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- Here, two problems:
  - recognizing a Garside group from a presentation;
  - working with a presented Garside group.

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- Principle: A Garside group is controlled by the finite lattice $\mathtt{Div}(\Delta)$.

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Δ_4

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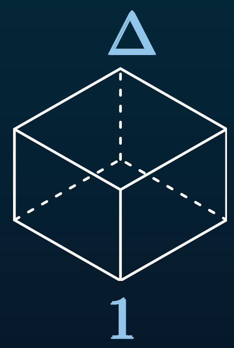

- More generally: spherical Artin–Tits groups
  $\rightsquigarrow$ lattice = weak order on the associated Coxeter group

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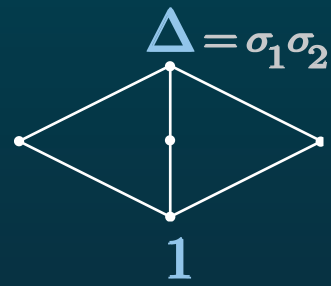

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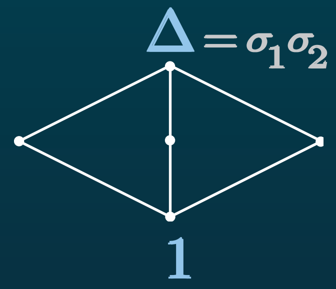

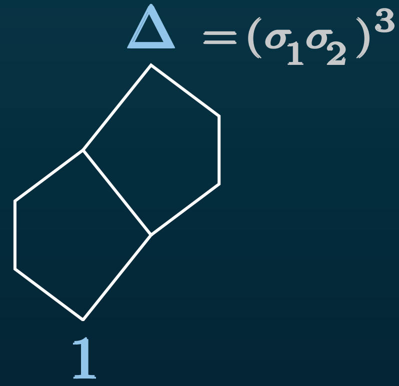

- Also, always for B_3: $\langle a, b; aba = b^2\rangle$:

EXAMPLES (2)

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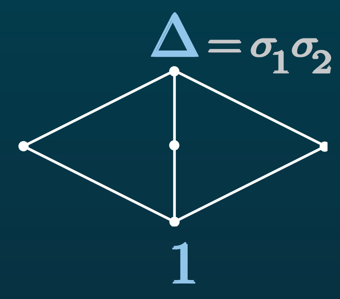

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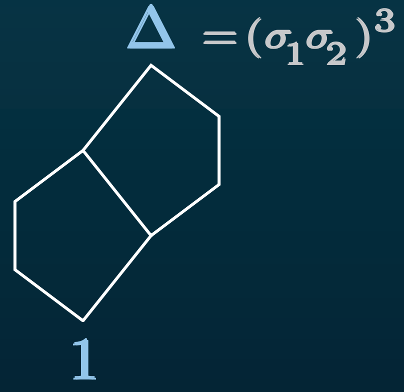


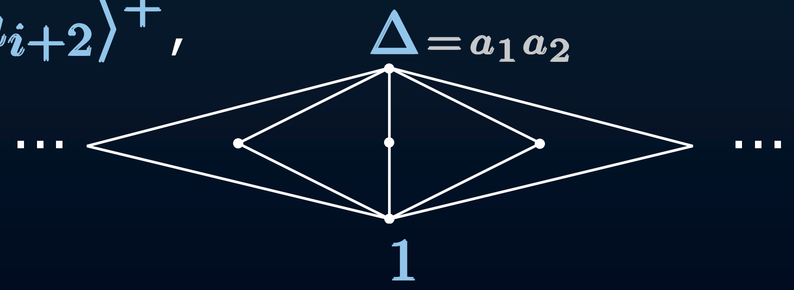
like Garside but with $\mathbf{Div}(\Delta)$ finite height only (not necessarily finite)

↓

- Free groups are quasi-Garside (Bessis, Brady-Crisp-Kaul-McCammond)

F_2: - monoid $\langle \ldots a_{-1}, a_0, a_1, \ldots; a_i a_{i+1} = a_{i+1} a_{i+2}\rangle^+$,

- quasi-Garside element $\Delta = a_i a_{i+1}$:



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- Assume that M is a Garside monoid generated by X, and R contains one relation $sv = tu$ representing $\mathbf{lcm}(s, t)$ for all s, t in X. Then (X, R) is a presentation of M (and of the group of fractions of M).

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- For such a presentation by construction:

- all relations of the form $u = v$ with u, v nonempty positive words (no s^{-1});
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- Remark 3: $w \curvearrowright w'$ implies $w \equiv w'$.

represents the same element in the group $\langle S; R \rangle$

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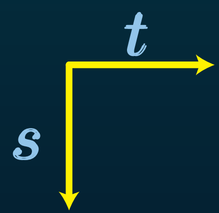

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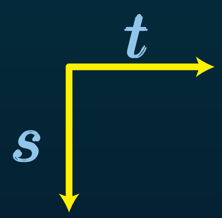
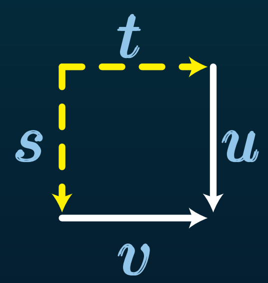
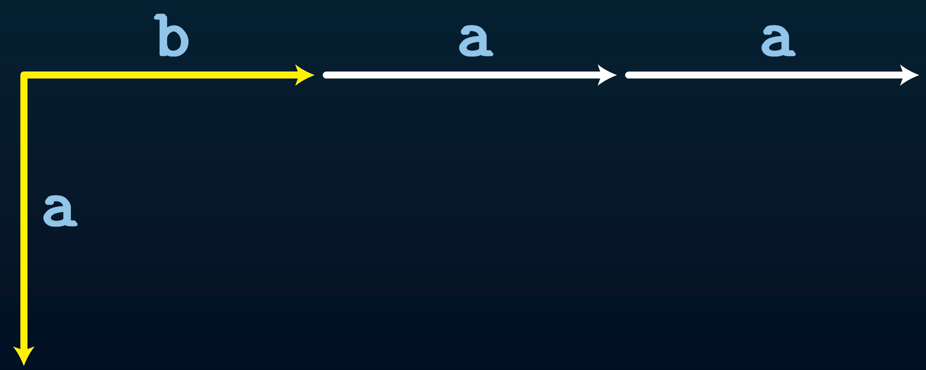
$s^{-1}t \curvearrowright vu^{-1} \rightsquigarrow$ replacing 

with 



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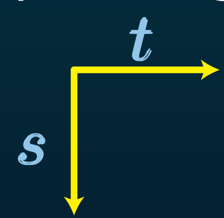
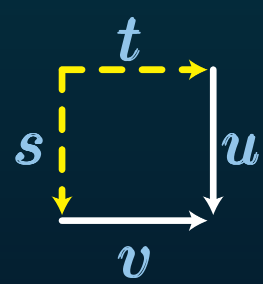
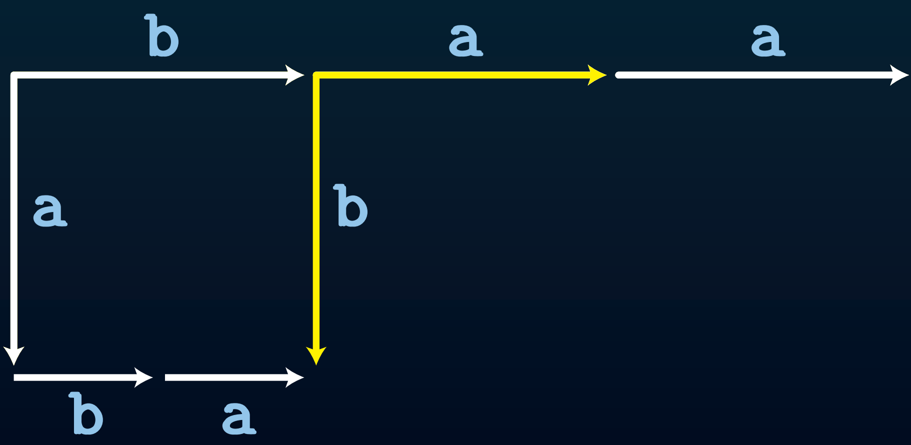
$s^{-1}t \curvearrowright vu^{-1} \rightsquigarrow$ replacing 

with 



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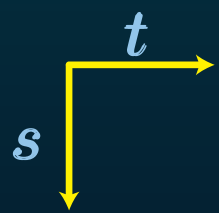
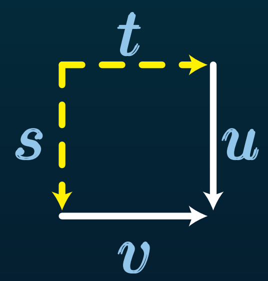
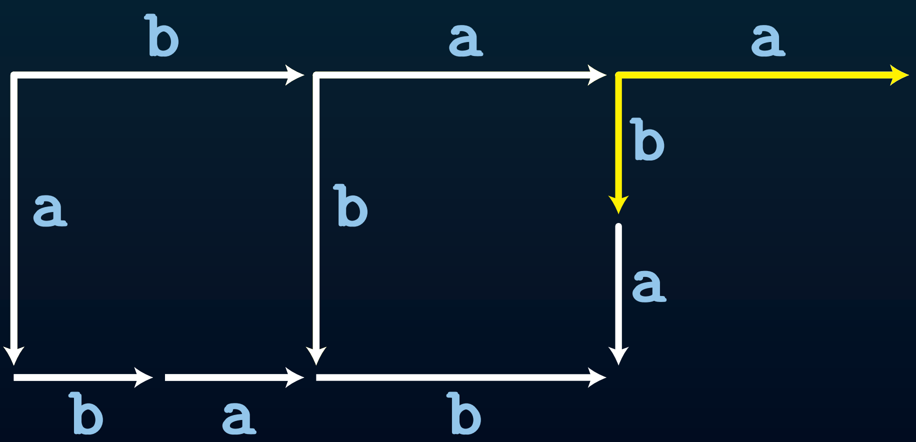
$s^{-1}t \curvearrowright vu^{-1} \rightsquigarrow$ replacing 

with 



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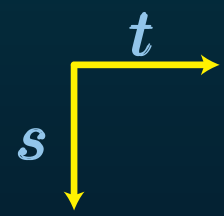
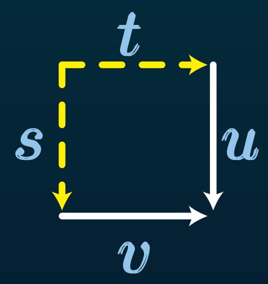
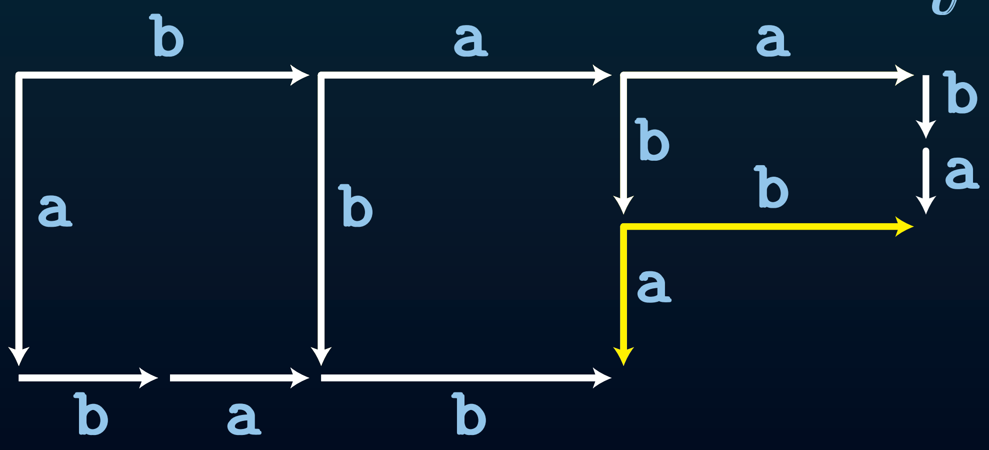
$s^{-1}t \curvearrowright vu^{-1} \rightsquigarrow$ replacing  with 



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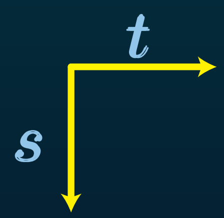
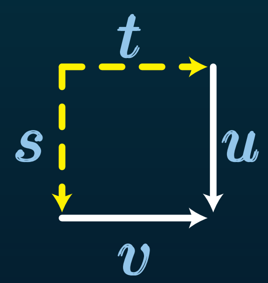
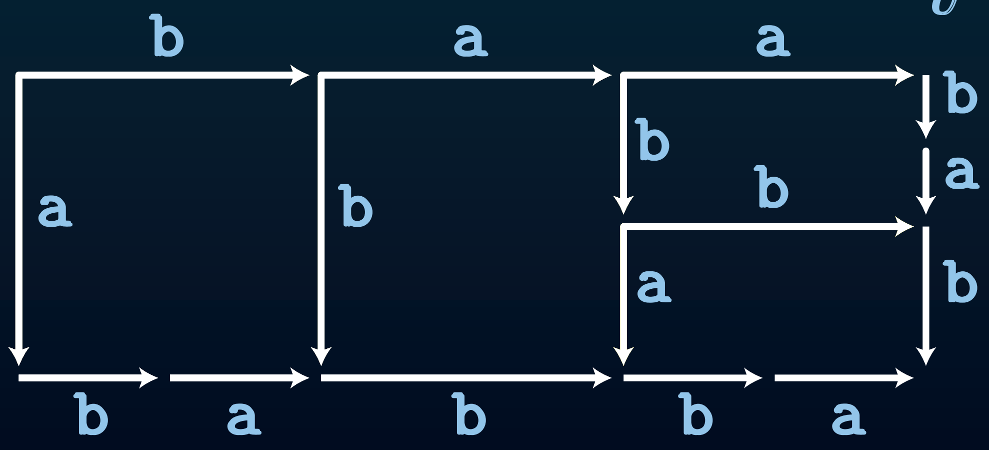
$s^{-1}t \curvearrowright vu^{-1} \rightsquigarrow$ replacing 

with 



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$\rightsquigarrow$ Criterion for recognizing completeness?

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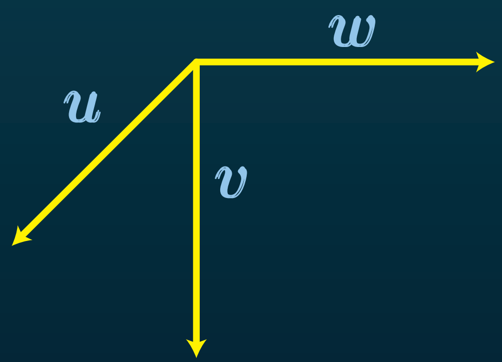

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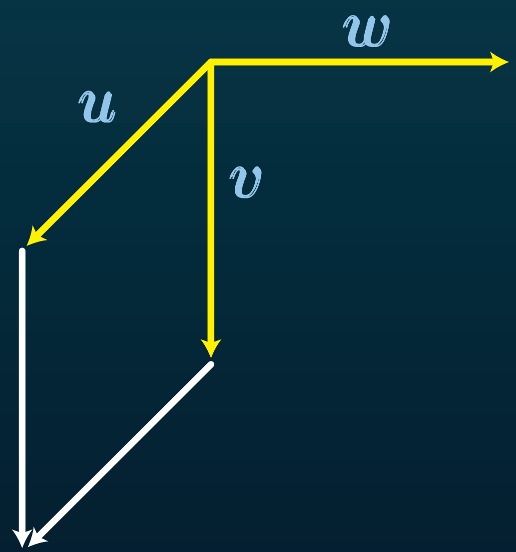

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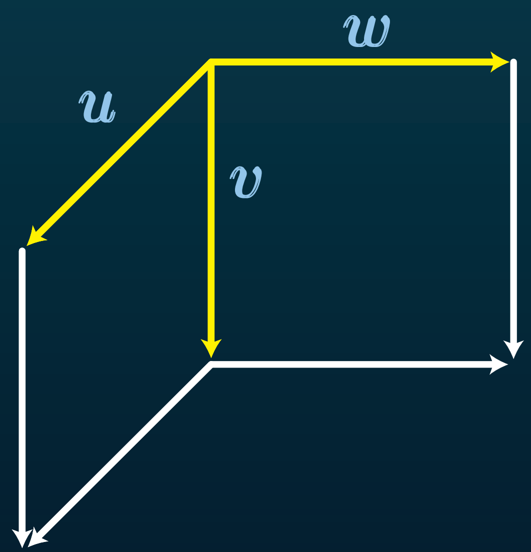

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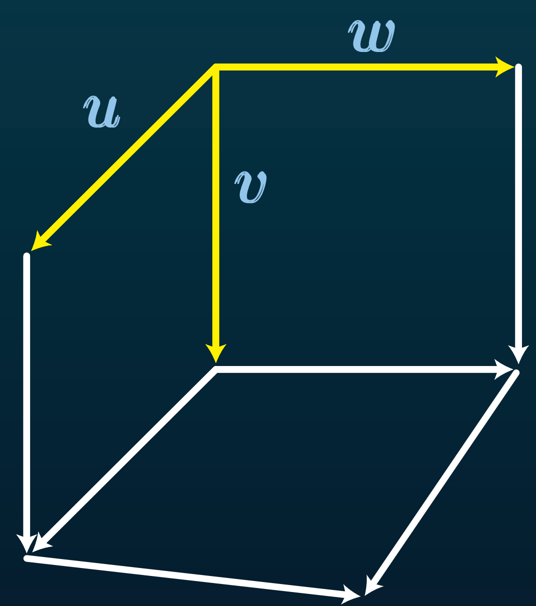

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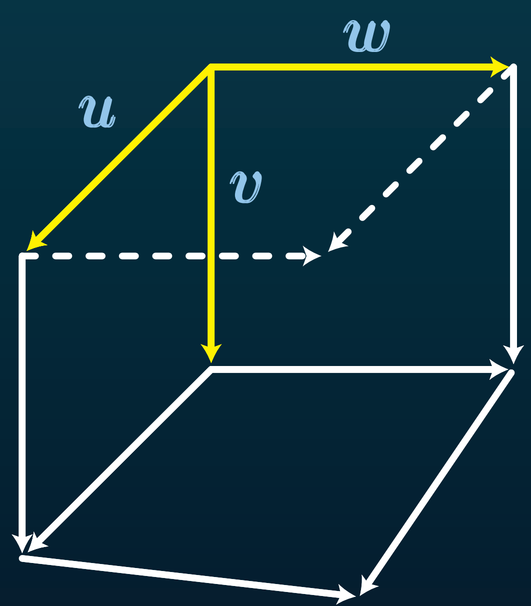

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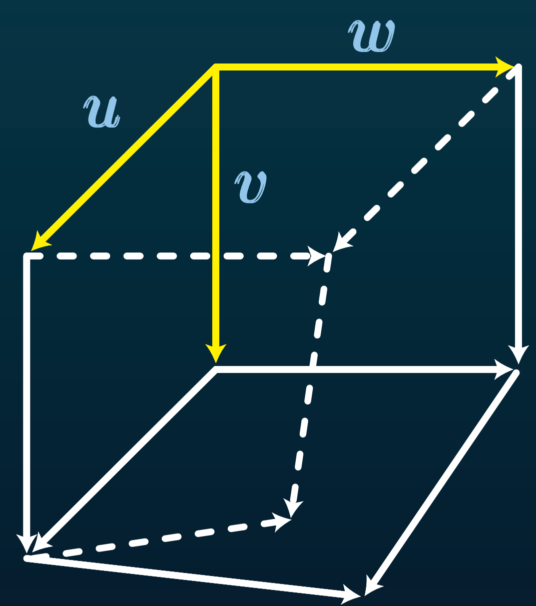

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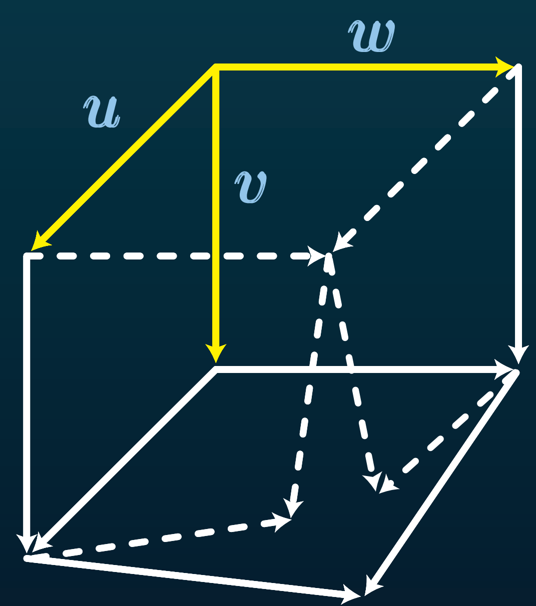

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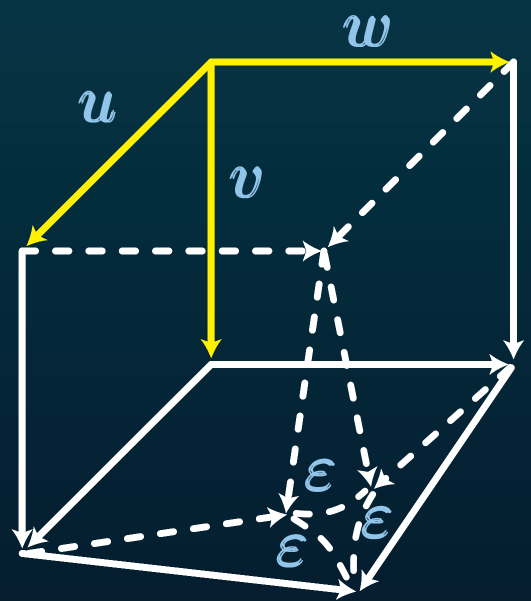

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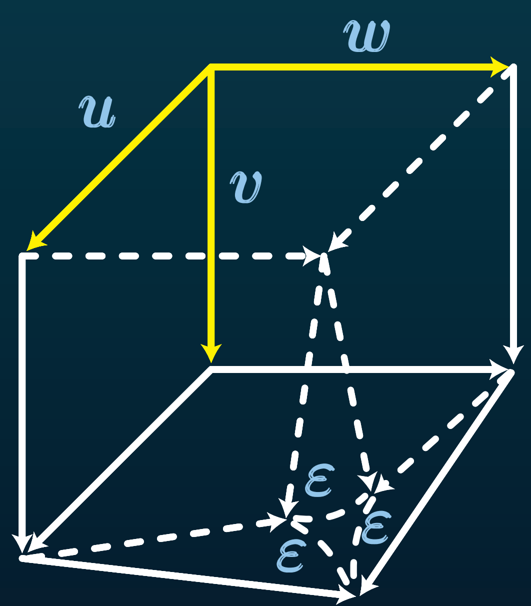

- Fact: A semigroup presentation (S, R) is complete for reversing iff the cube condition holds on S^* (***i.e.***, for all words).

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  For Criterion 2: $\widehat{S} := \{\varepsilon, \mathbf{a}, \mathbf{b}, \mathbf{ab}, \mathbf{bb}, \mathbf{ba}, \mathbf{bba}\}$; $\rightsquigarrow$ (in both cases) OK.

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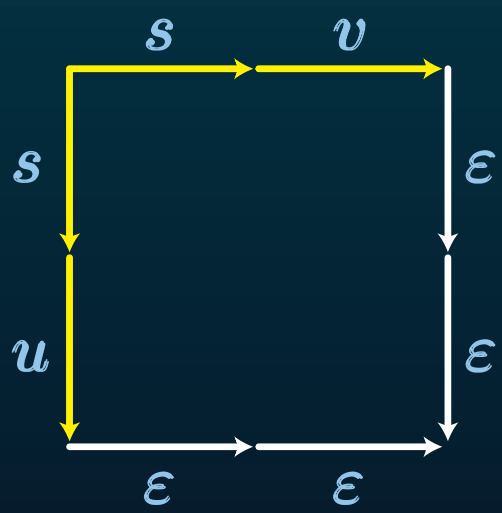

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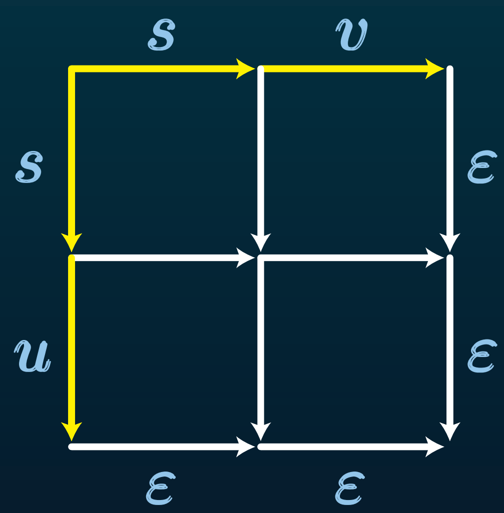

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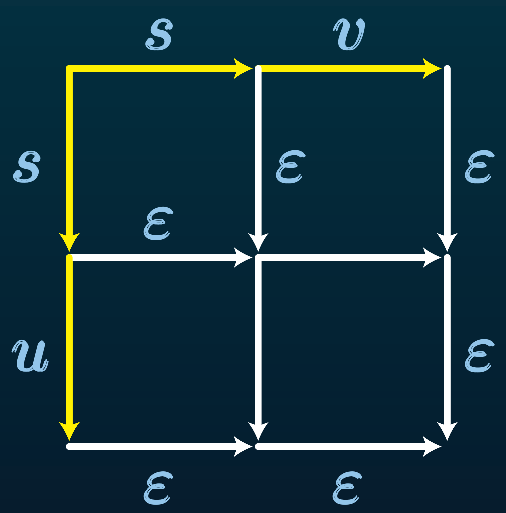

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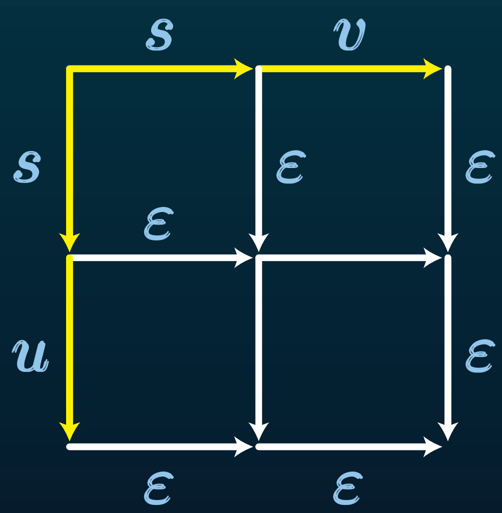

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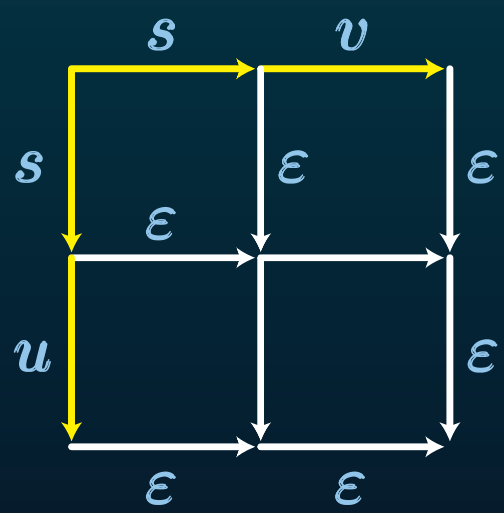

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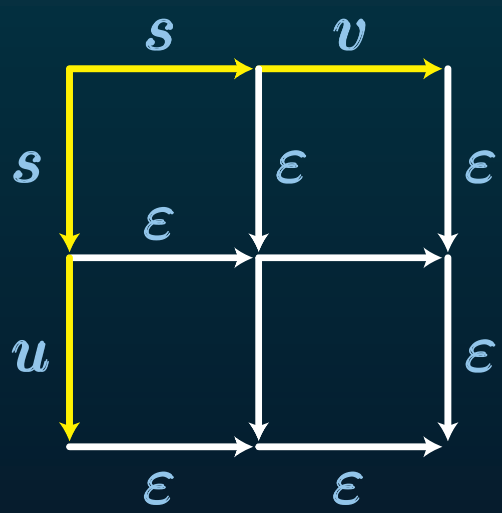

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- Prop.: Assume that (S, R) is complete for reversing, and R contains $\leqslant 1$ relation $su = tv$ for each pair s, t in S. Then $\langle S, R\rangle^+$ admits local right lcm's.

two elements with a common multiple admit a lcm

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RECIPE

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$\rightsquigarrow$ $\langle \mathbf{a}, \mathbf{b}; \mathbf{aba} = \mathbf{bb}\rangle^+$ is a Garside monoid with $\Delta = \mathbf{b}^3$ and 8 divisors of Δ.

• Proposition: Let G be a Garside group associated with (M, Δ). Then each element of G is uniquely expressed as gh^{-1} with $g, h \in M$ and $\mathrm{gcd}_R(g, h) = 1$.

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s.t. $\overline{u}'\overline{v}'^{-1}$ is the above expression of $\overline{w}$.

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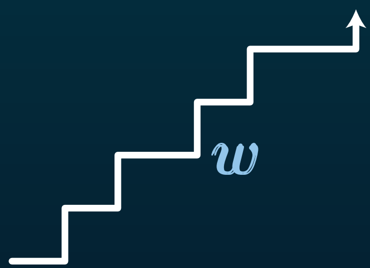

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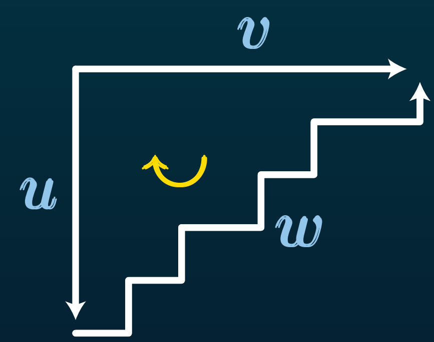

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• Corollary (solution to the word problem): w represents 1 iff $u' = v' = \varepsilon$.

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Output: uv' $(= \delta(u, v))$ and w,
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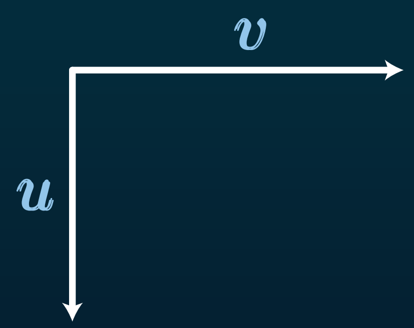

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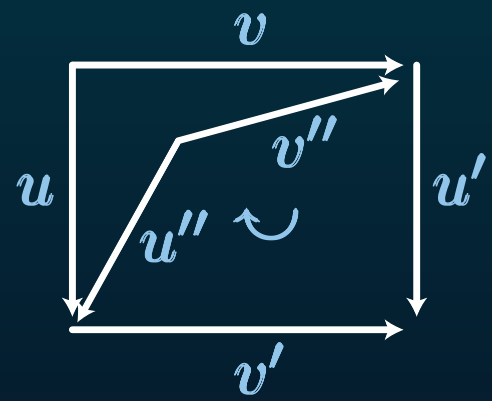

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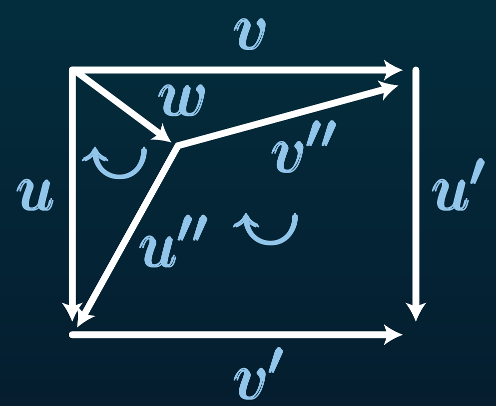

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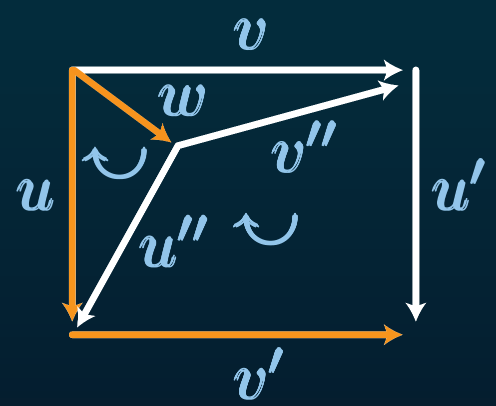

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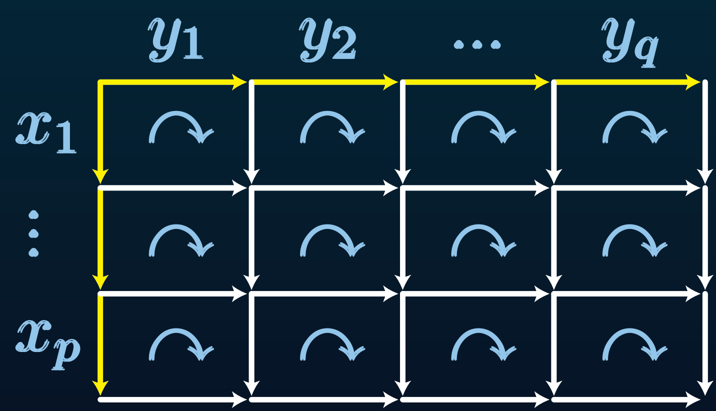

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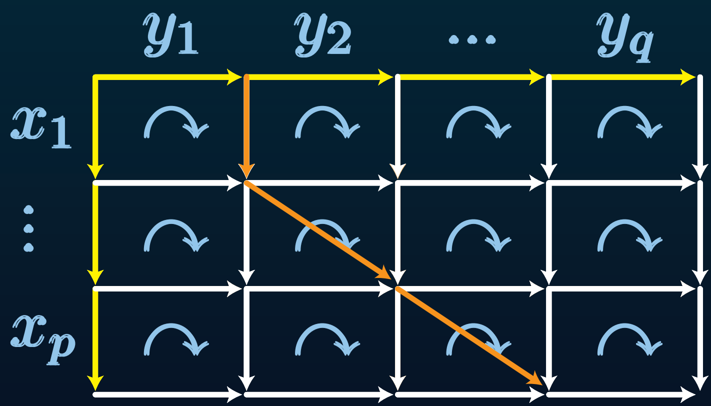

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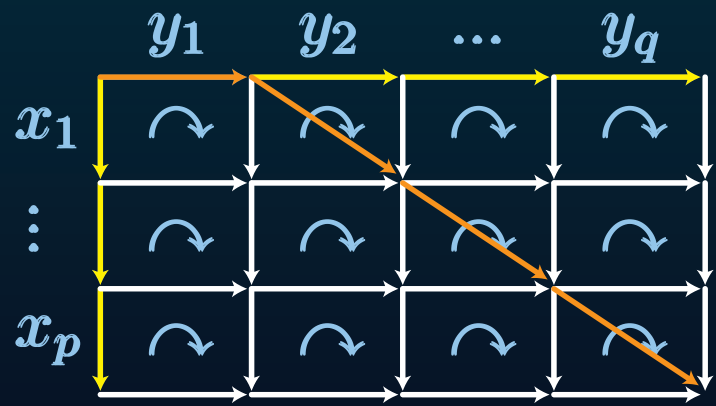

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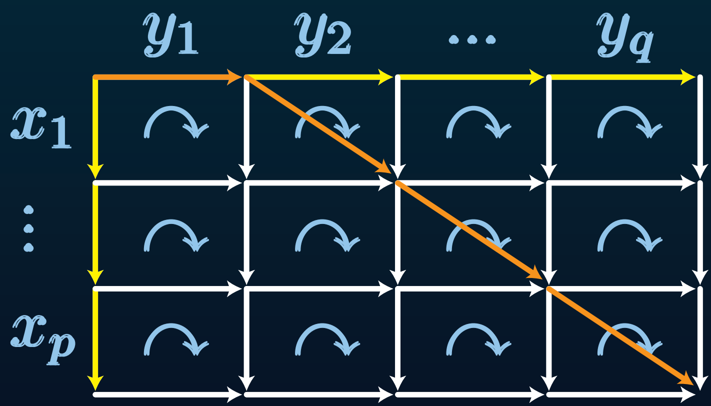

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$\rightsquigarrow$ Computation of the normal form of a product, or of an lcm.

- Word reversing is a convenient tool

  - for recognizing a Garside group from a (complemented) presentation,

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- Once completeness is granted, reversing $\approx$ computing lcm, but not a priori
  $\rightsquigarrow$ crucial to distinguish between words and the elements they represent.

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- Once completeness is granted, reversing $\approx$ computing lcm, but not a priori
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- References:

- Groupes de Garside; Ann. Scient. Ec. Norm. Sup. 35 (2002) 267–306.
- Complete positive group presentations; J. of Algebra 268 (2003) 156–197.

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