Recent progress about the Continuum Hypothesis after Woodin

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  Also remains the "essentially"...

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$\rightsquigarrow$ ZFC just incomplete

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⇝ restrict to axioms that do not contradict large cardinals.

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- **Question:** Can one have the same situation for larger fragments ?

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$\rightsquigarrow$ What about good axiomatizations for H_1, H_2, ... ?

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- Remark: CH deals with H_2, hence involves (possible) good axiomatizations of H_2.

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$$(\exists a_1)(\forall a_2)(\exists a_3)(\forall a_4)...(\overline{0{,}a_1a_2a_3...} \in A) \text{ or }$$

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- **Indeed: efficient** (by **Moschovakis** & al. PD implies projective sets are measurable, have Baire property, no choice needed), **compatible with large cardinals** (by **Martin–Steel** PD $\Longleftrightarrow$ exist ∞ many Woodin cardinals), **and provides forcing invariance** (by **Woodin** a proper class of Woodin cardinals makes properties of H_1 invariant under forcing).

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- **Theorem (Woodin, 1995)** If compatible with large cardinals, ZFC + WMM is a good axiomatization for H_2.

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$\rightsquigarrow$ invariance under forcing and empirically complete description; missing: compatibility with large cardinals.

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- When B becomes more and more complicated, there are less and less models with the desired property.

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"Everything that cannot be refuted by forcing has some witness in the family of universally Baire sets of reals."

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- Corollary: If the Ω-Conjecture is true, CH is essentially false.

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  $\rightsquigarrow$ **What Woodin (at least) proves: that CH is meaningful.**