BRAID-BASED CRYPTOLOGY

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- Introduction to braid groups;
- Description of some braid-based cryptographical protocols, after Sidelnokov & al. and Ko, Lee & al.;
- Length attack against the conjugacy problem, after Hofheinz–Steinwandt;
- A resisting protocol, after Sibert;
- New braid primitives: the shifted conjugacy problem;
- Discussion.

- A 4-strand **braid diagram** = 2D-projection of a 3D-figure

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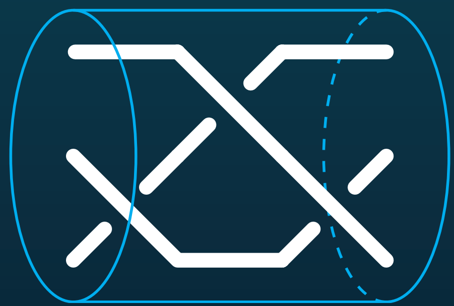

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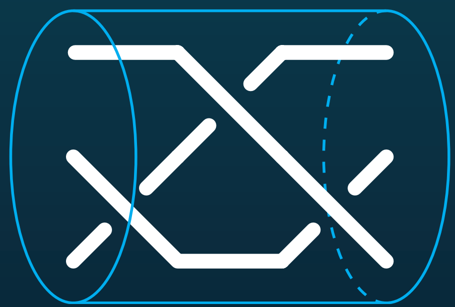

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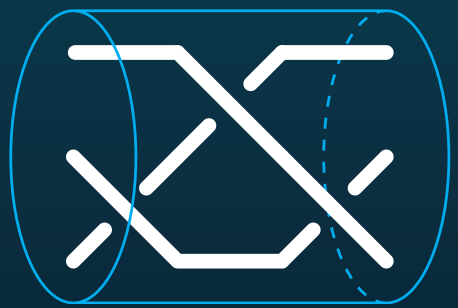

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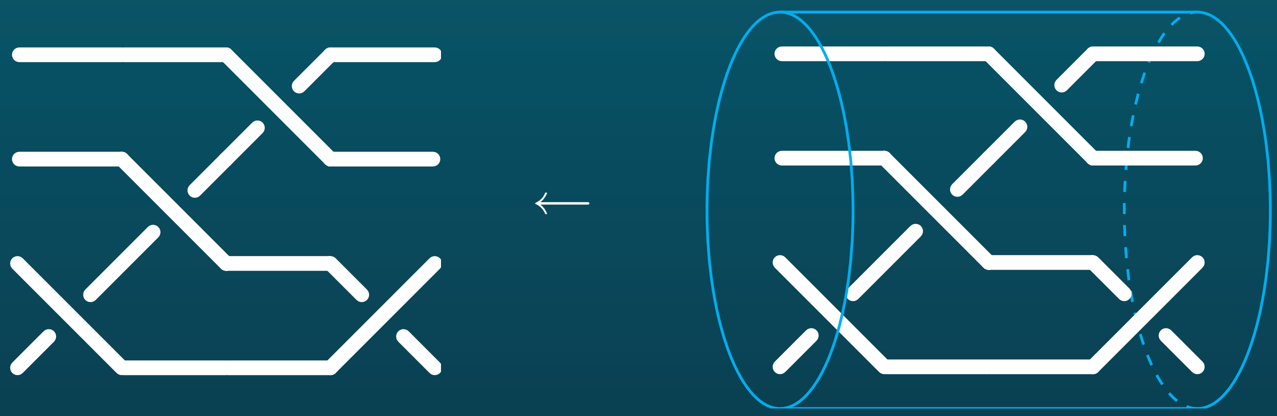

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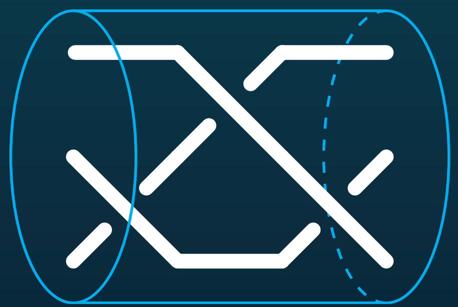

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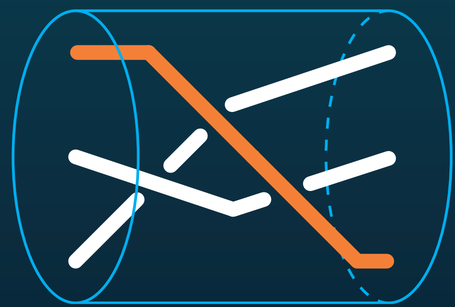

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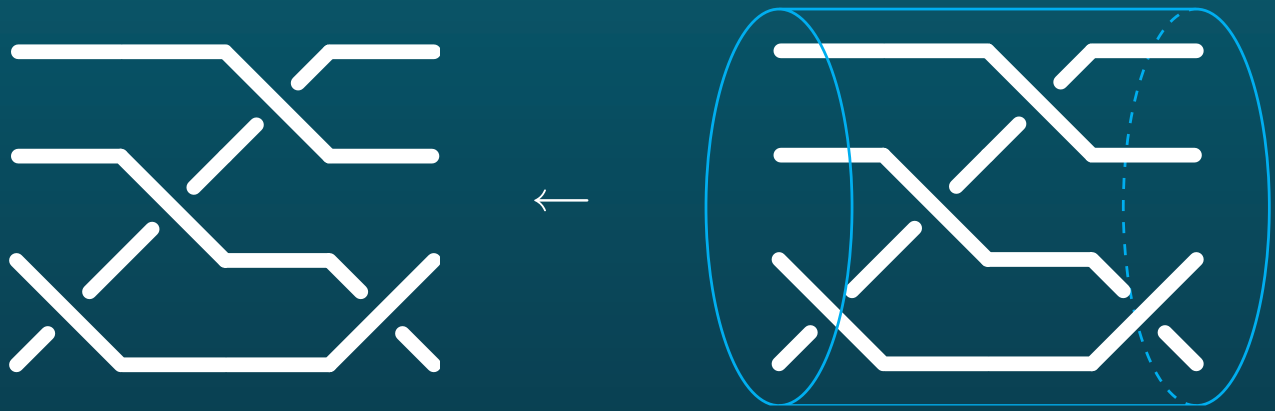

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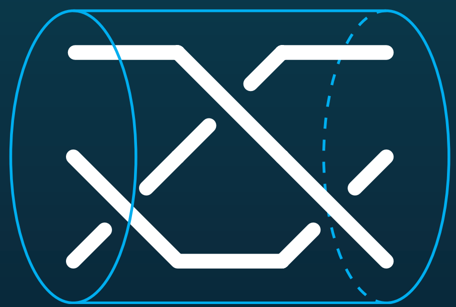

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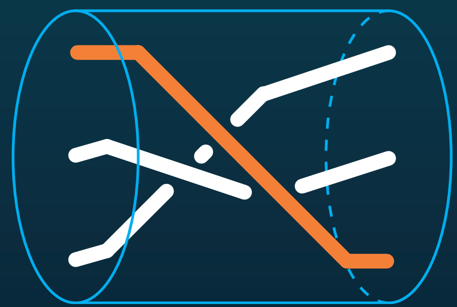

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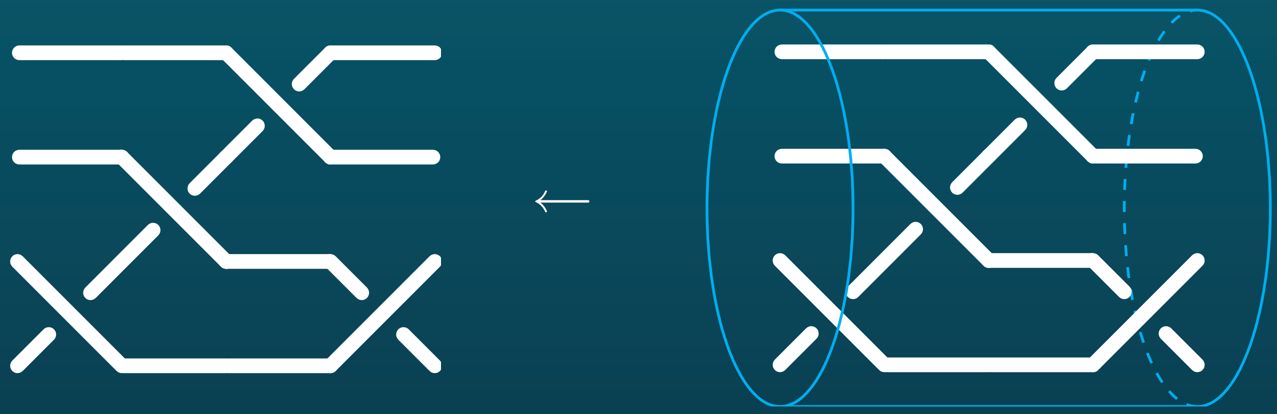

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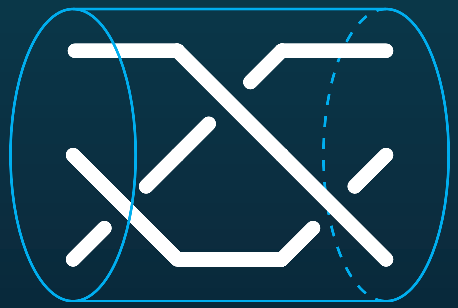

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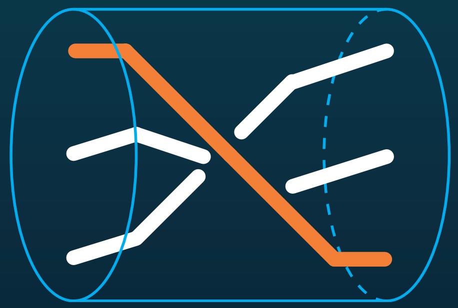

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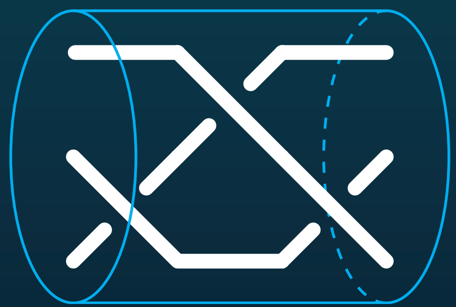

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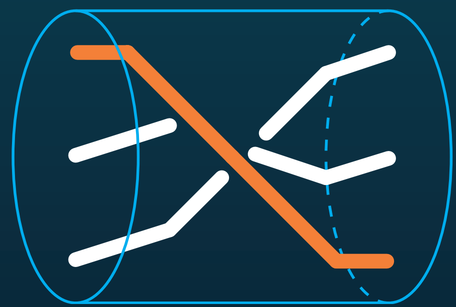

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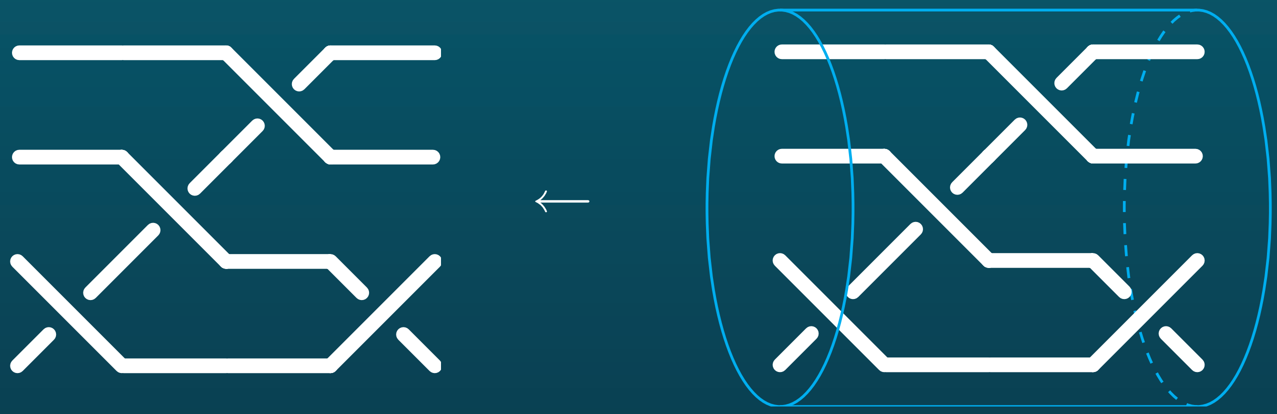

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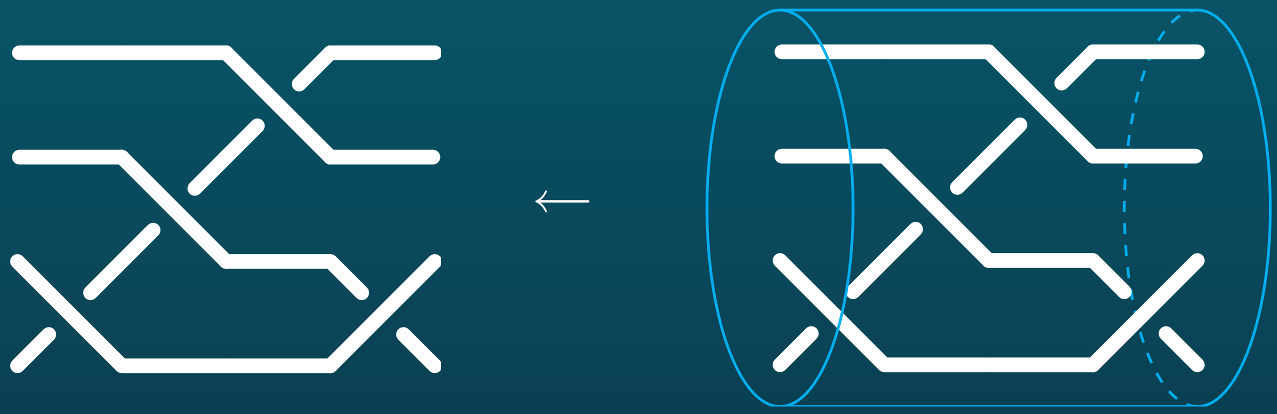

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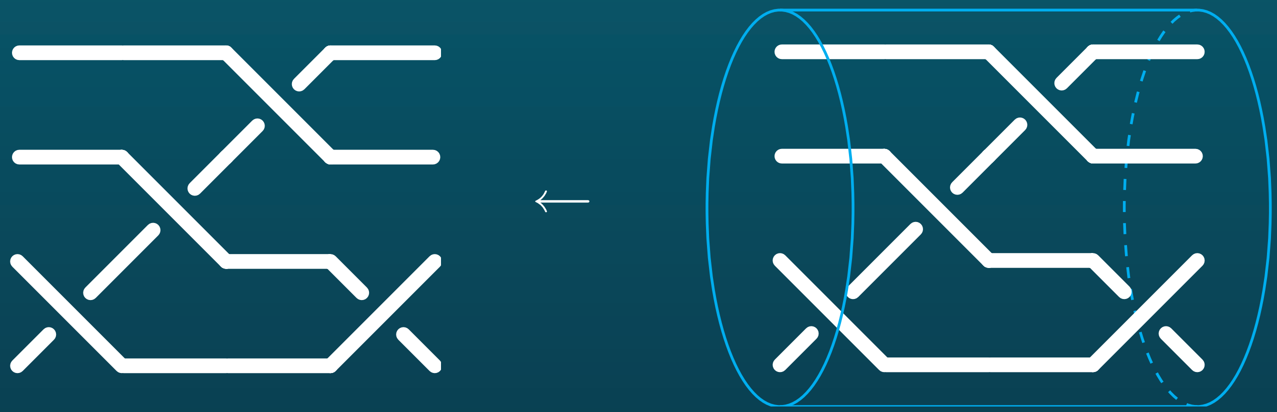

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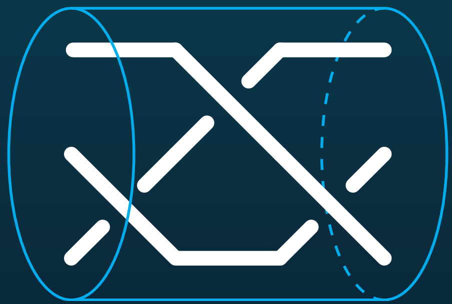

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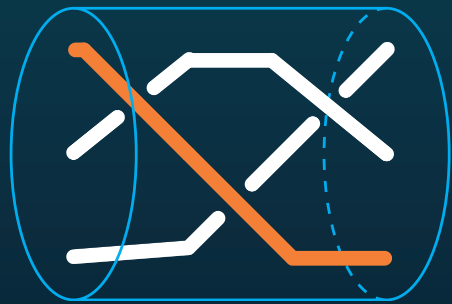

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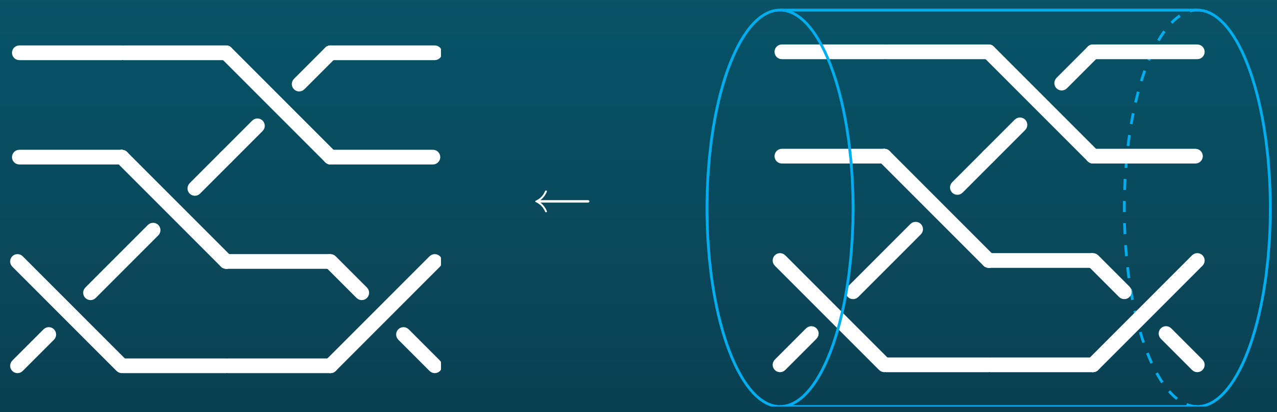

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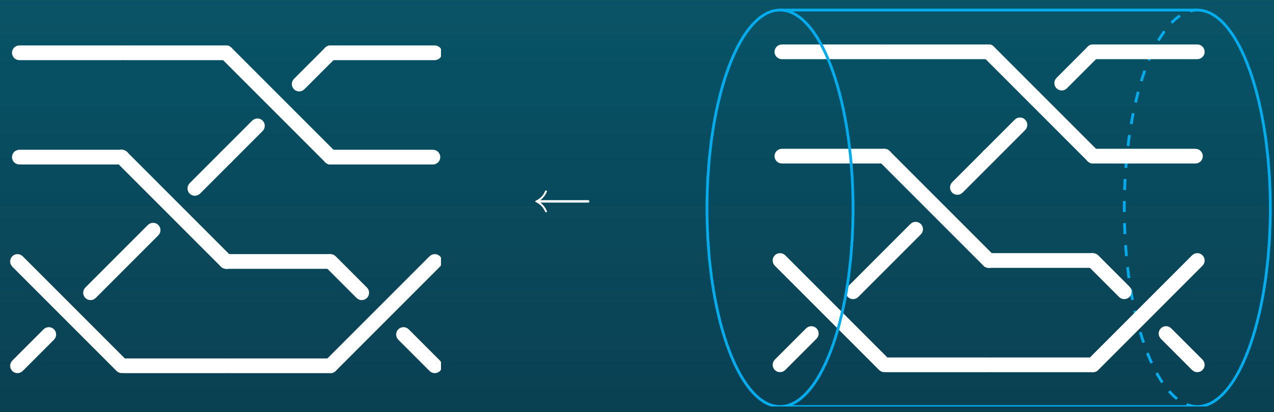

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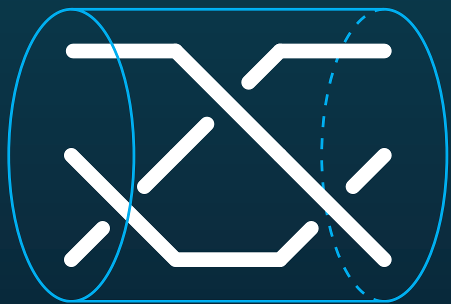

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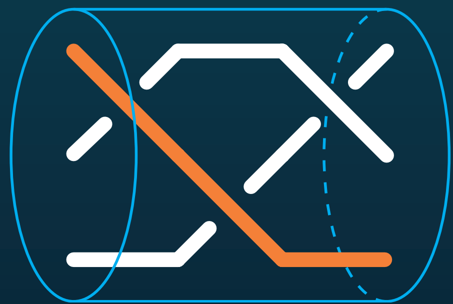
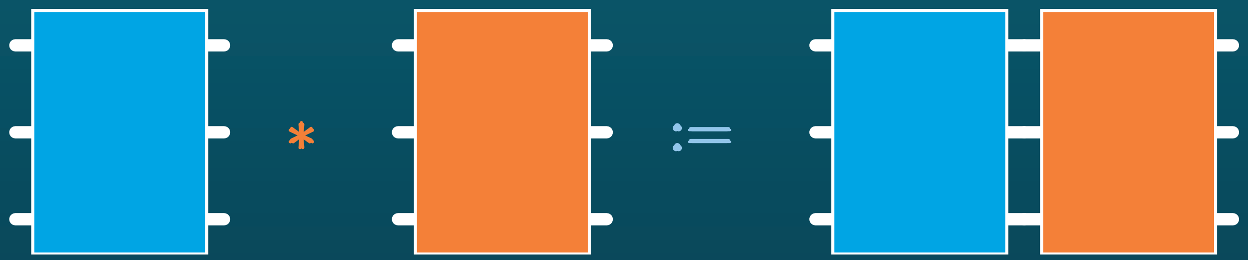


- a **braid** = an isotopy class

$\rightsquigarrow$ can be represented by 2D-diagram,

but different 2D-diagrams may give rise to the same braid.

- **The product of two braids:**



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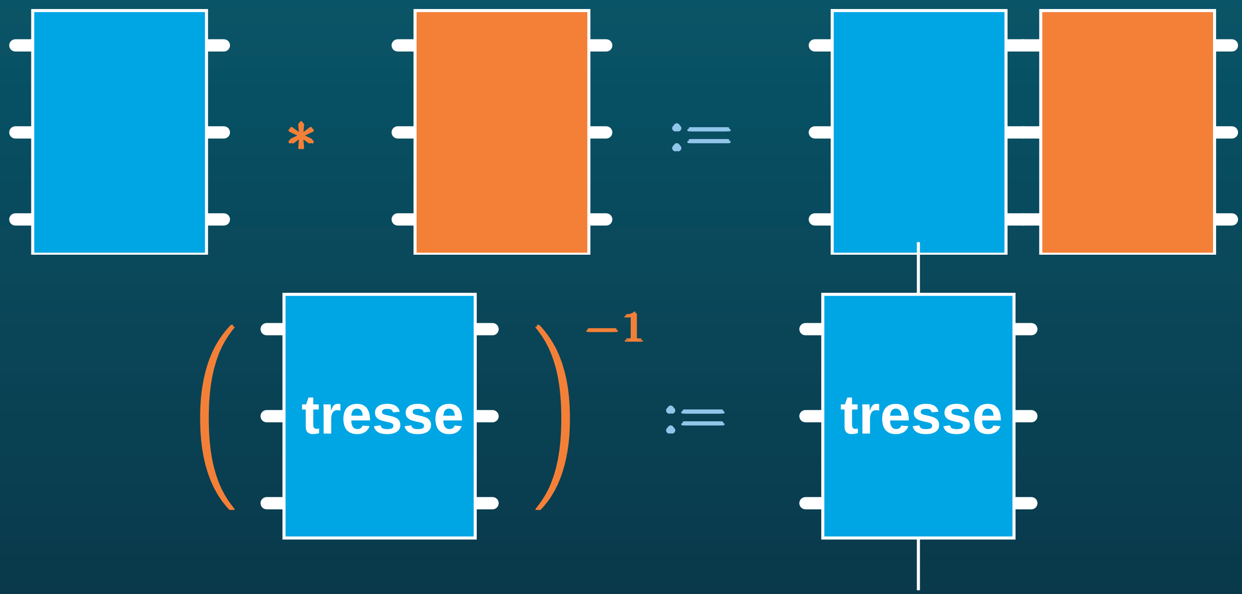

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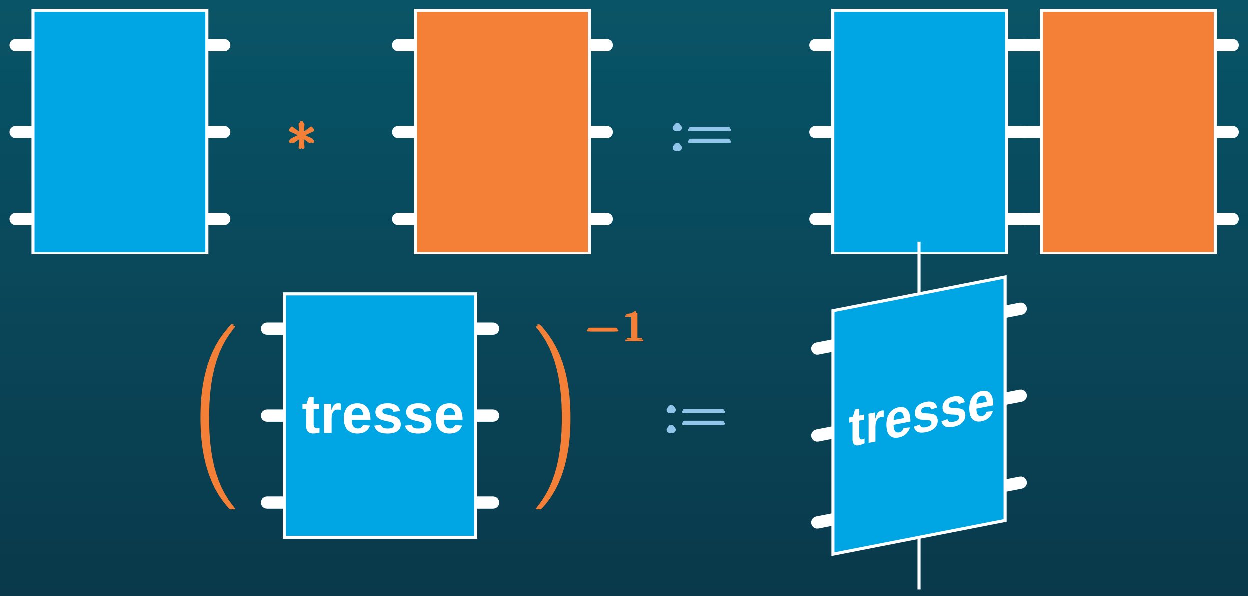

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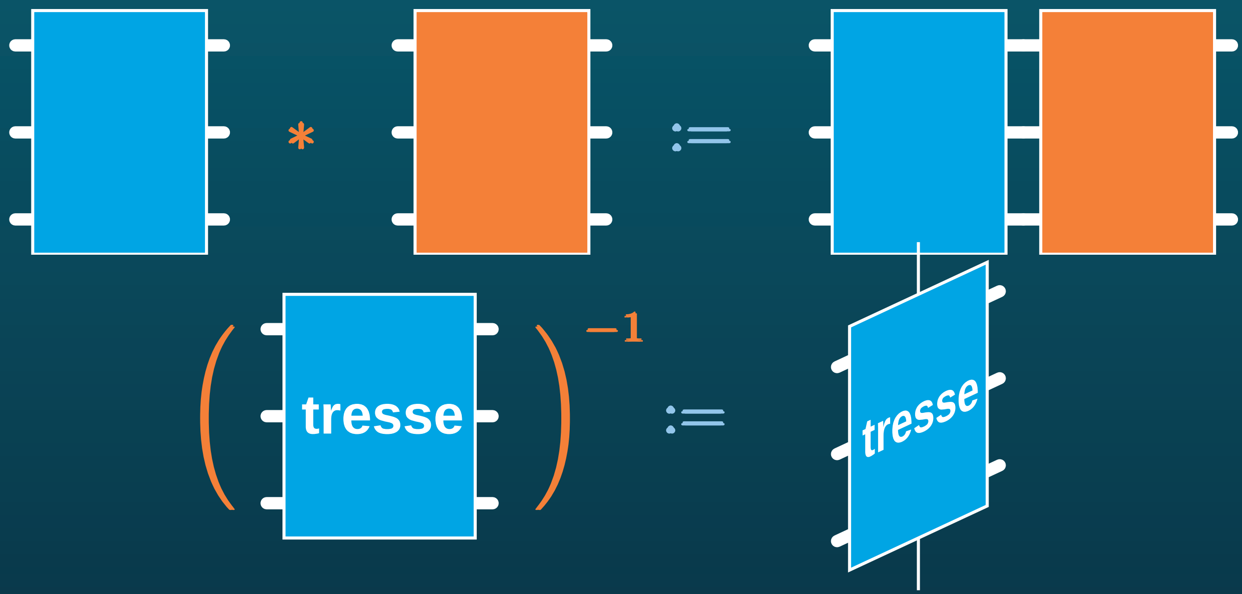

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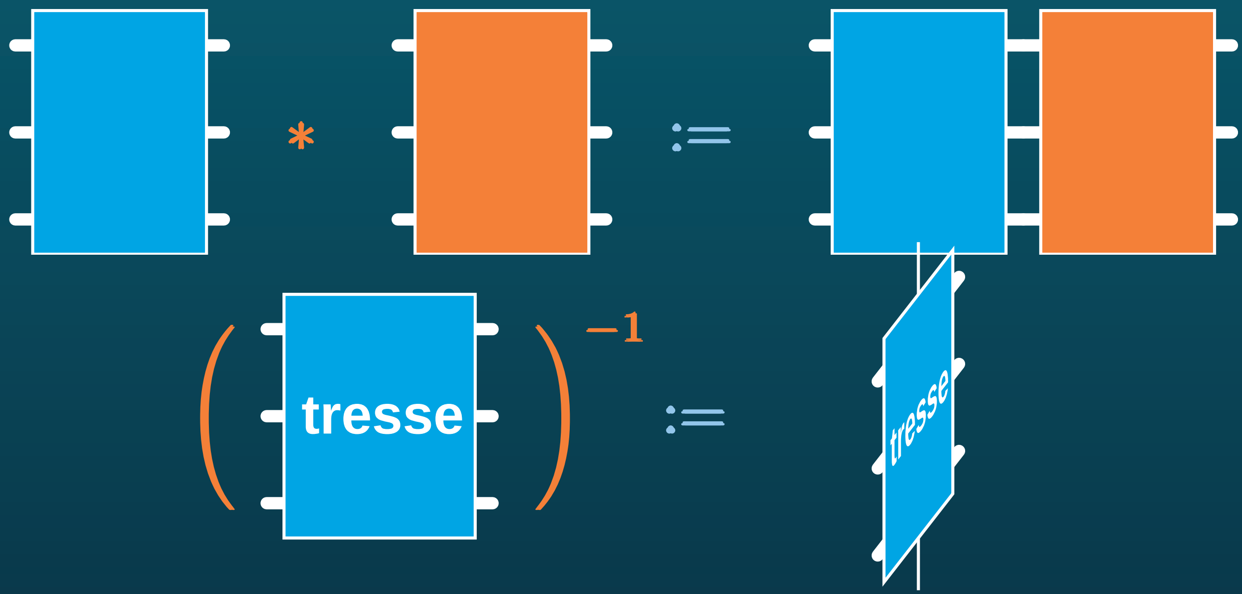

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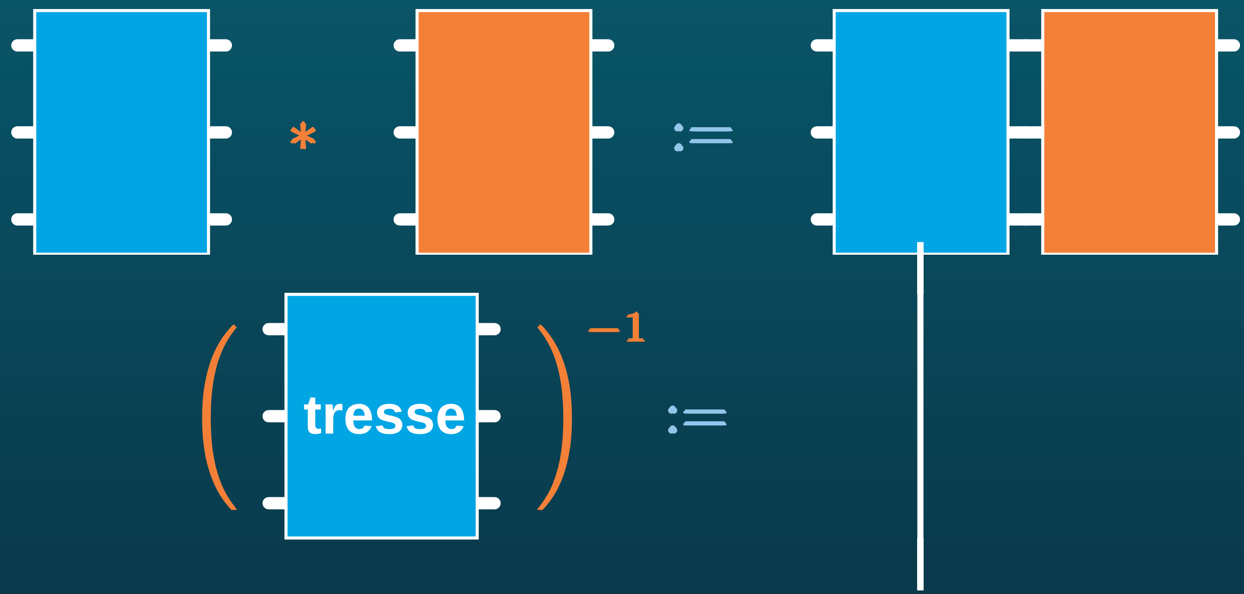

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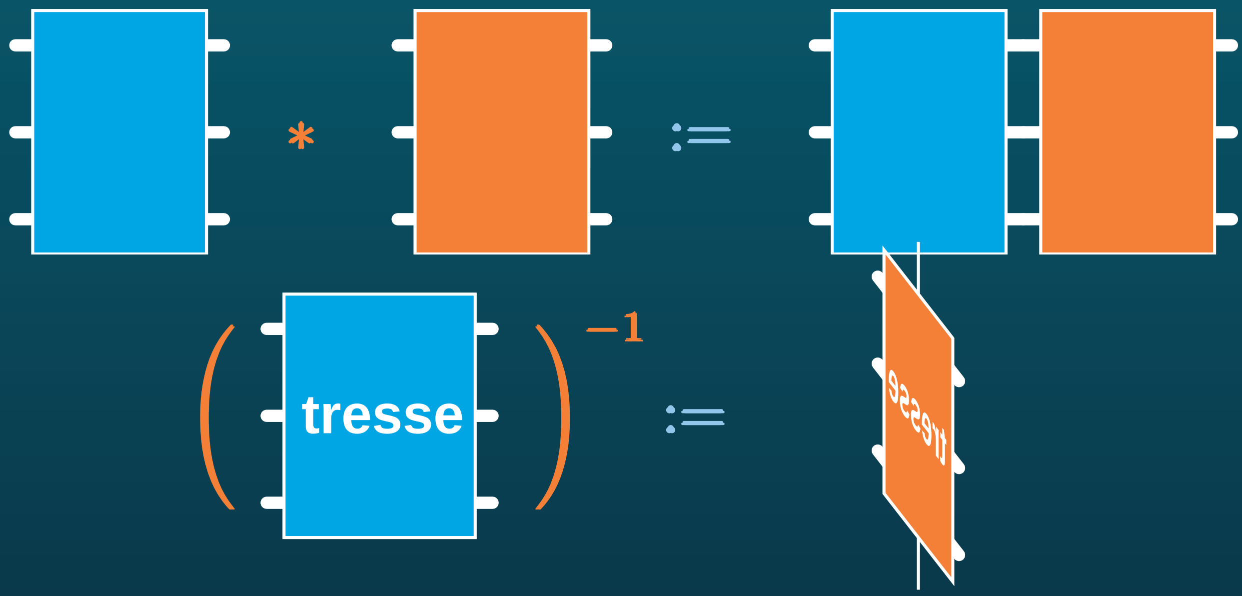

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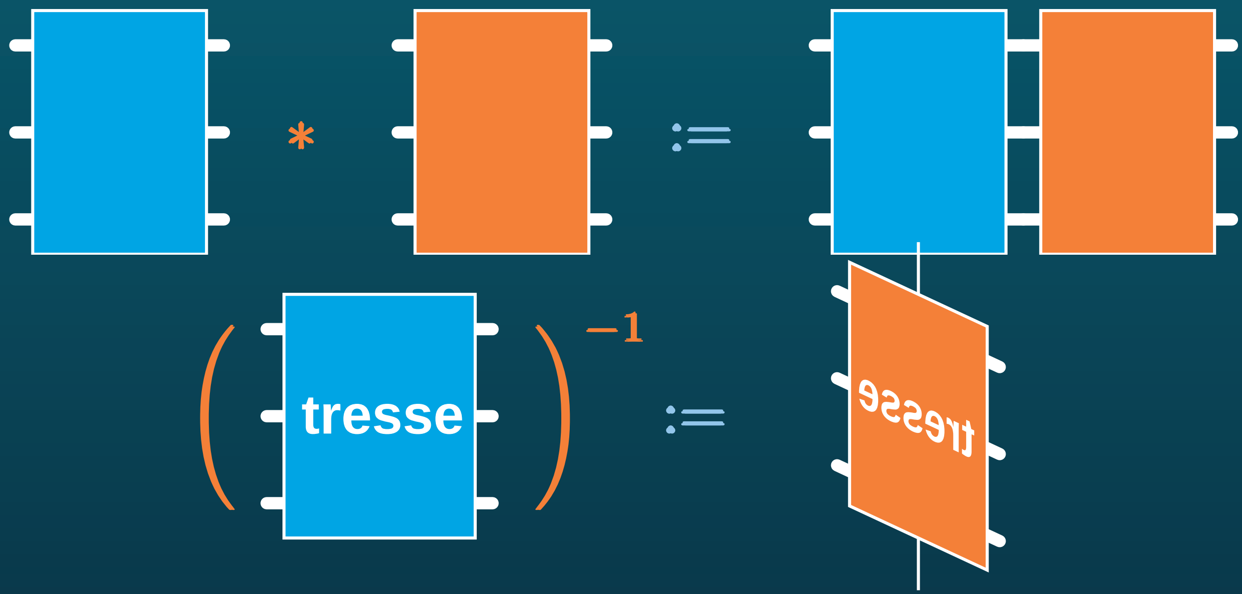

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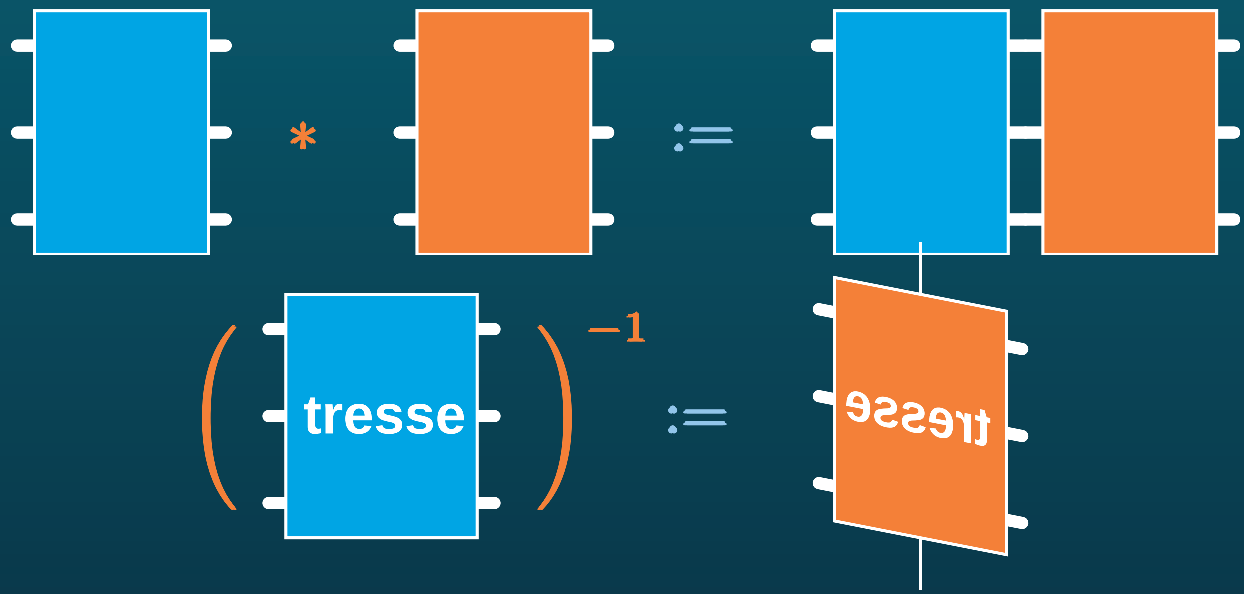

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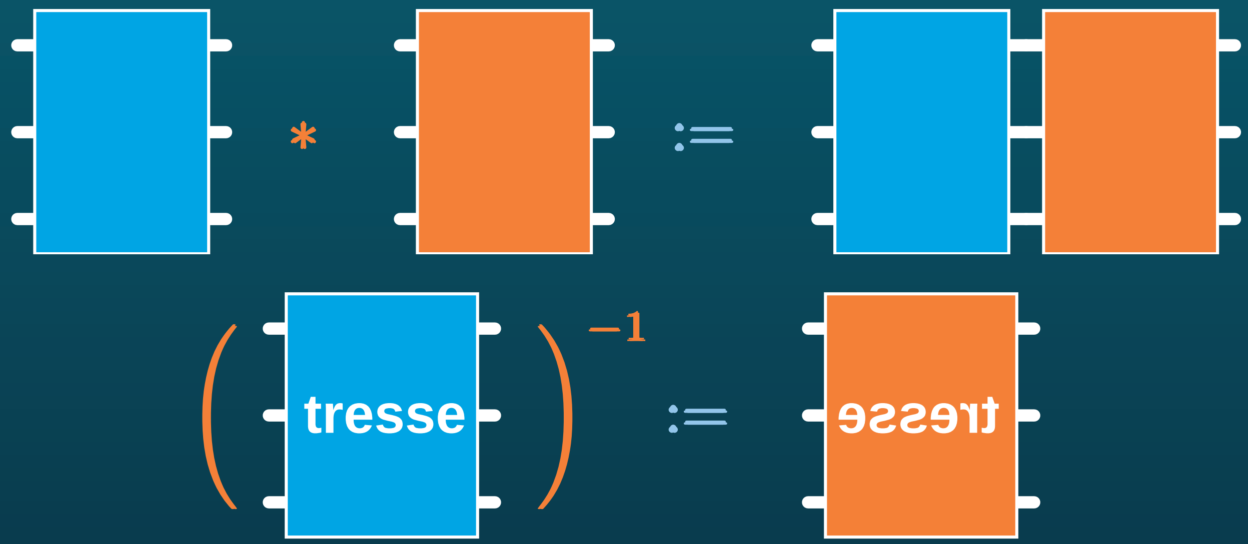

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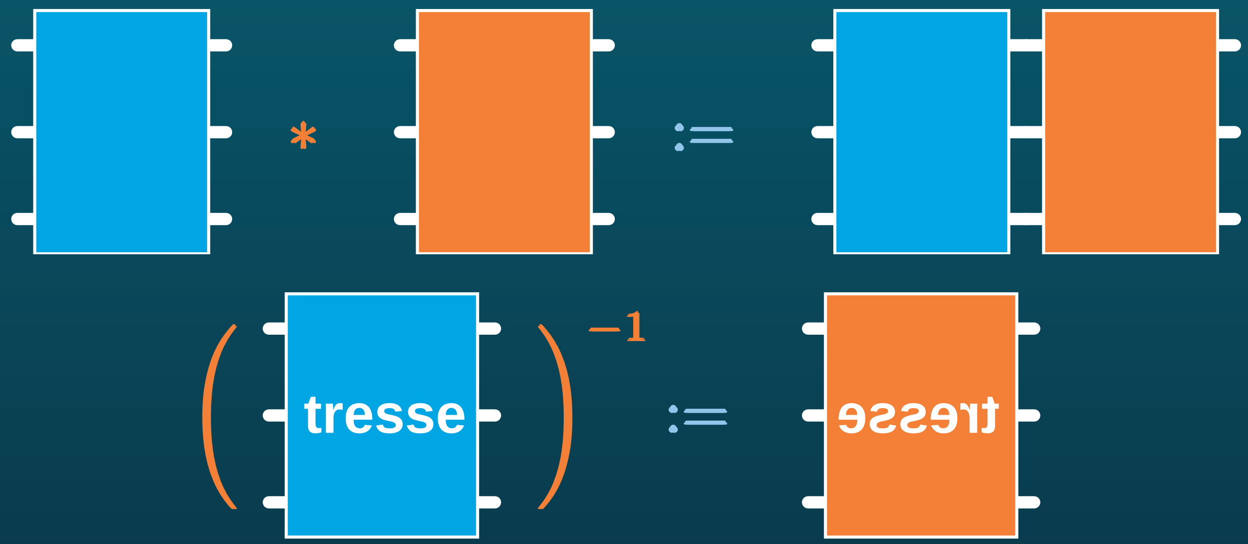

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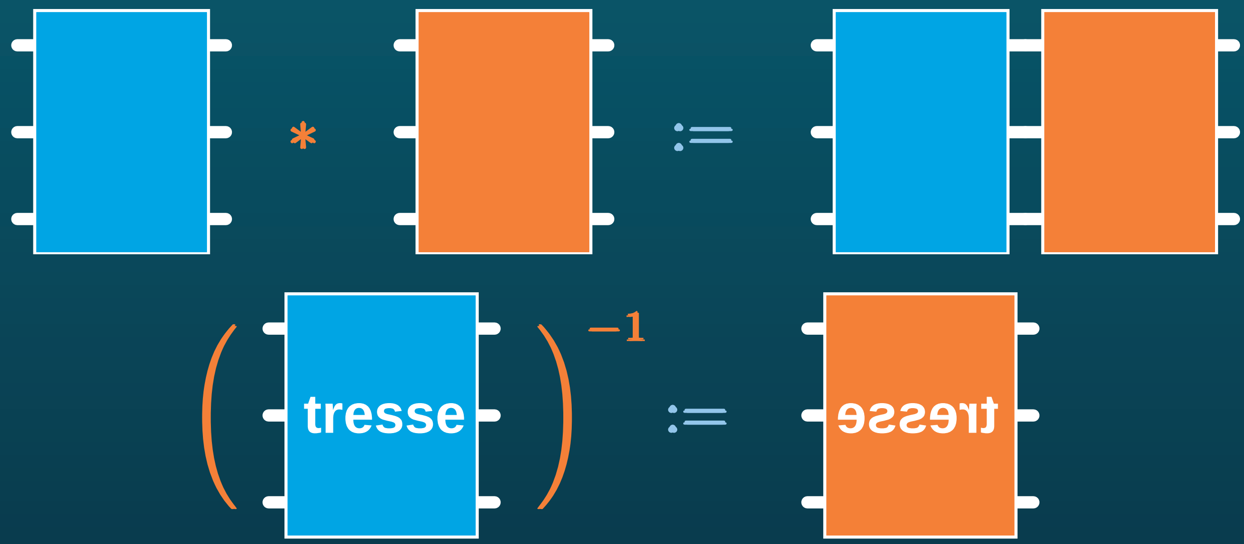

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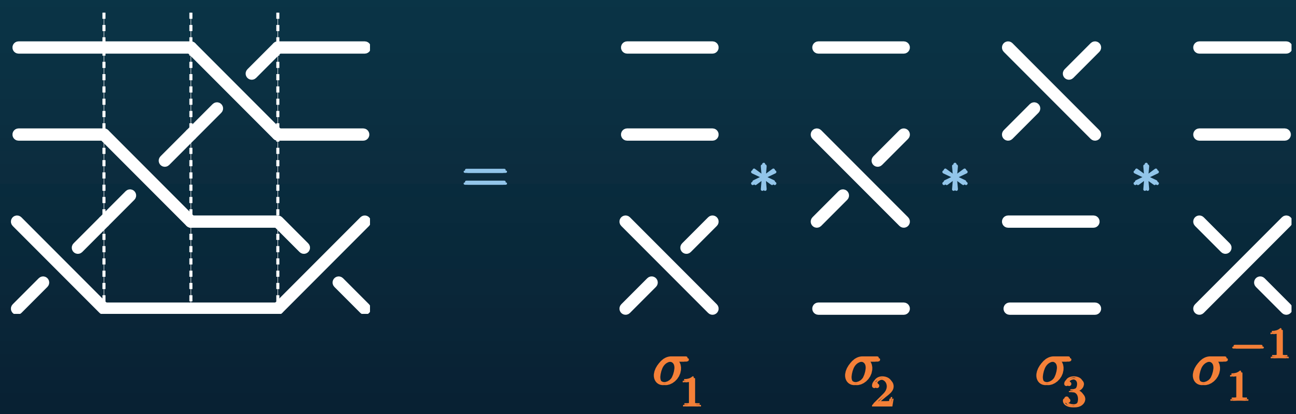
- **Presentation of B_n:**



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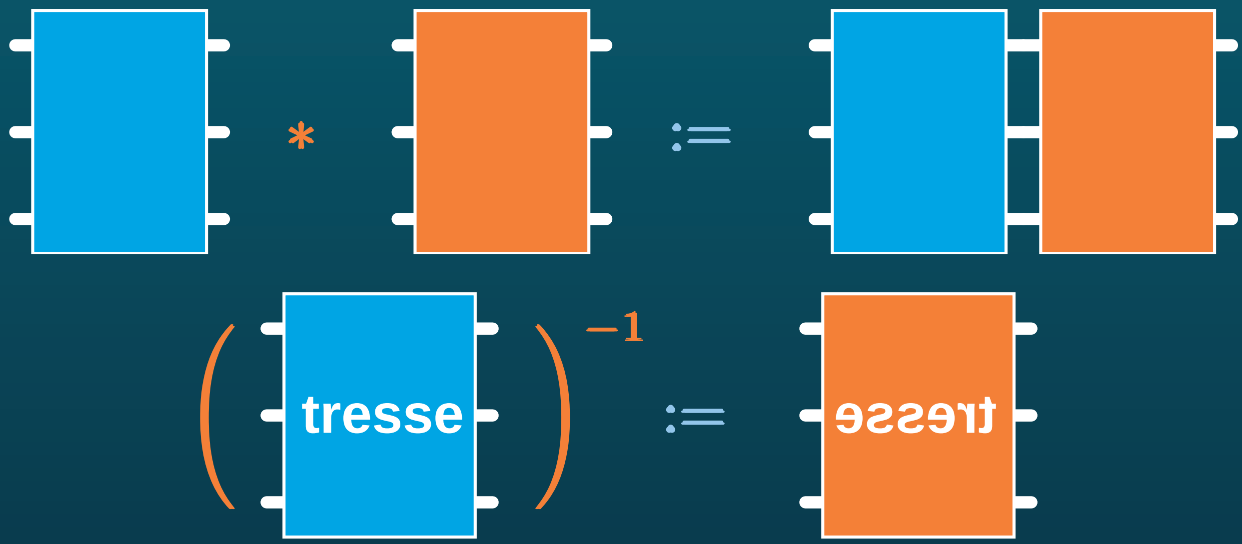

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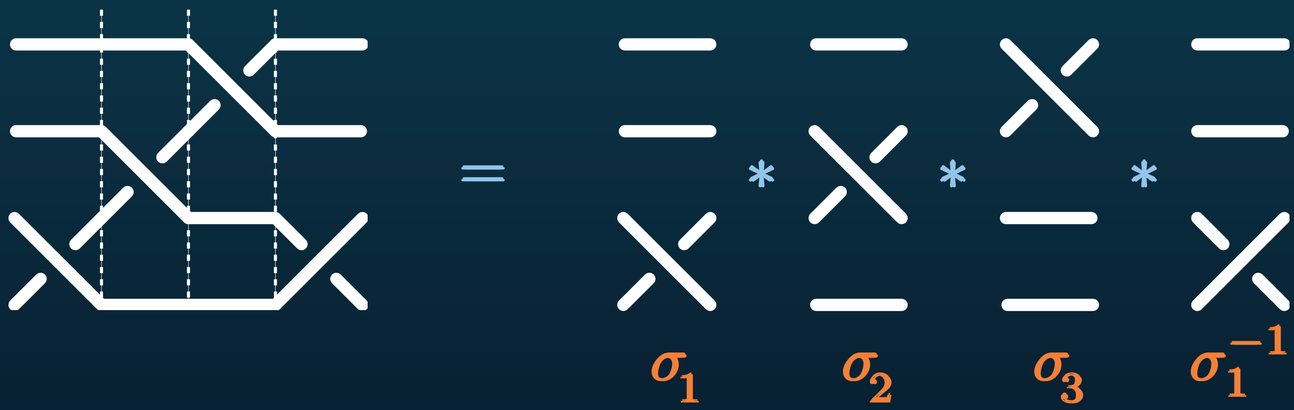

- **Theorem (Artin): The braid group B_n is generated by $\sigma_1, \ldots, \sigma_{n-1}$, subject to the relations**

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ with } |i-j| \geqslant 2, \quad \text{and} \quad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ with } |i-j| = 1.$$

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- **Improvement:** A sends $H(sxs^{-1})$, and B checks $y = H(rqr^{-1})$ with H a hash function.

AUTHENTICATION (bis)

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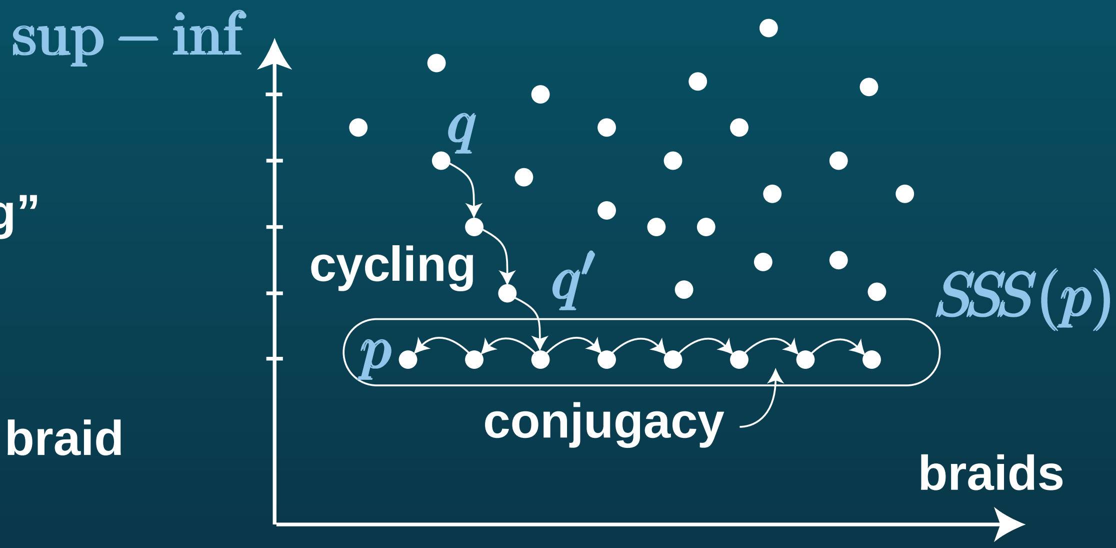

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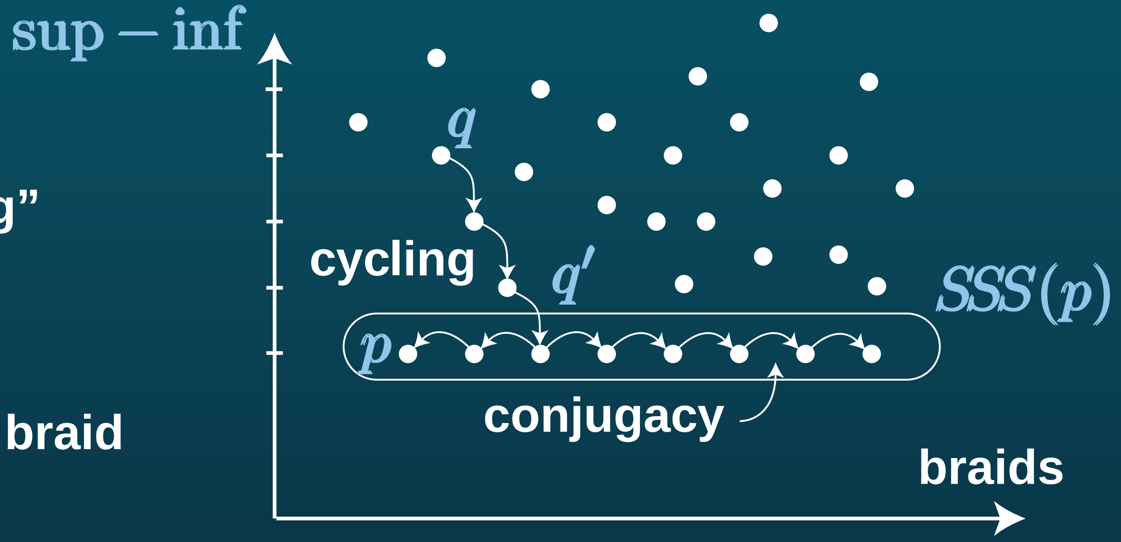

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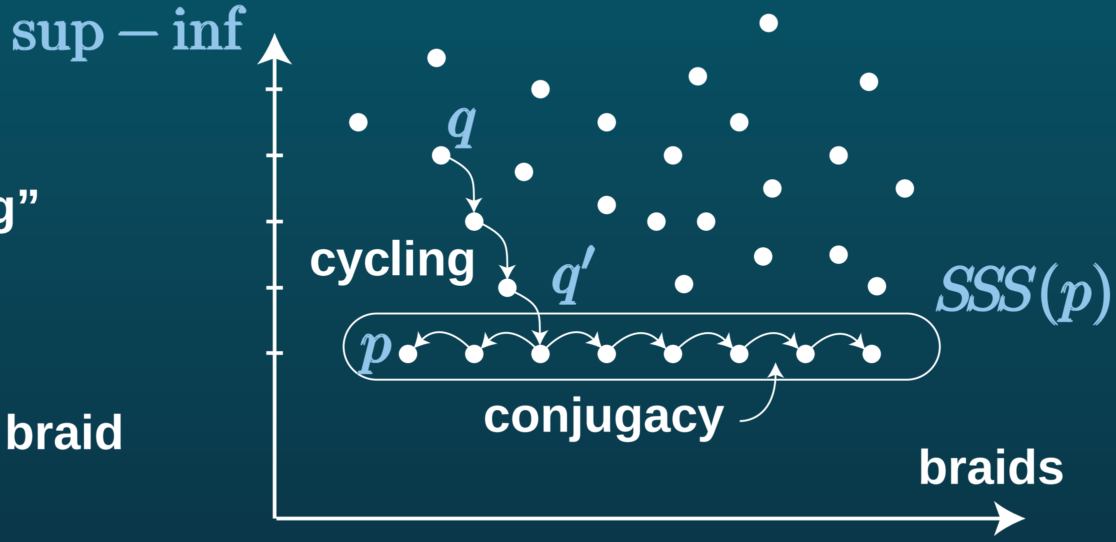

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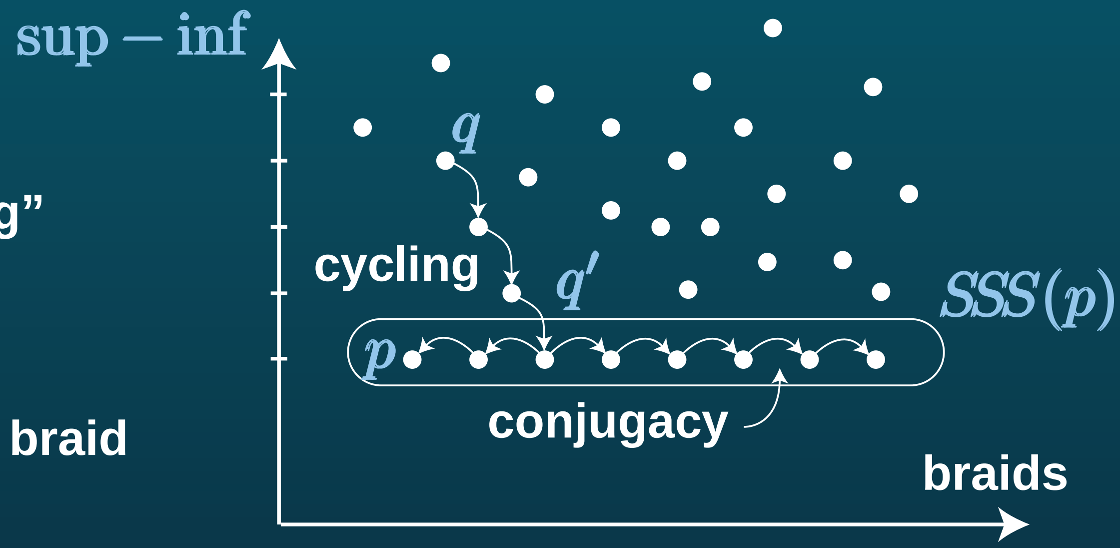

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⇝ main problem: choosing the instances (cf. RSA...)

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  ⇝ several options:

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  ⇝ Find families of braids with large SSS and USS (Ko, Lee);
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↑ the shift endomorphism $\sigma_i \mapsto \sigma_{i+1}$ for each i

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