Patrick Dehornoy

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**Connections between:**

- **the left self-distributive law LD: $x(yz) = (xy)(xz)$, and**
- **Artin's braid groups and related groups—*e.g.*, the group of parenthesized braids.**

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**Plan:**

- LD-systems
  - LD-systems from sets
    - LD-systems from braids
      - LD-systems from parenthesized braids
        - The geometry monoid of an algebraic law
          - The Laver tables

- A special case: LRD-systems ⇝ **LD** + **RD**: $(xy)z = (xz)(yz)$

  ⇝ Typical examples: - lattice $\inf$ and $\sup$;

  - mean: $x{*}y = (1-t)x + ty$.

  ⇝ All idempotent: $xx = x$.

- For $S$ an LRD-system, every element of $S(SS)$ and of $(SS)S$ is idempotent.

  ⇝ Proof: $(x(xx))(x(xx)) =_{\text{RD}} (xx)(xx) =_{\text{LD}} x(xx)$. □

- **Proposition:** If $S$ is an LRD-system and at least one left (or right) translation of $S$ is surjective, then $S$ is idempotent.

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- **Proposition:** (Belousov?, 1960's) Assume that $(L, +)$ is a commutative Moufang loop and $f, g$ are surjective endomorphisms s.t. $f(x) + g(x) = x$ and $x + f(x)$ lies in the nucleus of $L$, for each $x$. Then $L$ equipped with $x{*}y = f(x) + g(y)$ is a divisible LRD-system. Conversely, every divisible LRD-system is of this type.

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- **Lemma: Every LD-quasigroup satisfies $(xx)y = xy$.**
  - $\rightsquigarrow$ **Proof: Assume** $xz = y$**. Then** $(xx)y = (xx)(xz) = x(xz) = xy$**.** $\square$

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- Philosophy: General LD-systems $\neq$ LRD-systems and $\neq$ LD-quasigroups
  - $\rightsquigarrow$ more reminiscent of semigroups:
    *cf.* **LD**: $x(yz) = (xy)(xz)$ *vs.* associativity: $x(yz) = (xy)z$.
  - $\rightsquigarrow$ in particular free LD-system of rank $n$ vs. free semigroup $(\mathbb{Z}_{>0}, +)^n$:
    - the (transitive closure of) $(\exists z)(y = xz)$ defines a linear ordering in rank $1$;
    - the rank $n$ system is a lexicographical extension of the rank $1$ system.

- In Set Theory, most notions are definable from $\in$.

  $\rightsquigarrow$ relevant notion of endomorphism = elementary embedding:
  $F : X \to X$ s.t. for each formula $\Phi(\vec{x})$ and $\vec{a}$ in $X$, one has $\Phi(F(\vec{a})) \Leftrightarrow \Phi(\vec{a})$.

  $\rightsquigarrow$ An e.e. preserves
  - $=$ (*i.e.*, is injective): $F(a) = F(b) \Leftrightarrow a = b$;
  - $\in$: $F(a) \in F(b) \Leftrightarrow a \in b$;
  - $\subseteq$: $F(a) \subseteq F(b) \Leftrightarrow a \subseteq b$, as $x \subseteq y$ is $(\forall z)(z \in x \Rightarrow z \in y)$;
  - $\cup$: $F(a \cup b) = F(a) \cup F(b)$, as $z = x \cup y$ is...
  - "being the image under" : (*) $F(f(a)) = F(f)(F(a))$, as $y = f(x)$ is....

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- A rank is a set $R$ with the strange property that $f : R \to R$ implies $f \in R$.
- If $R$ is a rank and $F, G$ are e.e.'s of $R$, we can apply $F$ to $G$ ( because $G \in R$ ).
  - ⇝ Then (*) becomes $F(G(H)) = F(G)(F(H))$:
    - ⇝ $\mathrm{End}R$ ( all e.e.'s of $R$ ) equipped with $-(-)$ is an LD-system.

- An LD-system is called acyclic if left divisibility has no cycle, *i.e.*, $x \neq (\dots((xy_1)y_2)\dots)y_r$.

  ( ⇝ an idempotent LD-system is never acyclic: $x = xx$ )

- **Proposition:** (D., 1989) If there exists an acyclic LD-system, then the word problem of **LD** is decidable ( ⇝ there is an algorithm that detects LD-equivalence ).

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⇝ Provided there exists a self-similar rank ( ⇝ one s.t. $\mathrm{End}R \neq \{\mathrm{id}\}$ ),

...the word problem of **LD** is decidable.

⇝ Do self-similar ranks exist? NO: an unprovable logical axiom (“large cardinal”)

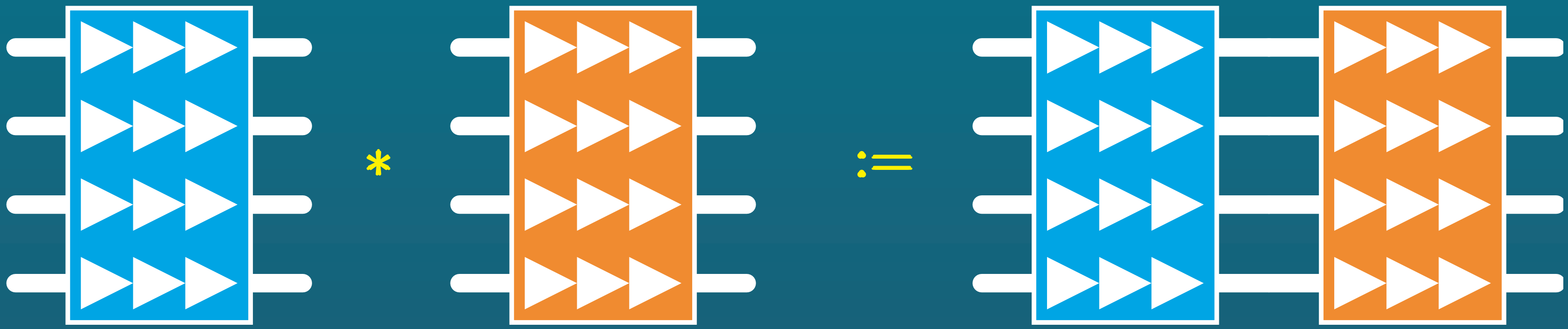
⇝ Find new examples...

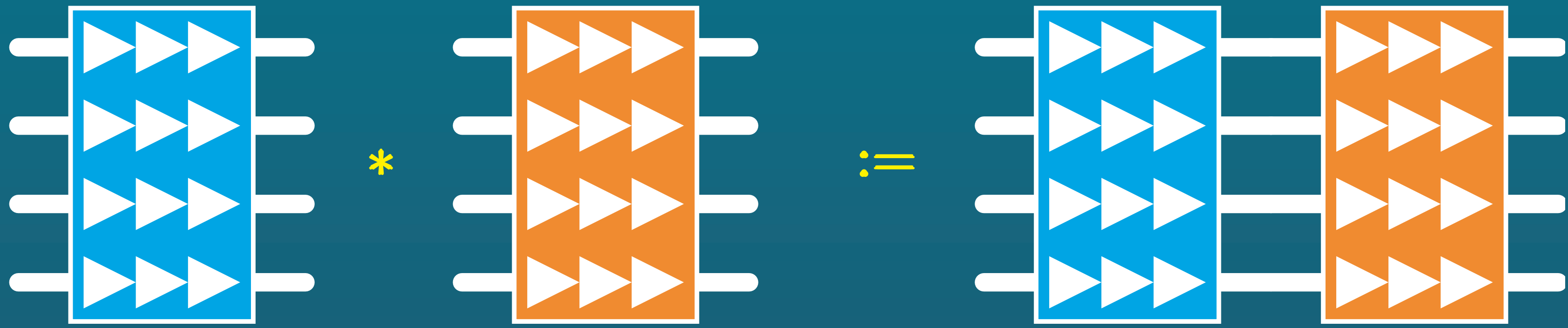
- A 4-strand **braid diagram**:

- Braid diagram as a **projection** of a 3D braid:

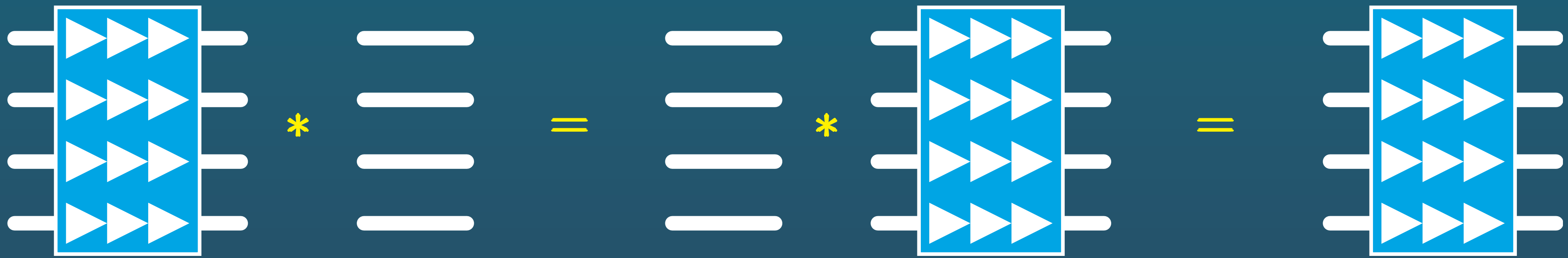
≡

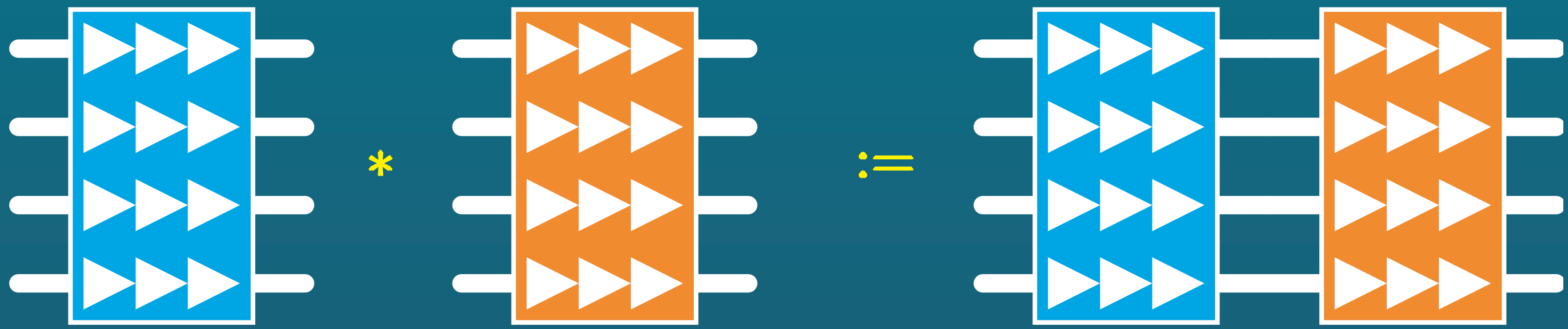
- **Projection: Isotopy of braid diagrams:**






- Up to isotopy, $n$ strand braids form a group: ⇝ Artin's braid group $B_n$





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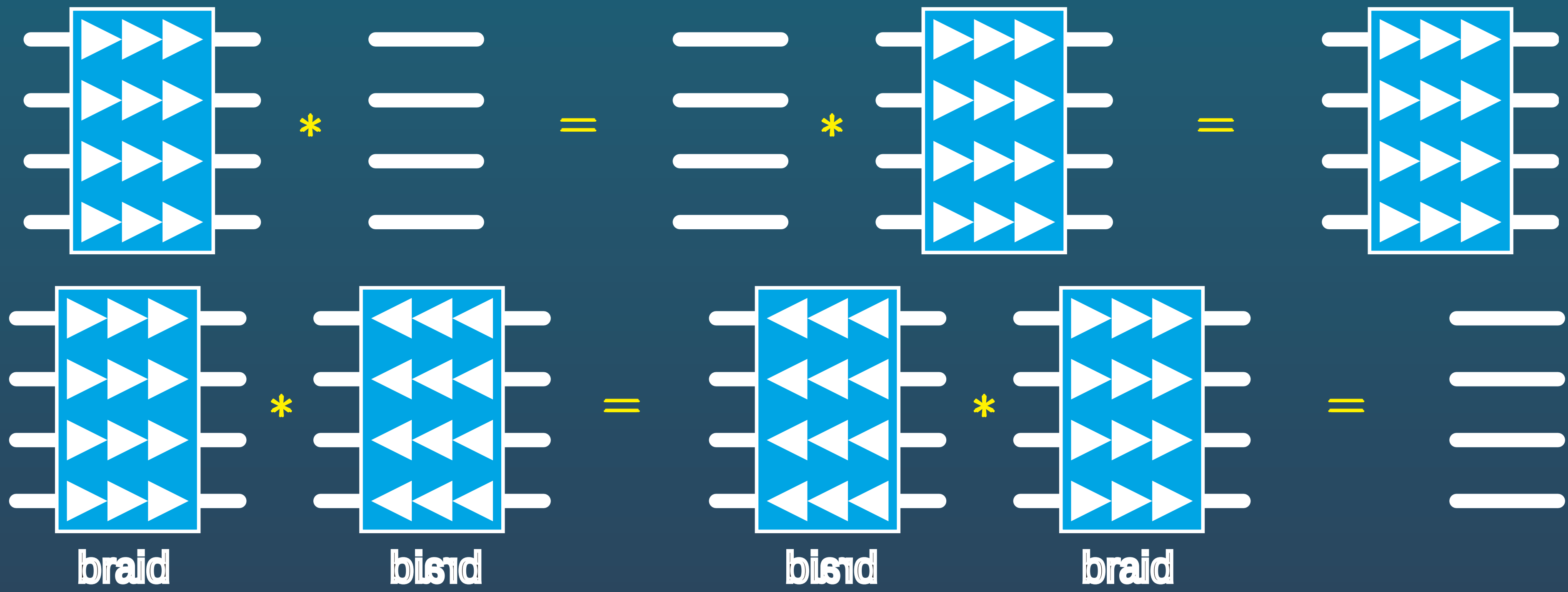

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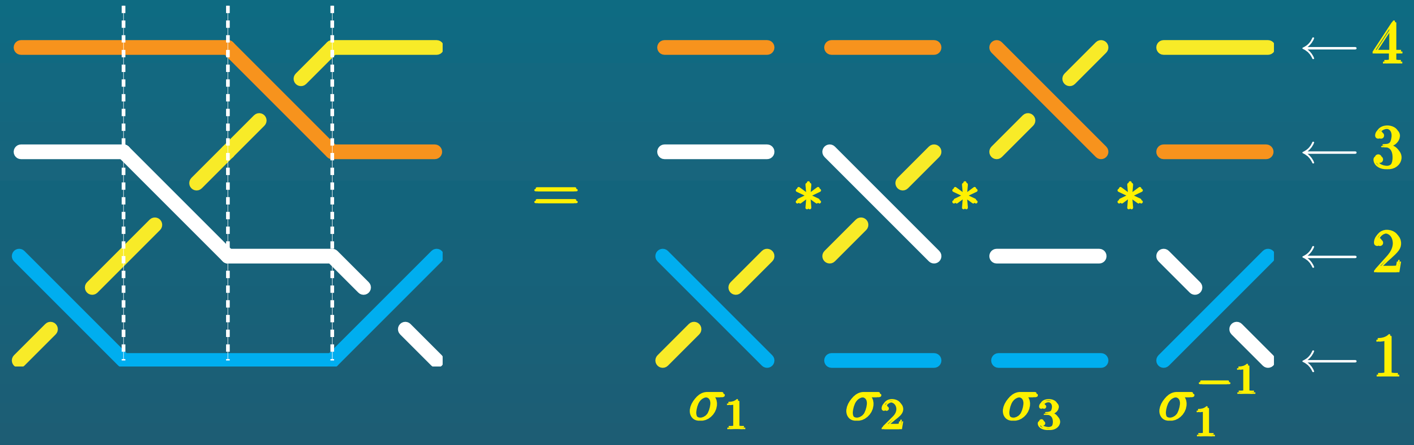

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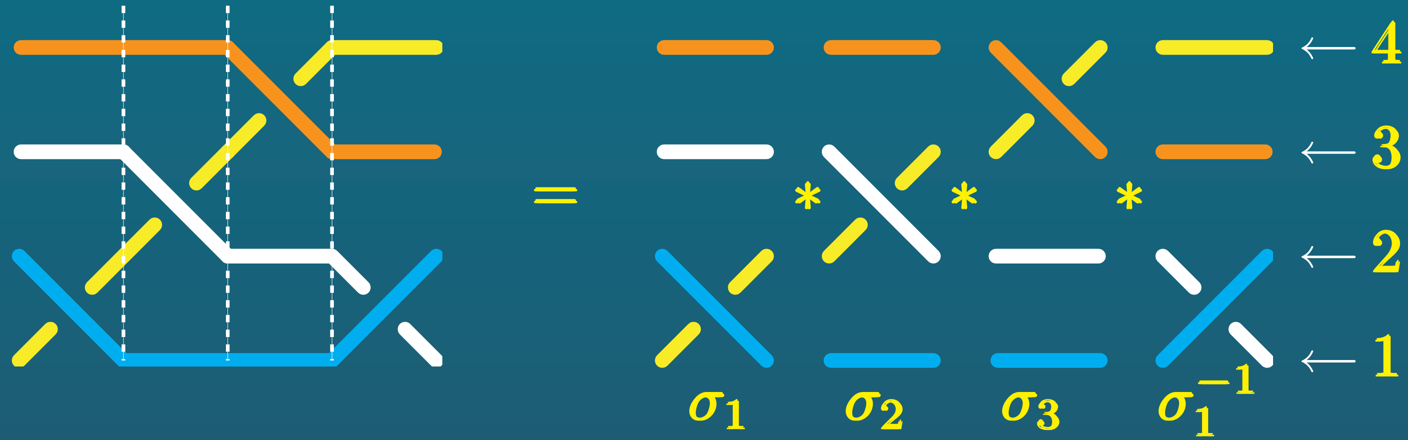


- **Proposition:** **(Artin, 1925)** The braid group $B_n$ admits the presentation $\langle \sigma_1, \ldots, \sigma_{n-1} \,;\, \sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geqslant 2,\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ for $|i-j| = 1\rangle$.

$\sigma_i\sigma_j = \sigma_j\sigma_i$

$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

$\rightsquigarrow$ Cf. Coxeter presentation of $S_n$: idem + $\sigma_i^2 = 1$.

- The **shift** endomorphism of $B_\infty$: $\partial : \sigma_i \mapsto \sigma_{i+1}$

- **Definition:** For $x, y$ in $B_\infty$, put $x * y := x \cdot \partial y \cdot \sigma_1 \cdot \partial y^{-1}$.

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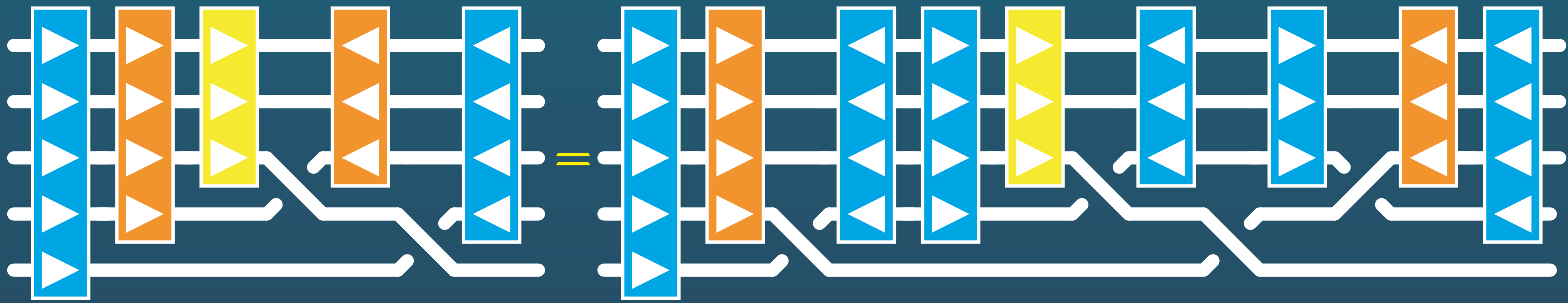
- Proposition: $(B_\infty, *)$ is an acyclic LD-system; under $*$, every braid (for instance $1$) generates a free LD-subsystem.



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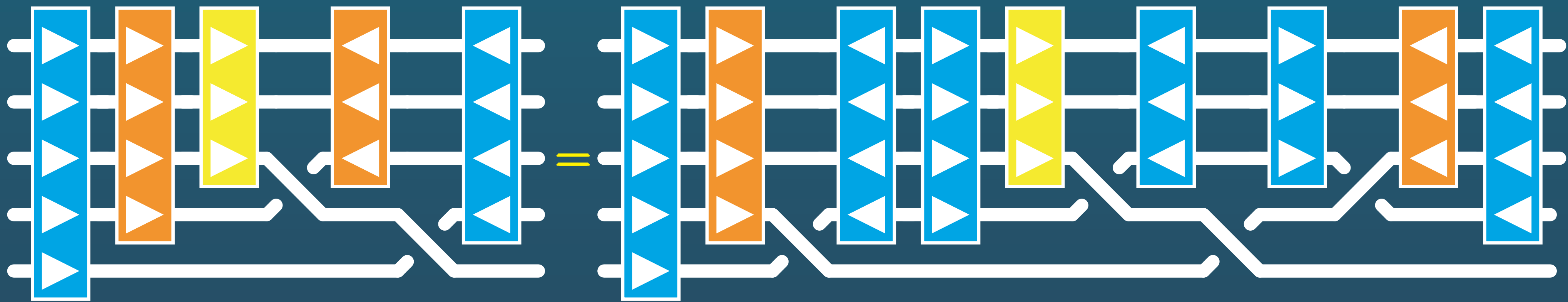

- Remark: For $G$ a group, $\partial$ in $\mathrm{End} G$ and $x * y = x \cdot \partial y \cdot \sigma \cdot \partial x^{-1}$,
then $(G, *)$ is an LD-system iff $\sigma, \partial\sigma, \partial^2\sigma, \ldots$ generate an image of $B_\infty$.

- Question: Does $(B_\infty, *)$ include free LD-systems of rank $\geqslant 2$ ?

- Many LD-systems admit a second operation $\circ$ satisfying
  - $(x \circ y) * z = x * (y * z)$, —*i.e.*, $L_{x \circ y} = L_x \circ L_y$,
  - $x * (y \circ z) = (x * y) \circ (x * z)$ —*i.e.*, $L_x$ is a $\circ$-homomorphism

  ⇝ Call this an **augmented** LD-system (ALD-system).

- If, in addition, $\circ$ is associative and one has $(x * y) \circ x = x \circ y$:

  ⇝ Call this an **LD-semigroup** (an **LD-monoid** if $1$ unit for $\circ$ with $x * 1 = 1$ and $1 * x = x$).

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- **Proposition:** For each LD-system $S$, there exists an LD-monoid $\widehat{S}$ extending $S$ and universal for this property.

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  ⇝ Proof: $\widehat{S}$ := finite sequences from $S$, quotiented under $(x*y) \circ x = x \circ y$. □

⇝ ”Real” question: For $(S, *)$ an LD-system, is there $\circ$ on $S$ so that $(S, *, \circ)$ is an ALD-system (or an LD-monoid) ?

  ⇝ Case of group conjugation: OK with $\circ$ = group product.

  ⇝ Case of $B_\infty$: impossible to have $x * (y * z) = t * z$.

- **Ordinary braid diagrams: equidistant strands indexed by positive integers;**
- **Parenthesized braid diagrams: non-uniform (infinitesimal) distances:**


**initial positions:** $(\bullet\bullet)\bullet$ **final positions** $\bullet(\bullet\bullet)$

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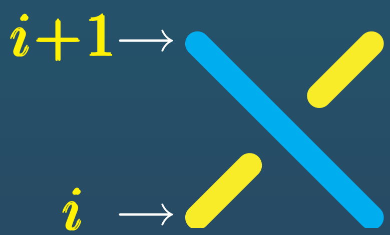
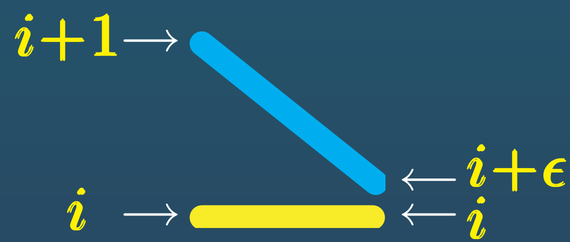
⇝ **Two types of elementary diagrams:**

**crossing $\sigma_i$ :**




**grouping $a_i$ :**




⇝ **A new group, the group of parenthesized braids $B_\bullet$.**

- **Commutation relations:**

$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

$$\sigma_i a_j = a_j \sigma_i$$

- **Commutation relations:**

$$\sigma_i\sigma_j = \sigma_j\sigma_i$$

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- **Thompson's relations:**

$$a_i a_j = a_{j+1} a_i$$

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- **Braid relations:**

$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

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- Def: For $x, y$ in $B_\bullet$, put $x * y := x \cdot \partial y \cdot \sigma_1 \cdot \partial x^{-1}$, and $x \circ y := x \cdot \partial y \cdot a_1$.

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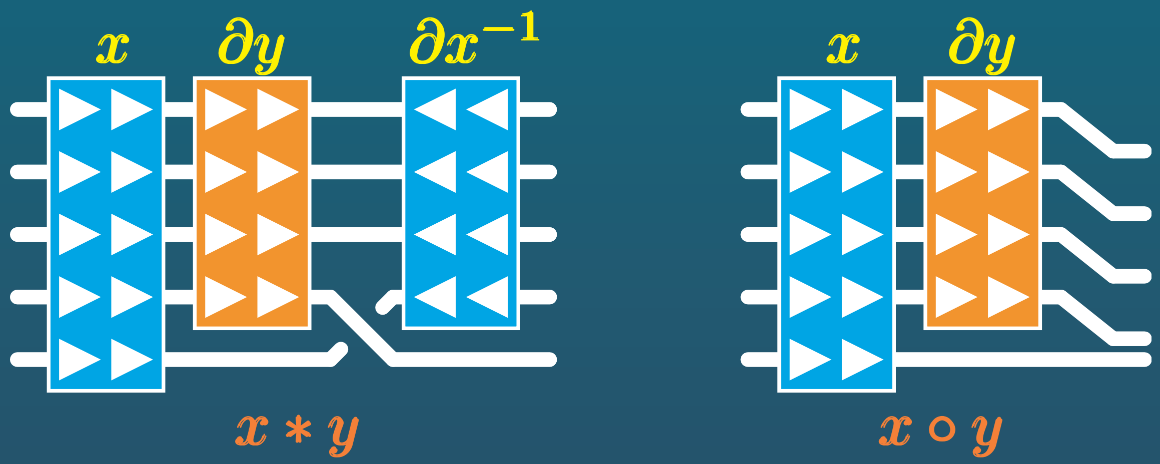

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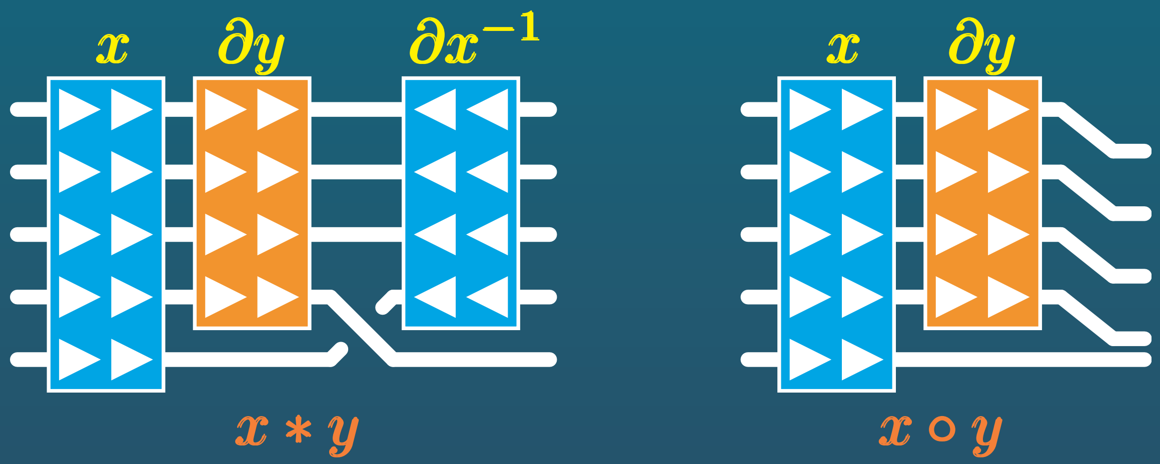


- **Proposition:** $(B_\bullet, *, \circ)$ is an acyclic ALD-system; under $*$ and $\circ$, every element of $B_\bullet$ (for instance $1$) generates a free ALD-subsystem.

$\rightsquigarrow$ OK, but where do these definitions come from ???

- Associate with LD a certain monoid that captures its geometry

  $\rightsquigarrow$ What means applying LD to a term ( = expression ) ?

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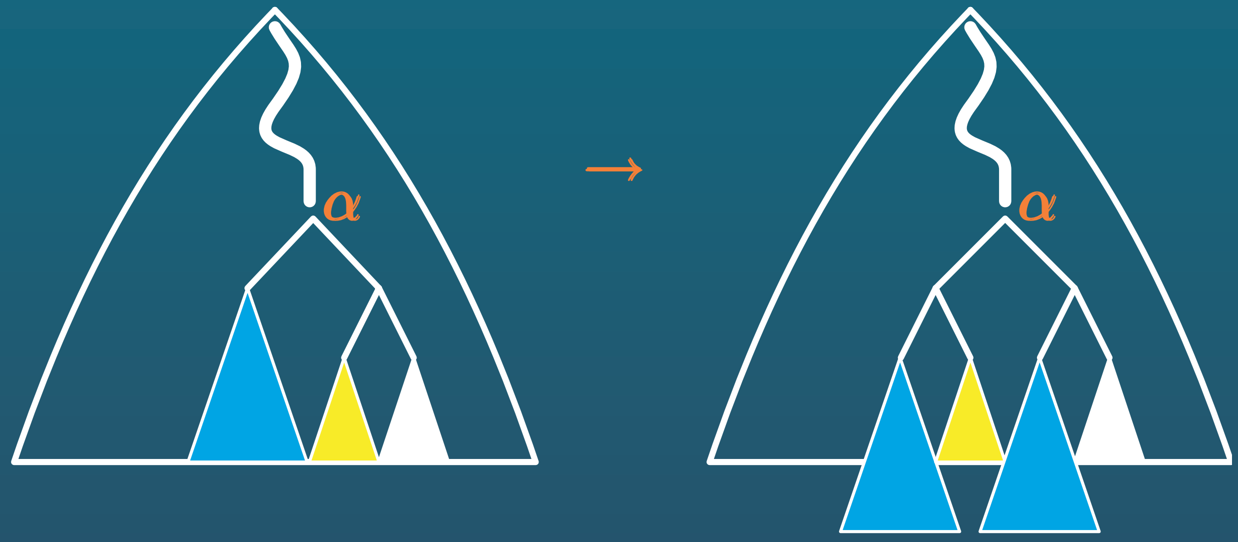


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- Def: $\Sigma_\alpha$ := the (partial) operator "apply LD at address $\alpha$ in the $\longrightarrow$ direction";

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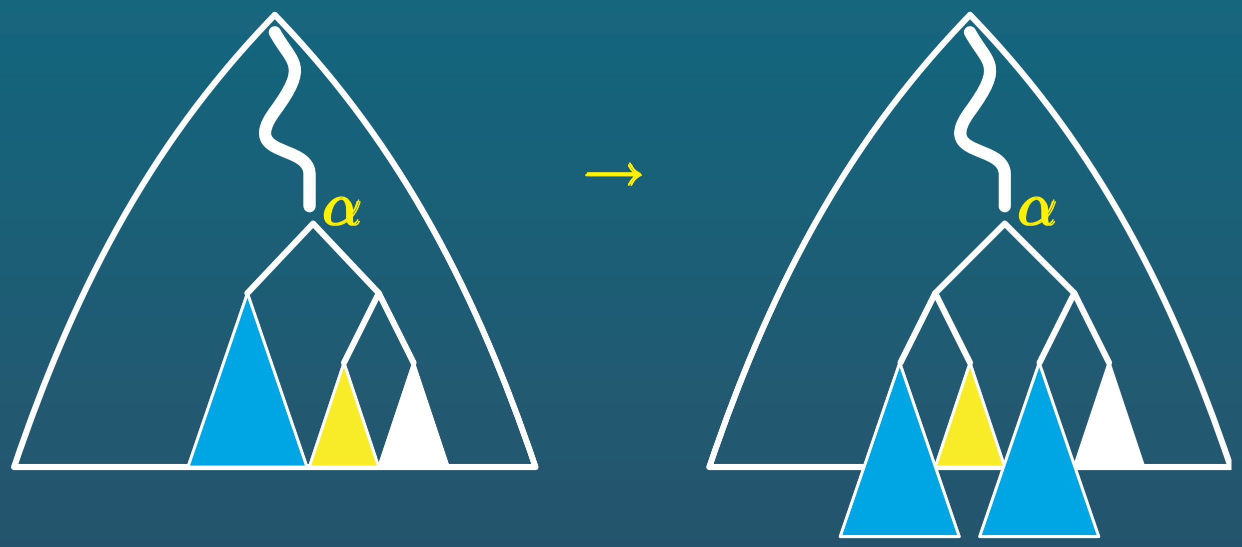


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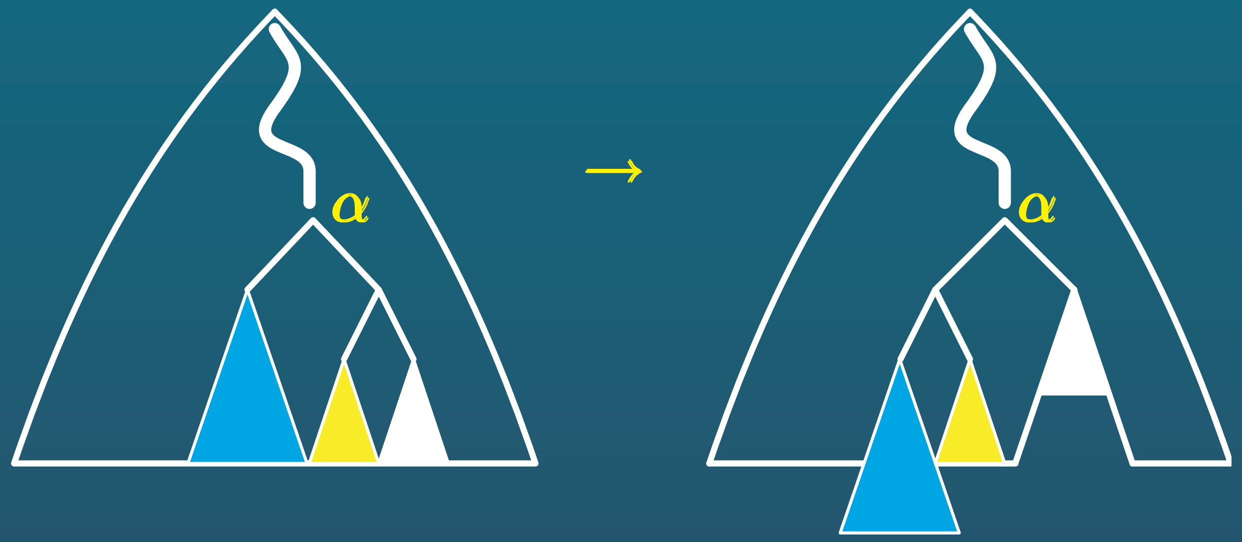
- Def: $\Sigma_\alpha$ := the (partial) operator "apply **LD** at address $\alpha$ in the $\longrightarrow$ direction";
- Def: $\mathcal{G}_{\mathbf{LD}}$ ("the geometry monoid of **LD**") := monoid generated by all $\Sigma_\alpha^{\pm 1}$.
- Fact: Two terms $t, t'$ are **LD**-equivalent iff some element of $\mathcal{G}_{\mathbf{LD}}$ maps $t$ to $t'$.

- **Same approach for each algebraic law (or family of algebraic laws):**
  - ⇝ **Example: associativity**



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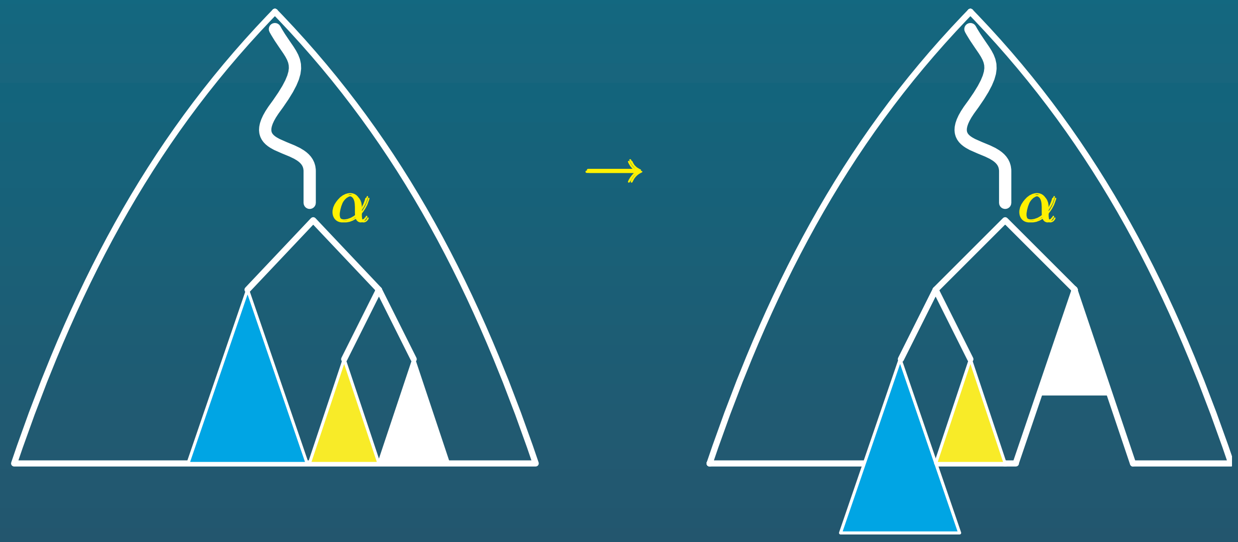


- Here, the geometry monoid acts transitively and the orbits are finite;

  ⇝ $\mathcal{G}_{\mathsf{A}}$ is nearly a group: $\mathcal{G}_{\mathsf{A}}/\!\approx$ is R.Thompson's group $F$ ( ⇝ dyadic homeo's of $[0, 1]$ ).

  ↑

  "to agree on at least one term"

- How to use $\mathcal{G}_{\mathsf{LD}}$ to construct an LD operation ?

  ⇝ Main idea: Associate with each term $t$ an element $\widehat{t}$ of $\mathcal{G}_{\mathsf{LD}}$ so that
  $t \equiv_{\mathrm{LD}} t \Leftrightarrow$ some relation connects $\widehat{t}$ and $\widehat{t'}$ ⇝ typically: $\widehat{t}^{-1} \cdot \widehat{t'} \in$ some sub(group).

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- Lemma: For every term $t$ in one variable $x$, we have $x^n \equiv_{\text{LD}} t{*}x^{n-1}$ for $n$ large enough.

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$x^n$ $\quad t_1{*}x^{n-1}$ $\quad t_1{*}(t_2{*}x^{n-2})$ $\quad (t_1{*}t_2){*}((t_1{*}x^{n-1}))$ $\quad (t_1{*}t_2){*}x^{n-1})$

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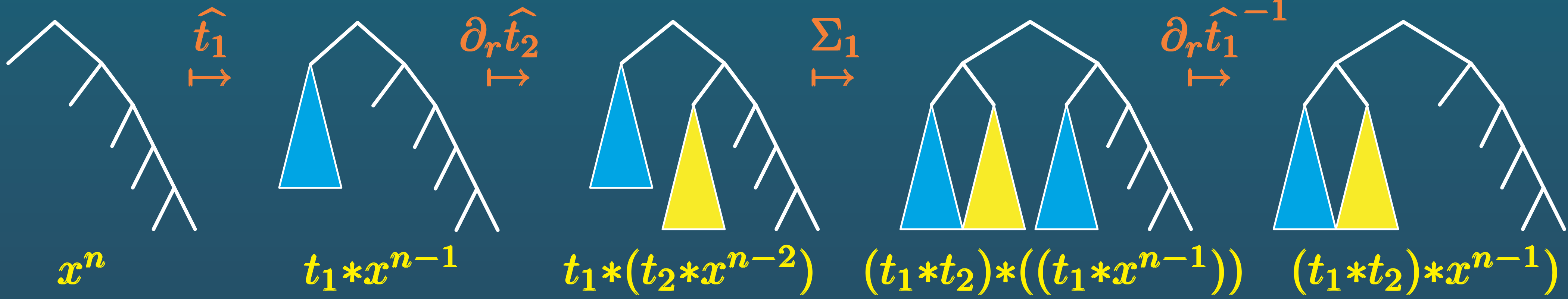
- Lemma: For every term $t$ in one variable $x$, we have $x^n \equiv_{\mathrm{LD}} t{*}x^{n-1}$ for $n$ large enough.

  ⇝ Use induction on $t$: OK for $t = x$; assume $t = t_1{*}t_2$. Then

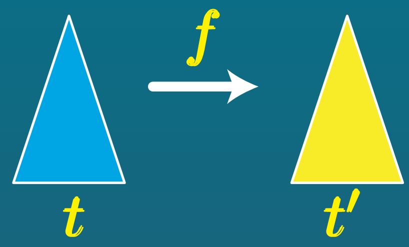



⇝ An operator $\widehat{t}$ of $\mathcal{G}_{\mathrm{LD}}$ that maps $x^n$ to $t{*}x^{n-1}$: then $\widehat{x} =$ **id**, and $\widehat{t_1{*}t_2} = \widehat{t_1} * \widehat{t_2}$ with

$$f * g = f \cdot \partial_r g \cdot \Sigma_1 \cdot \partial_r f^{-1}.$$

↑

"shift to the right subterm"

- Assume $t \equiv_{\text{LD}} t'$. Hence there exists $f$ in $\mathcal{G}_{\text{LD}}$ s.t. 

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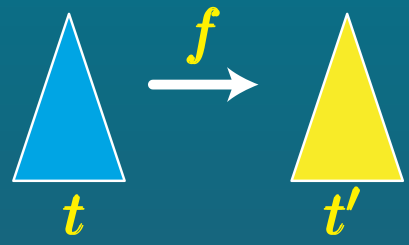
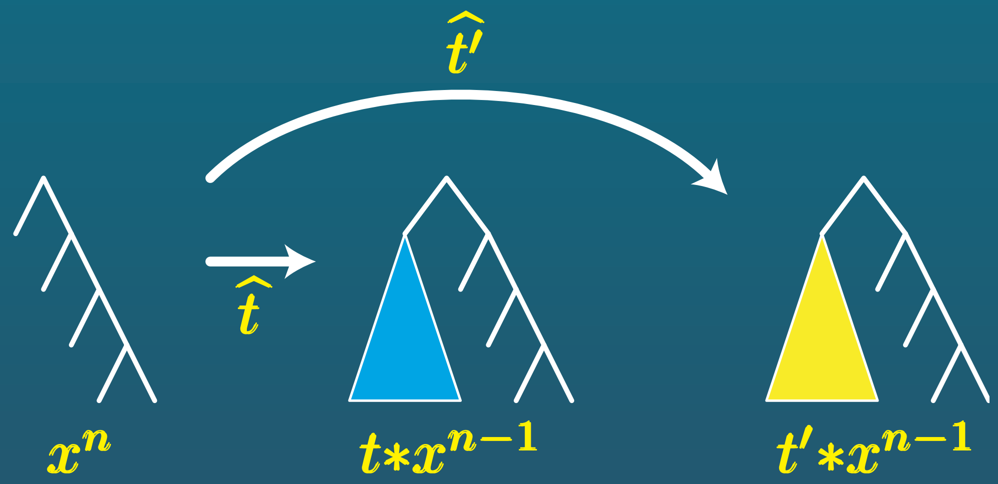
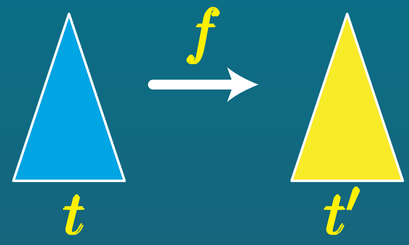
- Assume $t \equiv_{\mathrm{LD}} t'$. Hence there exists $f$ in $\mathcal{G}_{\mathrm{LD}}$ s.t.  . Then:





- Assume $t \equiv_{\text{LD}} t'$. Hence there exists $f$ in $\mathcal{G}_{\text{LD}}$ s.t. 

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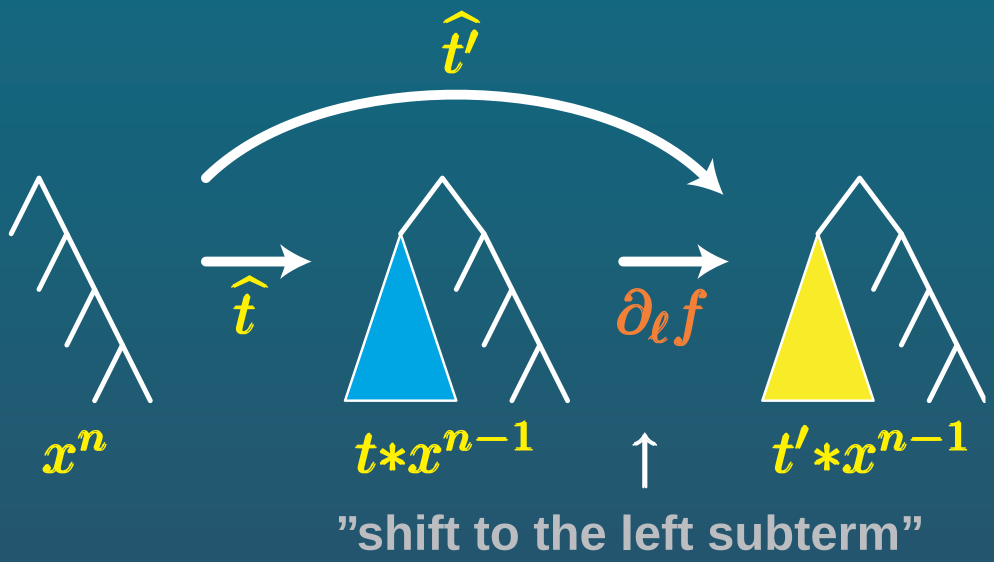

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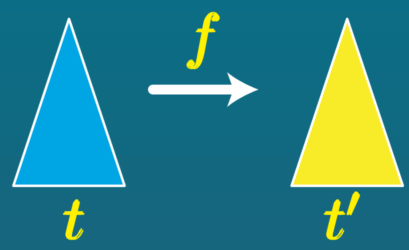


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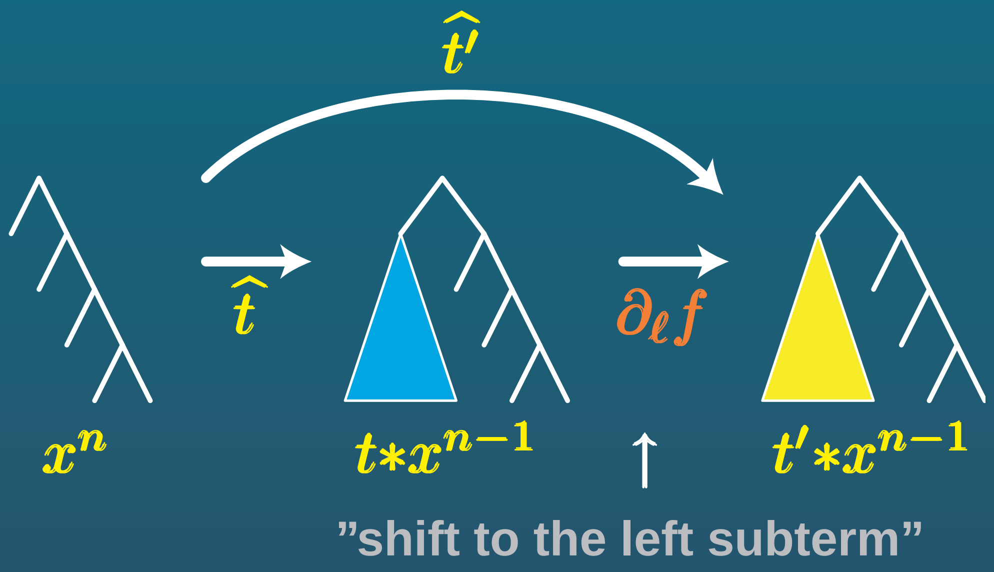


$\rightsquigarrow$ $\widehat{t}^{-1}\widehat{t'} \in \partial_\ell(\mathcal{G}_{\mathsf{LD}})$.

- **Proposition:** $*$ induces an LD-operation on $\mathcal{G}_{\mathsf{LD}}/\partial_\ell(\mathcal{G}_{\mathsf{LD}})$.

$\rightsquigarrow$ What is $\mathcal{G}_{\mathsf{LD}}/\partial_\ell(\mathcal{G}_{\mathsf{LD}})$ ?

- Relations between the operators $\Sigma_\alpha$ ? Lattice relations $\Sigma_\alpha \cdots = \Sigma_\beta \cdots$

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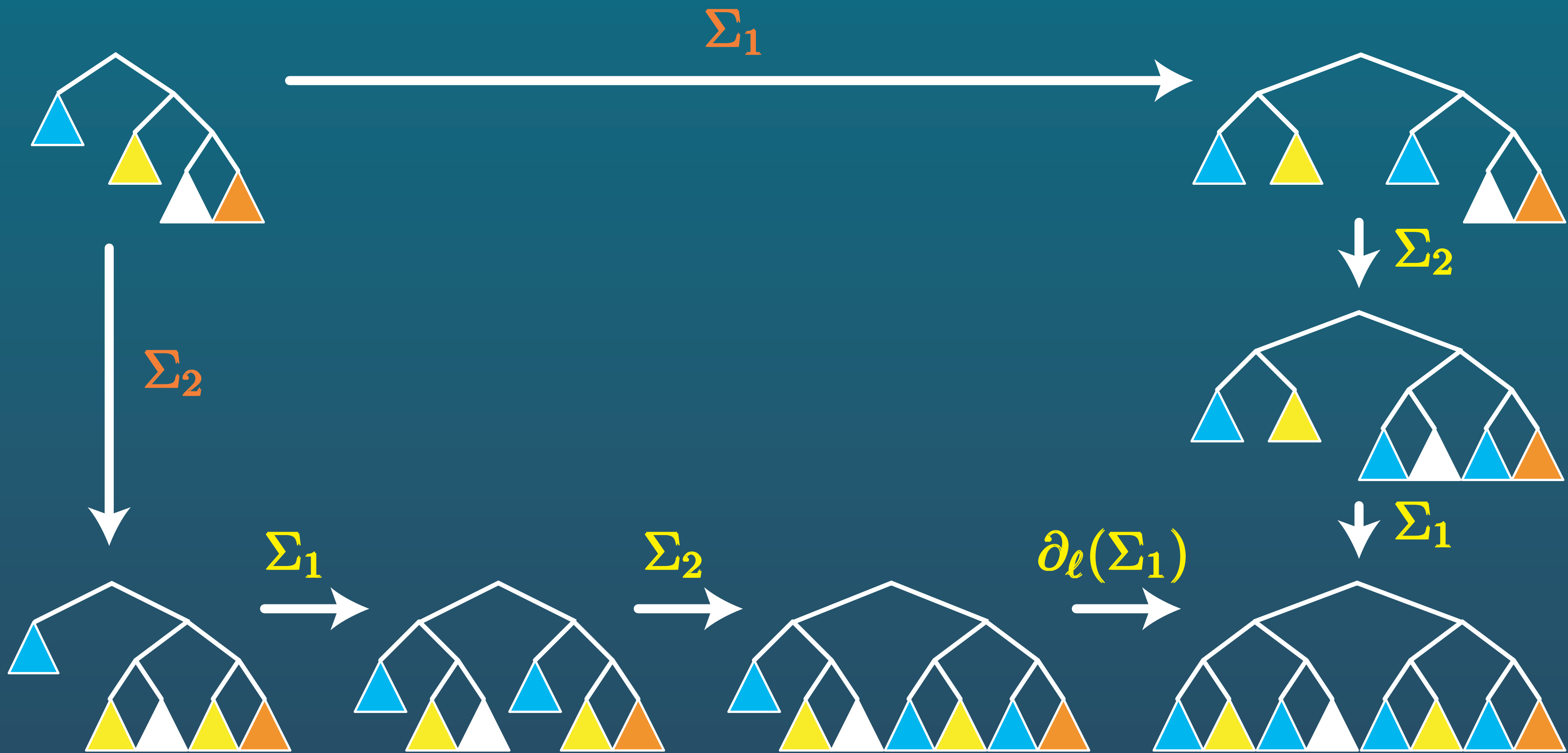

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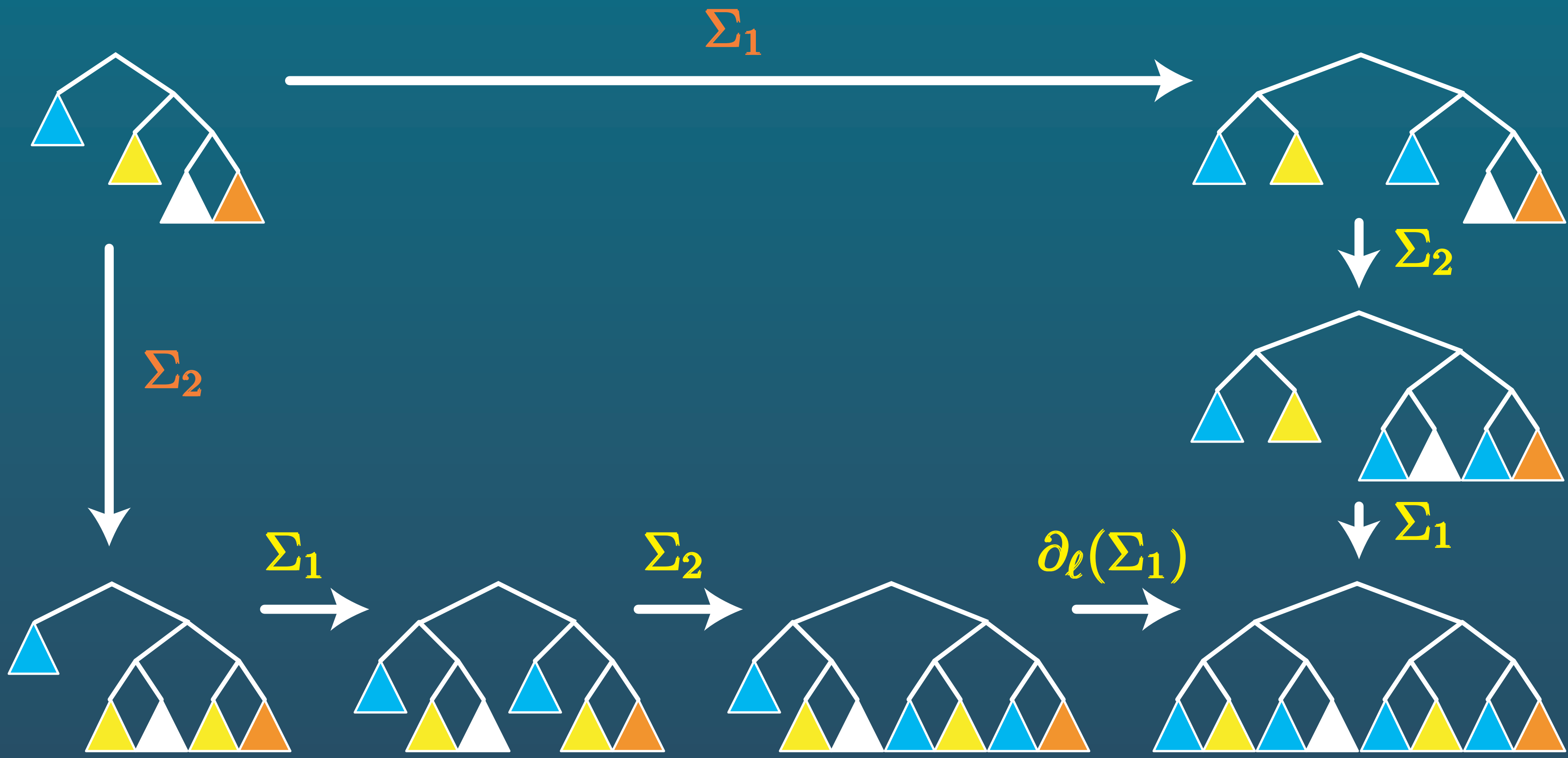


$$\rightsquigarrow \Sigma_1 \cdot \Sigma_2 \cdot \Sigma_1 = \Sigma_2 \cdot \Sigma_1 \cdot \Sigma_2 \cdot \partial_\ell(\Sigma_1)$$

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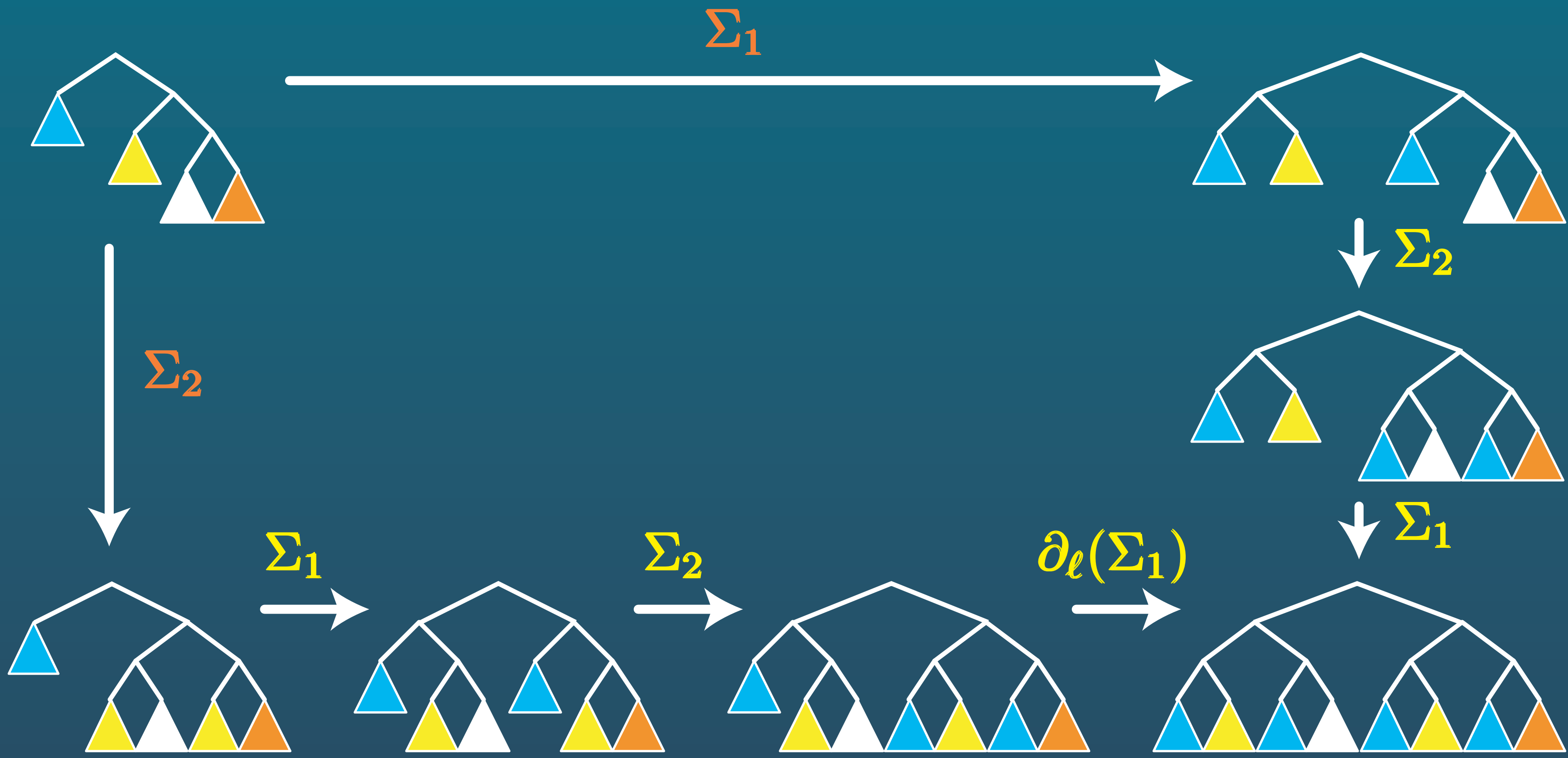


$\rightsquigarrow\ \Sigma_1 \cdot \Sigma_2 \cdot \Sigma_1 = \Sigma_2 \cdot \Sigma_1 \cdot \Sigma_2 \cdot \partial_\ell(\Sigma_1)$

- **When $\partial_\ell(\mathcal{G}_{\mathsf{LD}})$ is collapsed:** $\sigma_1 \cdot \sigma_2 \cdot \sigma_1 = \sigma_2 \cdot \sigma_1 \cdot \sigma_2$ $\rightsquigarrow$ **the braid relation**

$\rightsquigarrow\ \mathcal{G}_{\mathsf{LD}}/\partial_\ell(\mathcal{G}_{\mathsf{LD}}) = B_\infty$

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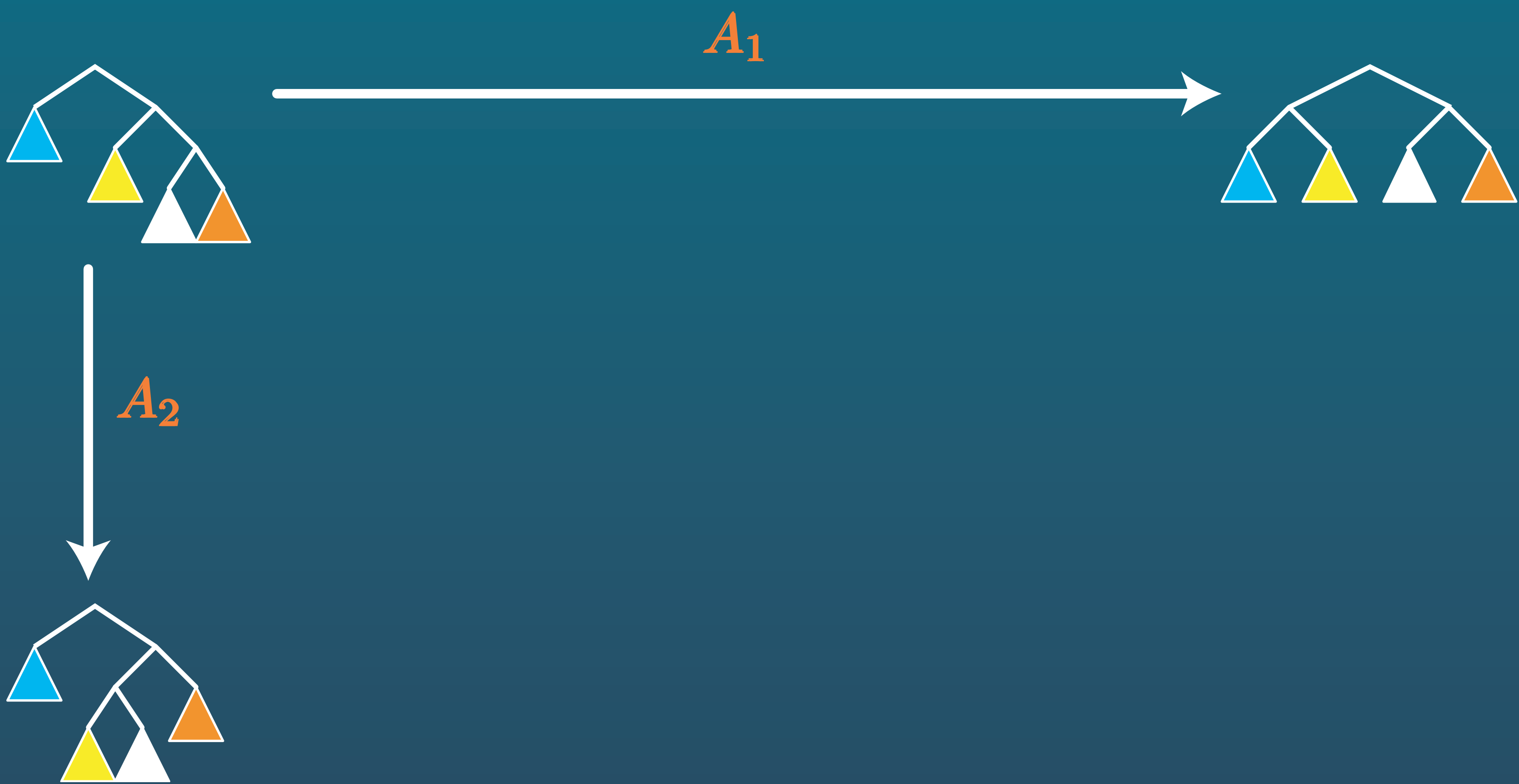

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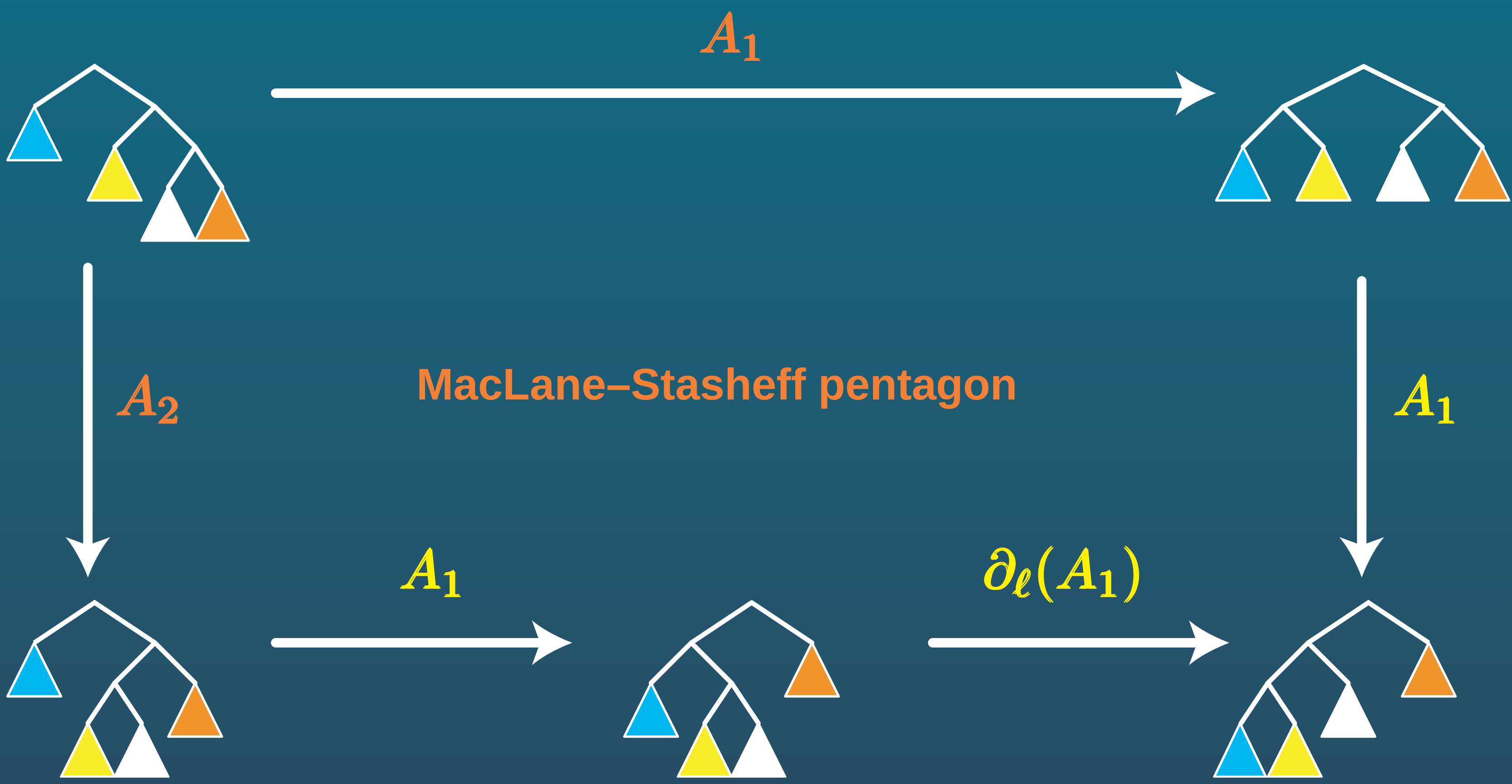


$\rightsquigarrow$ $A_1 \cdot A_1 = A_2 \cdot A_1 \cdot \partial_\ell(A_1)$ : **"geometric presentation" of R. Thompson's group $F$.**

- Start with

| | 1 | 2 | ... | $N$ |
|---|---|---|---|---|
| 1 | 2 | | | |
| 2 | 3 | | | |
| $\vdots$ | | | | |
| $N-1$ | $N$ | | | |
| $N$ | 1 | | | |

and try to construct an LD-table...

$\rightsquigarrow$ Example:

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 2 | | | |
| 2 | 3 | | | |
| 3 | 4 | | | |
| 4 | 1 | | | |

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| 4 | 1 | 2 | 3 | |

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|---|---|---|---|---|
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| 2 | 3 | | | |
| 3 | 4 | 4 | 4 | 4 |
| 4 | 1 | 2 | 3 | 4 |

- Start with

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|---|---|---|---|---|
| 1 | 2 | | | |
| 2 | 3 | 4 | 3 | 4 |
| 3 | 4 | 4 | 4 | 4 |
| 4 | 1 | 2 | 3 | 4 |

- Start with

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| 2 | 3 | 4 | 3 | 4 |
| 3 | 4 | 4 | 4 | 4 |
| 4 | 1 | 2 | 3 | 4 |

$\rightsquigarrow$ at most one solution for each $N$;

$\rightsquigarrow$ actually an LD-table iff $N$ is a power of $2$.

- Def: The $n$-th Laver table $A_n$ = the table with $2^n$ elements.

- Each row in $A_n$ is periodic, with period a power of $2$;
  - $A_n$ is the projection of $A_{n+1}$ mod. $2^n$.
    - $\rightsquigarrow$ period of the first row in $A_{n+1} \geqslant$ period of the first row in $A_n$.

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- **Proposition:** (Laver 1995) Assume that there exists a self-similar rank. Then the period of the first row in $A_n$ goes to $\infty$ with $n$.

⇝ Open problem: Prove that the period of the first row in $A_n$ goes to $\infty$ with $n$ !

- P.D., Braids and self-distributivity, PM 192, Birkhauser (1999)
- P.D., I.Dynnikov, D.Rolfsen, B.Wiest, Why are braids orderable?, PS 22, Soc. Math. France (2002)
- http://www.math.unicaen.fr/∼dehornoy