# HOMOLOGY OF GAUSSIAN GROUPS

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• Aim: to compute the homology of braid groups
⇝ done: Arnold, Fuks (1970's),

... or, more generally, of spherical Artin–Tits groups
(those associated with a finite Coxeter group)
⇝ done: Goryunov, Deligne, Salvetti, Cohen,

... using an algebraic approach (no differ. geom.)
⇝ done: Squier (1980, 1995),

... so what?
⇝ A simple method for constructing resolutions of $\mathbb{Z}$
- for still more general monoids,
- exclusively relying on least common multiples.

⇝ completes Squier's implicit program;
⇝ extends simultaneous indep. work by Charney, Meier, Whittlesey building on Bestvina (Garside groups);
⇝ gives a purely algebraic proof of the exactness of the Deligne–Salvetti resolution (braid groups).

• Remark (Cartan-Eilenberg, Squier): For $G$ group of fractions of $M$, we have $H_*(G,\mathbb{Z}) = H_*(M,\mathbb{Z})$.
$\rightsquigarrow$ Compute $H_*(M,\mathbb{Z})$ for some monoids $M$.
$\rightsquigarrow$ Construct resolutions of $\mathbb{Z}$ by free $\mathbb{Z}M$-modules.

• Which monoids? Those with good lcm (least common multiple) properties, typically Artin monoids (Squier), and, more generally:

• **Definition:** A monoid $M$ is Gaussian if
(i): cancellative,
(ii): division has no infinite $\searrow$-chain,
(iii): any two elements have a left and a right lcm.

• **Definition:** A monoid $M$ is locally Gaussian if
(i) + (ii) + (iii)$^-$: two elements with a common left (right) multiple admit a left (right) lcm.

• **Definition:** A monoid $M$ is Garside if Gaussian $+$ contains a Garside element $\Delta$,

$$\rightsquigarrow \begin{cases} \mathrm{Div}_{left}(\Delta) = \mathrm{Div}_{right}(\Delta), \\ \mathrm{Div}_{left}(\Delta) \text{ is finite,} \\ \mathrm{Div}_{left}(\Delta) \text{ generates } M. \end{cases}$$

• Examples:

- The braid monoids $B_n^+$ are Garside monoids; every finite Coxeter type Artin monoid is a Garside monoid (an Artin monoid or group is one defined by relations all of the form $abab... = baba...$);

- Every Artin monoid is locally Gaussian;

- Every singular braid monoid is locally Gaussian;

... of a different flavour:

- The dual braid monoids (Birman-Ko-Lee, Bessis-Digne-Michel) are Garside monoids;

- The monoids $\langle a, b, c, \ldots ; a^p = b^q = c^r = \ldots \rangle$ are Garside monoids;

- The monoids $\langle a, b ; aba = b^2 \rangle$, $\langle a, b ; ababa = b^2 \rangle$, ... are Garside monoids.

$\rightsquigarrow$ see M. Picantin's PhD thesis for many examples; (a conjecture: every finitely generated Gaussian monoid is a Garside monoid).

• Construction of a resolution (first method).

⇝ Hypothesis: $M$ is a locally Gaussian monoid (for instance, a Garside monoid). We construct

$$\cdots \to C_2 \to C_1 \to C_0 \to \mathbb{Z}$$

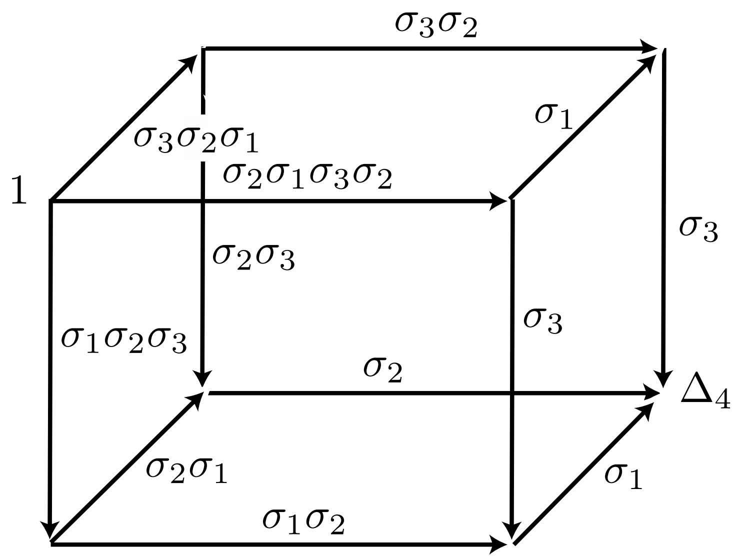
where the $C_*$ are free $\mathbb{Z}M$-modules and $\mathbb{Z}$ is a trivial $\mathbb{Z}M$-module.

⇝ Fix a set $\mathcal{X}$ of generators for $M$ that is closed under left complement, i.e., for $x$, $y$ in $\mathcal{X}$ with a common left multiple, hence a left lcm $z$ the elements $x'$ and $y'$ satisfying $z = xy' = yx'$ still belong to $\mathcal{X}$ (for $M$ Garside, can take $\mathcal{X} = \mathrm{Div}(\Delta)$ .

⇝ Then define $C_n$ to be the free $\mathbb{Z}M$-module based on $\mathcal{X}^{[n]}$, with

• **Definition:** Let $\mathcal{X}^{[n]}$ be the set of all $[\alpha_1, \ldots, \alpha_n]$ with $\alpha_1 < \ldots < \alpha_n \in \mathcal{X}$ and $\alpha_1, \ldots, \alpha_n$ admitting a left lcm (with $<$ a fixed linear ordering on $\mathcal{X}$).

• Idea: $[\alpha_1, \ldots, \alpha_n]$ is an $n$-cube associated with the computation of the left lcm of $\alpha_1, \ldots, a_n$ (which exists by hypothesis).



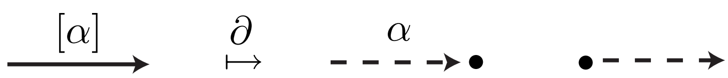

*The 3-cell $[\sigma_1, \sigma_2, \sigma_3]$ in the braid monoid $B_3^+$*

⇝ finite type modules whenever $\mathcal{X}$ is finite (so, in particular, for $M$ Garside).

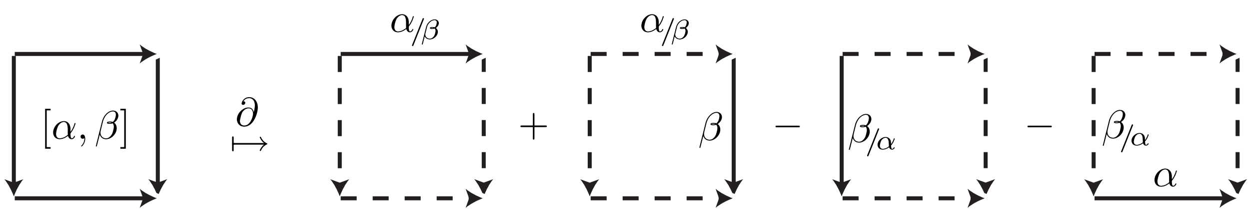
• We need:

- a boundary operator $\partial_n : C_n \to C_{n-1}$ satisfying $\partial^2 = 0$;
- a contracting homotopy $s_n : C_n \to C_{n+1}$ satisfying $\partial s + s\partial = \text{id}$.

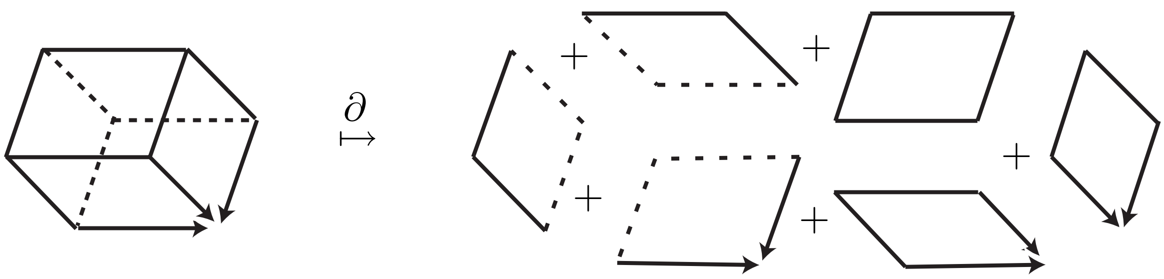
• The boundary is natural (“the boundary of a cube is the sum of its faces”) but it requires an *ad hoc* formalism.




$$\partial_1[\alpha] \quad = \quad \alpha[\emptyset] \quad - \quad [\emptyset]$$




$$\partial_2[\alpha,\beta] \quad = \quad [\alpha_{/\beta}] \quad + \quad \alpha_{/\beta}[\beta] \quad - \quad [\beta_{/\alpha}] \quad - \quad \beta_{/\alpha}[\alpha]$$



• The contracting homotopy is more difficult: how to invent a cube from one face?

⇝ use a normal form: in every (locally) Gaussian monoid $M$, there exists a good normal form, namely the greedy normal form (Deligne, Adyan, Thurston, ElRifai-Morton, Charney).

• Every element $x$ of $M$ has a unique maximal left divisor $x_1$ lying in $\mathcal{X}$, say $x = x_1 x'$, so, iteratively,

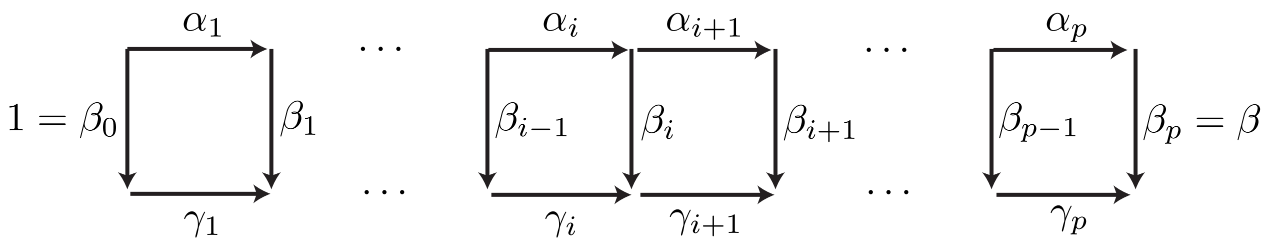
$$x = x_1 x_2 \cdots x_p$$

with $x_1, \ldots, x_p \in \mathcal{X}$ and $x_i$ the max. left div. of $x_i x_{i+1} \cdots x_p$ lying in $\mathcal{X}$.

• Point: The $\mathcal{X}$-normal form is local: $(x_1, \ldots, x_p)$ is $\mathcal{X}$-normal iff $(x_i, x_{i+1})$ is $\mathcal{X}$-normal for each $i$.

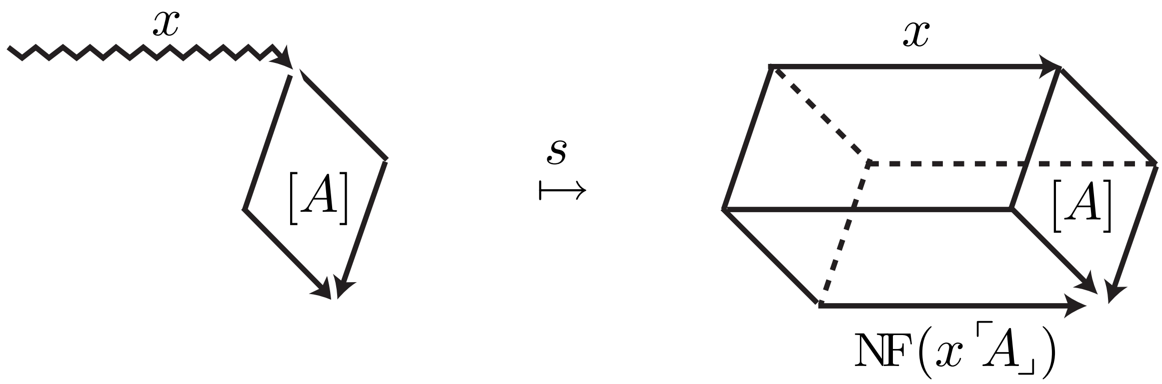
⇝ Corollary 1 (Charney for spherical Artin groups): Garside groups are automatic.

$\rightsquigarrow$ Corollary 2: The $\mathcal{X}$-normal form can be computed using left reversing:




$\rightsquigarrow$ Contracting homotopy for $C_*$:

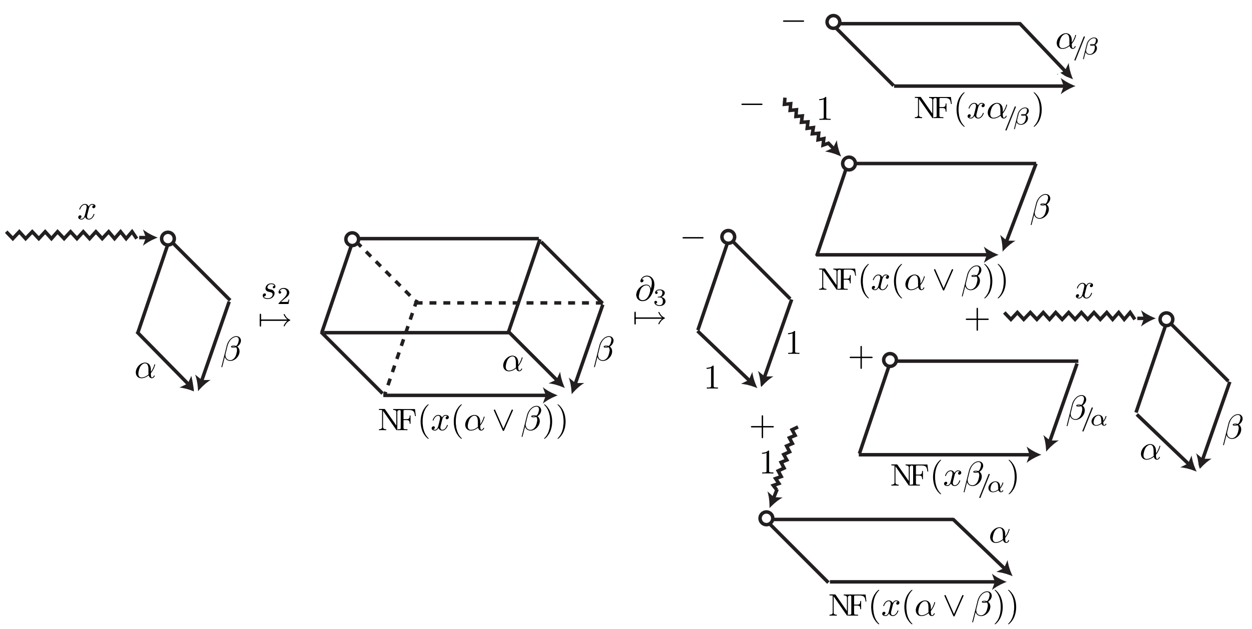
Problem: Starting from $x$ in $M$ and $[A]$ in $\mathcal{X}^{[n]}$, *i.e.*, from the $n$-cube $[A]$ translated by $x$ to define an $n+1$-parallelotope of which $x[A]$ is a face:

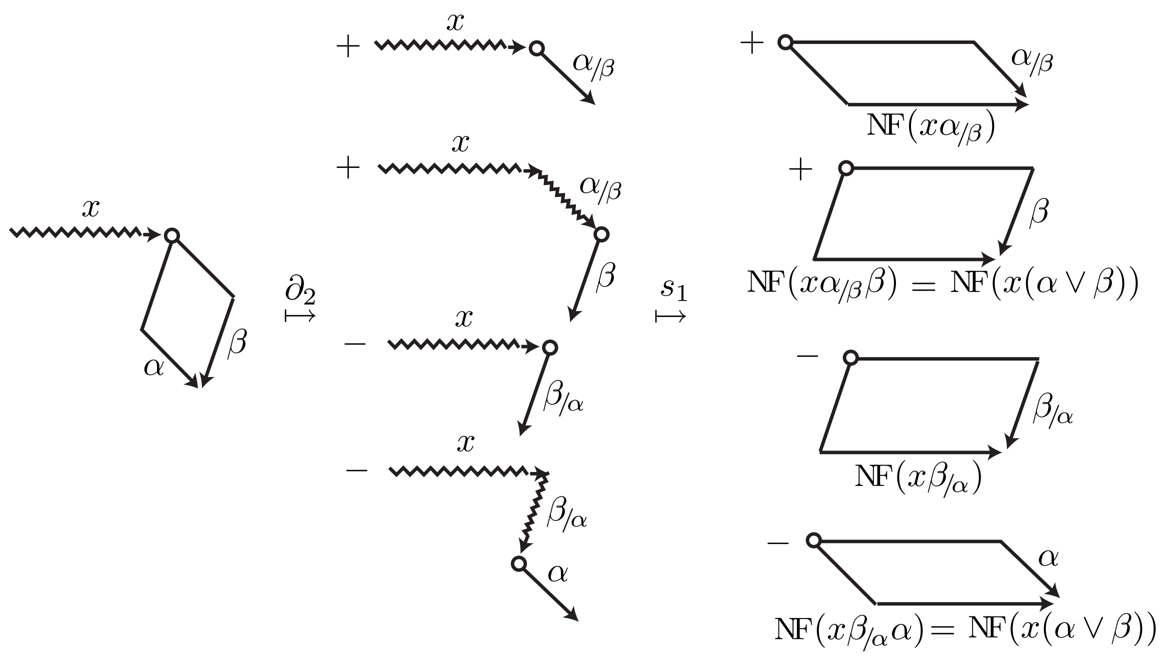



$\rightsquigarrow$ Solution: $s(x[A]) = [\mathrm{NF}(x\,\mathrm{lcm}(A)), A]$.

(Needs to define $[w, A]$ when $w$ is a word on $\mathcal{X}$, and not a single letter $\rightsquigarrow$ induction given by lcm's formulas:

$$[uv, A] = [u, A/v] + \overline{u/A}[v, A]).$$

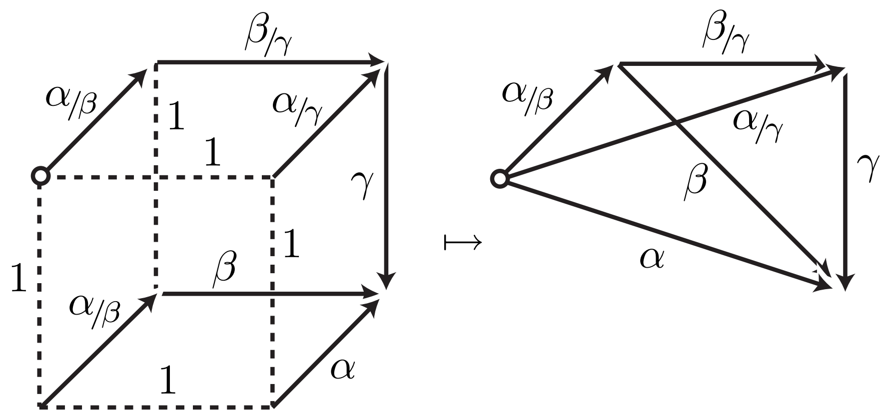


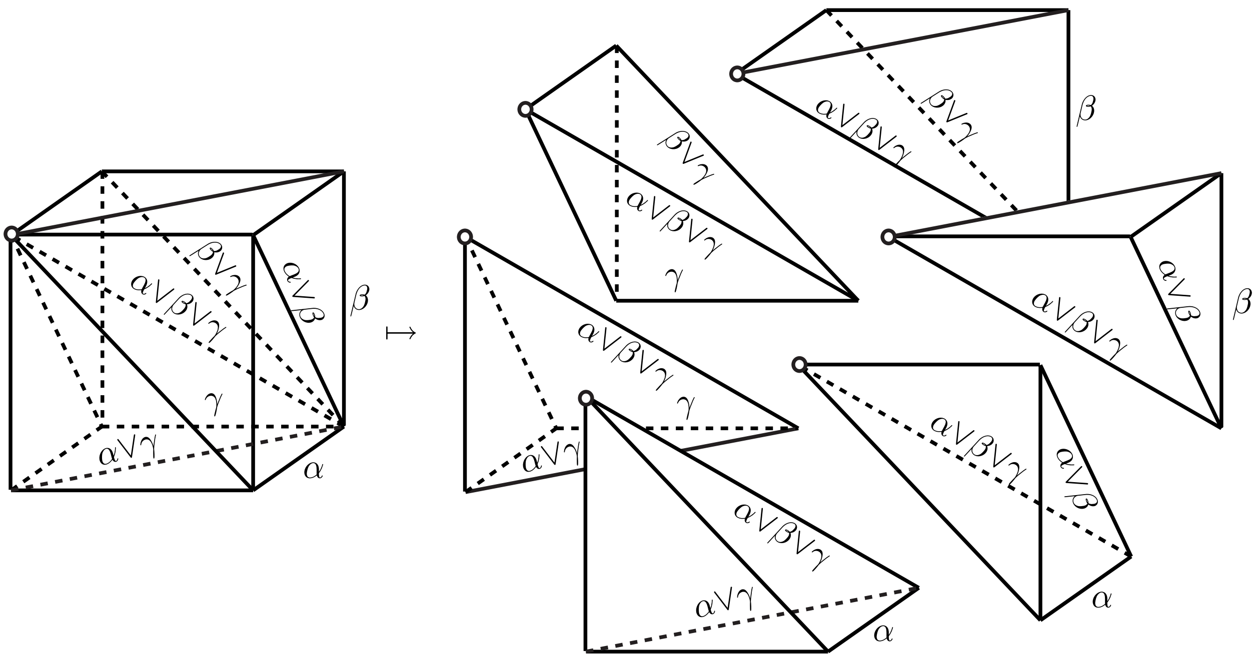

Proof of exactnes): works because normal forms everywhere.

• Improvement: Extract a smaller and shorter sub-complex by restricting to descending cubes: $[\alpha_1, \ldots, \alpha_n]$ s.t. $\alpha_n | \alpha_{n-1} | \ldots | \alpha_2 | \alpha_1$ (right division).

⇝ A descending $n$-cube is an $n$-simplex:




⇝ Still form a resolution, because an $n$-cube can be decomposed into a sum of $n!$ disjoint $n$-simplexes:



• **Proposition:** *Every Garside group (i.e., group of fractions of a Garside monoid) is of type FL, i.e., has a finite free resolution.*

• Still OK for a locally Gaussian monoid, but
(i) the set $\mathcal{X}$ may be infinite, and
(ii) if common multiples do not exist, there is no associated group of fractions.

• In the case of a Garside group $G$, the (improved) resolution is the one of Charney-Meier-Whittlesey, after Bestvina's construction of a flag complex whose 1-skeleton is the Cayley graph of $G$.

• Yves Lafont (Marseille): Construction of another resolution (second method, reminiscent of Kobayashi).

⇝ more general: $M$ locally left Gaussian: one-sided hypotheses: right cancellativity, left Noetherian, and any two elements with a common left multiple admit a left lcm;

⇝ more flexible: $\mathcal{X}$ arbitrary set of generators of $M$ (no closure requirement);

⇝ but less effective: inductive construction, no explicit formula, no geometric interpretation (so far).

• Method: (Pre)-well-order the chains:

$$x[A] \prec y[B]$$

if $x\,\mathrm{lcm}(A)$ proper left div. of $y\,\mathrm{lcm}(B)$,
or $x\,\mathrm{lcm}(A) = y\,\mathrm{lcm}(A)$ and $\mathrm{first}(A) < \mathrm{first}(B)$.

⇝ $\prec$ has no infinite $\searrow$-chain: allows $\prec$-induction.

- **Definition:**

$$\partial_{n+1}[\alpha, A] = \alpha/A[A] - s_{n-1}\partial_n(\alpha/A[A]),$$

$$s_n(x[A]) = \begin{cases} 0 & \text{for } x[A] \text{ irreducible,} \\ y[\alpha, A] + s_n(y s_{n-1}\partial_n(\alpha/A[A])) & \\ & \text{otherwise, with } \alpha \text{ min.left div.} \\ & \text{of } x \,\mathrm{lcm} A \text{ and } x = y(\alpha/A). \end{cases}$$

where $x[A]$ irreducible means:

$\alpha_1$ is the min. left div. of $x\,\mathrm{lcm}(A)$.

- The point: For $x[A]$ reducible, we have

$$s_{n-1}\partial_n(x[A]) \prec x[A].$$

⇝ induction possible (think of $s_{n-1}\partial_n(x[A])$ as a reduction of $x[A]$).

- **Proposition** (Lafont): ... makes a resolution of $\mathbb{Z}$.

• Question 1: Does solution 1 works with arbitrary generators, *i.e.*, when we do not assume the set $\mathcal{X}$ to be closed under lcm and complement?

• Question 2: Has Solution 2 a (natural) geometrical interpretation similar to that of Solution 1, *i.e.*, connected with some reversing process and some normal form?

**References**

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