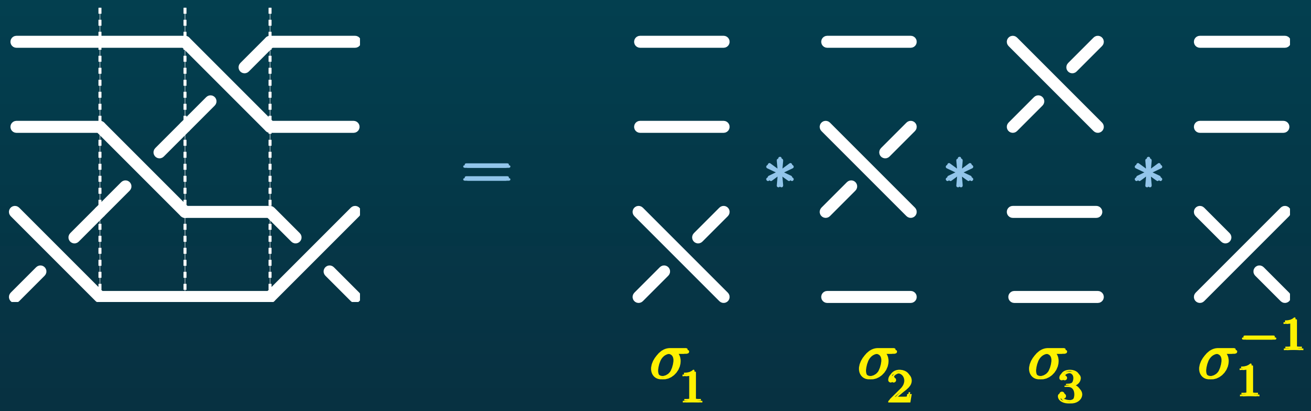

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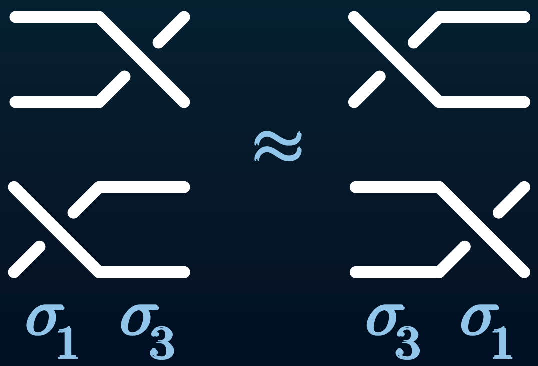

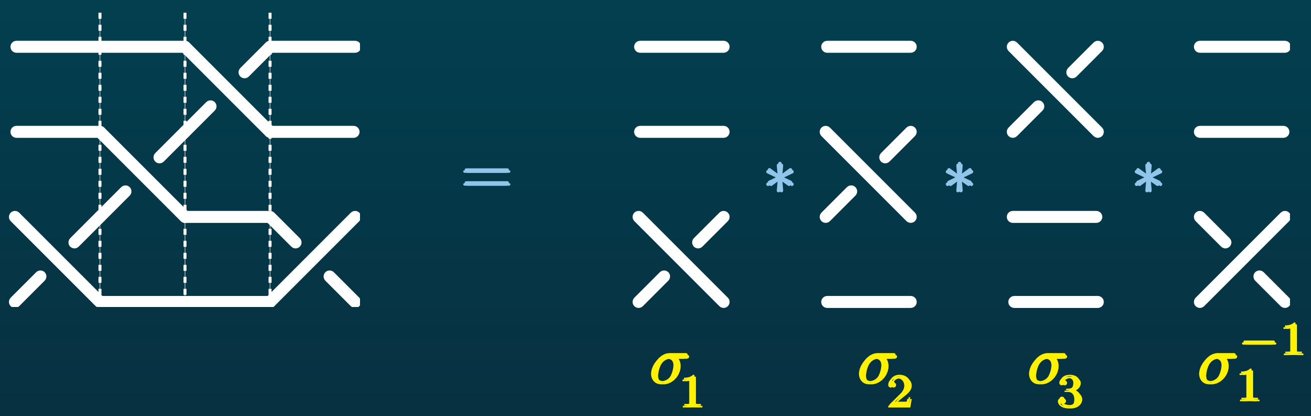

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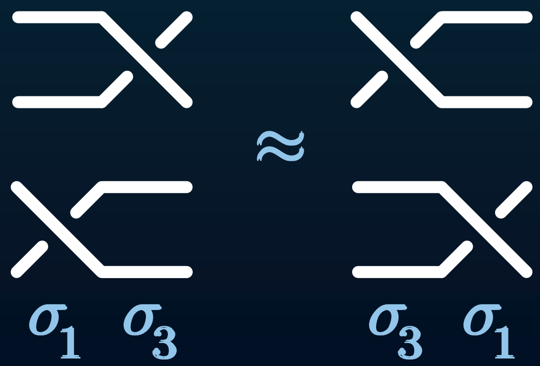

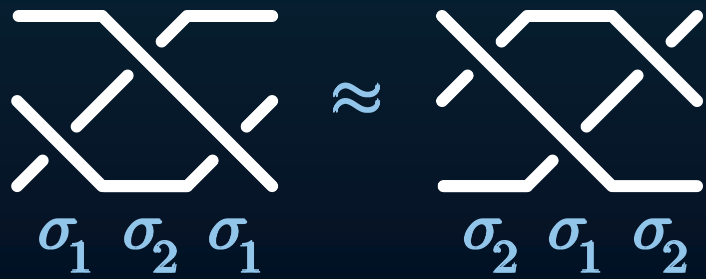

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  $\rightsquigarrow$ Question: Does some reduction work for B_n?

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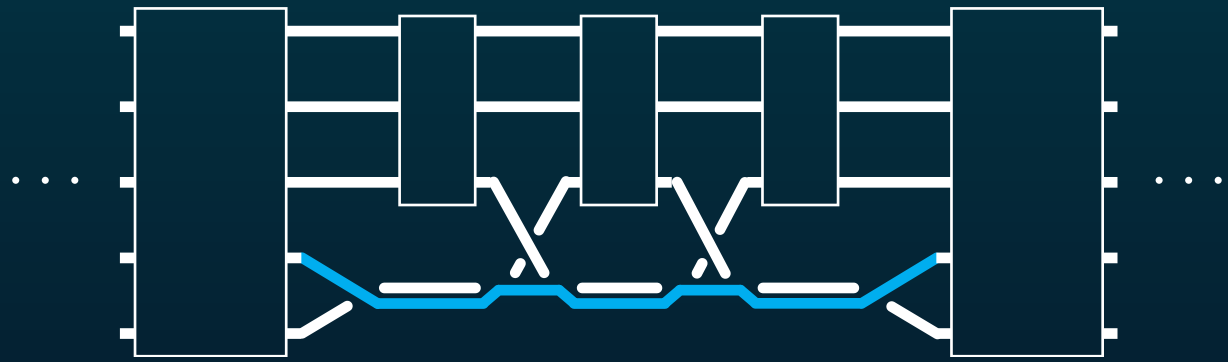

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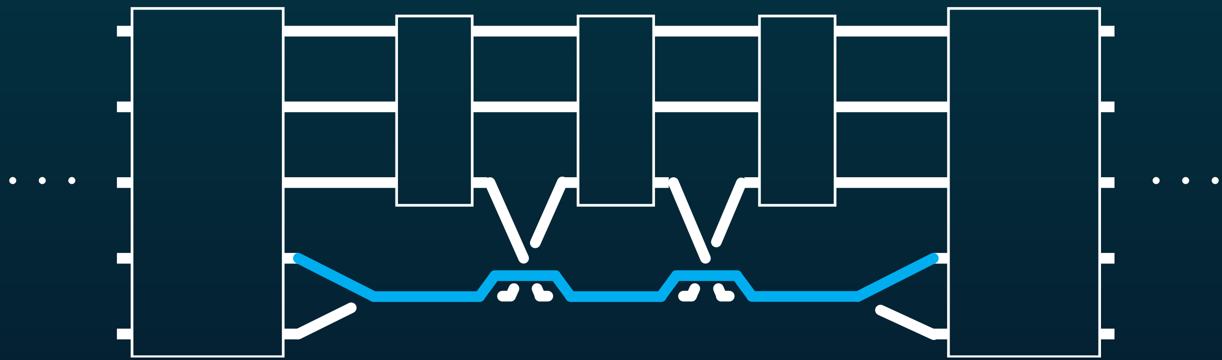

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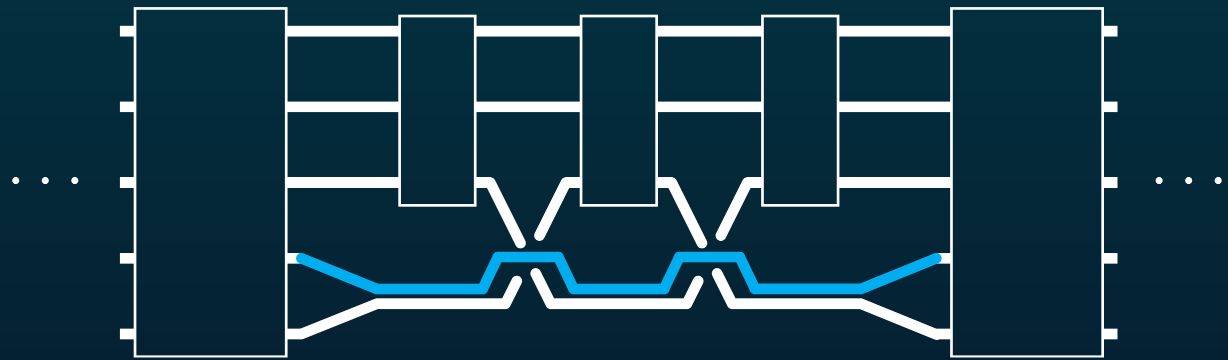

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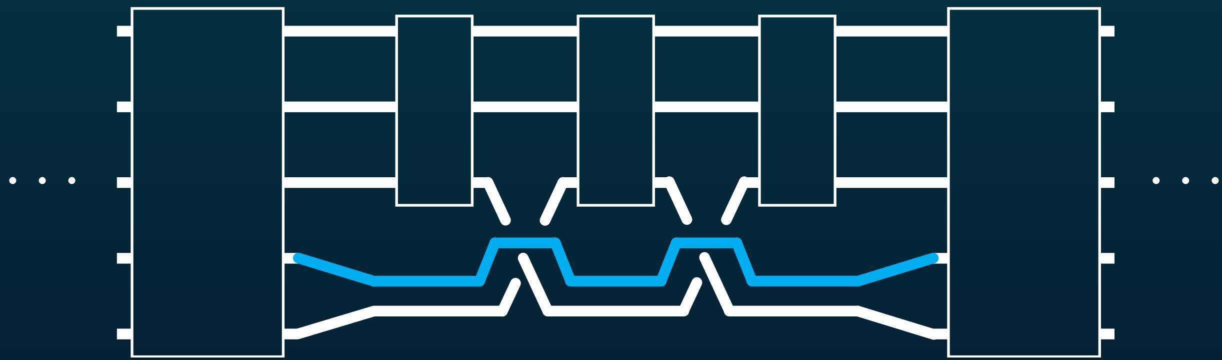

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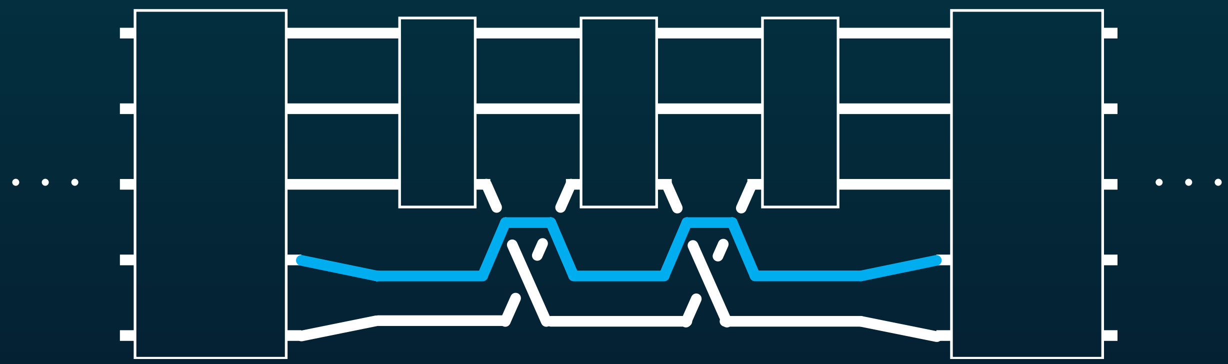

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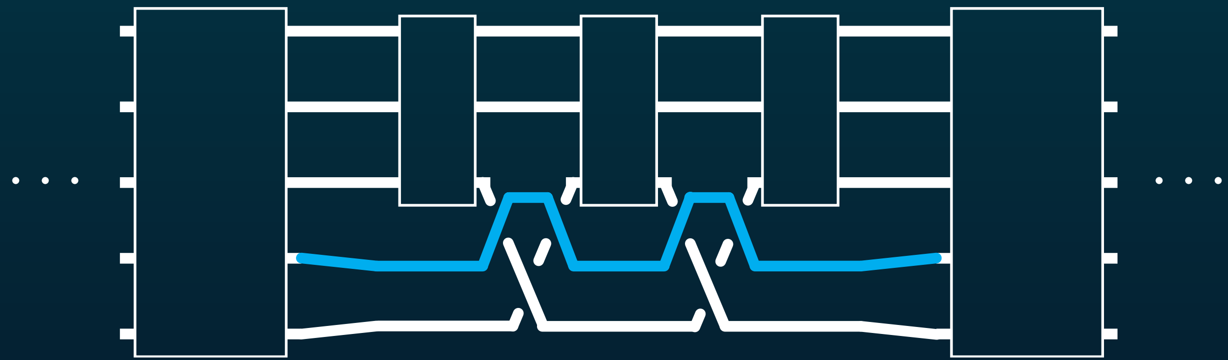

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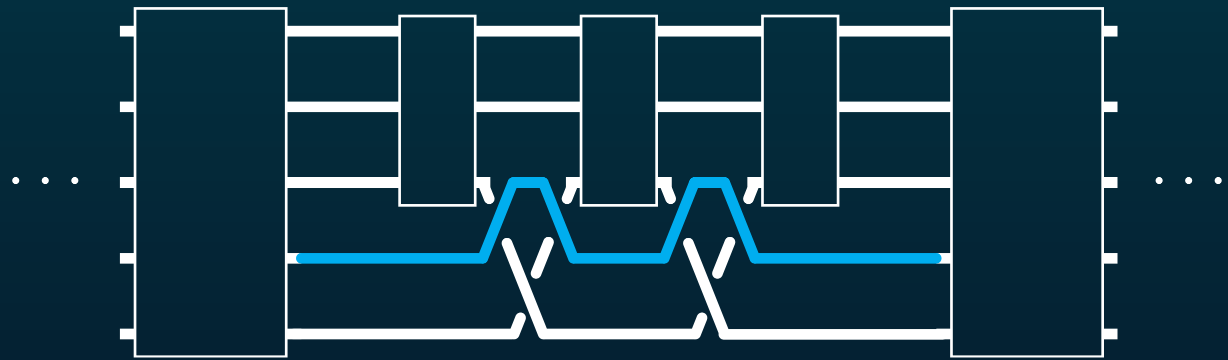

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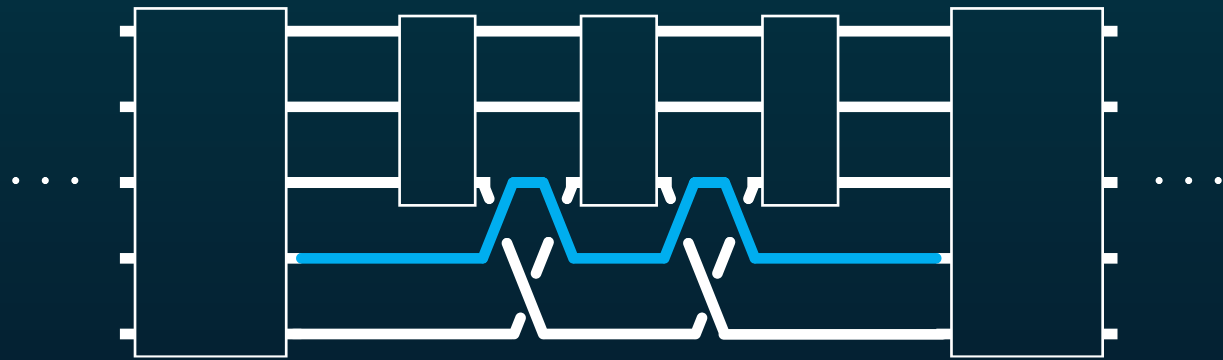

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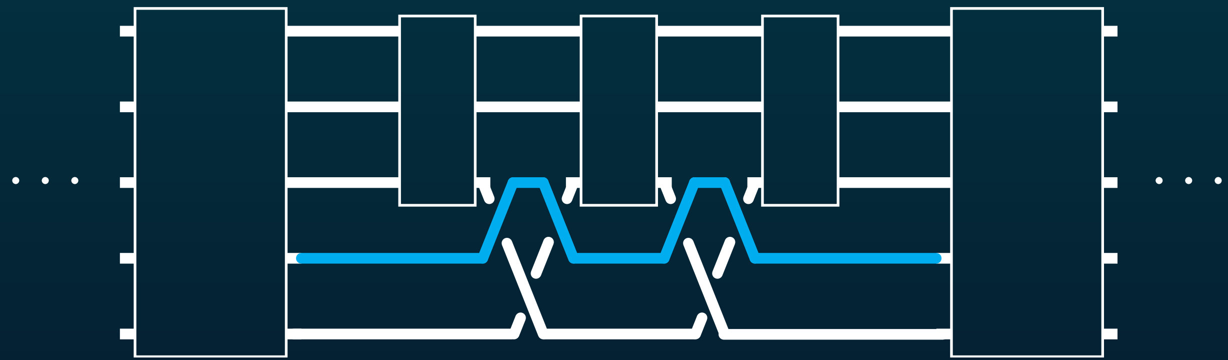

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  - deleting the initial and final σ_1,
  - replacing each $\sigma_2^{\pm 1}$ in w with $\sigma_2^{-e}\sigma_1^{\pm 1}\sigma_2^{e}$.

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⇝ Question: Where does this reduction come from, why does it work?

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$\rightsquigarrow$ Question: Where does this braid ordering come from?

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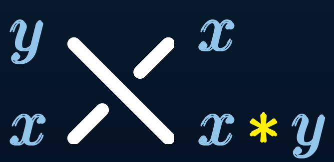

where $*$ is some binary operation on S

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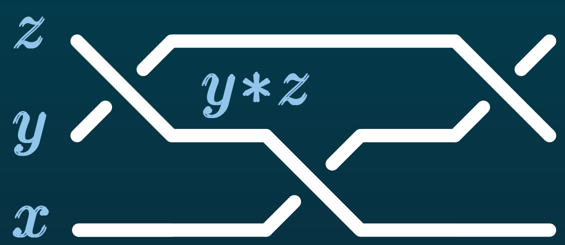

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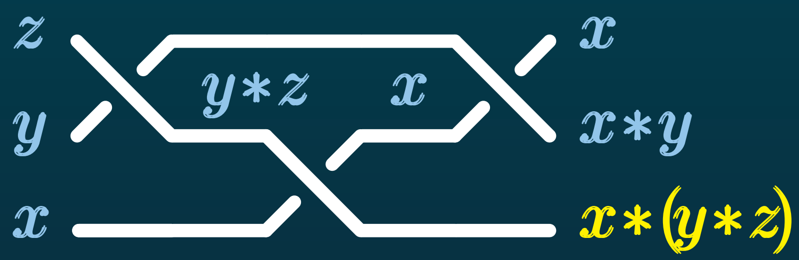

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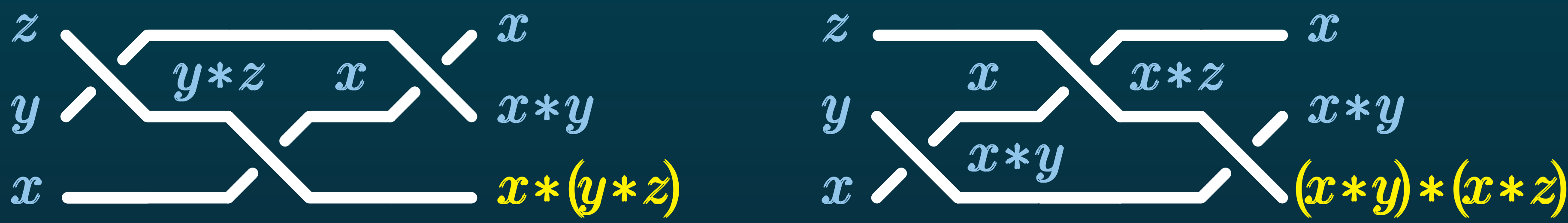

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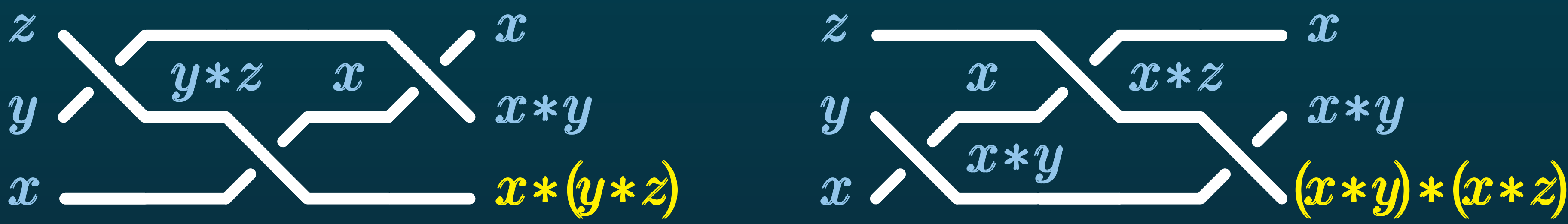

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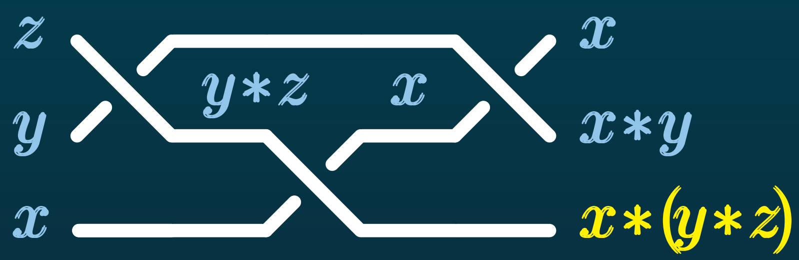

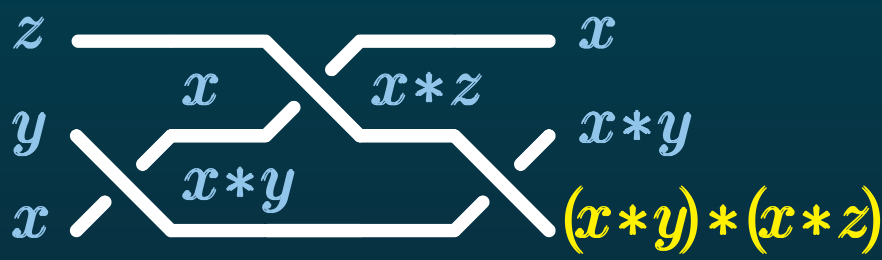

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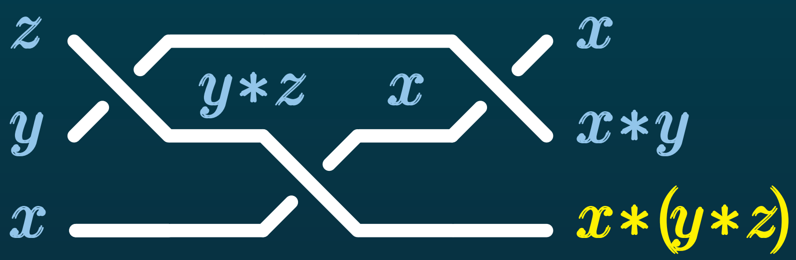

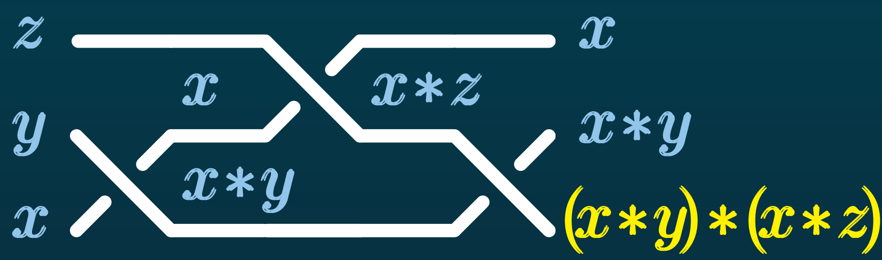

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LD-SYSTEMS

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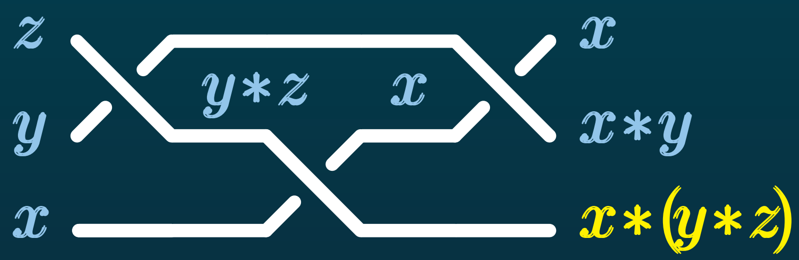

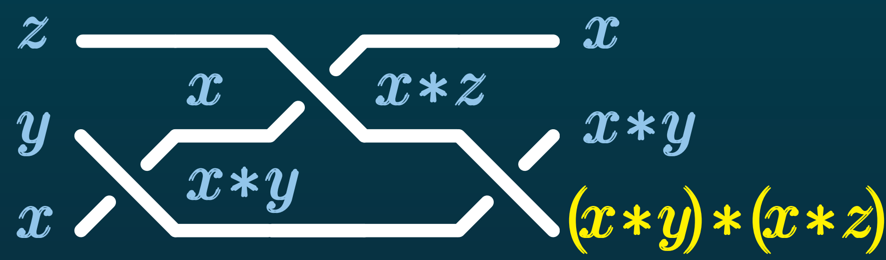

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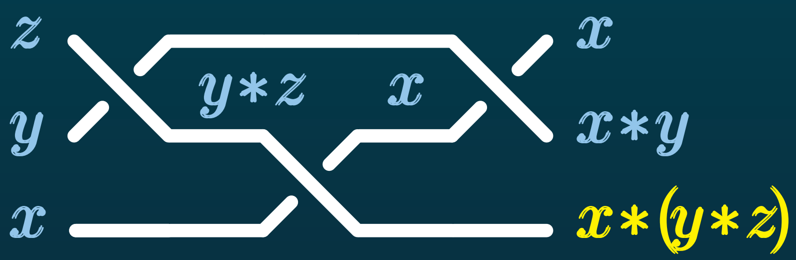

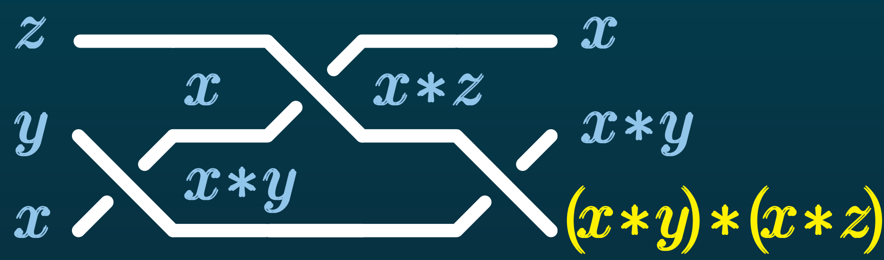

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Note: in these examples, $x * x = x$ always holds.

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- Theorem 2 comes from Theorem 3:
  Use an orderable LD-system to colour braids. The points are:
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  - (C): Linearity of the ordering.

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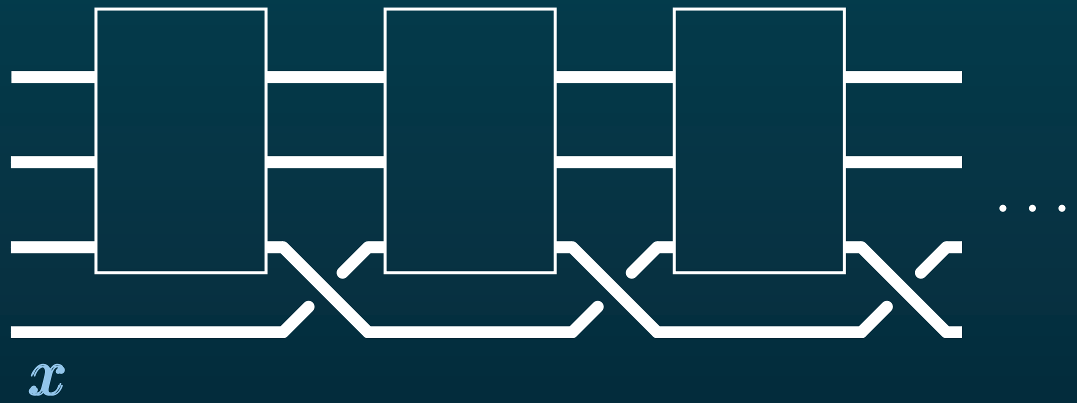

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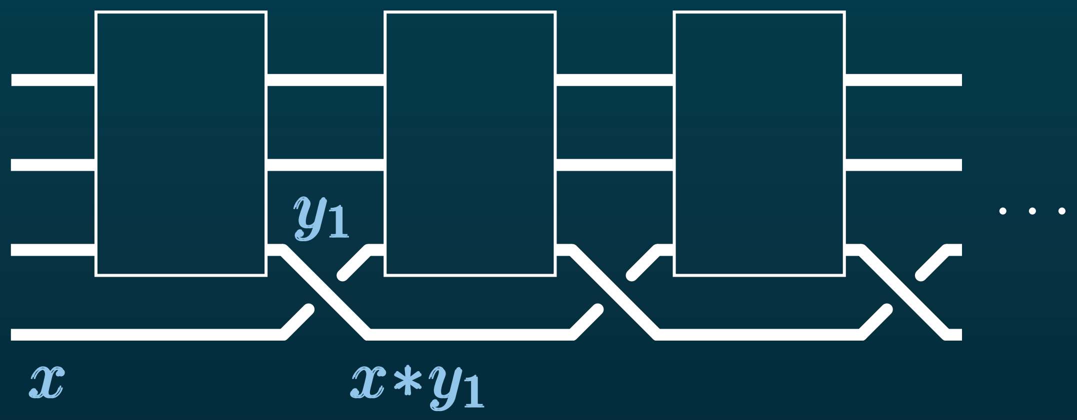

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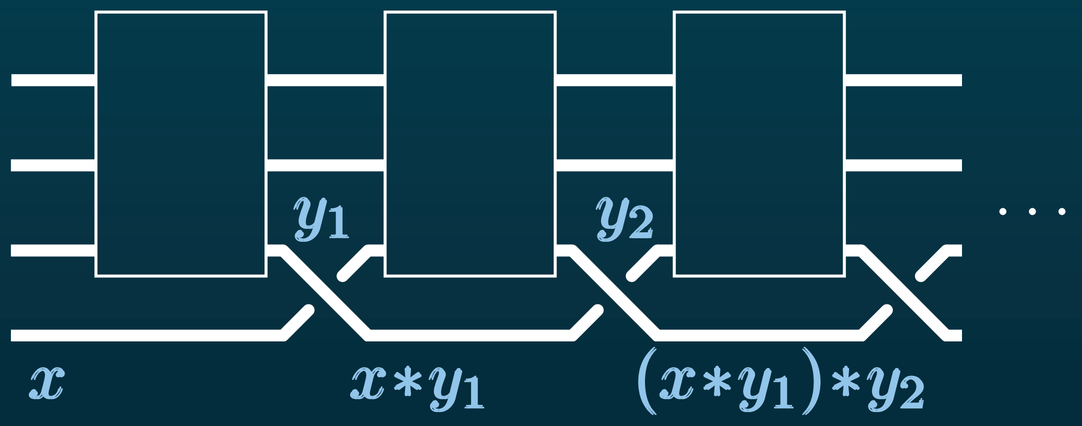

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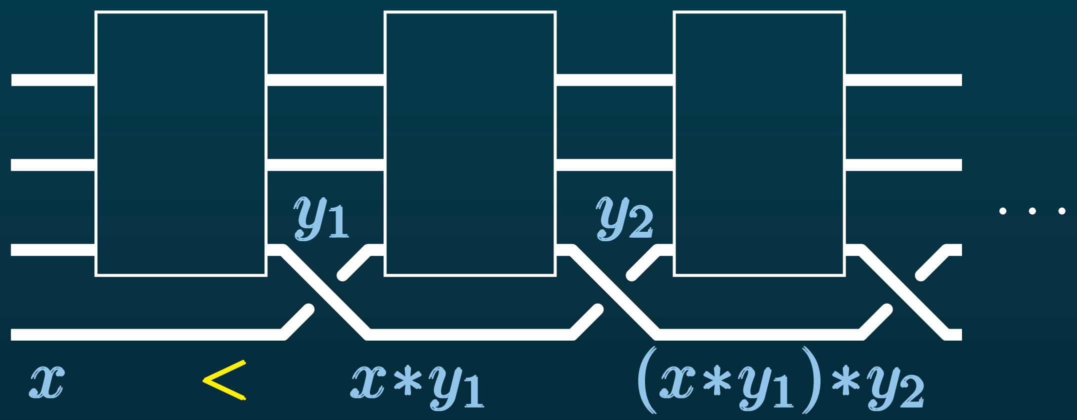

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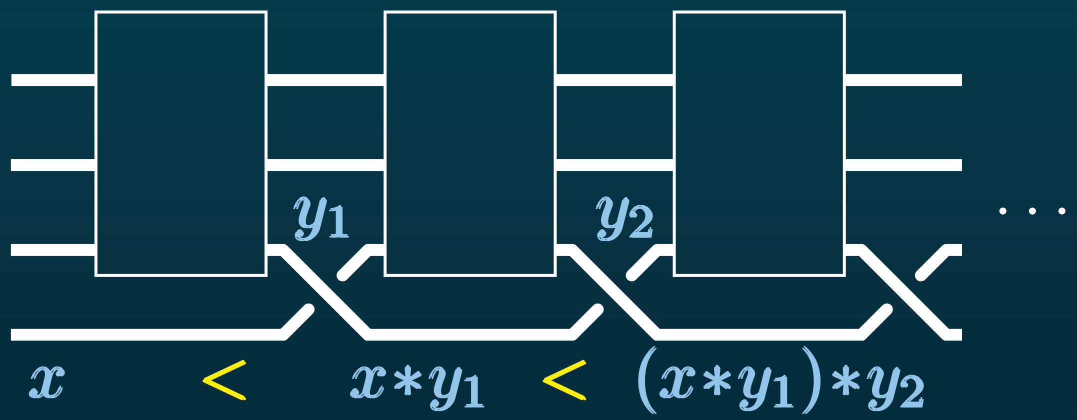

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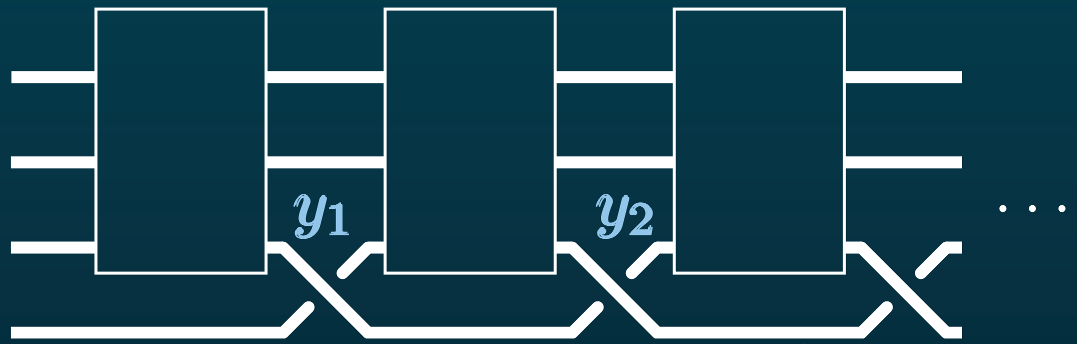

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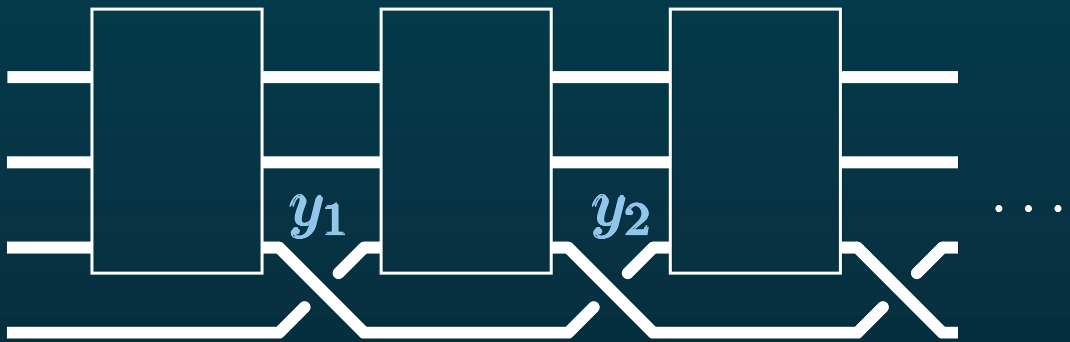

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PROOFS

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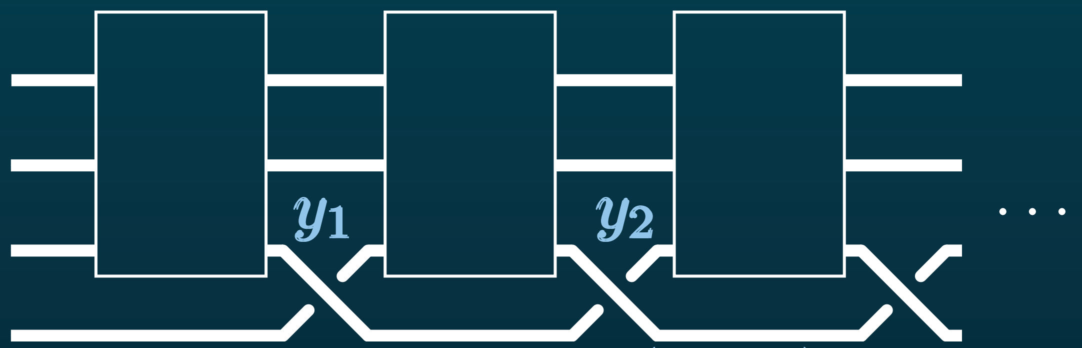

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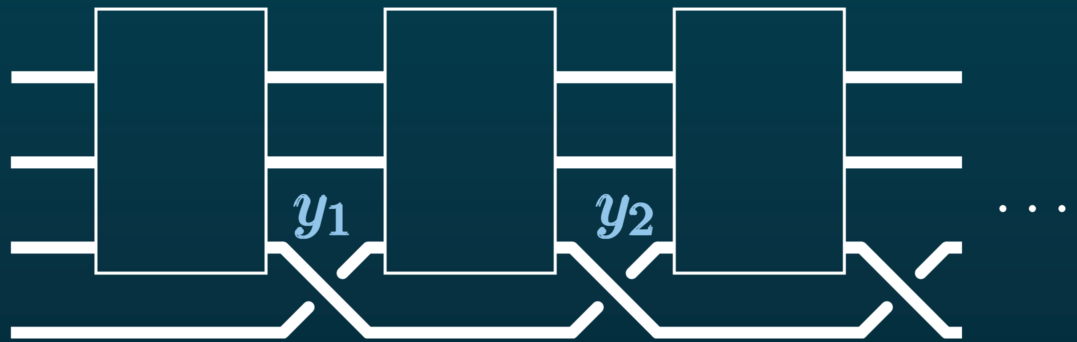

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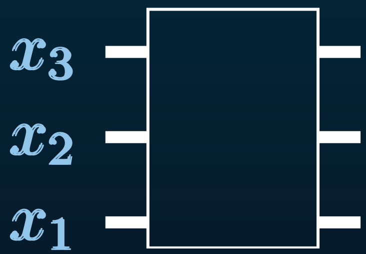

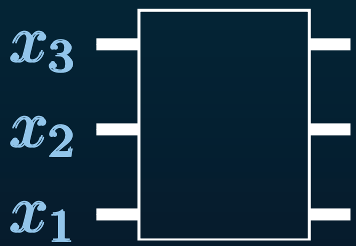

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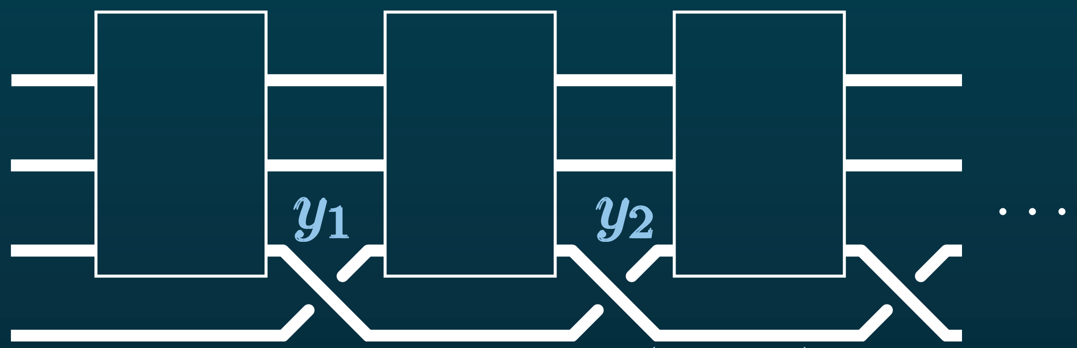

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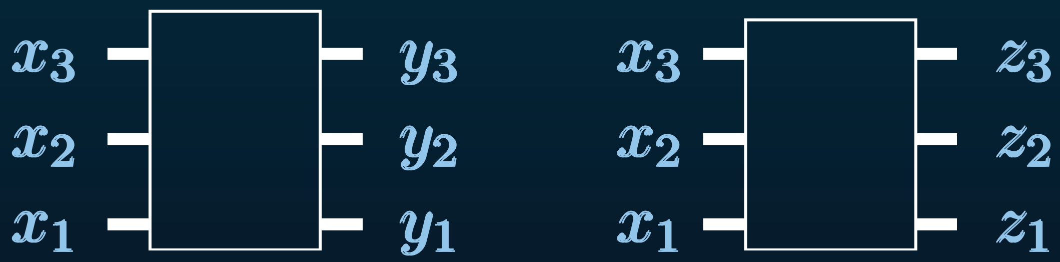

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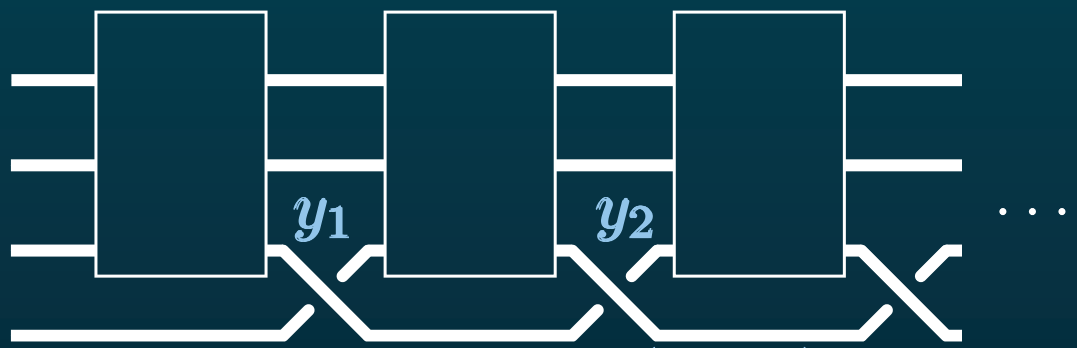

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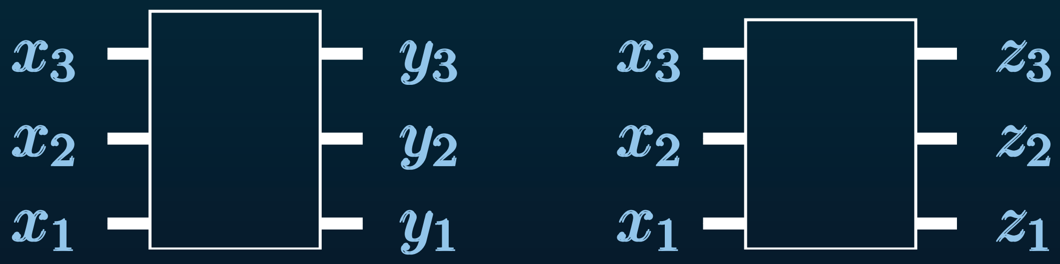

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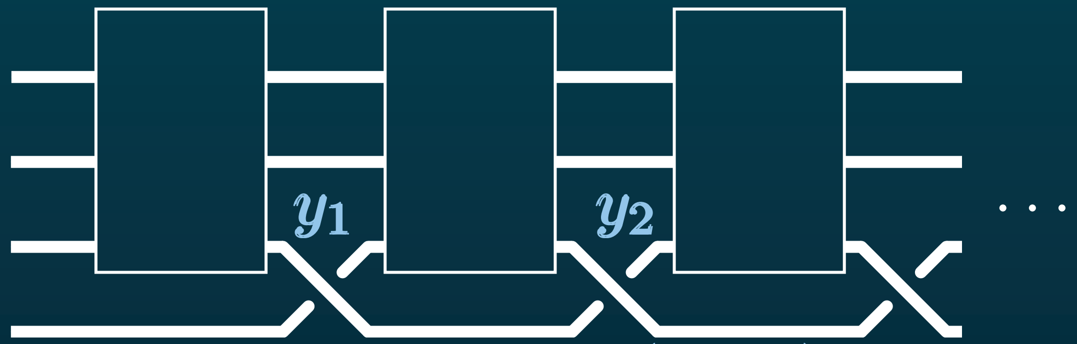

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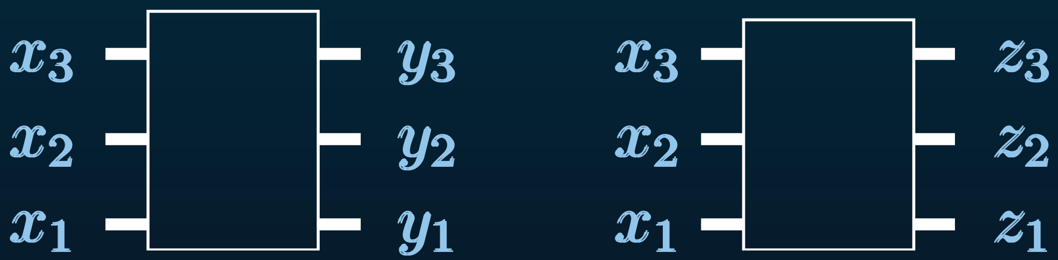

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- If R is a self-similar rank, and i, j are e.e.'s of R, we can apply i to j.
  - "Being an e.e." is definable from $\in$, so $i(j)$ is an e.e. too;
    $\rightsquigarrow$ a binary operation on e.e.'s defined on R.
  - "Being the image under" is definable from $\in$,
    so $\ell = j(k)$ implies $i(\ell) = i(j)(i(k))$, ***i.e.***, $i(j(k)) = i(j)(i(k))$.
    $\rightsquigarrow$ this operation satisfies the LD law.

$\rightsquigarrow$ For every e.e. j, a new LD-system $I(j)$, the iterates of j: $j(j)$, $j(j)(j)$...

Proposition: (D. 1986) If j is an e.e. of a self-similar rank, then the LD-structure of $I(j)$ implies Π^1_1-determinacy. $\rightsquigarrow$ "$I(j)$ is not trivial."

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        - $\rightsquigarrow$ As the latter extends Artin's braid group: braid applications

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- In essence, yes: if Set Theory had not shown that the LD law is involved in deep phenomena, and made the existence of orderable LD-systems plausible, it is unlikely that such objects would have been investigated...

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  - guess some statement, then
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$\rightsquigarrow$ Even if one does not believe in the existence of (hyper)infinite sets, one should agree that, in this case, they led to applicable mathematics.

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$\rightsquigarrow$ Define the n-th Laver table A_n to be the one with 2^n elements.

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- Only known negative result (Dougherty 1995):

  Not provable in Primitive Recursive Arithmetic (double recursion needed).

P. Dehornoy; Braids and Self-Distributivity; PM 192, Birkhauser (2000).

P. Dehornoy, I. Dynnikov, D. Rolfsen, B. Wiest; Why are braids orderable?; Panoramas & Syntheses vol. 14, Soc. Math. France (2002).

• http://www.math.unicaen.fr/$\sim$ dehornoy