THE ISOTOPY PROBLEM OF BRAIDS

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- A problem of medium difficulty:
  many efficient solutions,
  but all involving (requiring)
  some nontrivial theory behind.

- Artin's braid group B_n : $\left\langle \sigma_1, ..., \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle$

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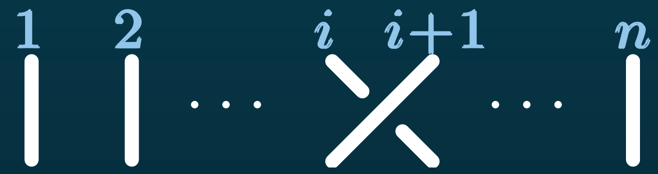
  $\simeq \{$ braid diagrams $\}/$ isotopy

$\sigma_i \quad \leftrightsquigarrow$



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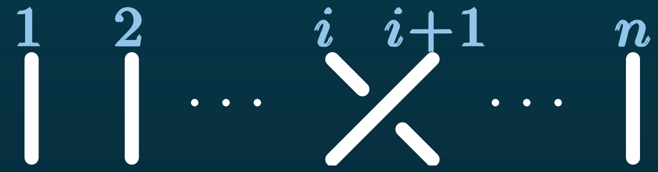
  $\sigma_i \leftrightsquigarrow$ 

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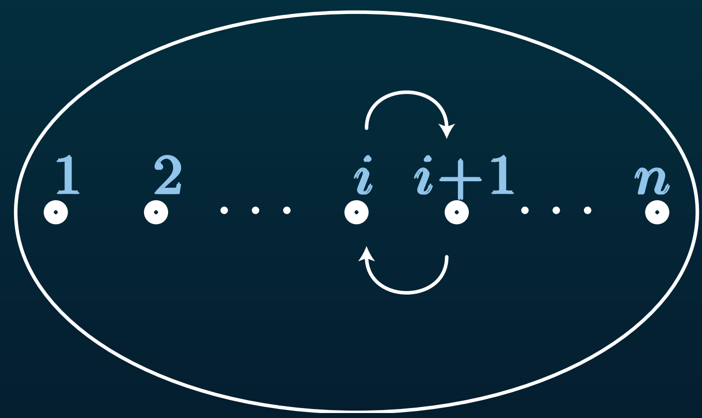
  $\sigma_i \leftrightsquigarrow$ 

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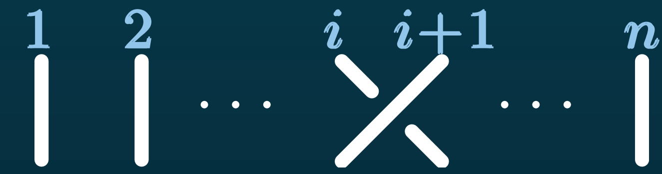

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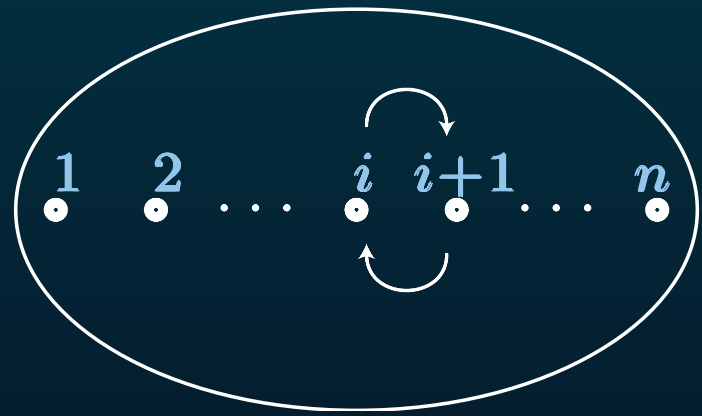

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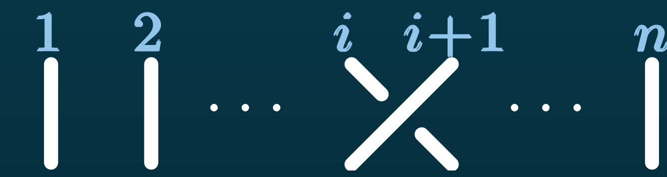
  σ_i ⇝ 

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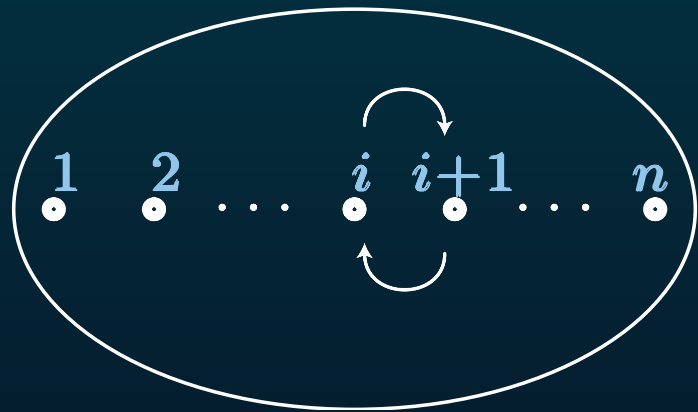
  σ_i ⇝ 

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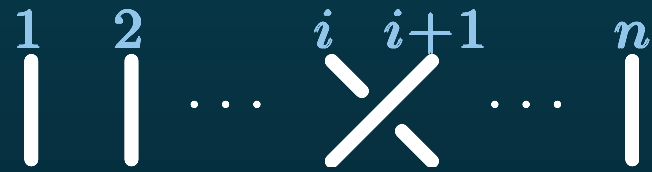
  σ_i ⟿ 

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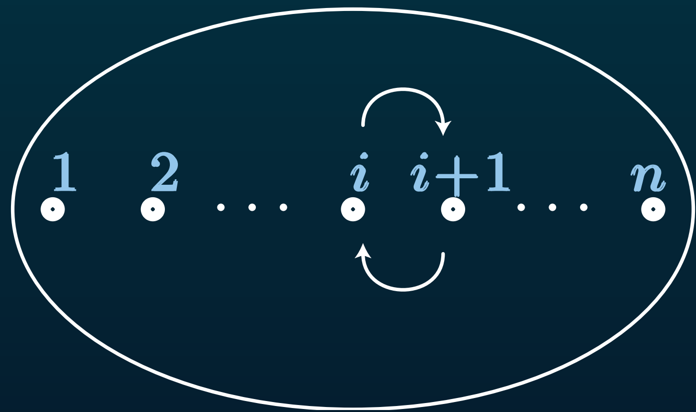
  σ_i ⟿ 

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- Behind: PB_n = semi-direct product of free groups:

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$\nwarrow$ a free group of rank $n-1$

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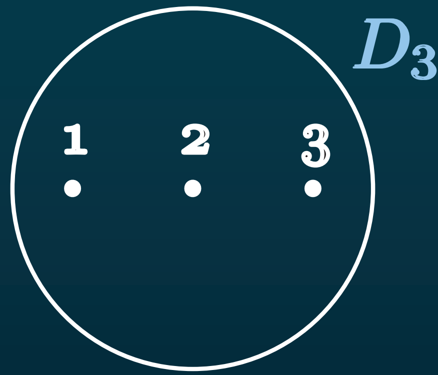

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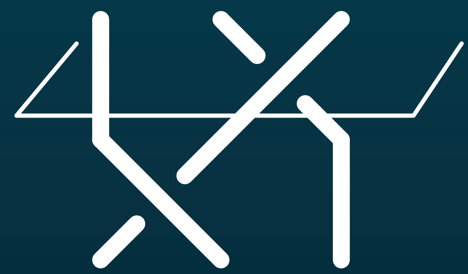

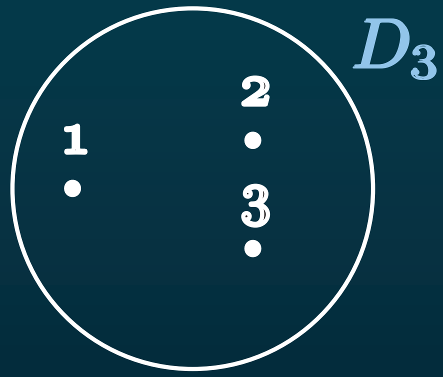

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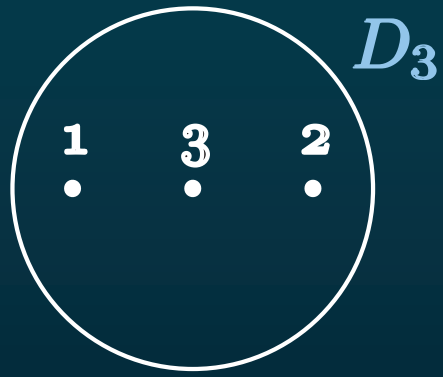

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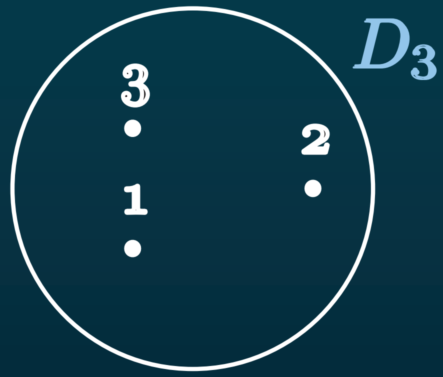

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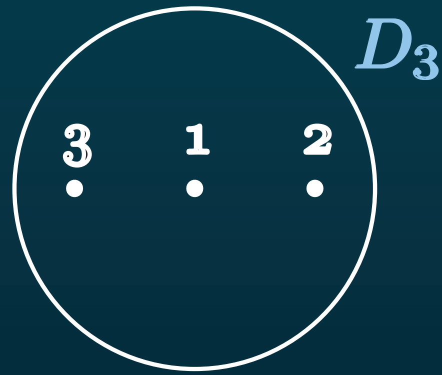

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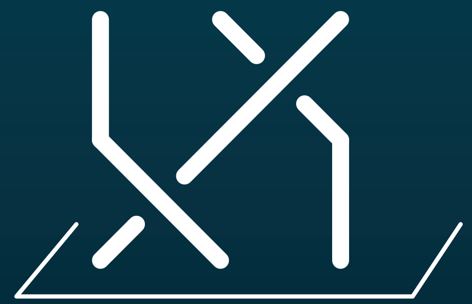

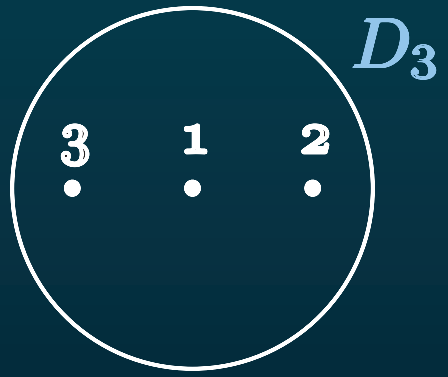

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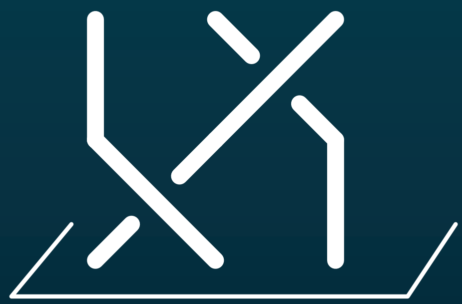

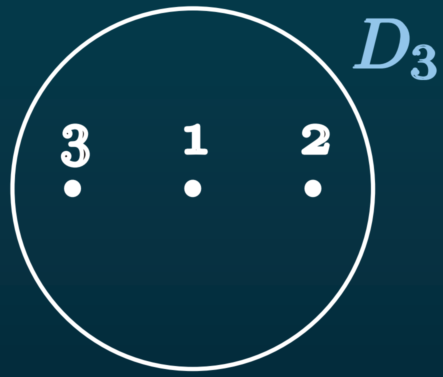

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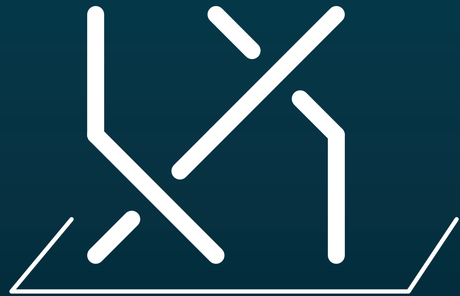

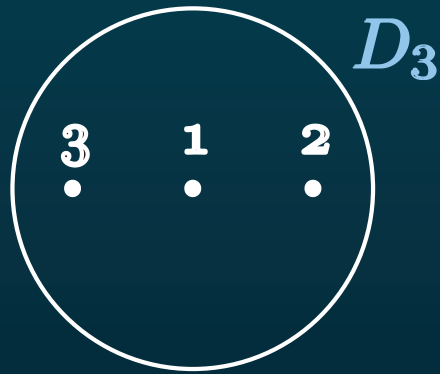

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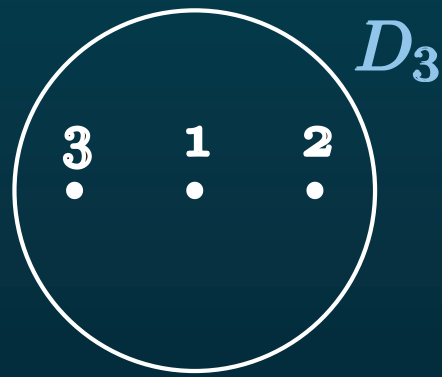

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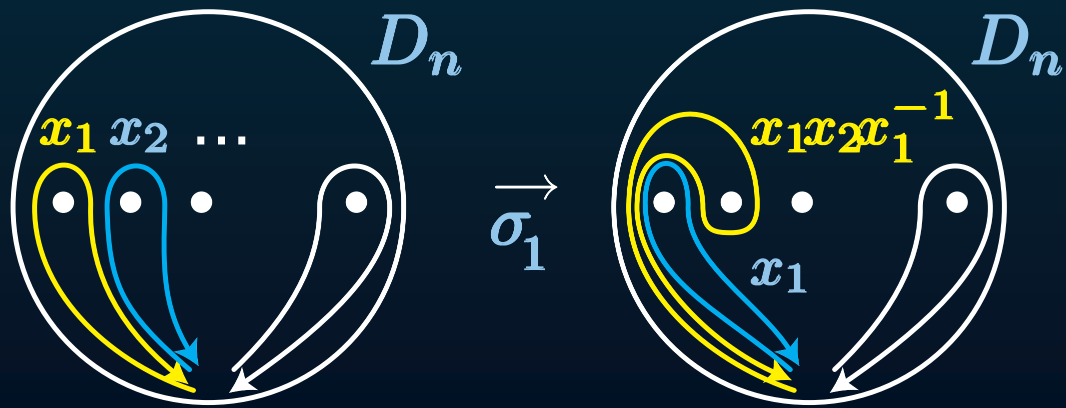

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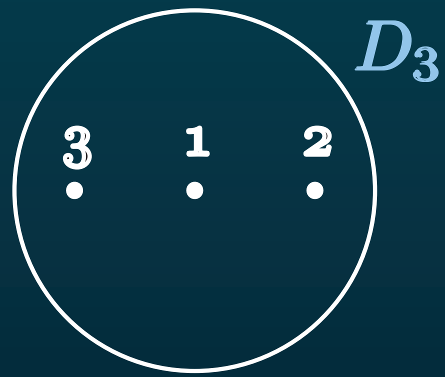

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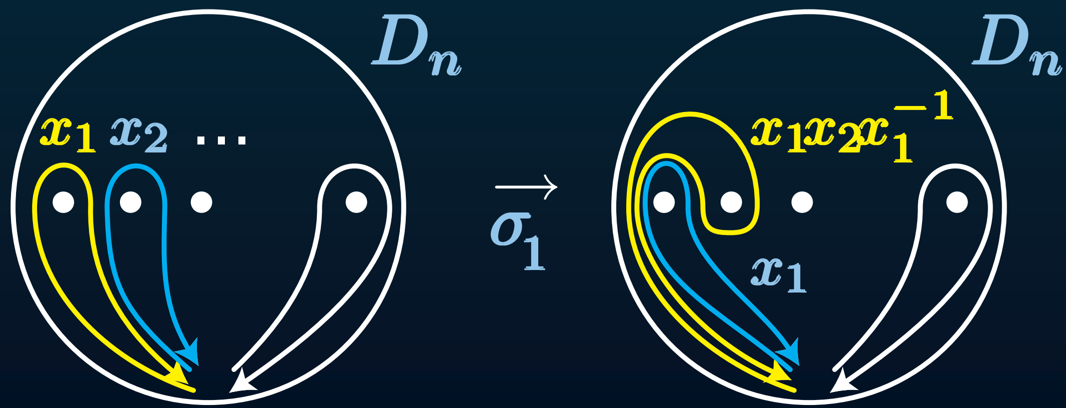

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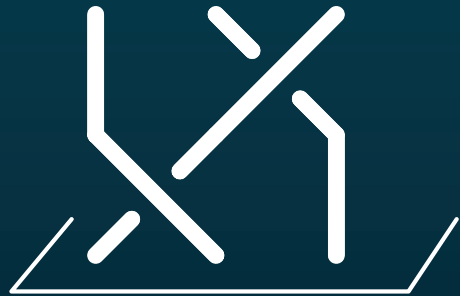

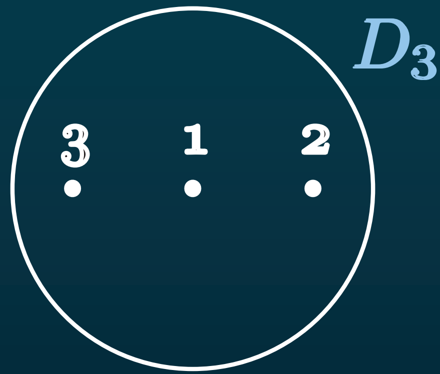

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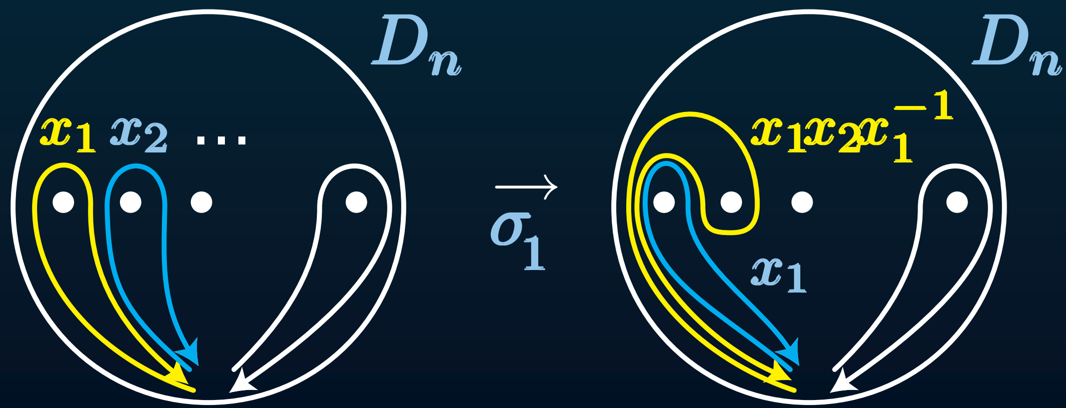

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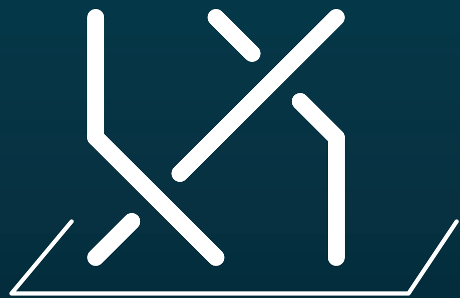

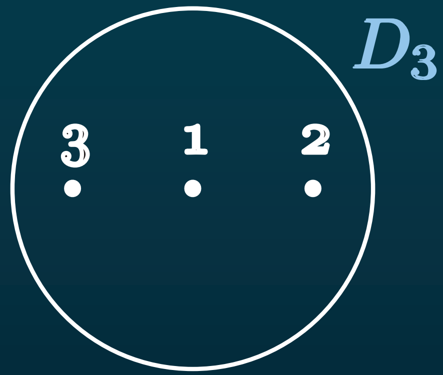

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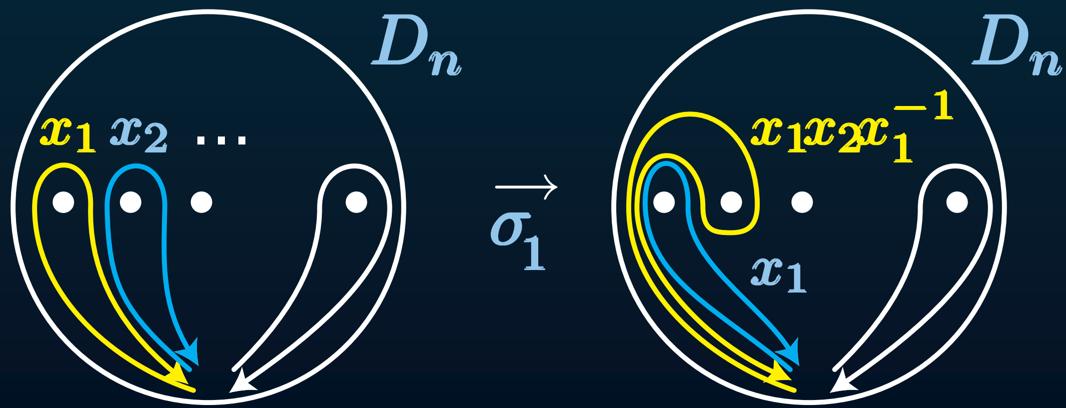

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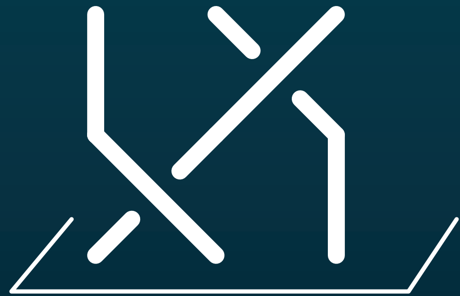

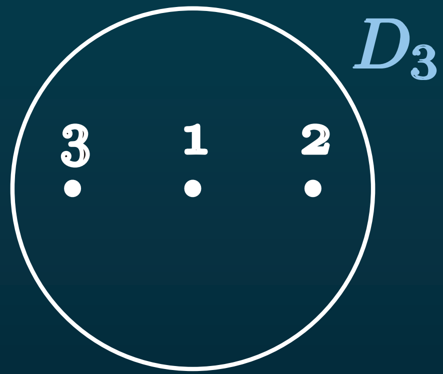

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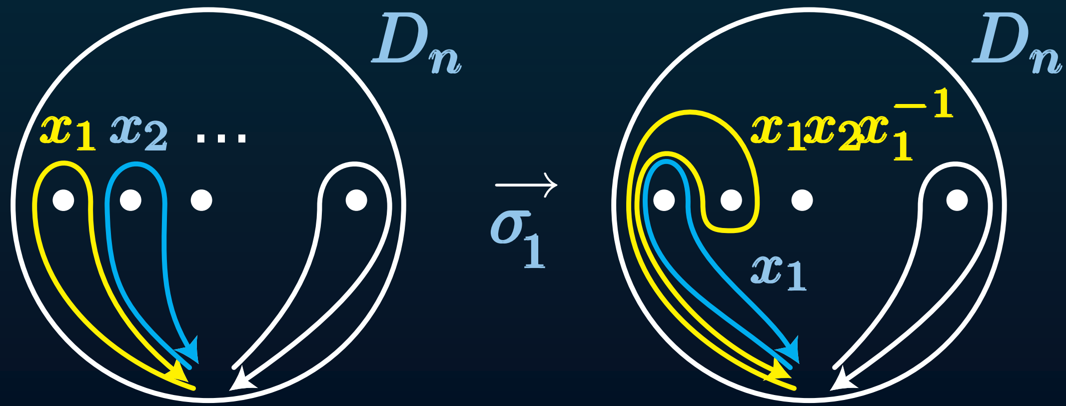

- Then: $B_n \hookrightarrow \mathtt{Aut}(F_n)$, hence solution to the braid isotopy problem.

SOLUTION 3: GREEDY NORMAL FORM

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$\rightsquigarrow$ more precisely: (B_n^+, Δ_n) s.t.
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- The group B_n is a group of fractions for the monoid B_n^+ ($:= \langle \ldots \rangle^+$)

  $+$ there exists some "universal" denominator Δ_n.

  ⇝ more precisely: (B_n^+, Δ_n) s.t.

  - B_n^+ is cancellative, has no invertible element, admits lcm's,
  - left divisors (Δ_n) $=$right divisors (Δ_n), and generate B_n^+.

  "Δ_n is a Garside element in B_n^+"

  ⇝ B_n is a Garside group.

- Then: Every element of B_n has a unique expression $x_p^{-1}...x_1^{-1}y_1...y_q$ with
  - $x_1, ..., x_p, y_1, ..., y_p$ divisors of Δ_n,
  - $x_i = \mathbf{gcd}(x_i x_{i+1}, \Delta_n)$ for each i, ***id.*** for y_j's, and $\mathbf{gcd}(x_1, y_1) = 1$.

(can be computed in quadratic time, for each fixed n)

SOLUTION 3: GREEDY NORMAL FORM

(Garside, 1967, Deligne, Adjan, Thurston, Morton-El Rifai...)

- The group B_n is a group of fractions for the monoid B_n^+ ($:= \langle \ldots \rangle^+$)
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- Behind: automatic structure of B_n (Cannon, Thurston)

"redresses to"

$\downarrow$

- Say that $w \curvearrowright w'$ holds if w' obtained from w by (iteratively)

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- Say that $w \curvearrowright w'$ holds if w' obtained from w by (iteratively)
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- Behind: Garside theory again.

- Free group reduction: delete $\sigma_i \sigma_i^{-1}$ or $\sigma_i^{-1} \sigma_i$;

  ⇝ if w reduces to ε, then w represents 1; no converse, as B_n not free.

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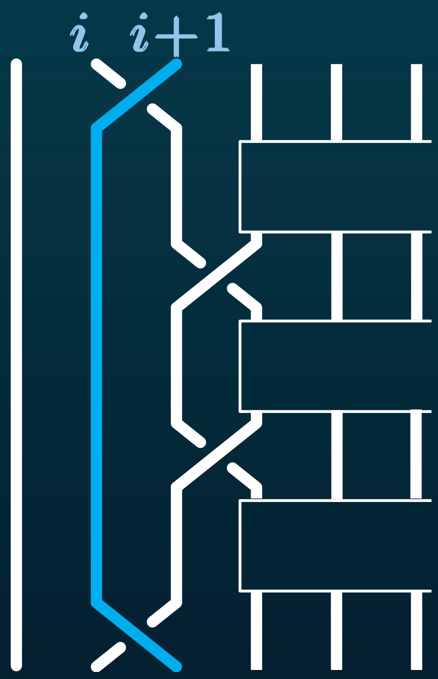

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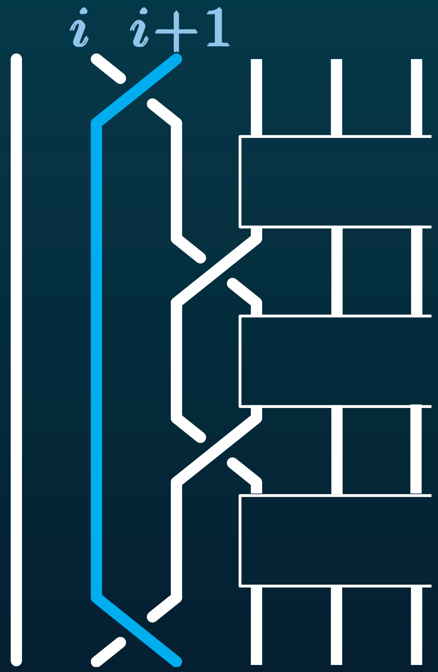

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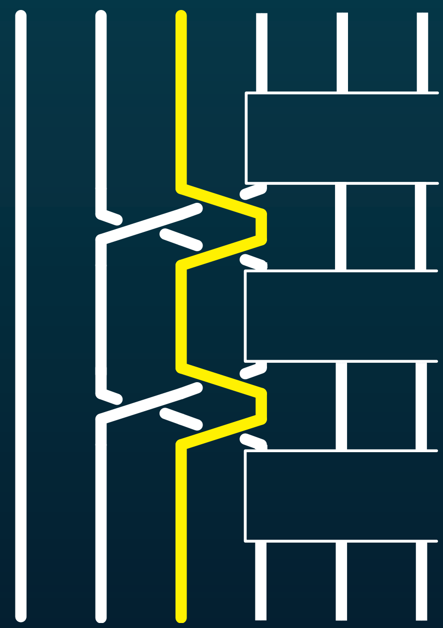

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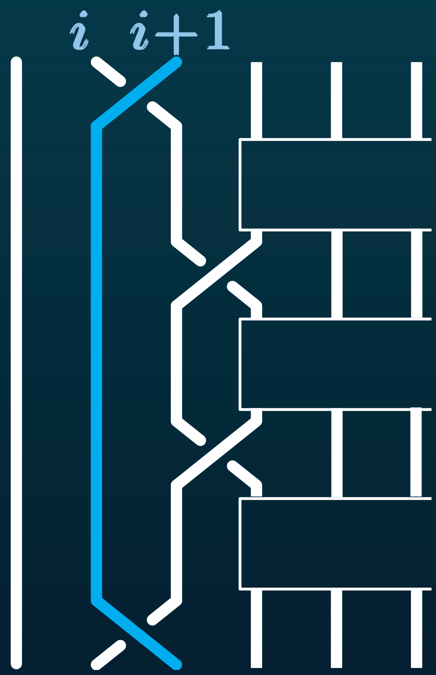

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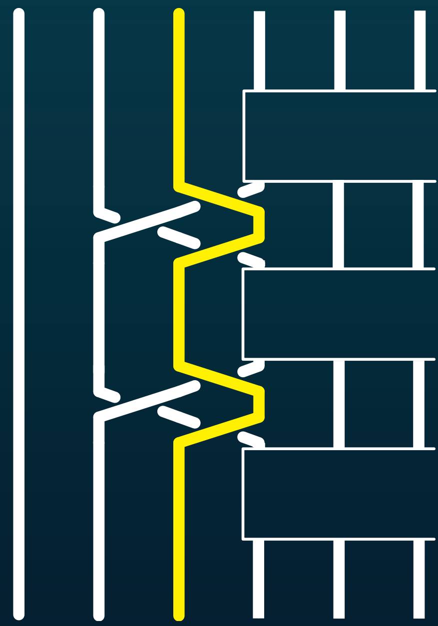

- Then: A braid word represents 1 iff it reduces to ε.

SOLUTION 5: HANDLE REDUCTION

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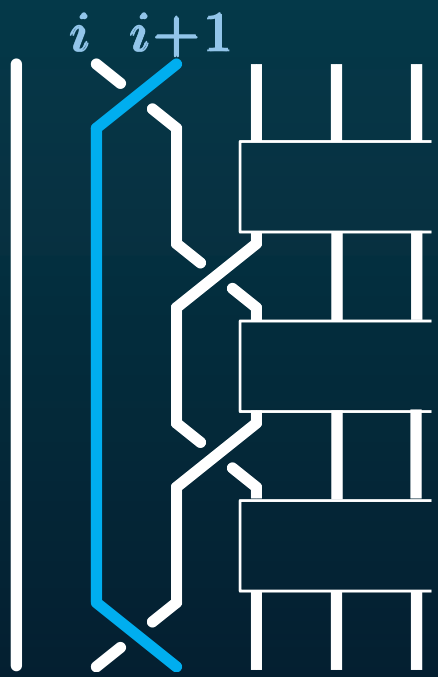

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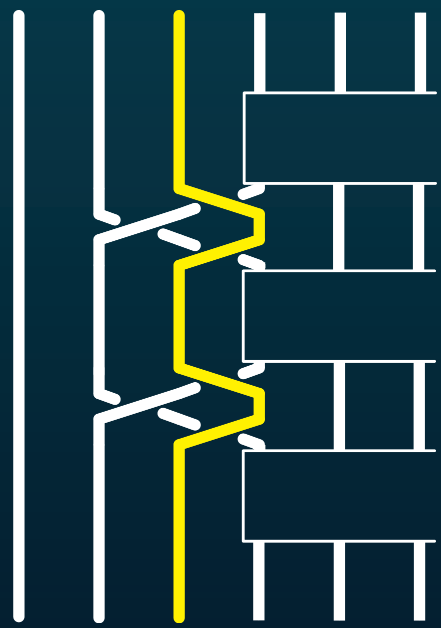

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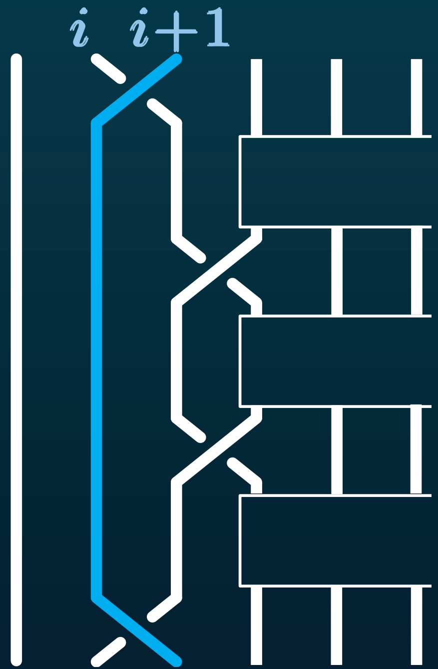

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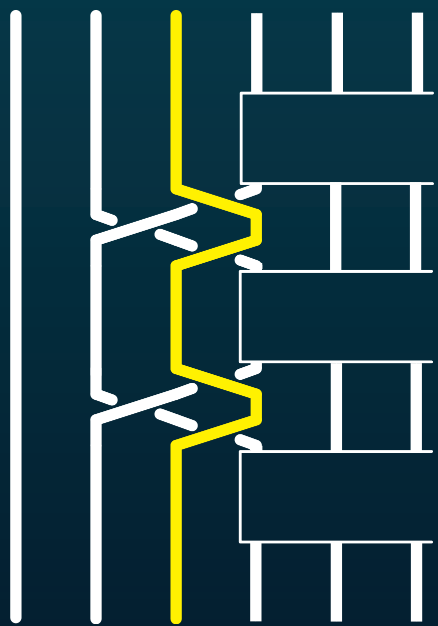

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- Behind: Garside theory $+$ order properties.

- Cayley graph of B_n: vertices = braids; edges: $\overset{x}{\bullet}\xrightarrow{\sigma_i}\overset{y}{\bullet}$ for $y = x\sigma_i$.

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$\rightsquigarrow$ **Cayley(Δ_n^d)** := restriction of the Cayley graph of B_n to divisors of Δ_n^d

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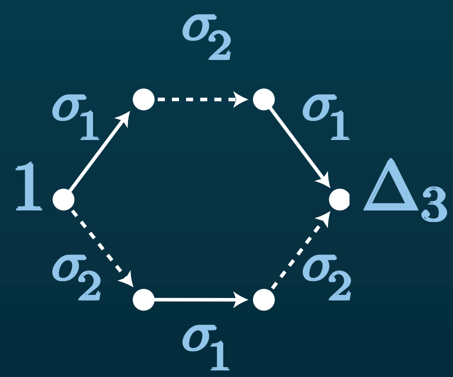
Example: Cayley(Δ_3) = 

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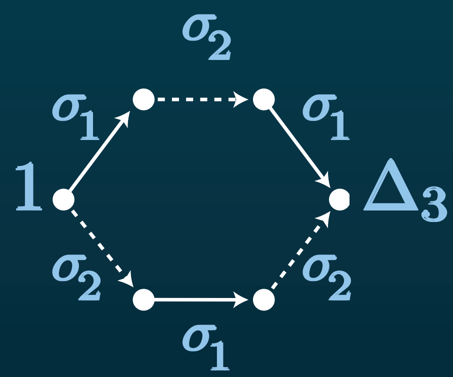

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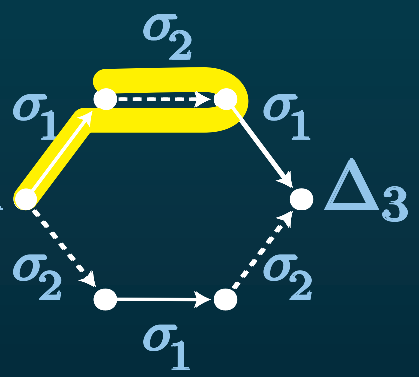
Example: Cayley(Δ_3) = 1 

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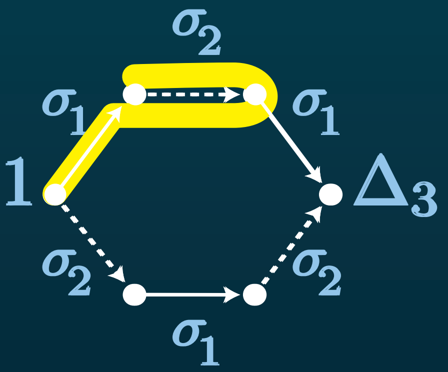
Example: Cayley(Δ_3) = 

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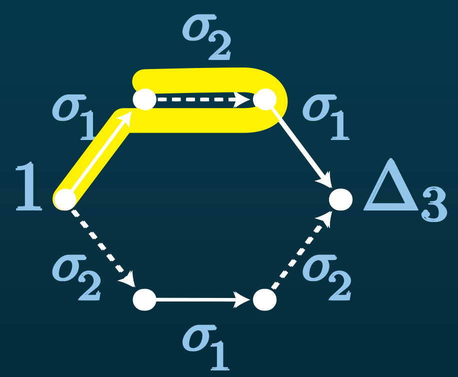

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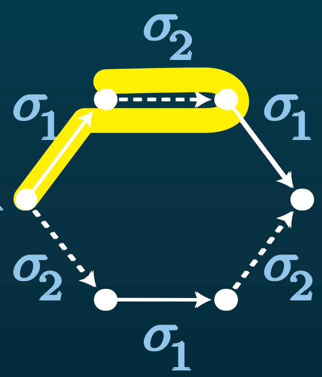
Example: Cayley(Δ_3) = 1  Δ_3

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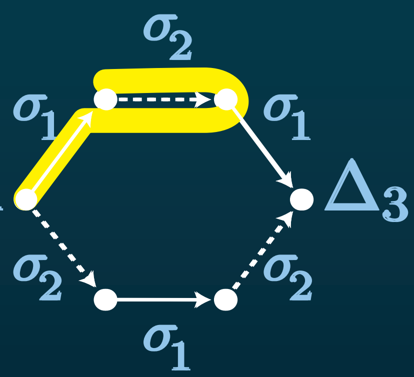

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CONVERGENCE OF HANDLE REDUCTION (1)

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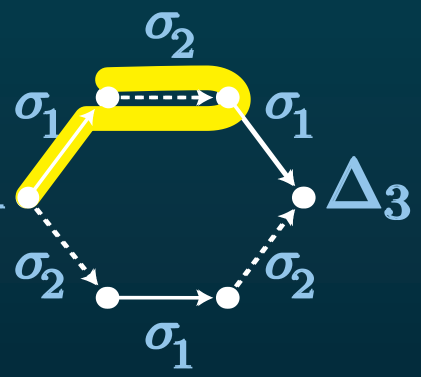

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⇝ a boundedness result: when reduction is performed,
all words are drawn in some fixed finite subgraph of the Cayley graph.

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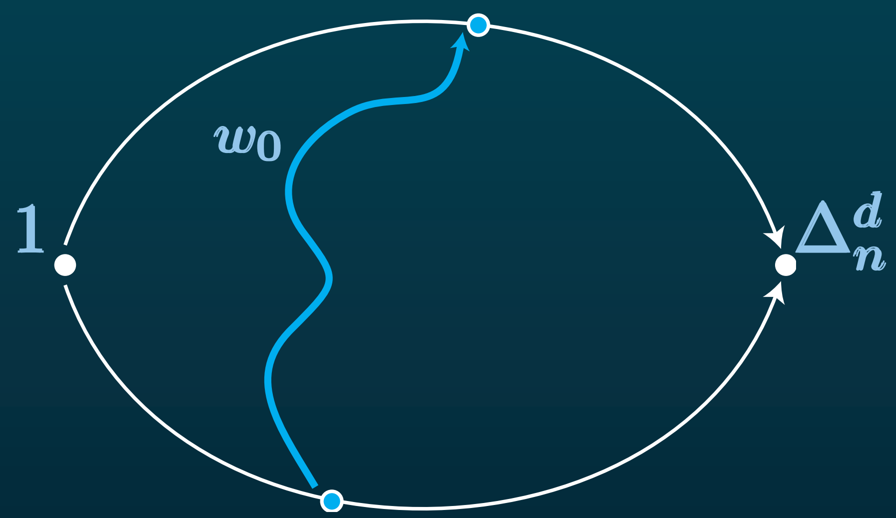

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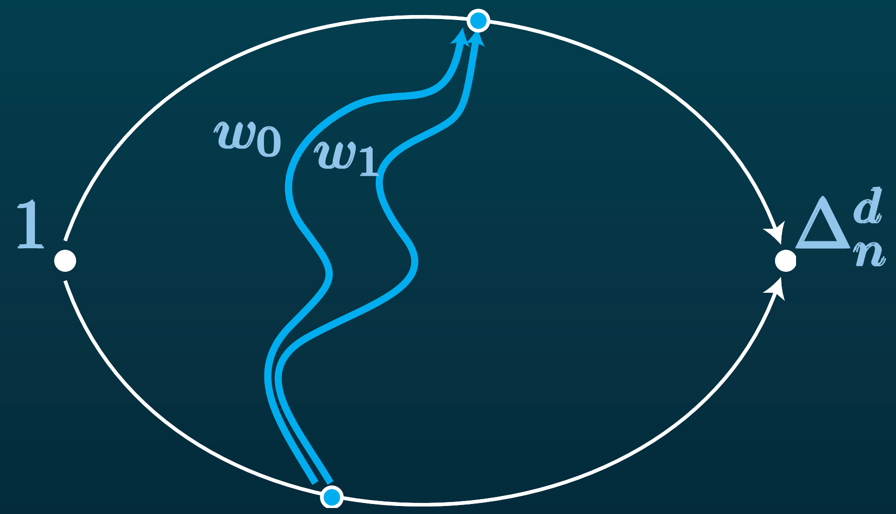

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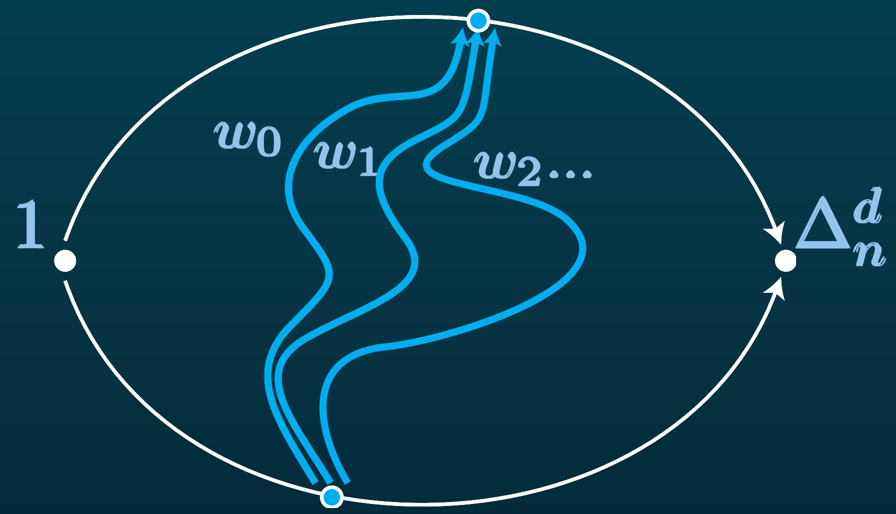

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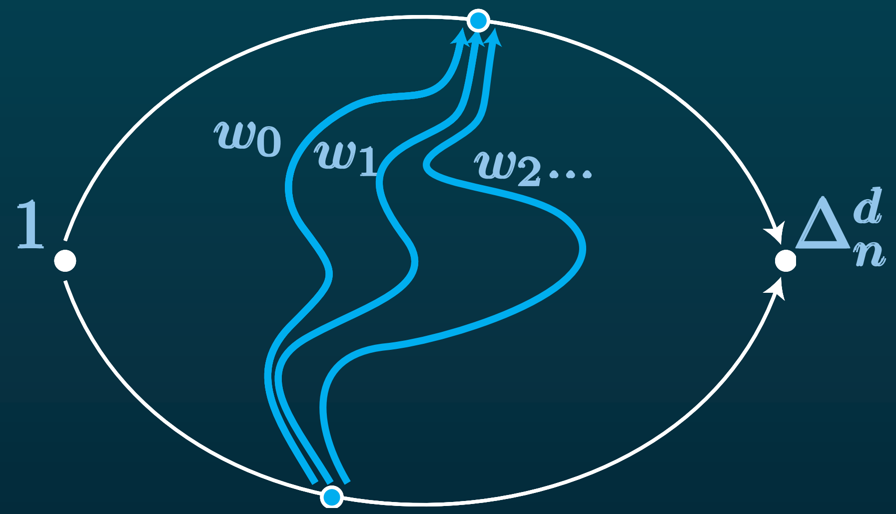

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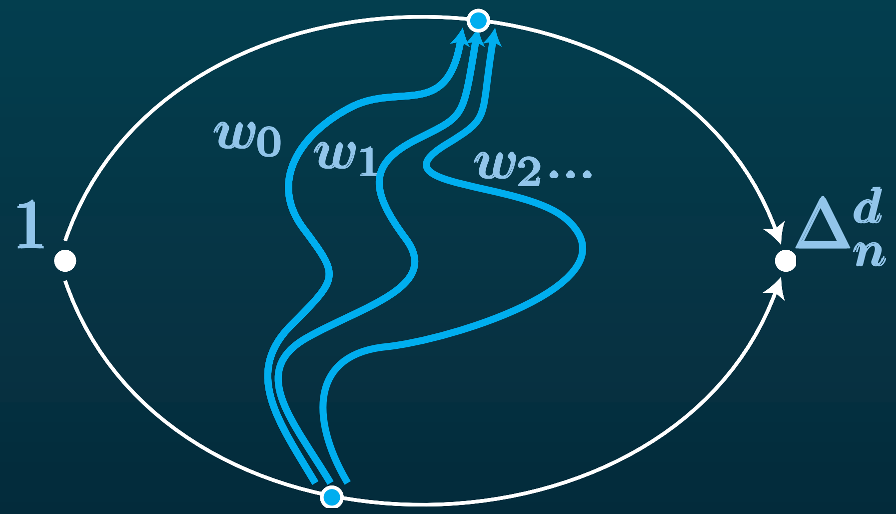

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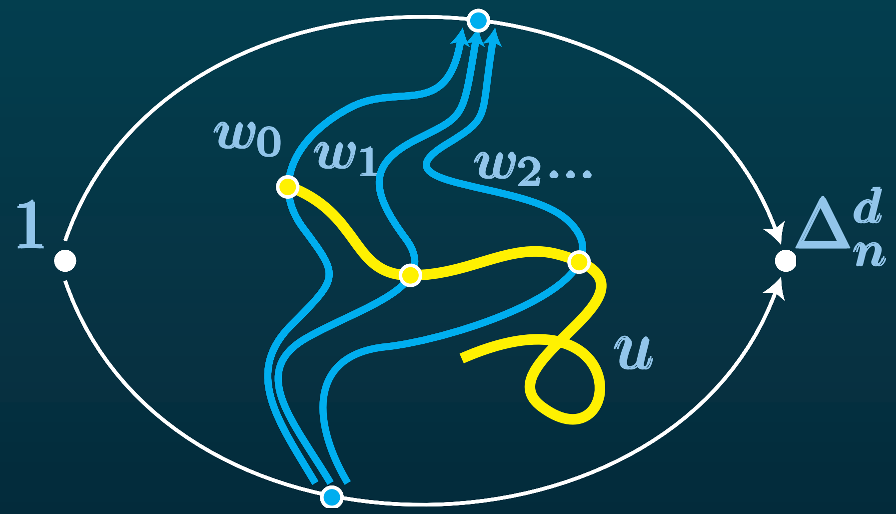

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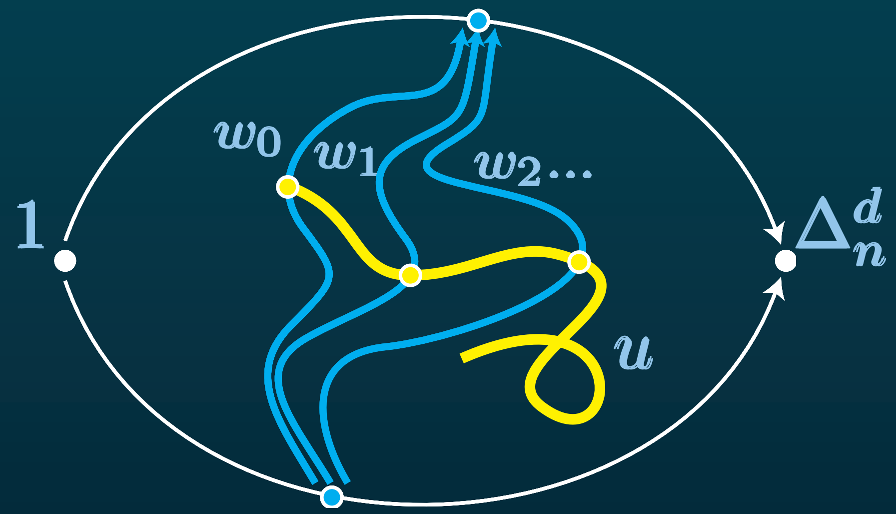

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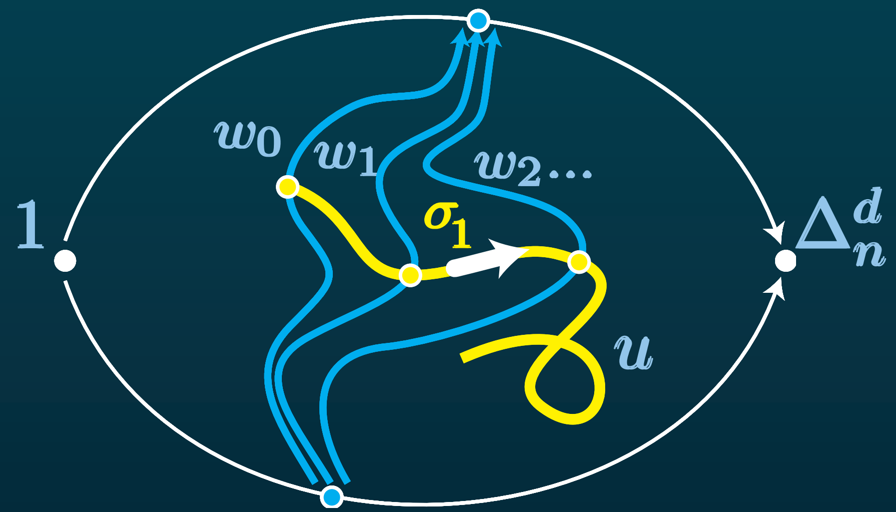

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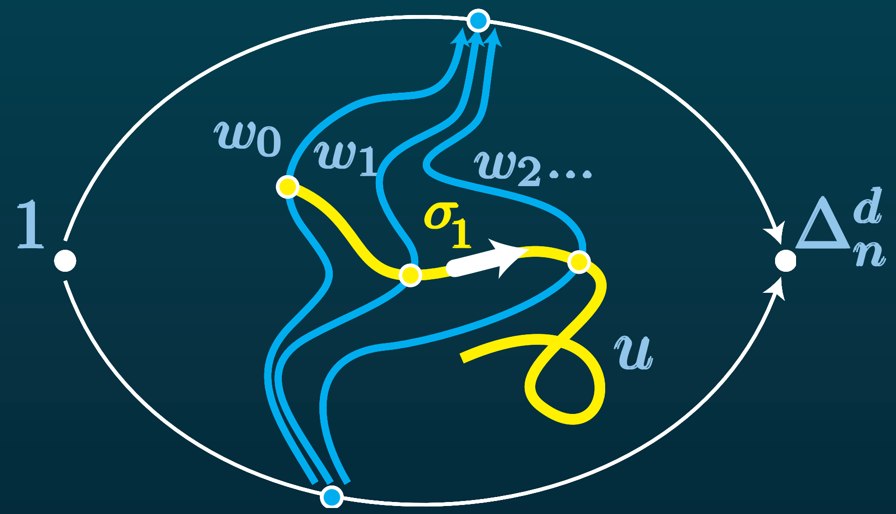

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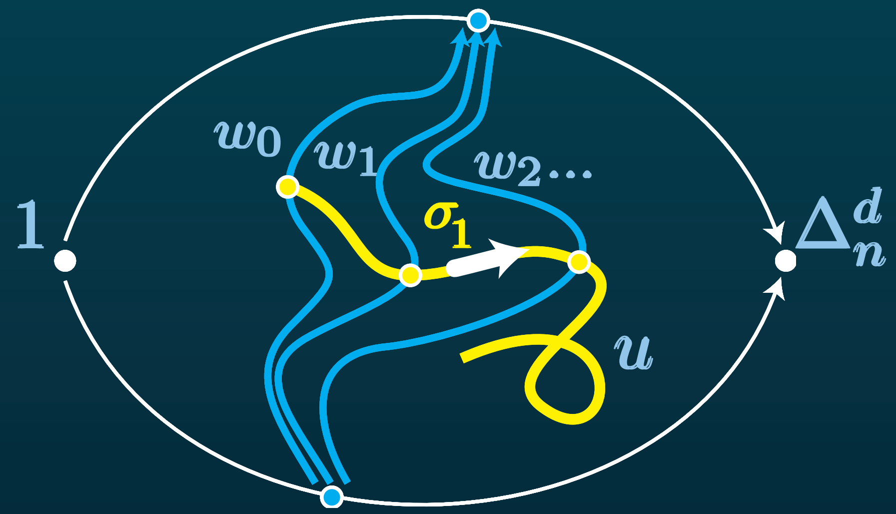

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⇝ convergence of reduction □

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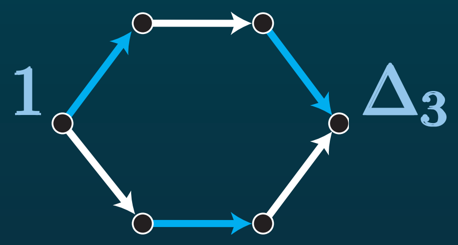

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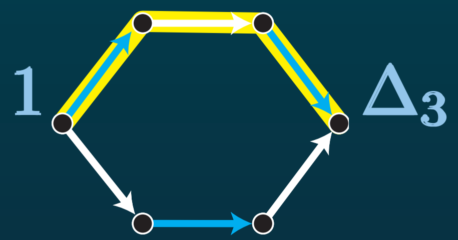

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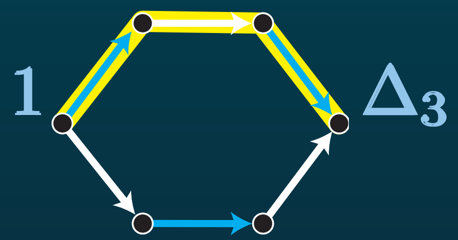

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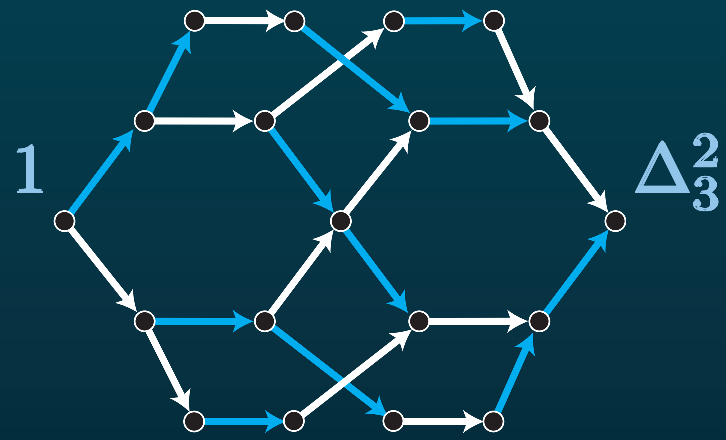

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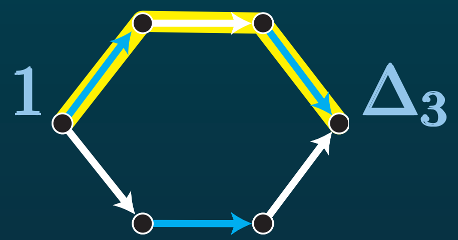

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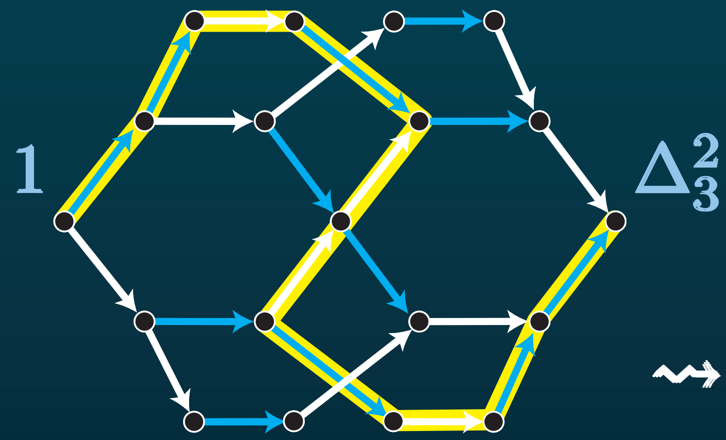

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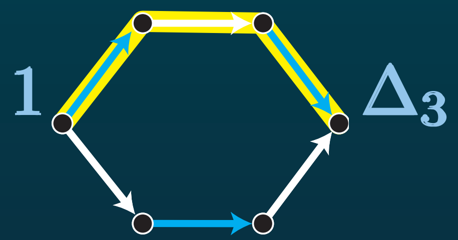

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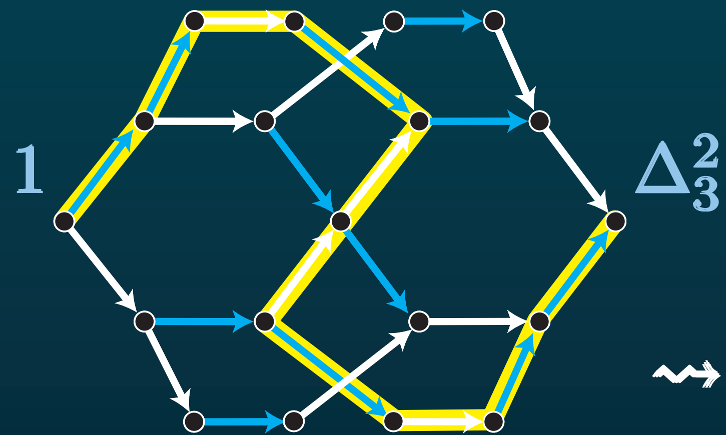

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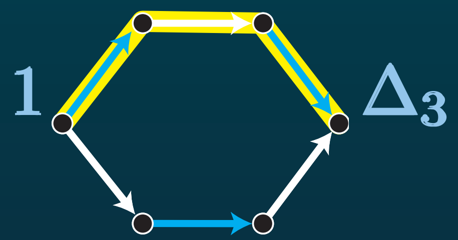

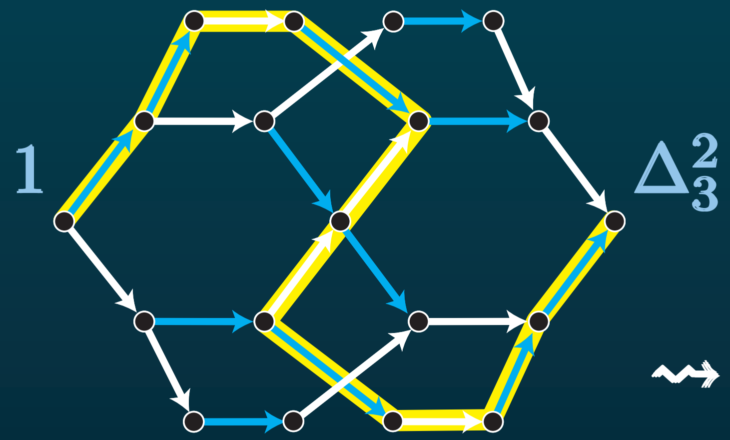

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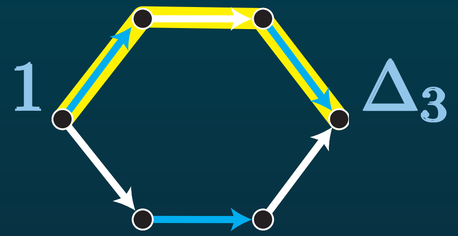

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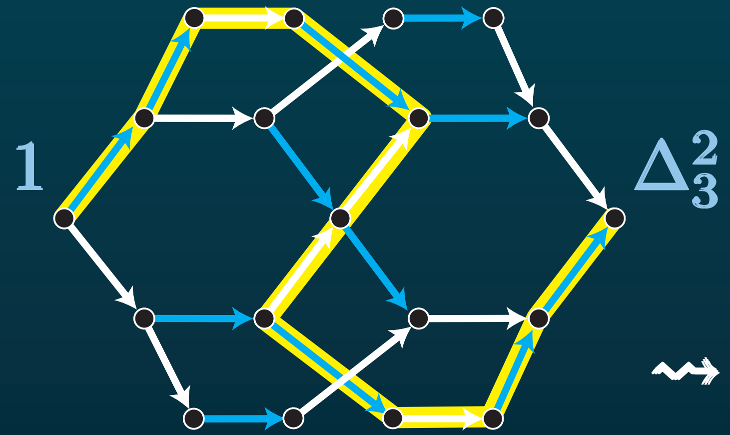

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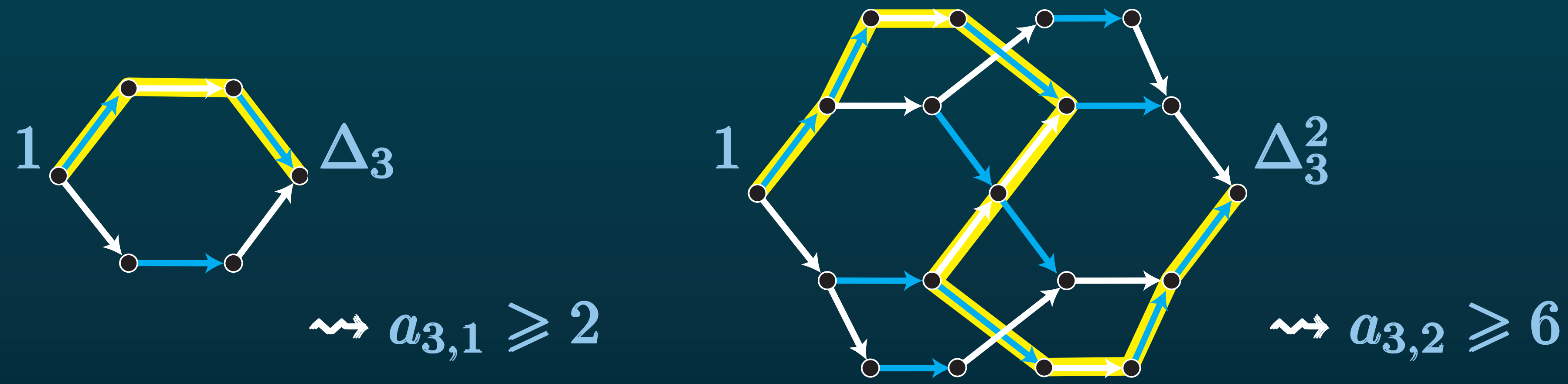

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- Corollary: $a_{n,d}$ expressed from M_n^d, where M_n is the $n! \times n!$ matrix

$$(M_n)_{f,g} := \begin{cases} 1 & \text{if } \{\text{ descents of } f^{-1}\} \supseteq \{\text{ descents of } g\}, \\ 0 & \text{otherwise.} \end{cases}$$

# of partitions of $\{1, ..., n\}$

↓

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- Question: What is the asymptotic behaviour of $\lambda_{max}(M_n)$?

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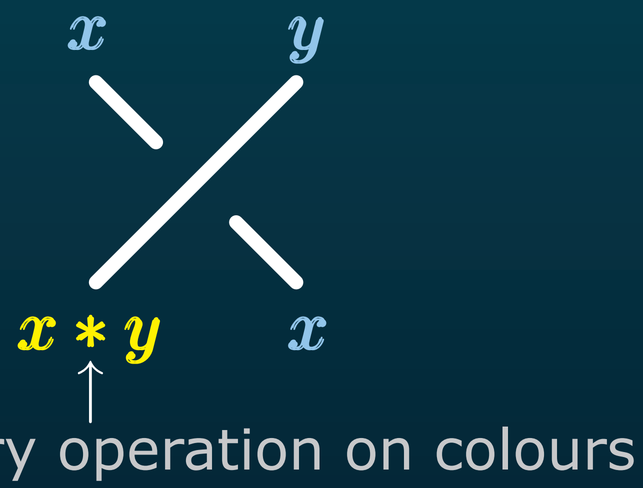

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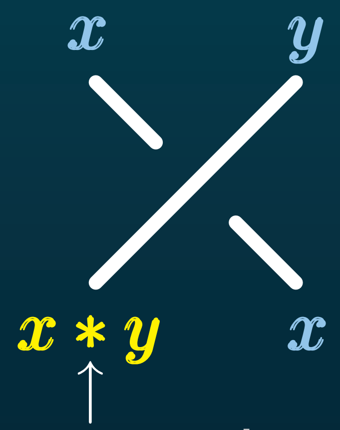

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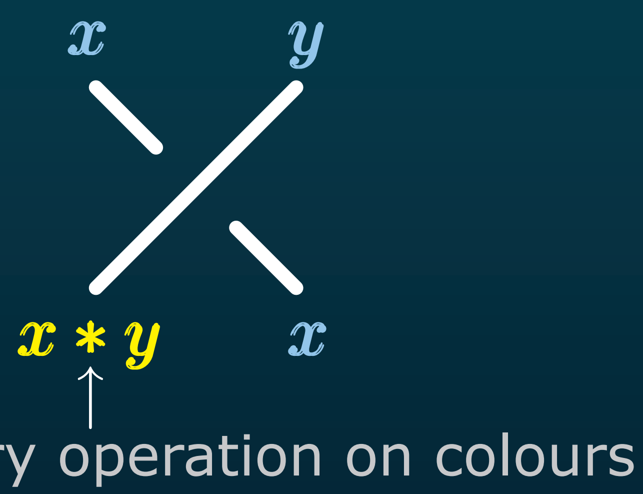

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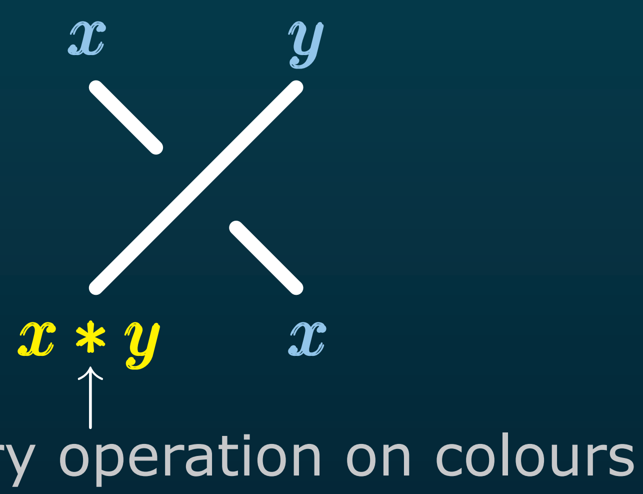

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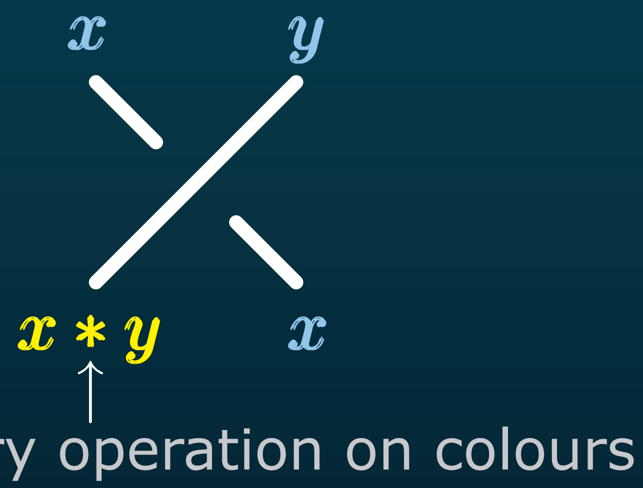

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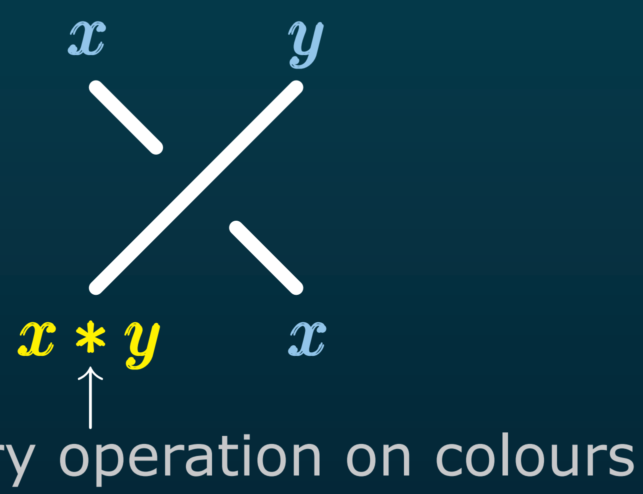

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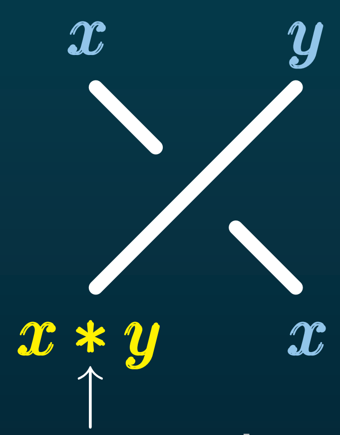

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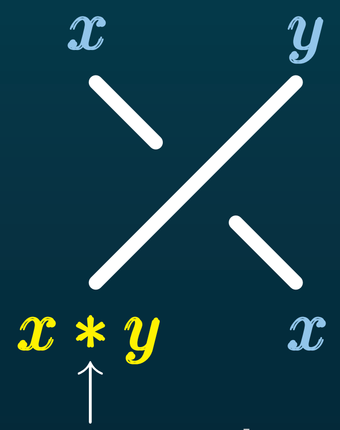

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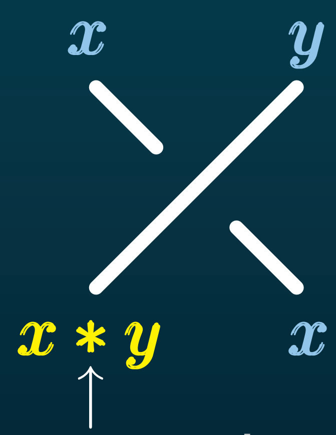

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(all satisfy $x * x = x$)

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a set with a binary operation satisfying...

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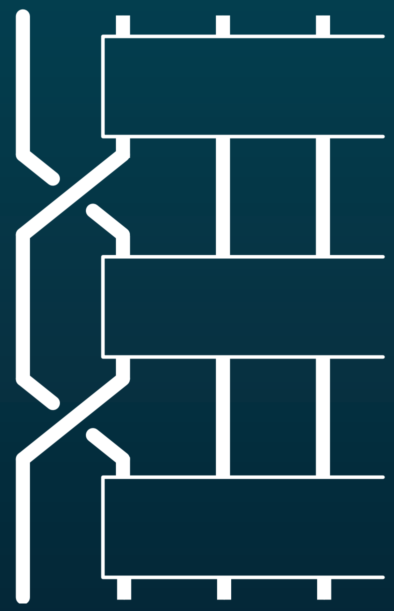

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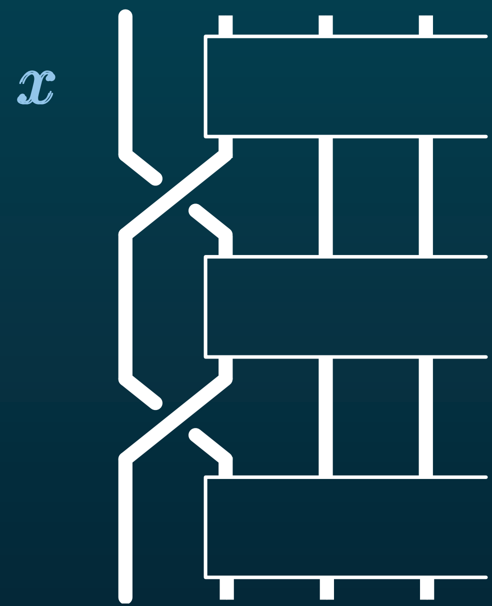

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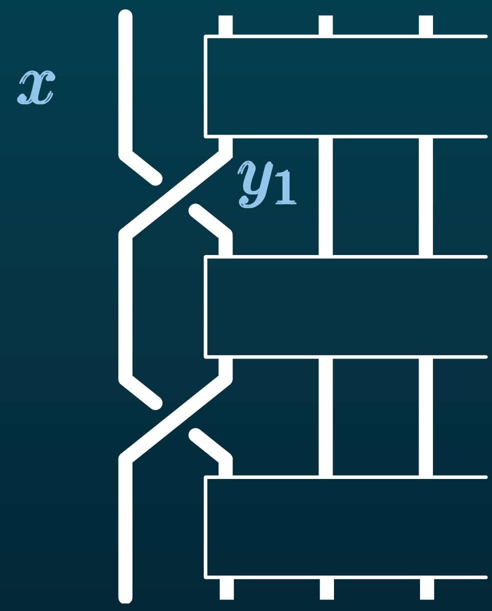

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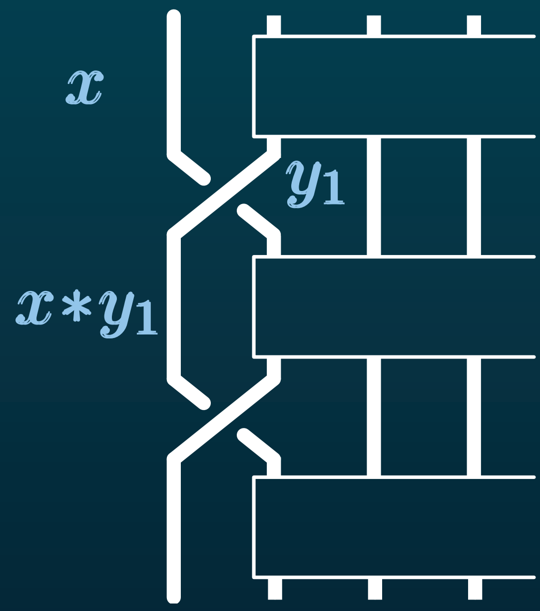

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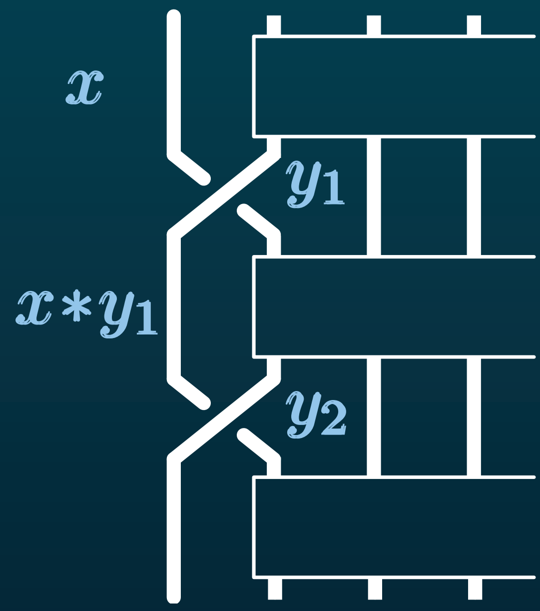

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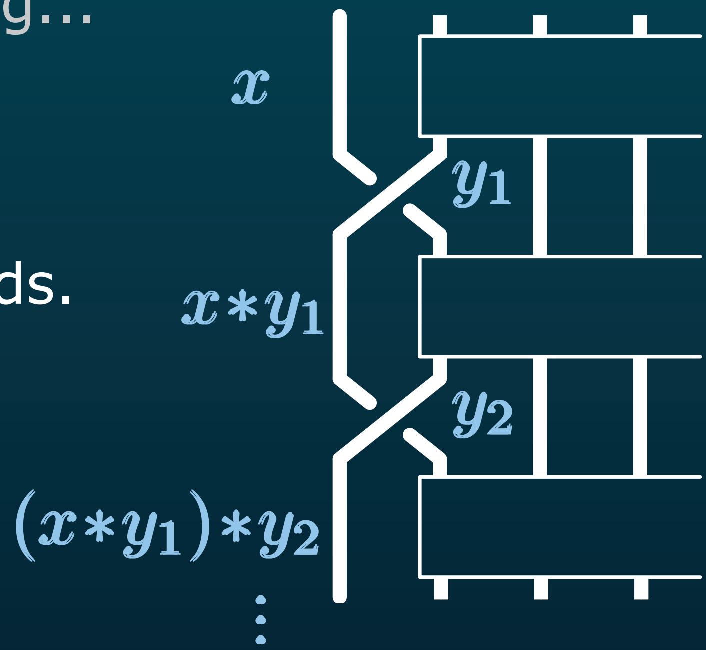

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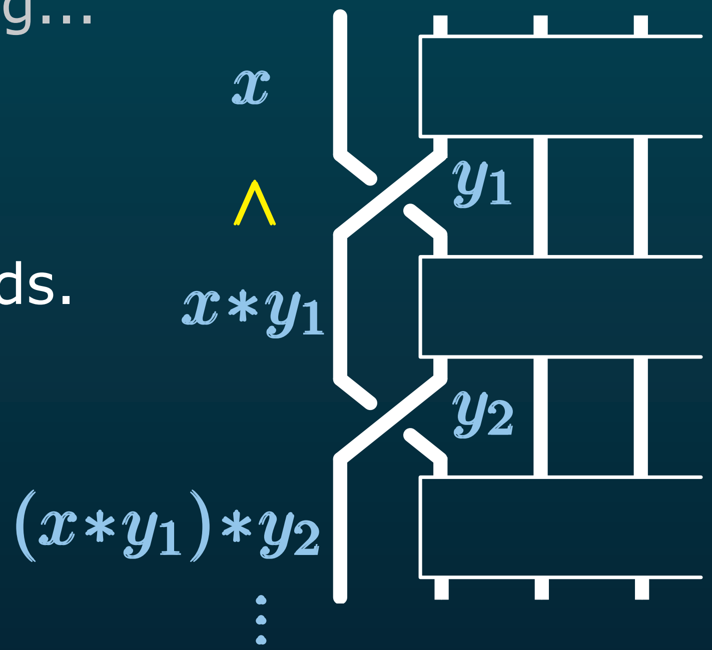

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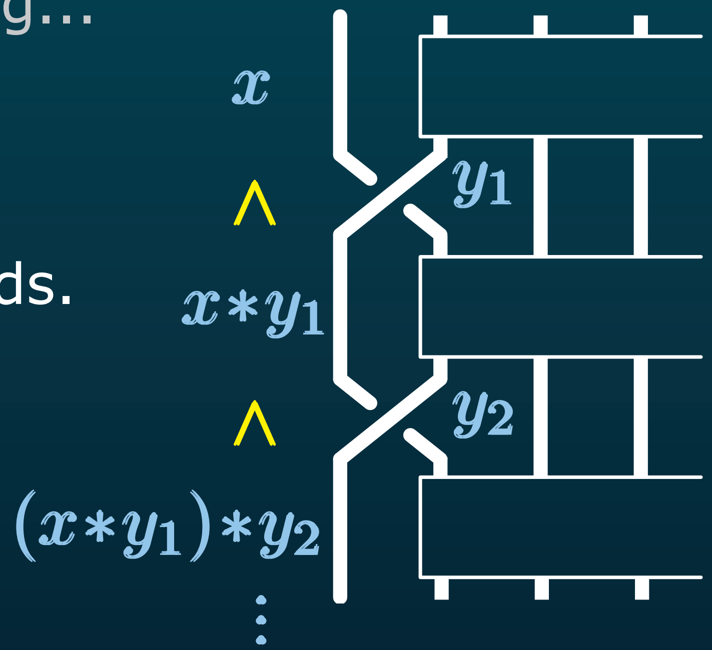

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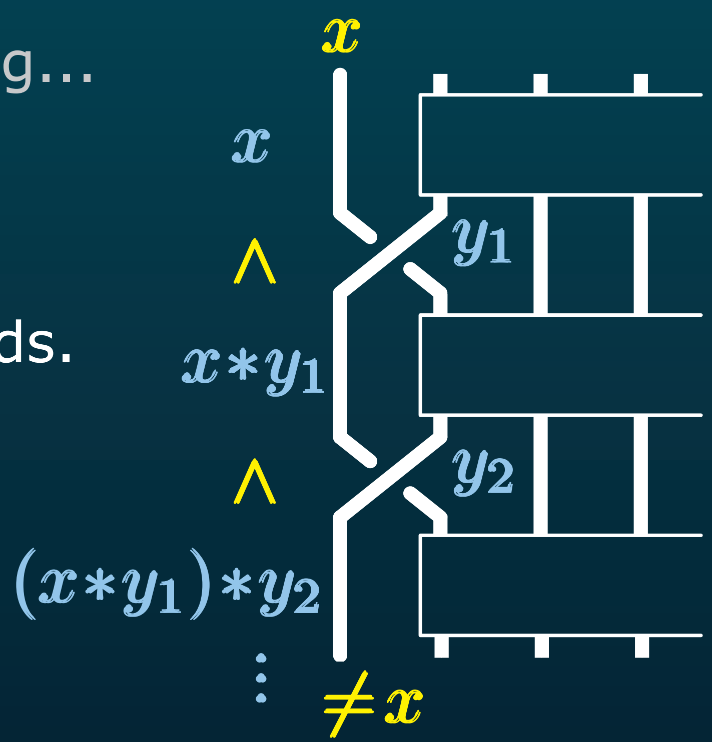

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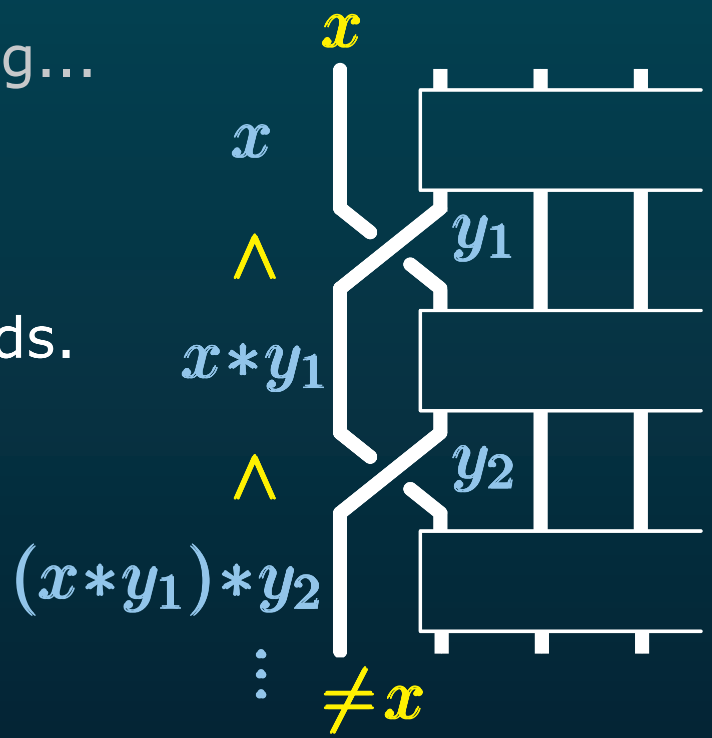

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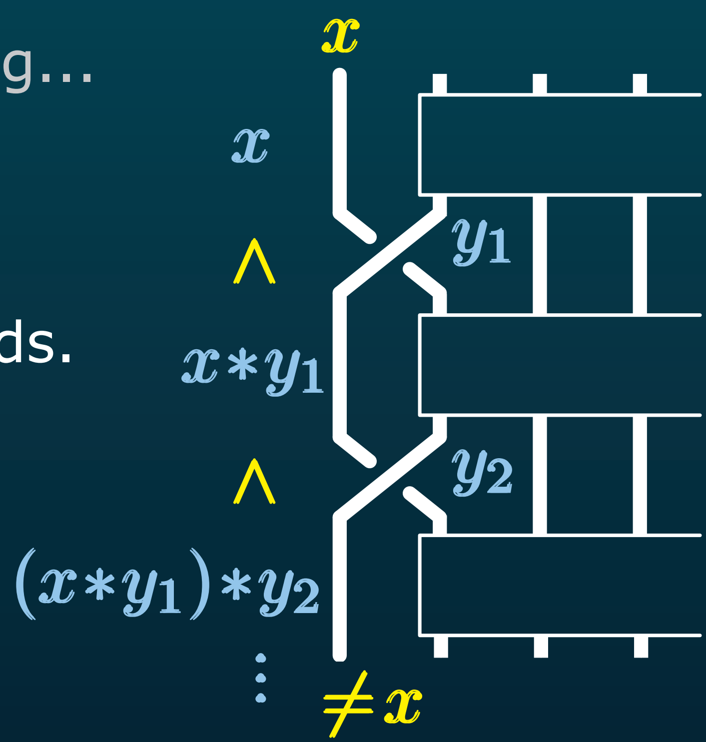

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by Gödel's incompleteness thrm, an unprovable logical assumption

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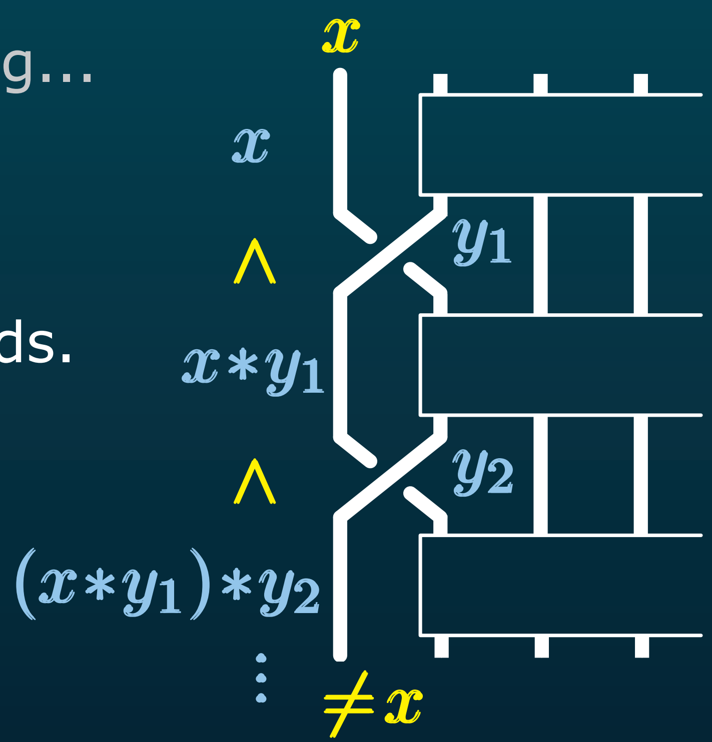

- Proposition: If there exists an orderable LD-system, then Property A is true.

by Gödel's incompleteness thrm, an unprovable logical assumption

↓

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a set with a binary operation satisfying...
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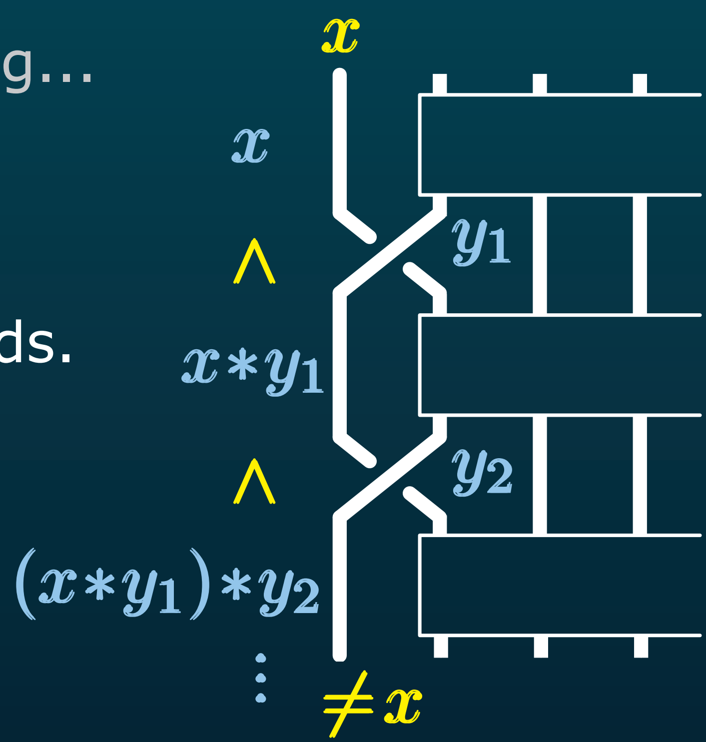

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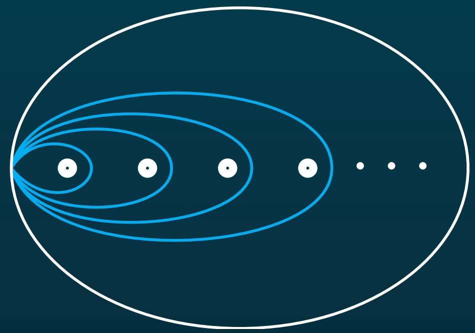
$\rightsquigarrow$ Handle reduction is an application of Set Theory (?)

(I. Dynnikov, 1999)

- View B_n as $\mathbf{MCG}(D_n)$, and let the homeo act on a fixed lamination L:



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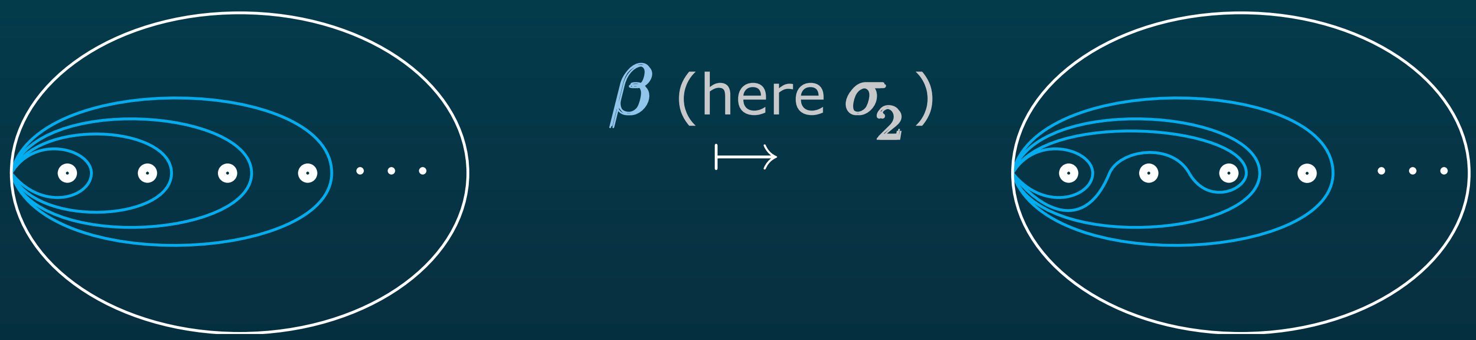

SOLUTION 6: SINGULAR TRIANGULATIONS

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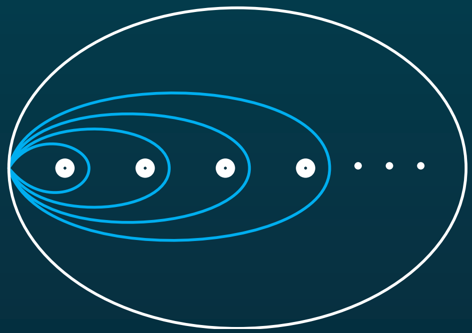

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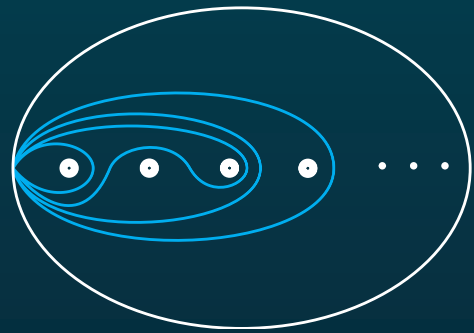

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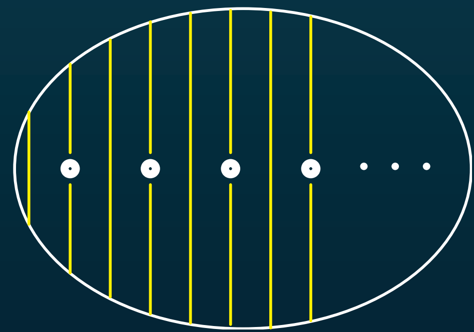

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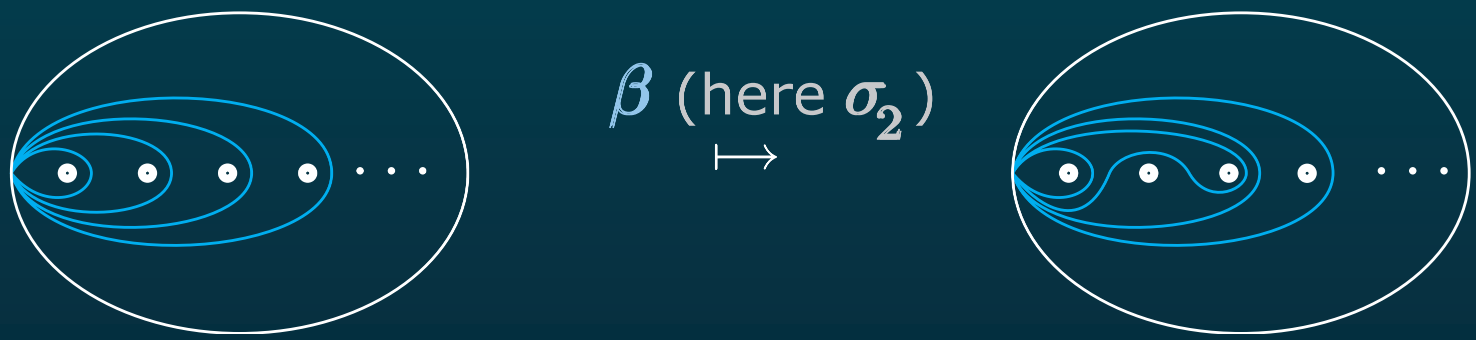

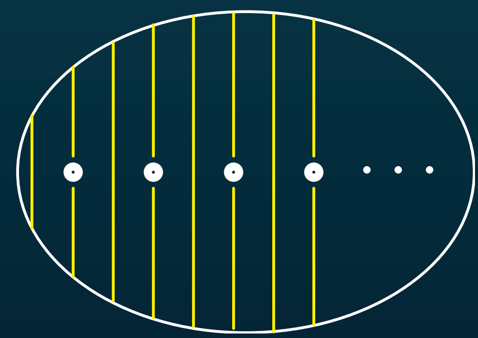

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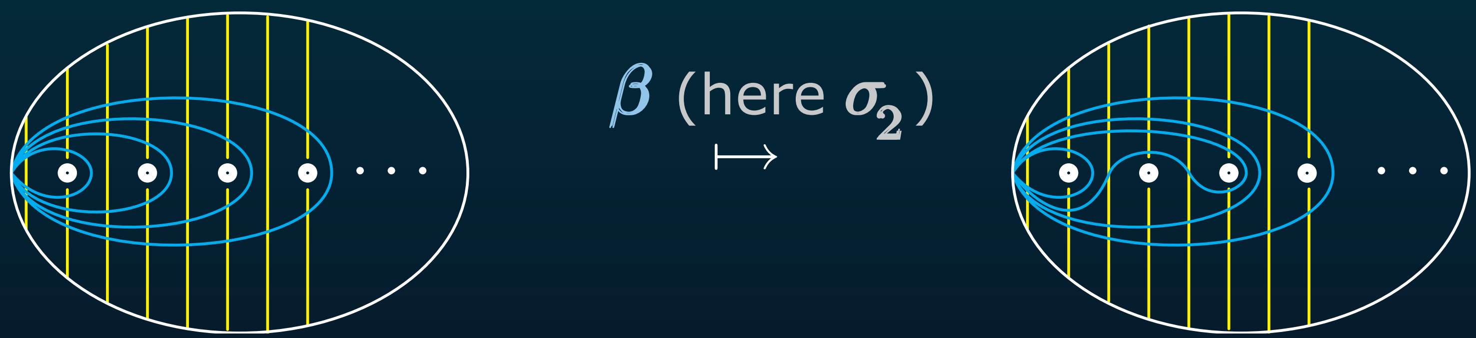

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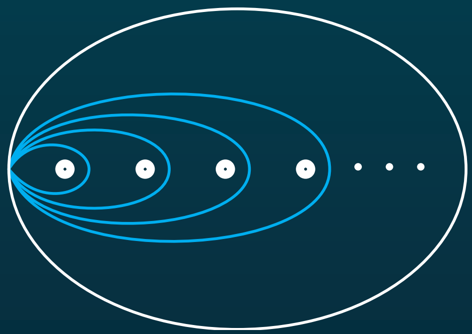

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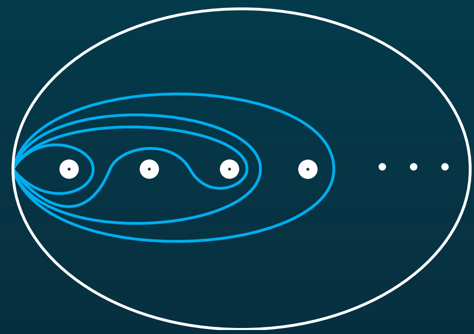

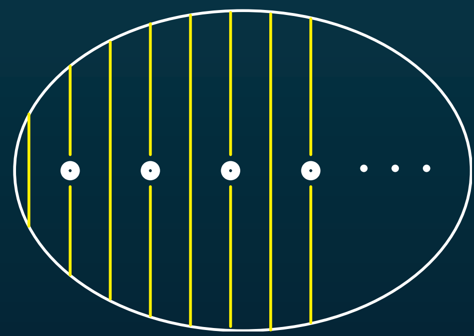

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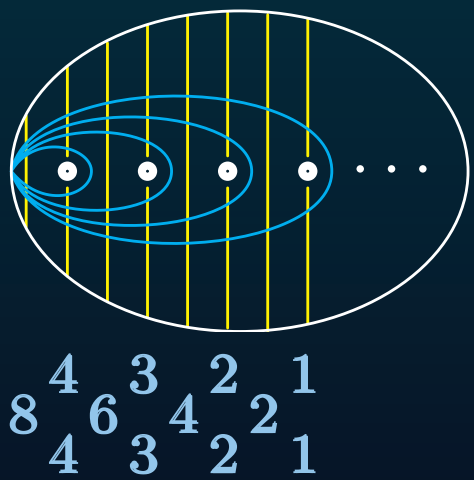

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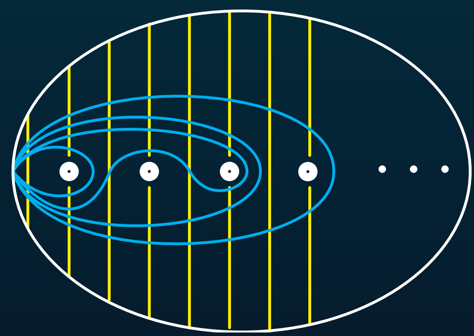

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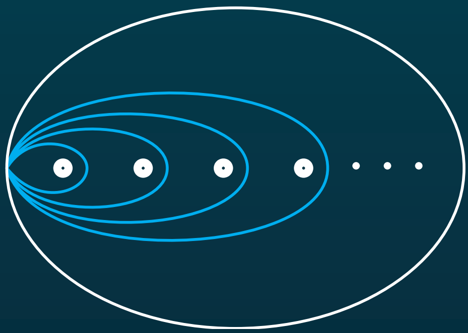

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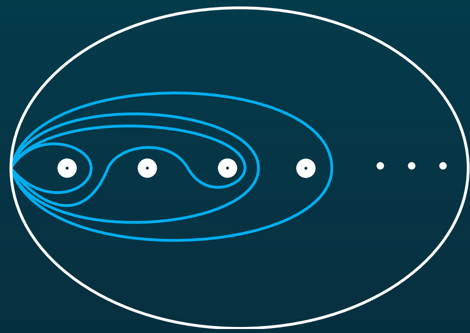

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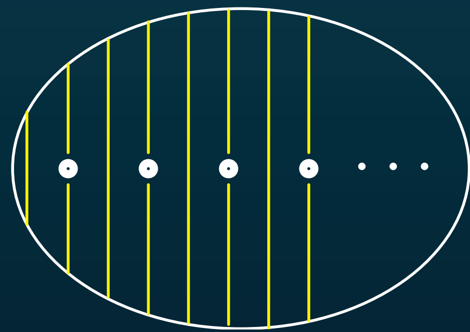

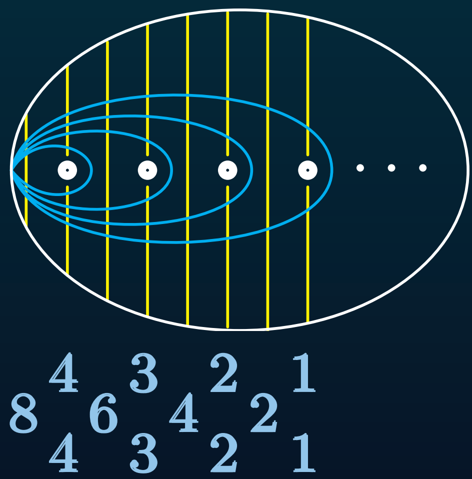

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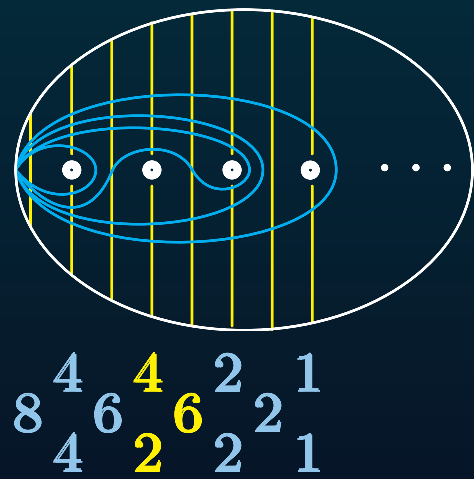

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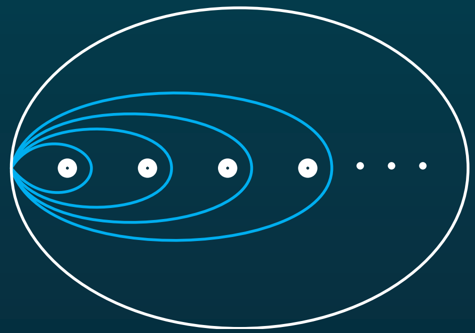

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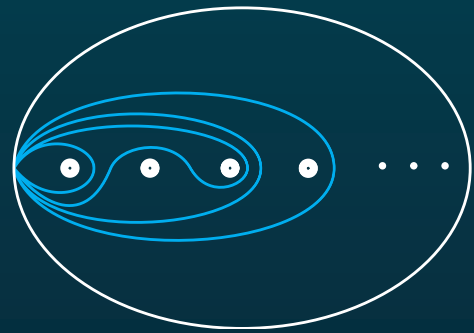

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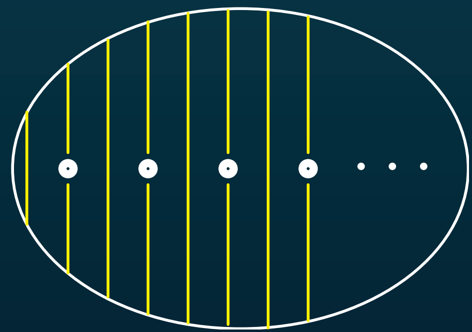

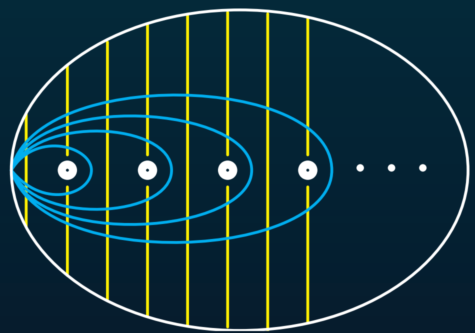

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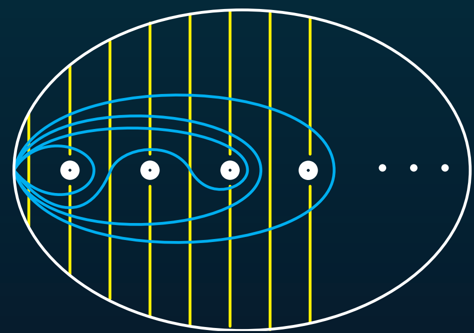

$8\,{}^{4}_{4}\,6\,{}^{3}_{3}\,4\,{}^{2}_{2}\,2\,{}^{1}_{1}\ldots \rightsquigarrow (0,1,0,1,0,1,0,\ldots)$ $\quad$ $8\,{}^{4}_{4}\,6\,{}^{4}_{2}\,6\,{}^{2}_{2}\,2\,{}^{1}_{1}\ldots \rightsquigarrow (0,1,1,0,0,2,0,\ldots)$

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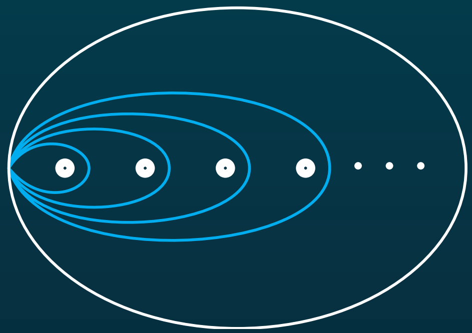

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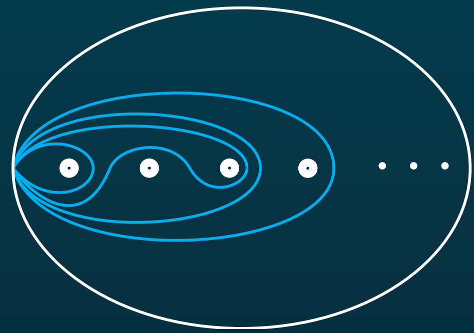

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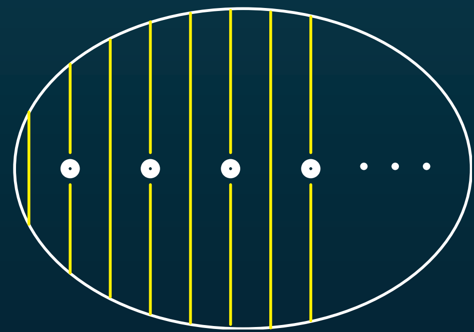

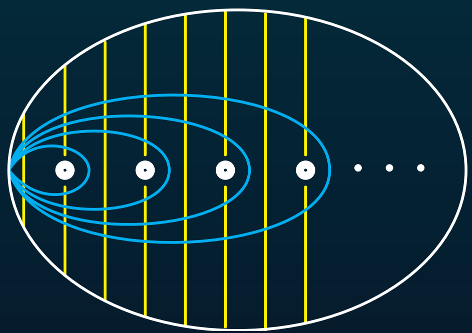

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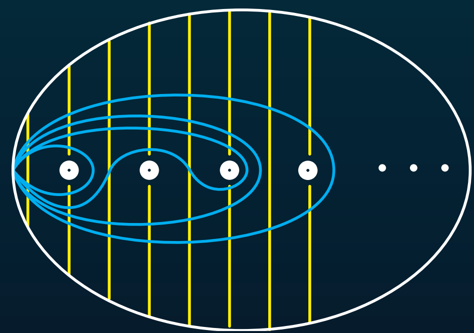

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$\rightsquigarrow$ Explicit injection $B_n \hookrightarrow \mathbb{Z}^{2n}$: coordinates for $L \cdot \beta$.

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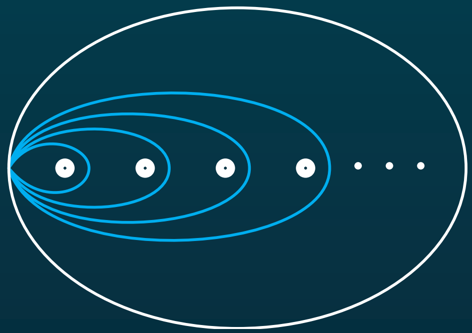

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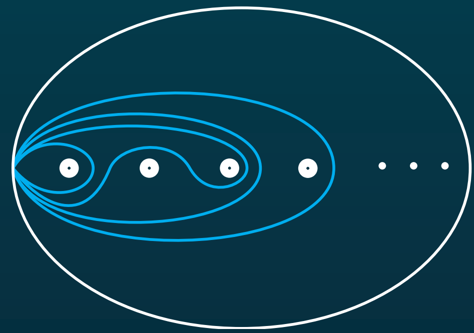

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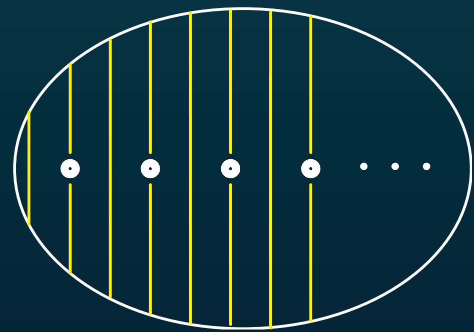

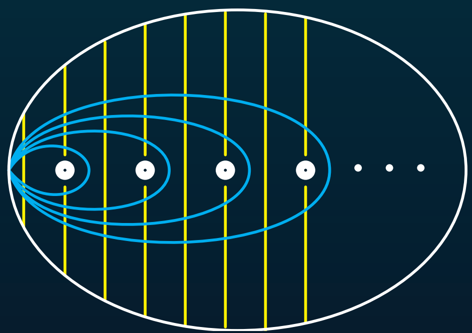

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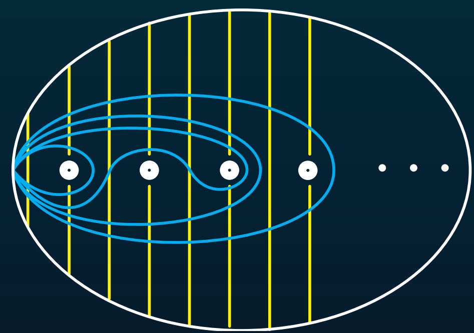

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- Behind: automatic structure for mapping class groups (Mosher)

(D., 2007)

- Another unique normal form for (positive) braids.

SOLUTION 7: ALTERNATING DECOMPOSITION

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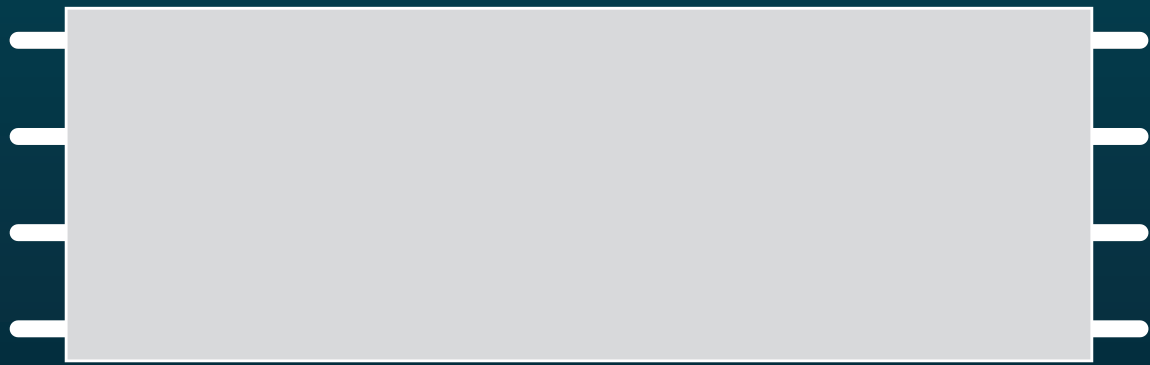

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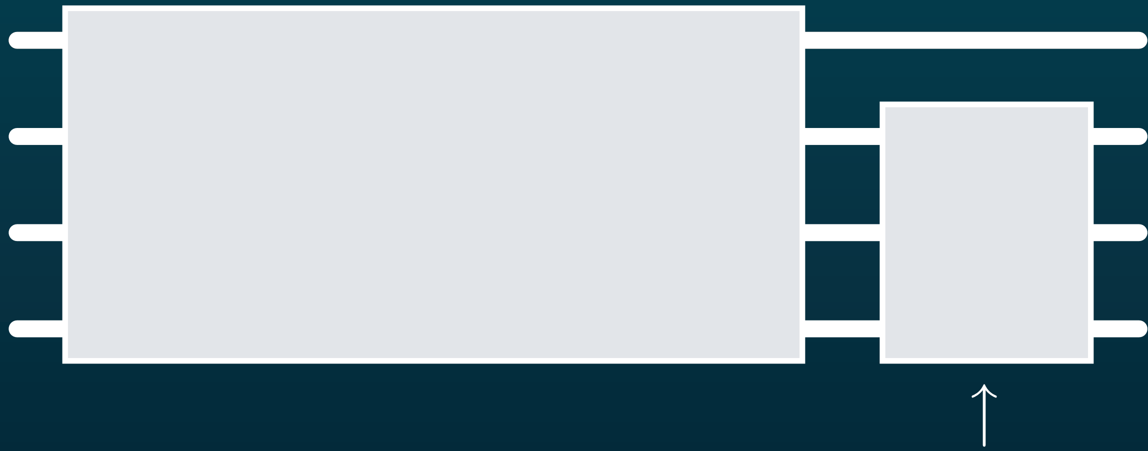

max. right divisor lying in B^+_{n-1}

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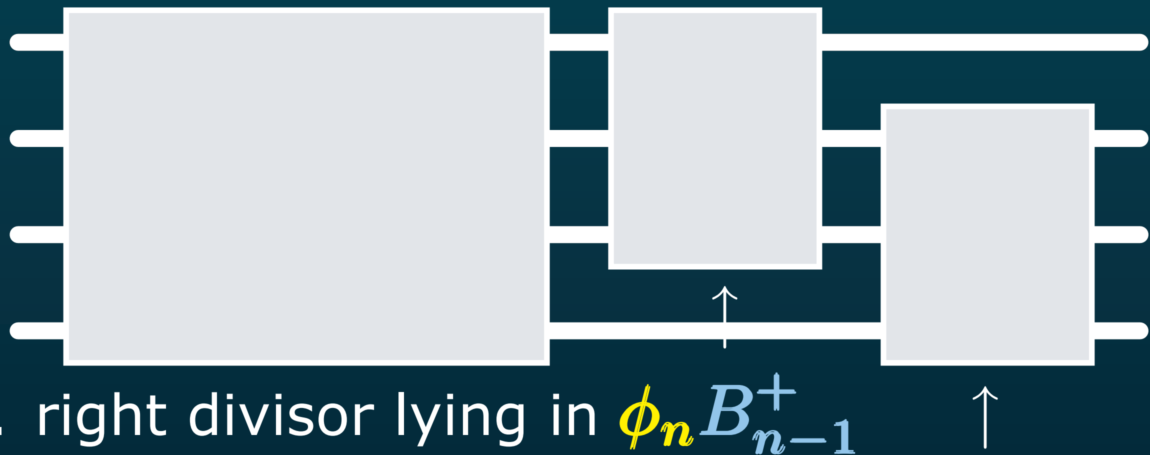

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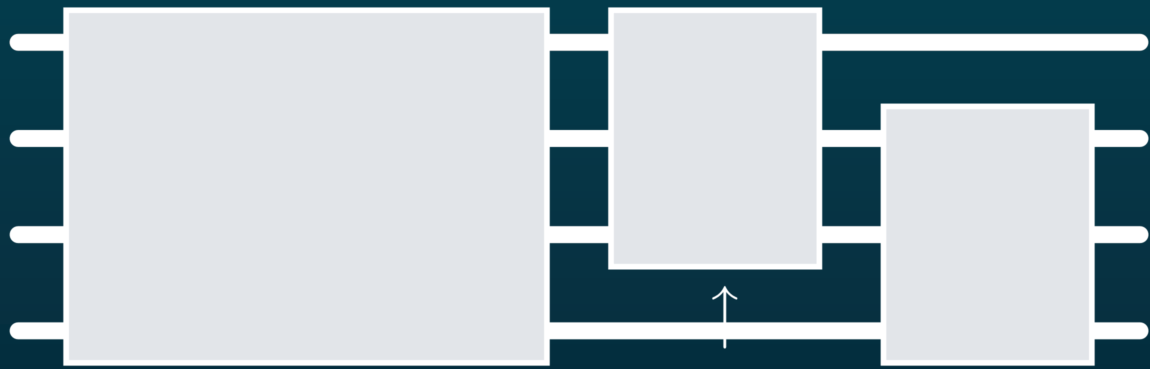

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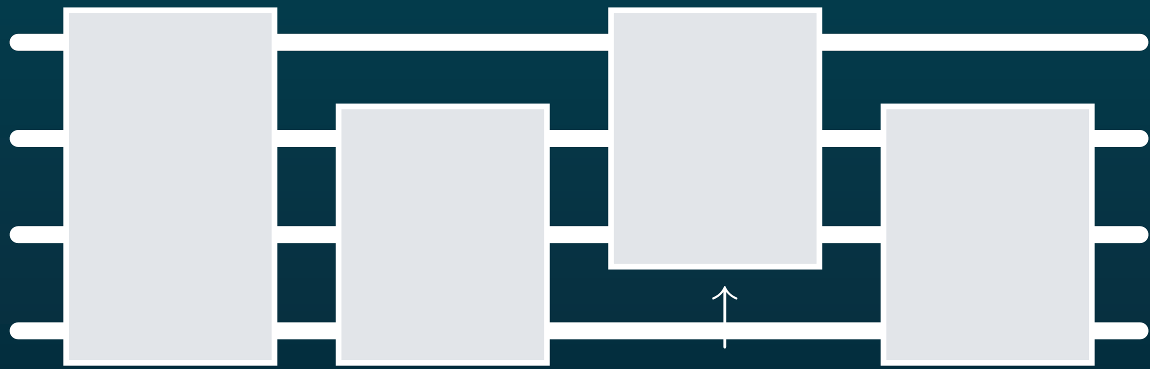

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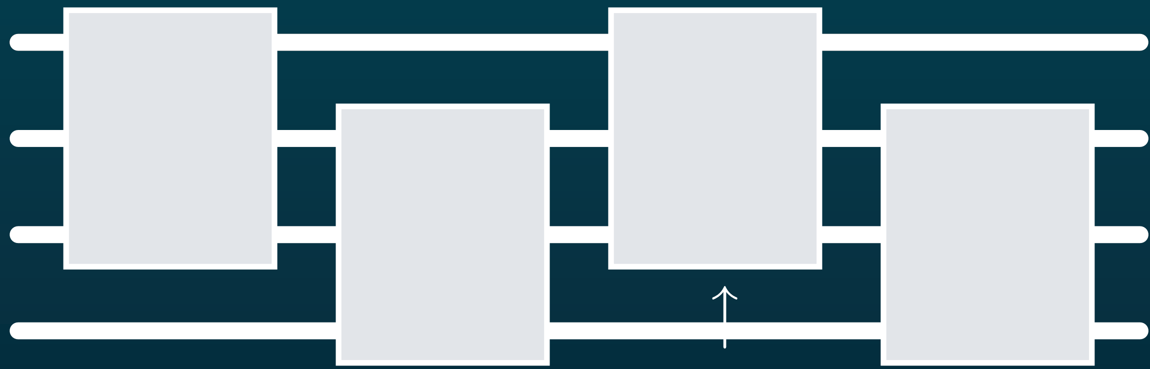

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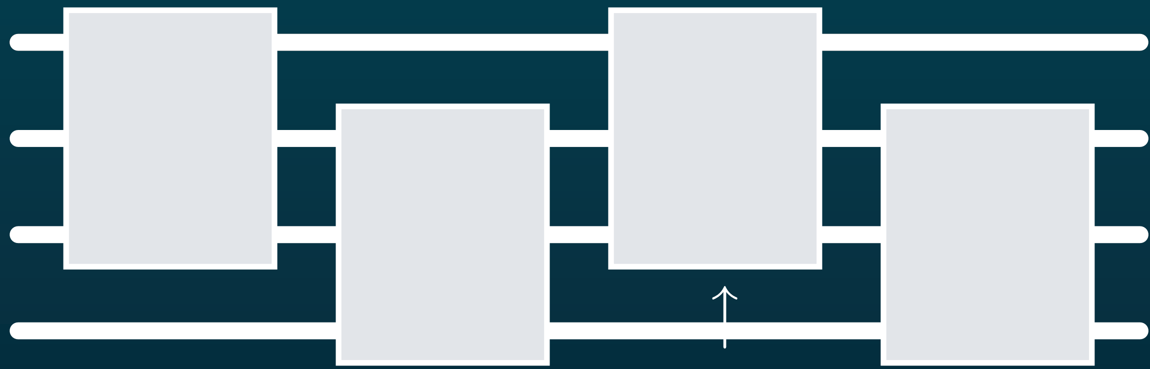

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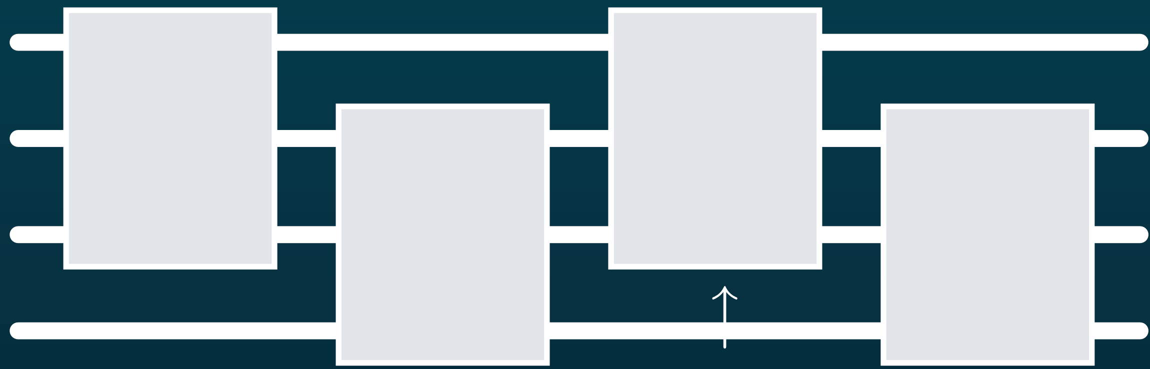

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SOLUTION 6: FLIP NORMAL FORM VS. ORDER

- By iterating: flip normal form

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$\rightsquigarrow$ quadratic time solution to the braid isotopy problem.

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- Behind: Standard braid order (“Dehornoy order"), Burckel's approach

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- Definition: For x, y braids, say that $x < y$ holds if, among all word expressions of $x^{-1}y$, there is at least one where the generator σ_i with higher index occurs only positively (σ_i occurs, σ_i^{-1} does not).

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  $\rightsquigarrow$ completely defines the order on B_n^+ from the order on B_{n-1}^+

SOLUTION 8: THE DUAL BRAID MONOID

(Birman–Ko–Lee, 1997)

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- New (redundant) family of generators: for $1 \leqslant i < j \leqslant n$, put

$$a_{i,j} = \sigma_{j-1} ... \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} ... \sigma_{j-1}^{-1}.$$

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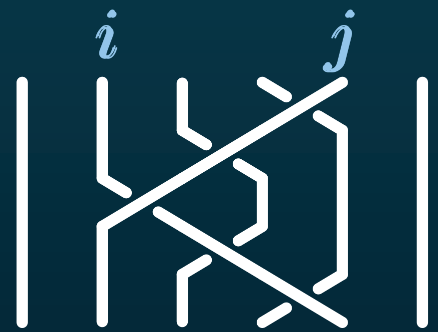
$a_{i,j} = \cdots$



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- New (redundant) family of generators: for $1 \leqslant i < j \leqslant n$, put

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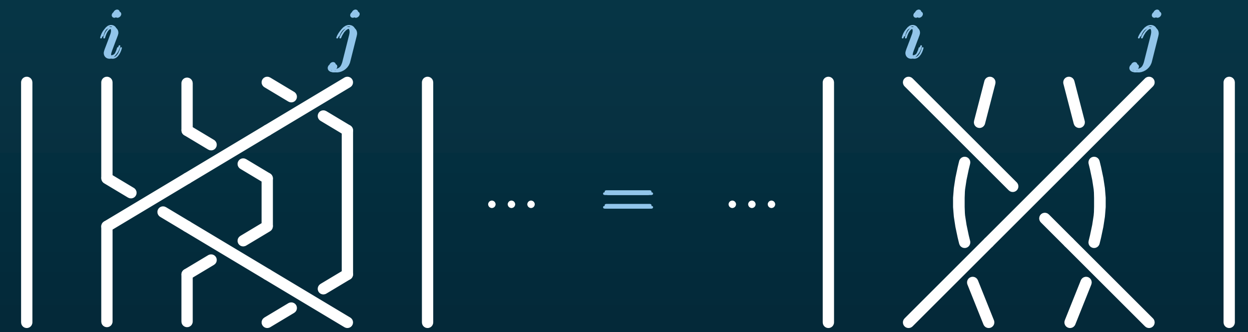
$a_{i,j} = \cdots$  $\cdots$

- Def: BKL_n^+ = submonoid of B_n generated by all $a_{i,j}$'s.

  ⇝ Another Garside structure, with Garside element $\delta_n = \sigma_{n-1} \ldots \sigma_2 \sigma_1$:

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    - ⇝ Greedy normal form
      - ⇝ new quadratic solution to the braid isotopy problem
    - ⇝ Automatic structure, etc.

SOLUTION 9: THE CYCLING NORMAL FORM

(Fromentin, 2007)

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- Mix the ideas of the flip normal form and the Birman-Ko-Lee monoids

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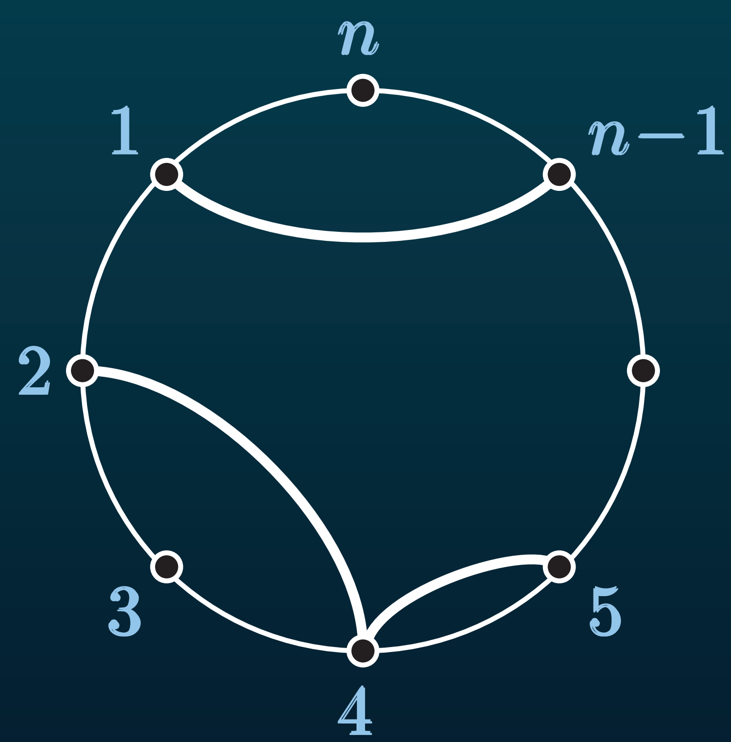
⇝ Generators $a_{i,j}$ of BKL_n^+
= chords of a circle



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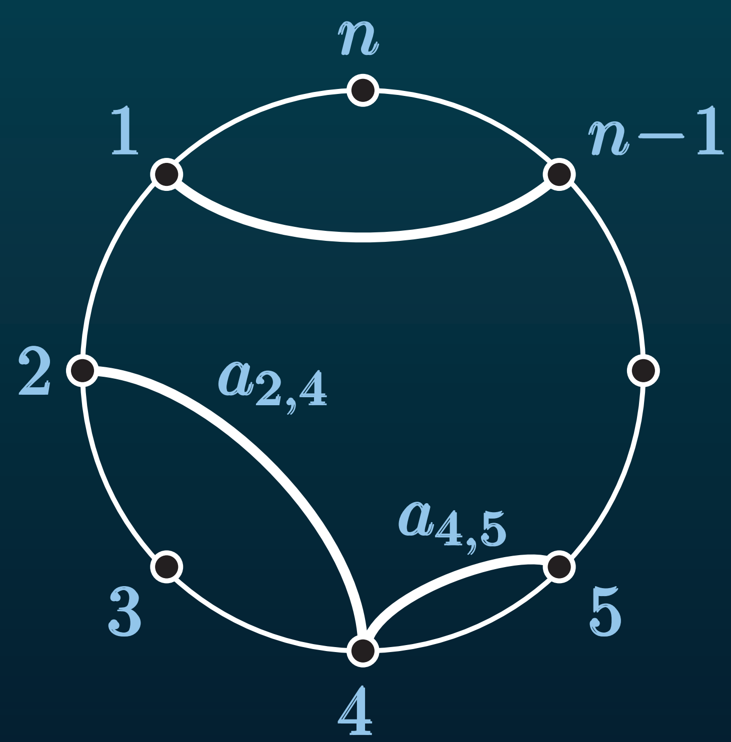

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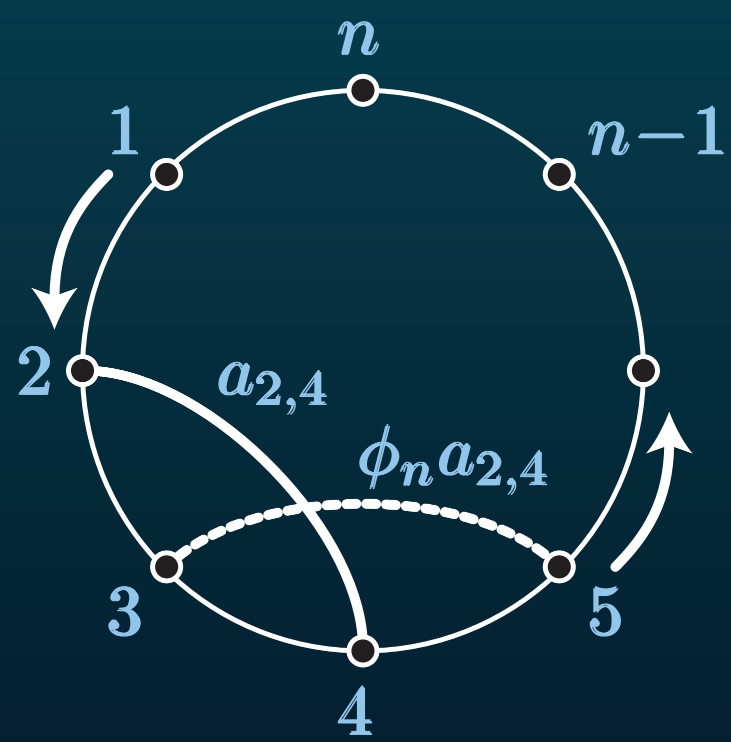
  - ⇝ Conjugation by δ_n
    = rotation by $2\pi/n$



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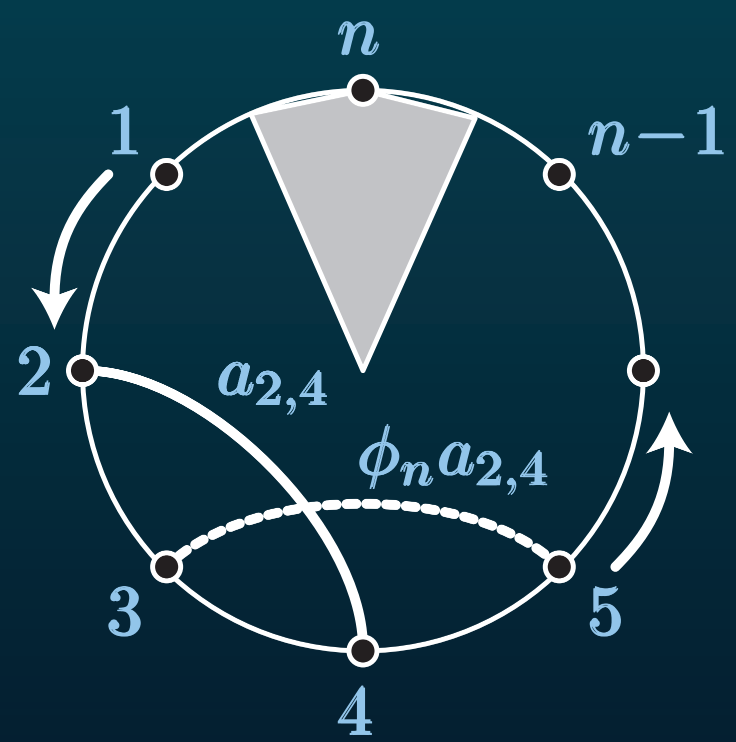

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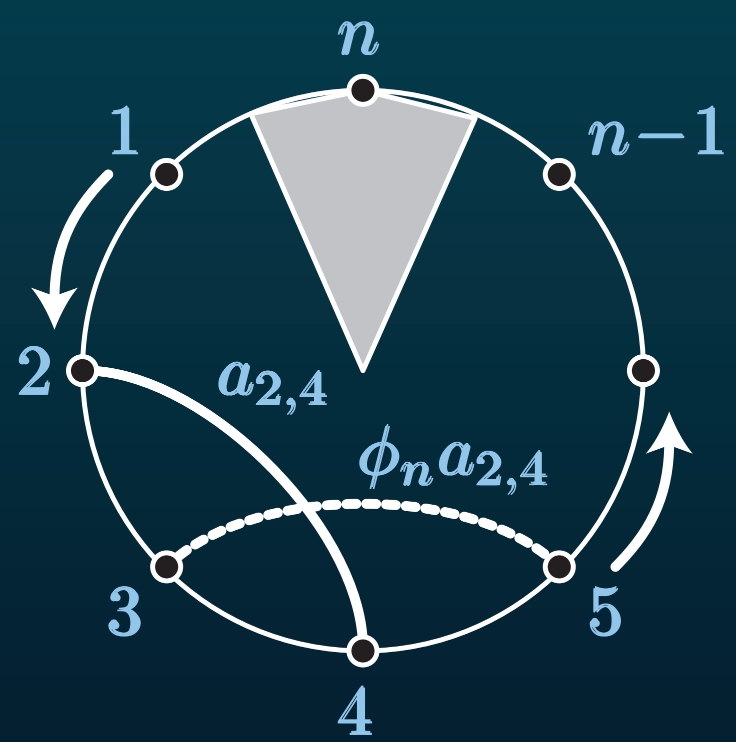

- Then: Every braid in BKL_n^+ admits a unique decomposition

$$x = \phi_n^{p-1} x_p \cdot \ldots \cdot \phi_n^2 x_3 \cdot \phi_n x_2 \cdot x_1$$

s.t. $x_p, \ldots, x_1$ lie in BKL_{n-1}^+ and the only $a_{i,j}$'s that is are right divisors of $\phi_n^{p-k} x_p \cdot \ldots \cdot \phi_n x_{k+1} \cdot x_k$ are $a_{i,n-1}$'s.

SOLUTION 8: CYCLING NORMAL FORM VS. ORDER

⇝ One more quadratic time solution to the braid isotopy problem,

$\rightsquigarrow$ One more quadratic time solution to the braid isotopy problem,
but also, mainly:

• Theorem (Fromentin, 2007) Assume $x, y \in BKL_n^+$, and let $(x_p, ..., x_1)$, $(y_q, ..., y_1)$ be the cycling decompositions of x and y. Then $x < y$ holds iff
- either $p < q$,
- or $p = q$ and $(x_p, ..., x_1)$ is lexicographically smaller than $(y_q, ..., y_1)$.

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⇝ New simple existence proof for the braid order;

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$\rightsquigarrow$ New simple existence proof for the braid order;

$\rightsquigarrow$ The restriction of the braid order to BKL_n^+ is a well-order of ordinal type $\omega^{\omega^{n-2}}$.

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⇝ New simple existence proof for the braid order;

⇝ The restriction of the braid order to BKL_n^+ is
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⇝ a distinguished element in each nonempty subset,

⇝ One more quadratic time solution to the braid isotopy problem,
but also, mainly:

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⇝ New simple existence proof for the braid order;

⇝ The restriction of the braid order to BKL_n^+ is
a well-order of ordinal type $\omega^{\omega^{n-2}}$.

⇝ a distinguished element in each nonempty subset,
typically in each conjugacy class.