ANOTHER APPROACH TO THE BRAID ORDERING

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- There exists a distinguished order on B_n, with many equivalent approaches
  (self-distributive algebra, homeomorphisms of a puntured disk, hyperbolic geometry...)

$\rightsquigarrow$ a new combinatorial approach, based on the connection between
- the order on B_n, - the Garside structure on B_n^+.

$\rightsquigarrow$ leads to - new questions/results about the combinatorics of DivΔ_n^d,
- new construction of $<$,
- new proof of the (not yet understood) well-ordering property
  $\rightsquigarrow$ completed for B_3 only.

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Sigma-positive braid words

- **Definition:** A braid word is **σ-positive** if the main generator (⇝ the one with maximal index) occurs only positively.

  ⇝ **Example:** $\sigma_1 \sigma_2 \sigma_1^{-1}$ is σ-positive, $\sigma_2^{-1} \sigma_1 \sigma_2$ is not (and not σ-negative either)

- **Property A:** (⇝ Acyclicity) A braid with a σ-positive expression is not 1.

- **Property C:** (⇝ Comparison) Each braid not equal to 1 admits a σ-positive expression, or a σ-negative expression.

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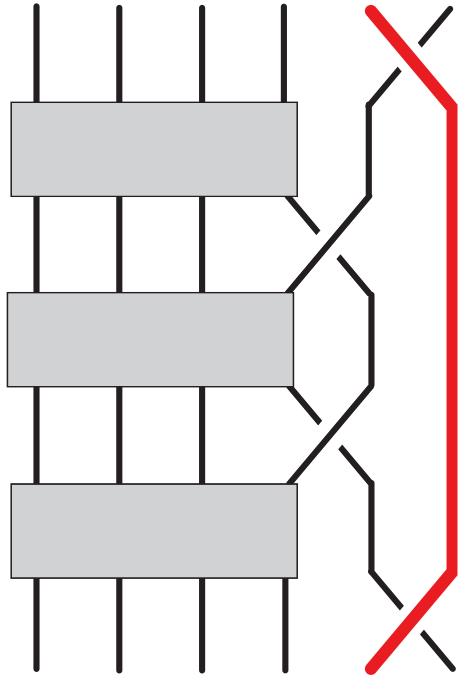
- **Proposition:** For x, y in B_∞, say that $x < y$ holds if $x^{-1}y$ admits a σ-positive expression. Then $<$ is a linear ordering, compatible with multiplication on the left.

(Property A $\Rightarrow$ no cycle; Property C $\Rightarrow$ linear)

Handle reduction

If a braid word w is
neither σ-positive nor σ-negative,
it contains both σ_m and σ_m^{-1}

$\rightsquigarrow$ a **handle**:



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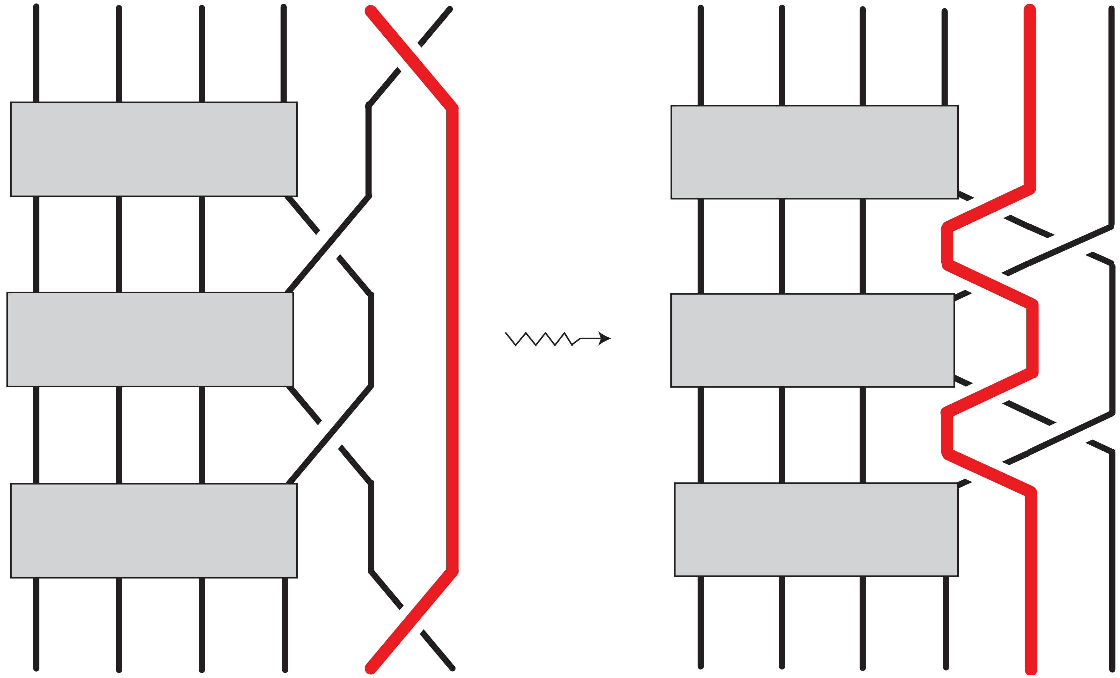
⇝ a **handle**:

⇝ **reduce** the handle:



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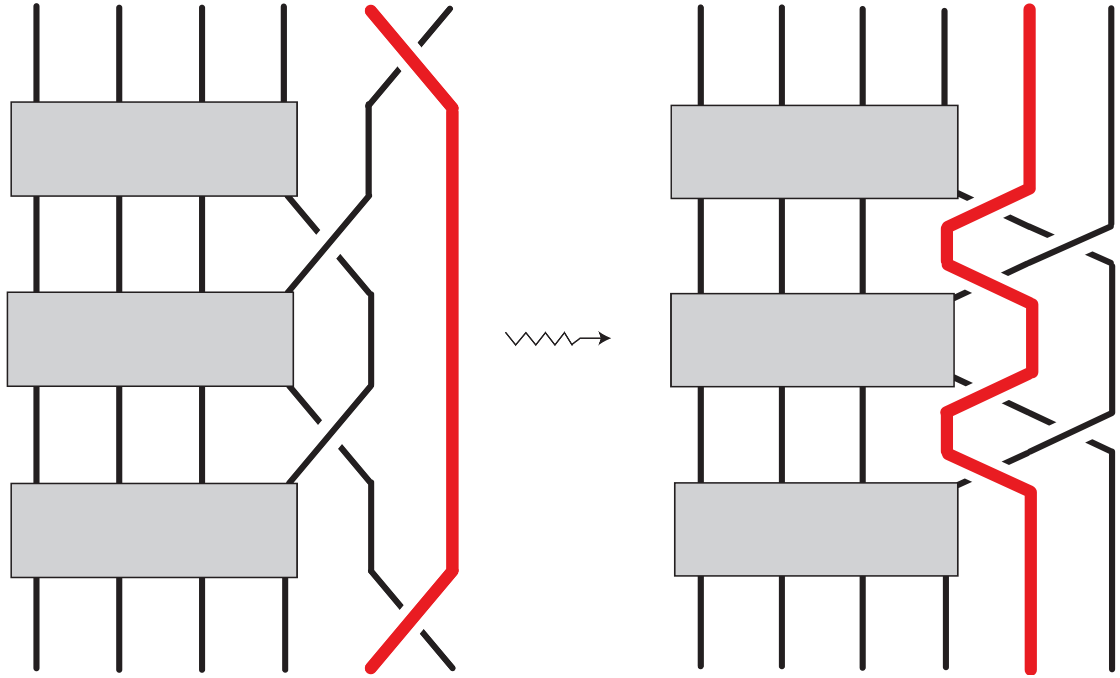

- **Proposition:** Handle reduction always terminates.

⇝ Corollary: Property C.

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The complexity of a braid

For z a positive braid, let **Div**z := the left divisors of z.

$\rightsquigarrow$ "the word w is drawn in **Div**z"

= is drawn in the restriction of the Cayley graph of B_∞^+ to **Div**z.

- **Lemma:** Assume w is drawn in **Div**z and S be a sequence of handle reductions from w. Then there exists a σ-positive word $\widetilde{S}$ ("witness-word") drawn in **Div**z s.t.

$$\# \text{ of main reductions in } S \quad = \quad \# \text{ of } \sigma_m\text{'s in } \widetilde{S}.$$

- **Definition:** The complexity of z:

$$c(z) := \text{maximal } \# \text{ of } \sigma_m\text{'s in a } \sigma\text{-positive word drawn in } \mathbf{Div}z.$$

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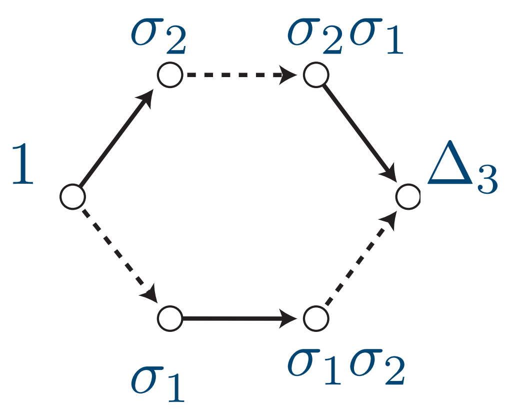
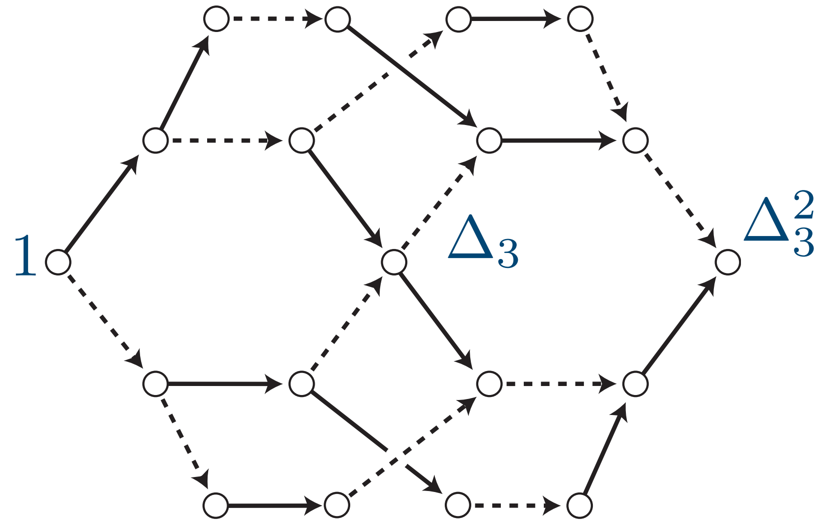
The complexity of a braid (cont'd)

- Property A $\Rightarrow$ $c(z) < \infty$.
- For w drawn in **Div**(z), at most $c(z)$ main reductions in any reduction sequence from w.
- Every n strand braid word drawn in **Div**Δ_n^d for some d

  $\rightsquigarrow$ **Main question** : What is $c(\Delta_n^d)$?





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- **Property A** $\Rightarrow$ $c(z) < \infty$.
- **For** w **drawn in Div**(z)**, at most** $c(z)$ main **reductions in any reduction sequence from** w**.**
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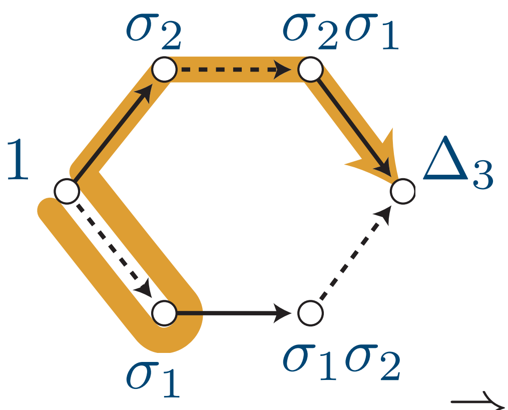
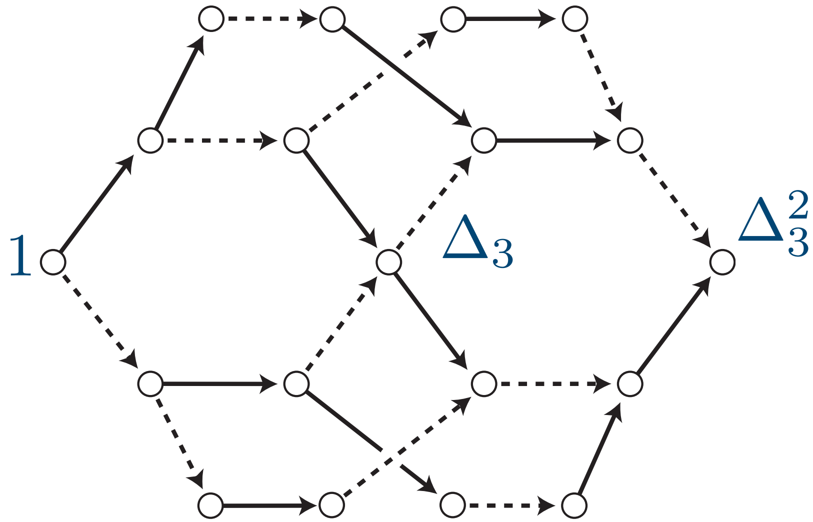


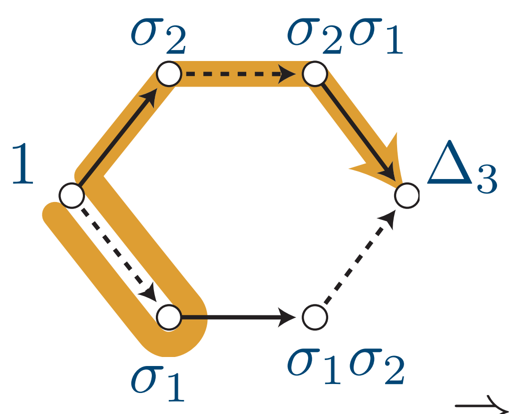
$\Rightarrow c(\Delta_3) \geqslant 2$



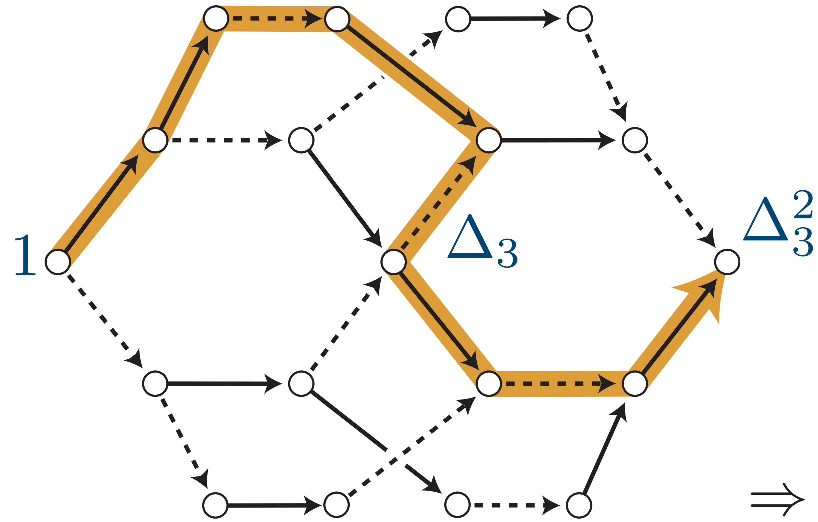
The complexity of a braid (cont'd)

- **Property A** $\Rightarrow$ $c(z) < \infty$.
- **For w traced in Div(z), at most $c(z)$ main reductions in any reduction sequence from w.**
- **Every n strand braid word traced in DivΔ_n^d for some d**

  $\rightsquigarrow$ **Main question : What is $c(\Delta_n^d)$?**



$\Rightarrow c(\Delta_3) \geqslant 2$



$\Rightarrow c(\Delta_3^2) \geqslant 5$

Remark: $\Delta_3 = \sigma_2^{k+1}\sigma_1\sigma_2\sigma_1^{-k}$.

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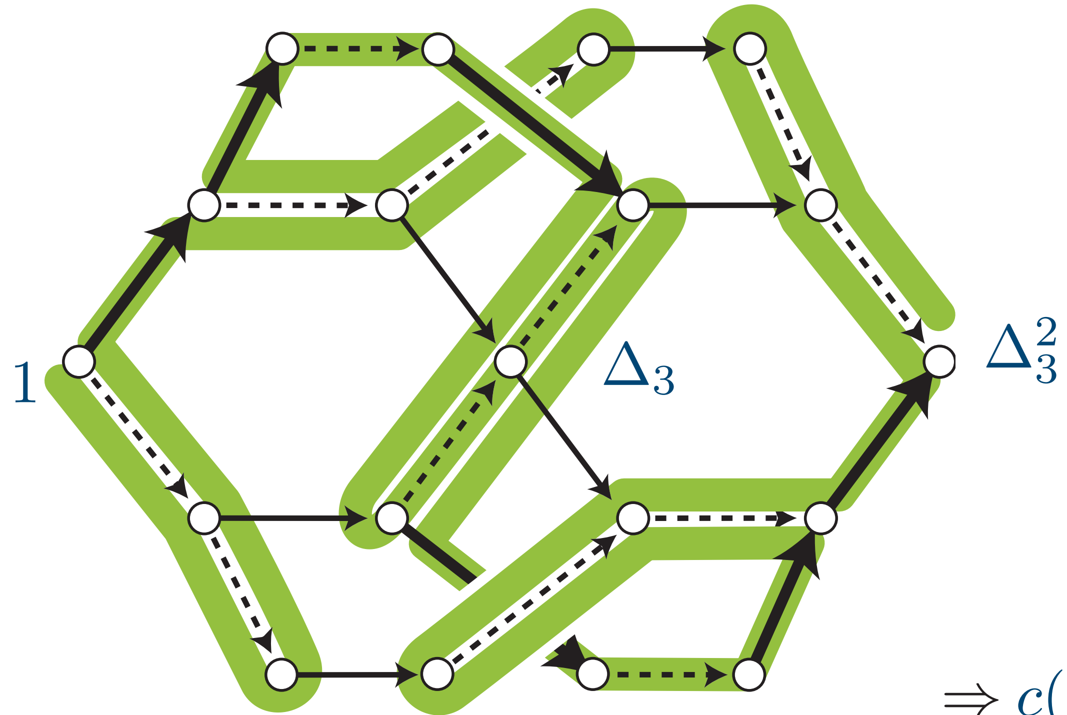
Computation of $c(\Delta_n^d)$, step 1

● **Proposition:** $c(z)$ **is the** $\#$ **of main jumps** ($\rightsquigarrow$ the main generator in the quotient of two successive entries is the maximal one) **in the increasing enumeration of Div** z.

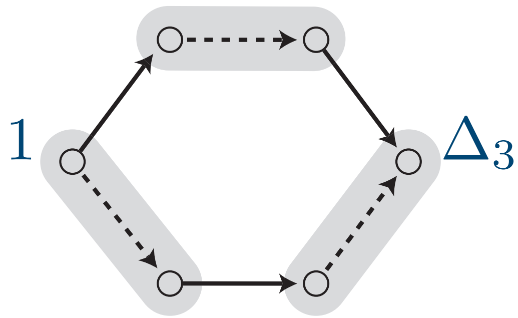


$\Rightarrow c(\Delta_3^2) = 6$

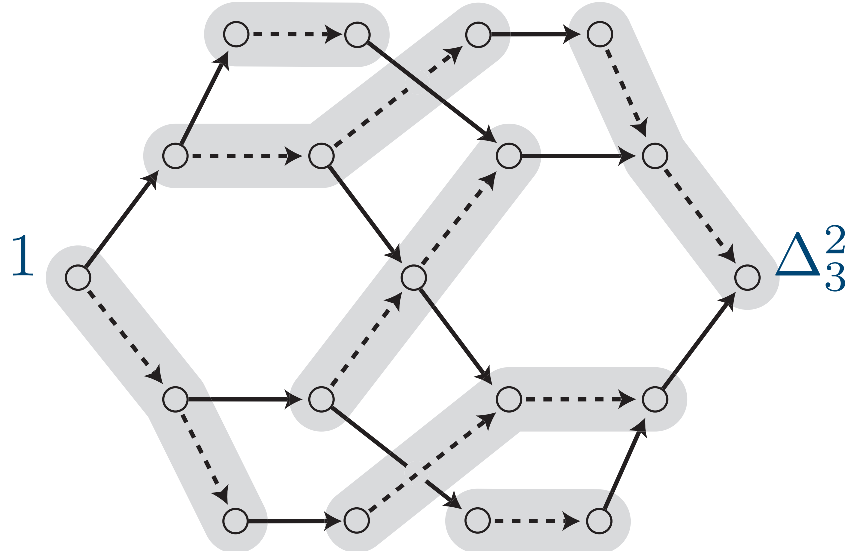
Computation of $c(\Delta_n^d)$, step 2

- **Proposition:** $c(z) + 1$ **is the $\#$ of B_m-classes (⇝ w.r.t. $x^{-1}y \in B_m$) in Div$z$.**

Examples:



3 B_2-**classes** $\Rightarrow c(\Delta_3) = 2$



7 B_2-**classes** $\Rightarrow c(\Delta_3) = 6$

Computation of $c(\Delta_n^d)$, step 3

- **Proposition:** $c(\Delta_n^d) + 1$ = $\#$ of elements in DivΔ_n^d whose dth factor (of the normal form) is right divisible by Δ_{n-1}.

⇝ Compute $N_{n,d}(s)$:= $\#$ of (positive) braids of degree $\leqslant d$ whose dth factor is s
= $\#$ of normal sequences $(s_1, \ldots, s_{d-1}, s)$.

Computation of $c(\Delta_n^d)$, step 3

- **Proposition:** $c(\Delta_n^d)$ = $\#$ of elements in DivΔ_n^d whose dth factor (of the normal form) is right divisible by Δ_{n-1}, minus 1.

⇝ Compute $N_{n,d}(s)$:= $\#$ of (positive) braids of degree $\leqslant d$ whose dth factor is s
= $\#$ of normal sequences $(s_1, \ldots, s_{d-1}, s)$.

⇝ Easy, because normality is local:

- **Proposition:** (Charney, folklore) Let M_n be the $n! \times n!$ matrix with entries indexed by divisors of Δ_n (or permutations of $\{1, ..., n\}$) s.t.

$$(M_n)_{s,t} = \begin{cases} 1 & \text{if } (s,t) \text{ is normal,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $N_{n,d}(s)$ is the s-entry in $(1, ..., 1) \cdot M_n^{d-1}$.

Counting results

⇝ (new) questions / results about braid combinatorics:

⇝ (Solomon) $N_{n,d}(s)$ only depends on the partition associated with the descents of s

⇝ reduces the size of the adjacence matrix from $n!$ to $p(n)$.

- **Proposition:** $c(\Delta_3^d) = 2^{d+1} - 2$, $c(\Delta_4^d) = \ldots \sim (3 + \sqrt{6})^d$, etc.

$$c(\Delta_n) = n - 1, \quad c(\Delta_n^2) = 2^n - 2, \quad c(\Delta_n^3) = \sum_{k=1}^{n-1} \frac{n!}{k!} - 1.$$

Proof: Let $I, J \subseteq \{1, \ldots, n-1\}$, with $(p_1, \ldots, p_k)$, $(q_1, \ldots, q_\ell)$ the block compositions of I and J. Then the $\#$ of divisors of Δ_n that are left divisible by all σ_i's with $i \in I$ and right divisible by all σ_j's with $j \in J$ is the $\#$ of $k \times \ell$-matrices with $\mathbb{N}$-entries s.t. $\sum$ ith row $= p_i$ and $\sum$ jth column $= q_j$. Then $c(\Delta_n^3) + 1 = \sum_{p_1 + \cdots + p_k = n} \frac{n!}{p_1! \ldots p_k!} p_1(p_2 - 1) \ldots (p_{k-1} - 1) p_k$.

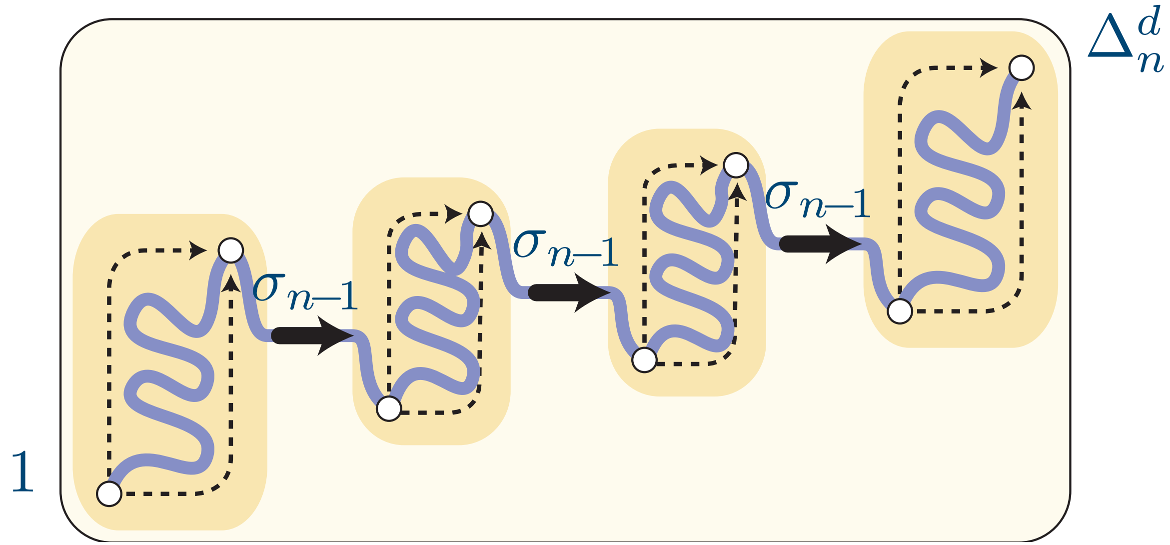
More counting

- **Proposition:** (Carlitz, Scoville, Vaughan) Let a_n be the $\#$ of n-braids of degree $\leqslant 2$. Then
$$1 + \sum a_n \frac{z^n}{(n!)^2} = \frac{1}{J_0(\sqrt{z})}$$
with $J_0(x)$ the Bessel function.

- **Conjecture:** The characteristic polynomial of M_{n-1} divides that of M_n.

Canonical decomposition of $(\mathbf{Div}\Delta_n^d, <)$

● **Proposition:** $(\mathbf{Div}\Delta_n^d, <)$ **is the concatenation of $c(\Delta_n^d) + 1$ intervals, each of which is a lattice for divisibility, isomorphic to some initial subinterval of $(\mathbf{Div}\Delta_{n-1}^d, <)$.**



⇝ **Inductive construction of $(\mathbf{Div}\Delta_n^d, <)$ from $(\mathbf{Div}\Delta_{n-1}^d, <)$ — and $(\mathbf{Div}\Delta_n^{d-1}, <)$?**

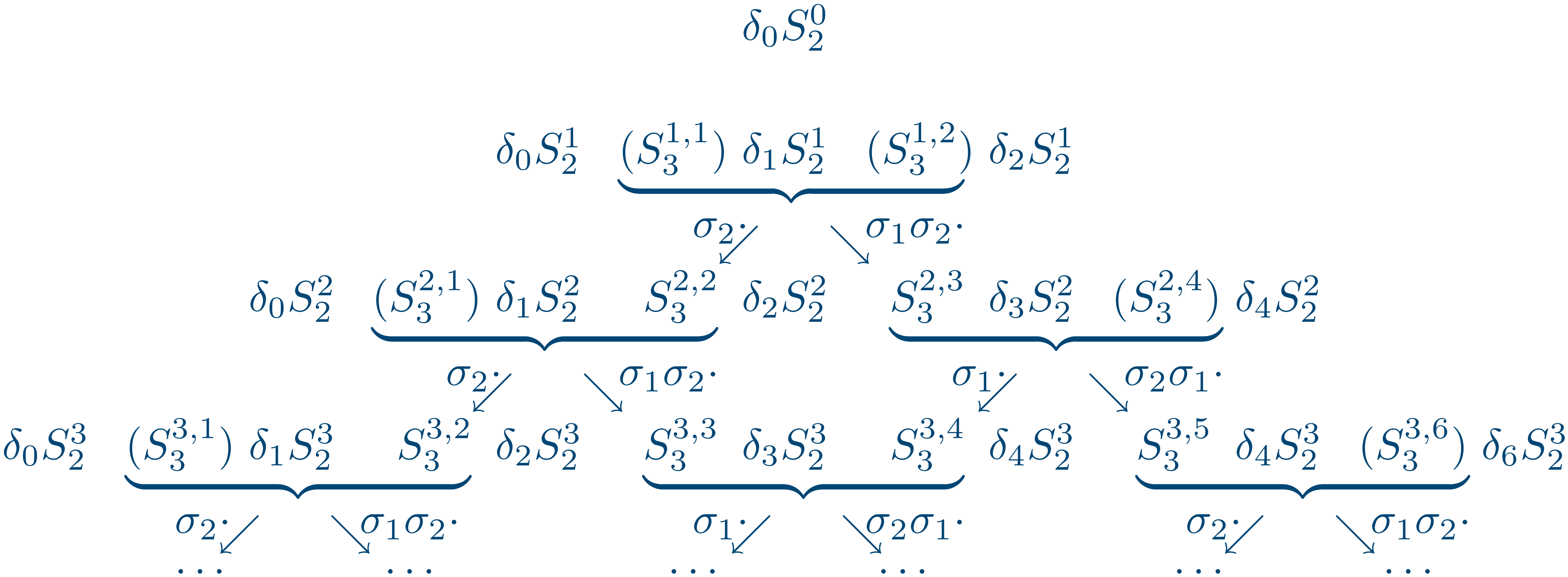
Genetic sequences

⇝ **Method: look at the successive quotients in the increasing enumeration of** $(\mathbf{Div}\Delta_n^d, <)$**:**

- **degree** 1**:** a Ab a bA a **(with** $\mathtt{a} = \sigma_1$, $\mathtt{A} = \sigma_1^{-1}$, $\mathtt{b} = \sigma_2$, **etc.)**
- **degree** 2**:** a a AAb a a AAb a bAA a a AAb a bAA a a bAA a a
- **degree** 3**:** a a a AAAb a a a AAAb a a AAb a bAAA a a a AAAb a a AAb a bAAA a a ...

- **Proposition:** **The increasing enumeration of** $(\mathbf{Div}\Delta_3^d, <)$ **is given by ...**

The Pascal triangle



with δ_k = length k suffix of $\ldots\sigma_1^2\sigma_2^2\sigma_1^2\sigma_2$ and $S_2^d = (1, \sigma_1, \ldots, \sigma_1^d)$ (i.e., the increasing enumeration of $($Div$(\Delta_2^d), <)$).

(Proof: Property A $\Rightarrow$ injectivity; cardinalities $\Rightarrow$ surjectivity.)

Applications

$\rightsquigarrow$ **Corollary: - New and easy proof of Property C for B_3;**

- New and easy proof of the existence of the Burckel normal form for 3-braids;

- New and easy proof of the fact that $(B_3^+, <)$ is a well-ordering of ordinal type ω^ω.

$\rightsquigarrow$ **Can this be done for $n \geqslant 4$?**

$\rightsquigarrow$ **Replace the induction $u_d = 2u_{d-1} + 3d + 1$**

with $u_d = 6u_{d-1} - 3u_{d-2} + 32 \cdot 2^d - 12d - 34$.