COMBINATORICS OF NORMAL SEQUENCES OF BRAIDS

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- Many different induction schemes occur.

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  ⇝ degree of a positive braid := this d; (*e.g.*, simple $\Leftrightarrow$ degree $\leqslant 1$)

- Corollary: Each braid admits a unique expression $t_e^{-1}...t_1^{-1}s_1...s_d$ with $(s_1, ..., s_d)$, $(t_1, ..., t_e)$ normal sequences s.t. $s_d, t_e \neq 1$ and $\gcd(s_1, t_1) = 1$.

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• Remark: *id.* in every Garside group: group of fractions for a monoid M with an element Δ s.t.

- the left and right divisors of Δ coincide (⇝ simple) and generate M;
- the family of all simple elements is a finite lattice w.r.t. left divisibility;
- if s, t are simple and divide x in M, then $\mathrm{lcm}(s,t)$ divides x as well.

Normality condition:“each simple left divisor of $s_i s_{i+1}$ is a left divisor of s_i”.

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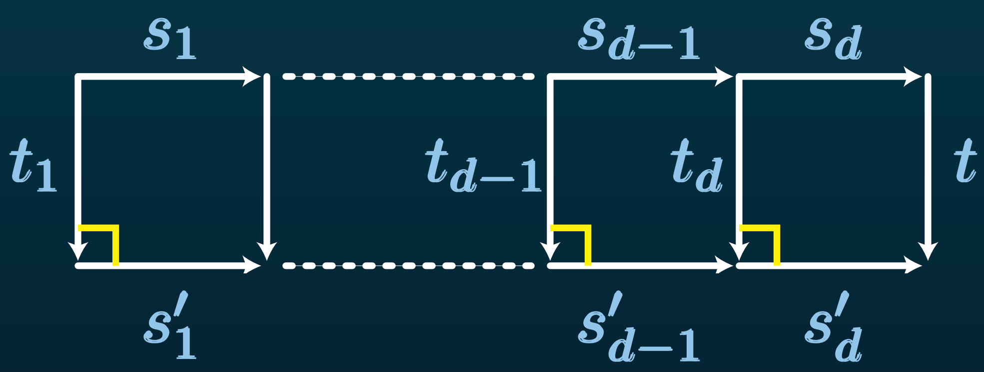

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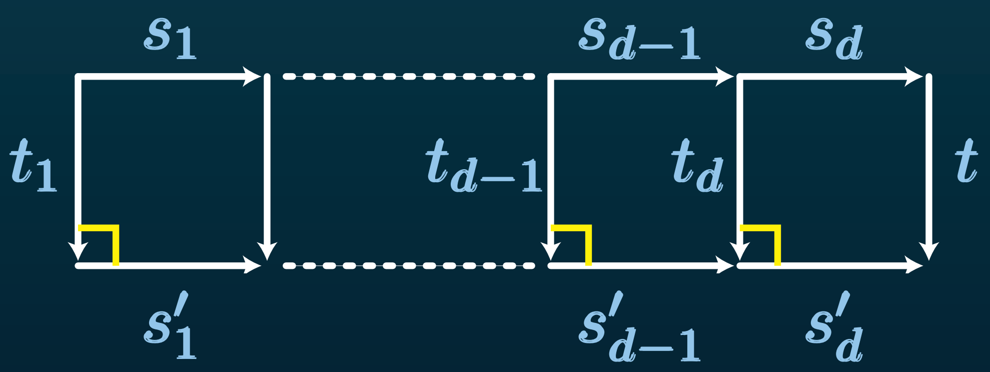

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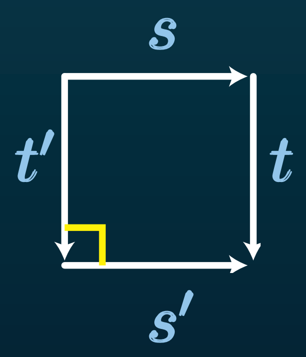

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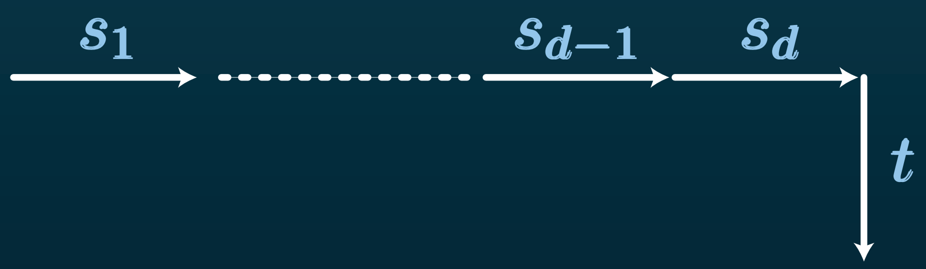

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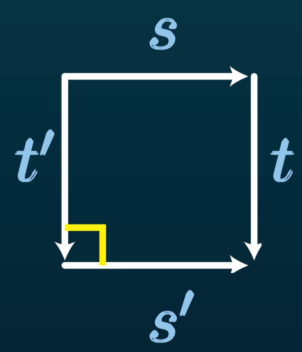

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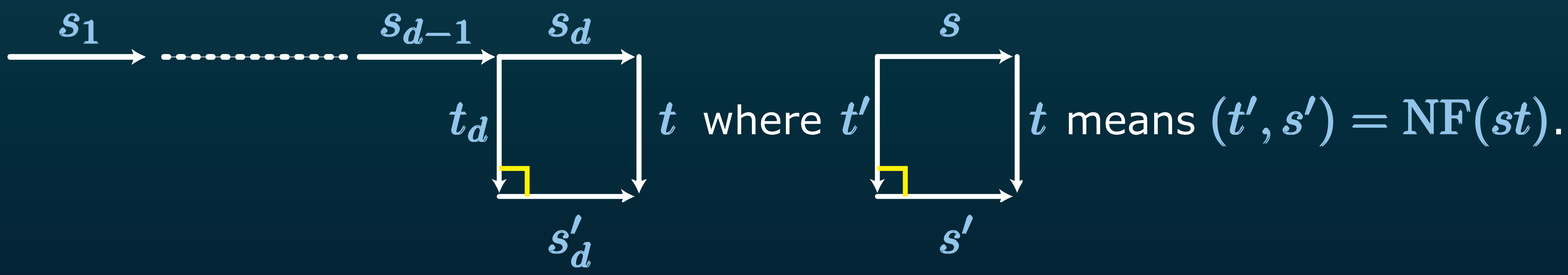

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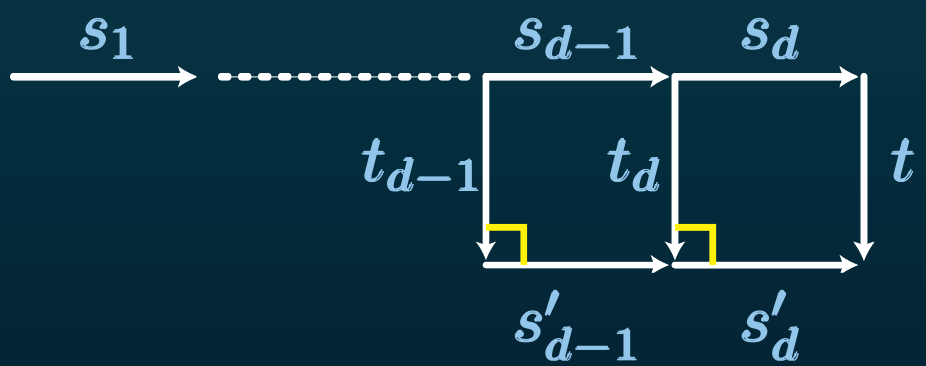

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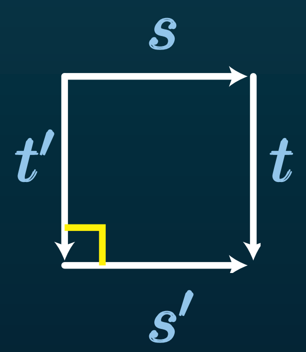

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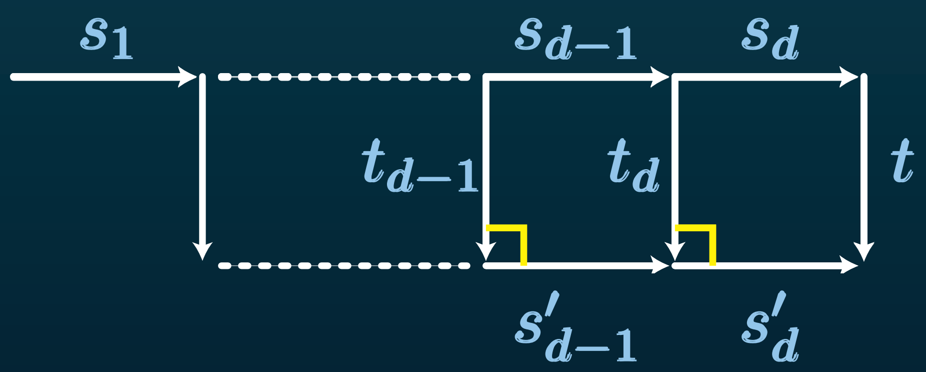

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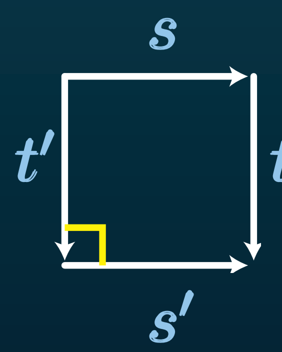

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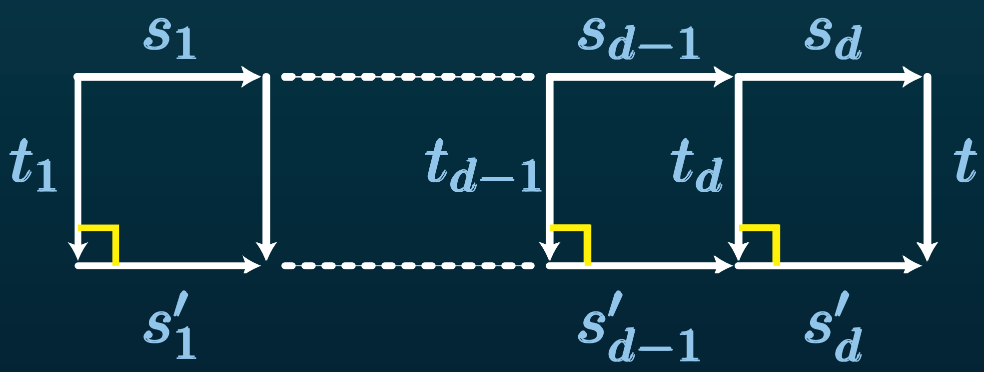

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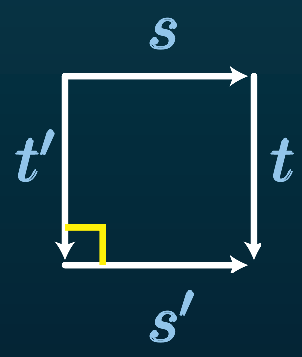

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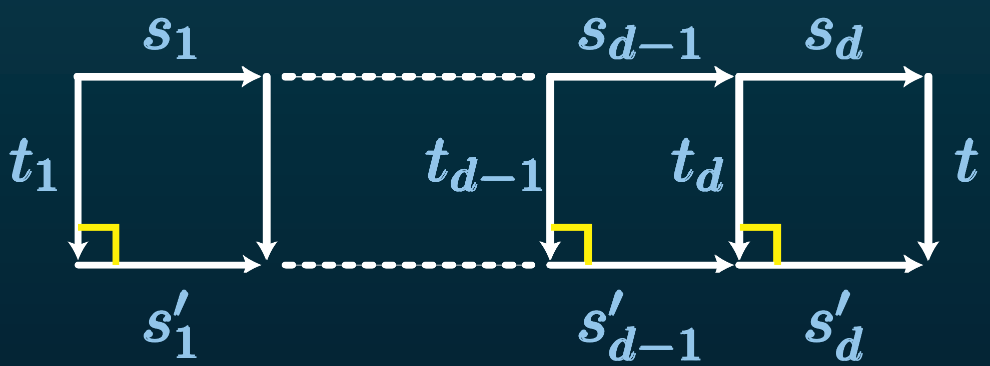

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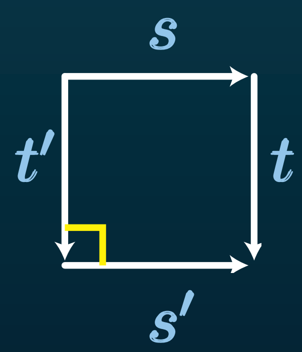

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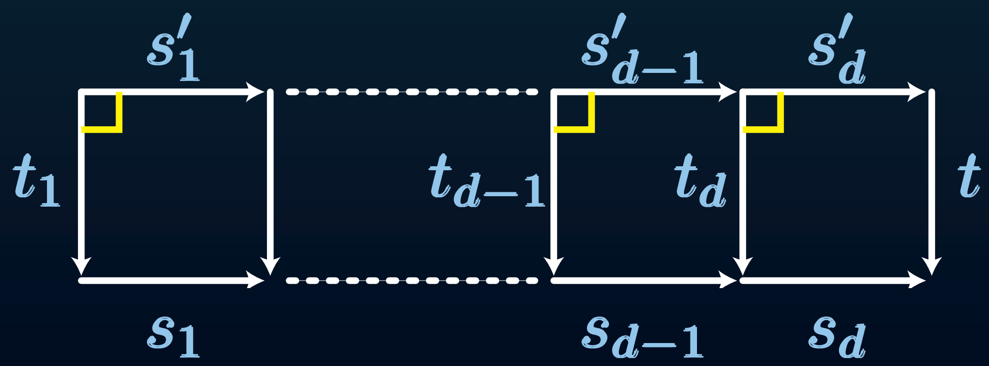

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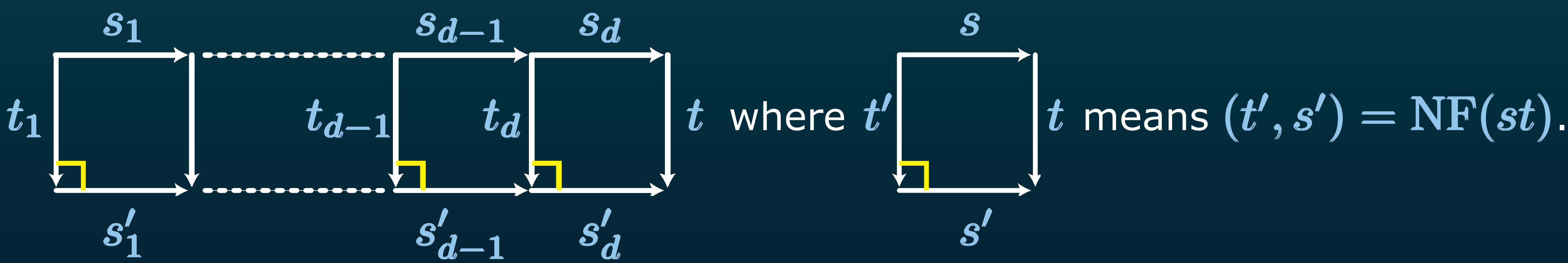

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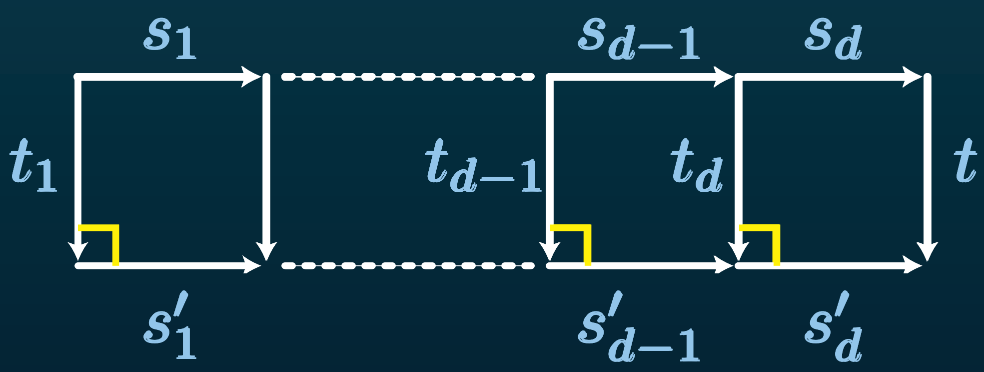

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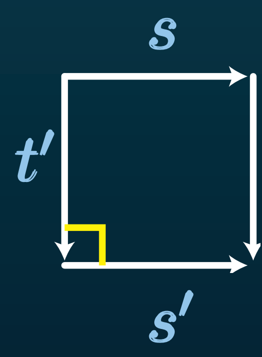

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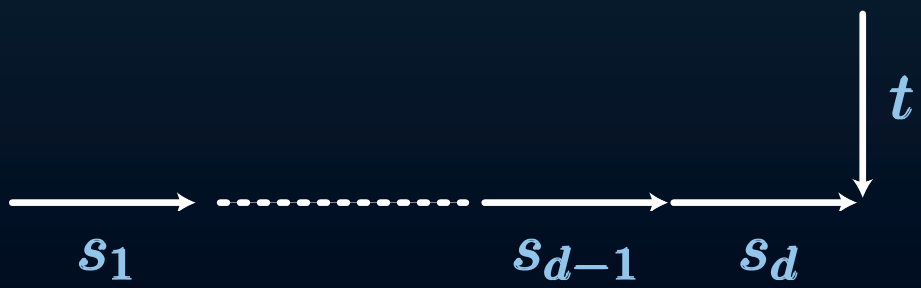

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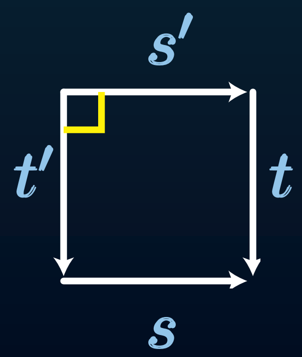

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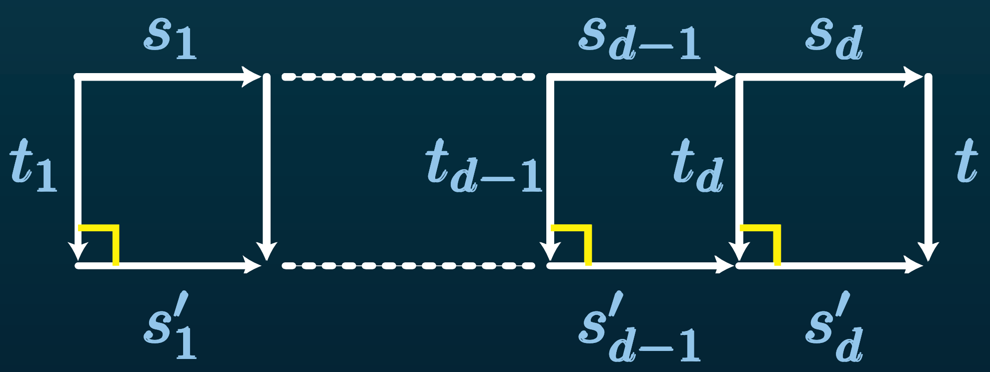

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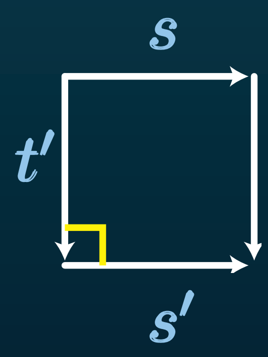

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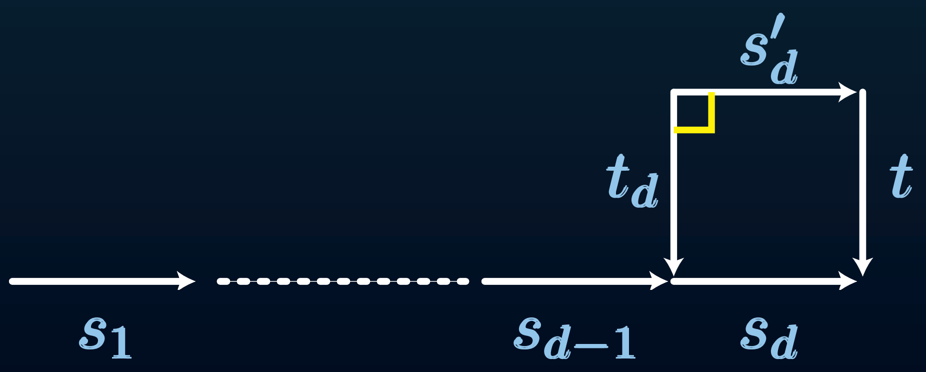

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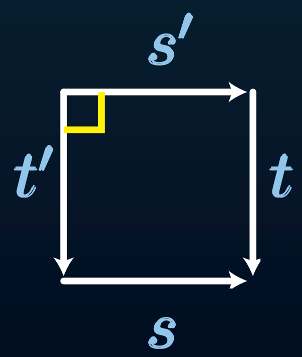

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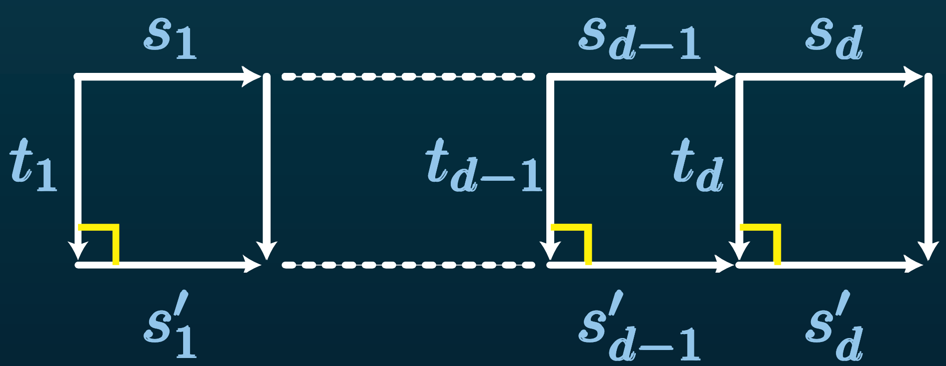

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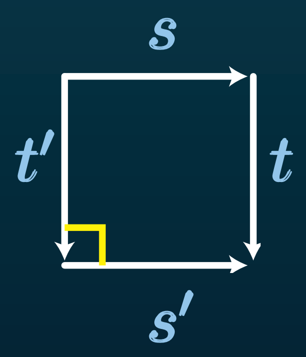

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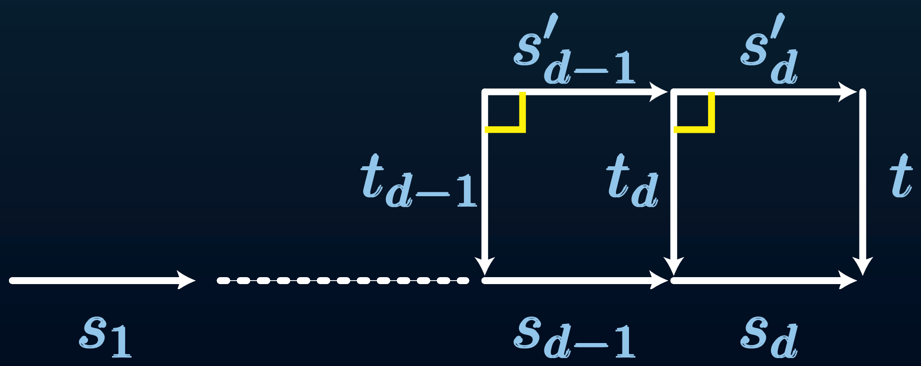

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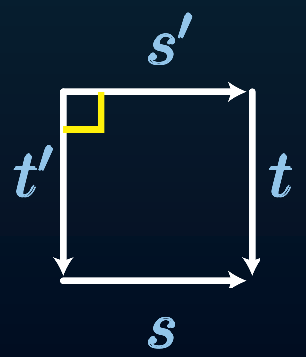

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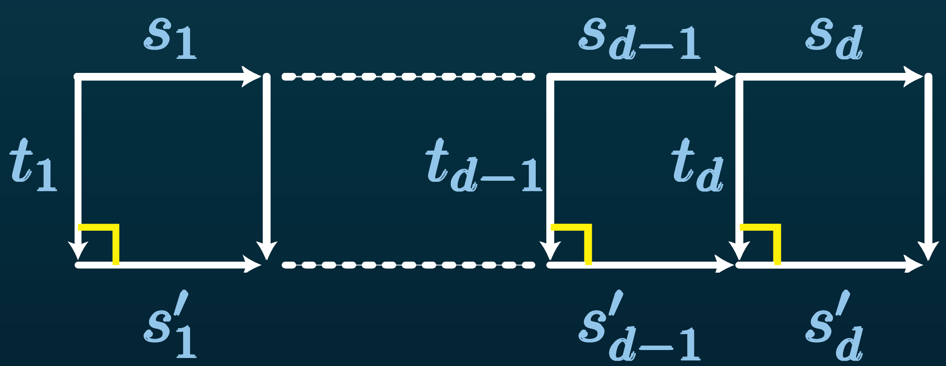

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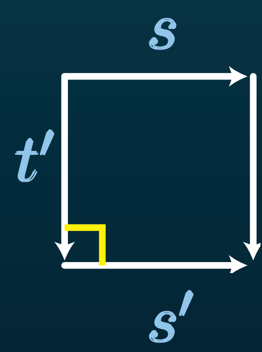

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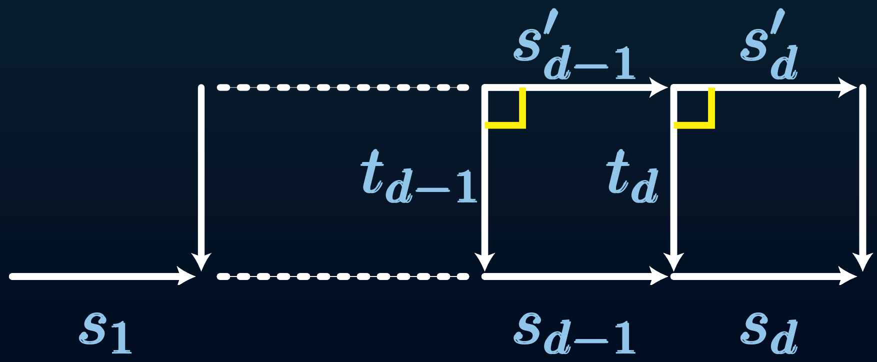

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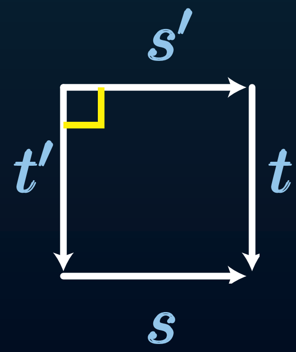

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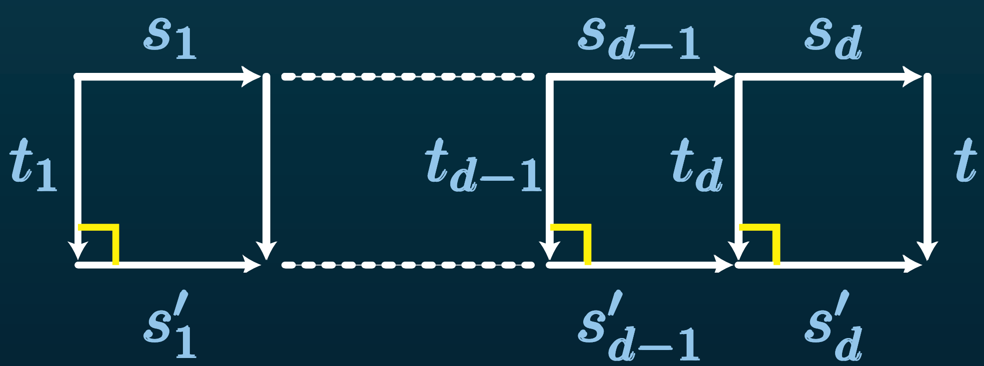

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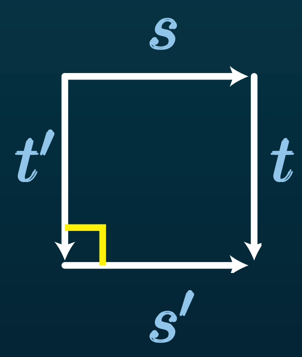

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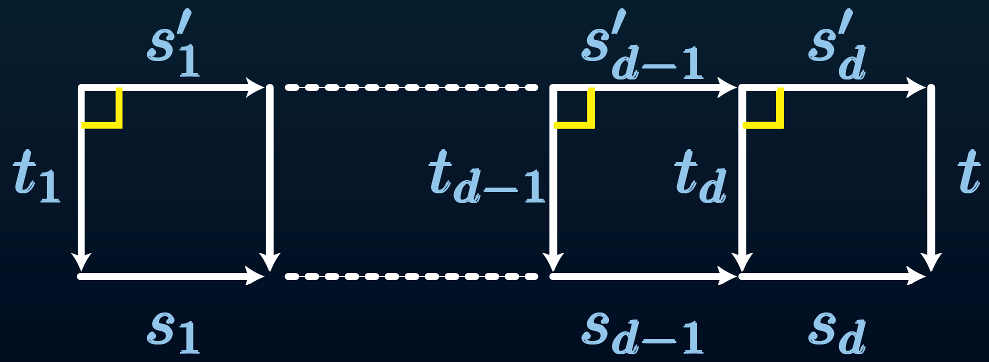

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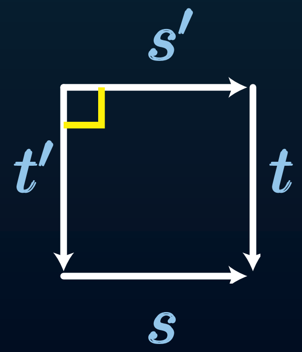
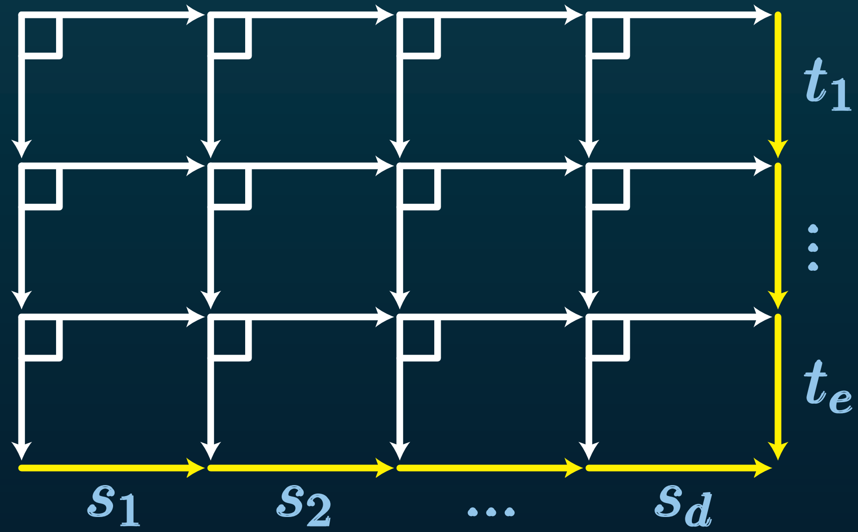


means $s't = \mathrm{lcm}(s, t)$.

• Proposition: Let (M, Δ) be a Garside system, and $(s_1, \ldots, s_d)$, $(t_1, \ldots, t_e)$ be normal. Then every diagonal–then–horizontal and diagonal–then–vertical sequence in



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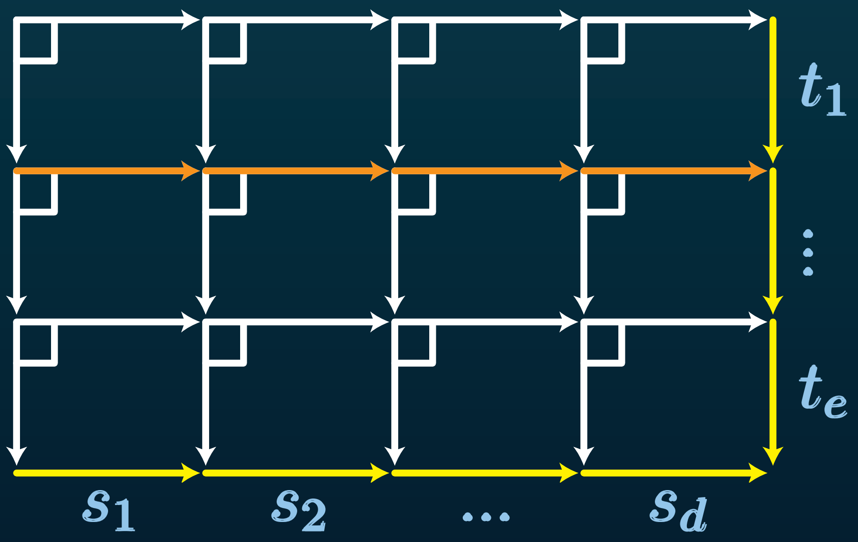

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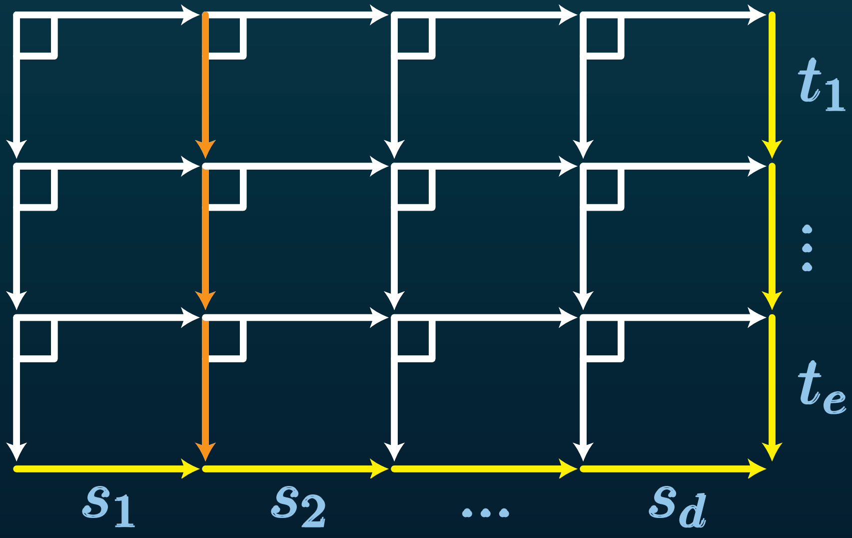

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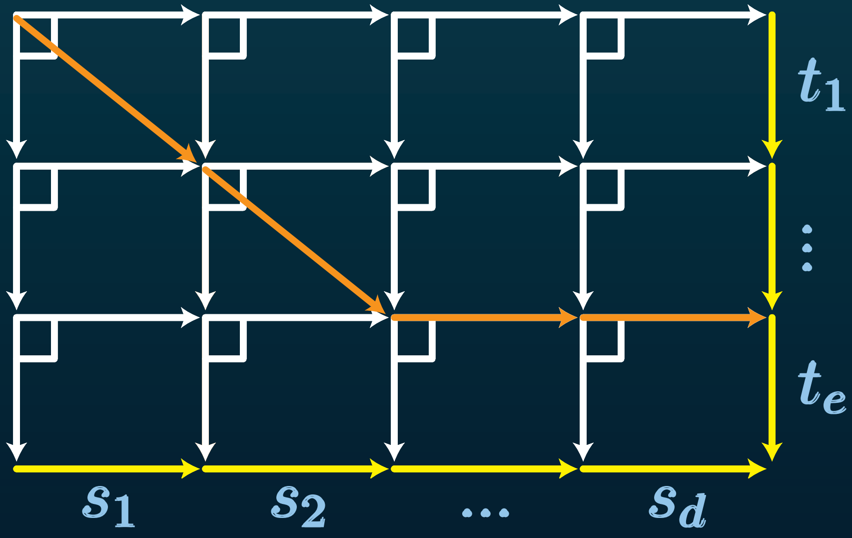

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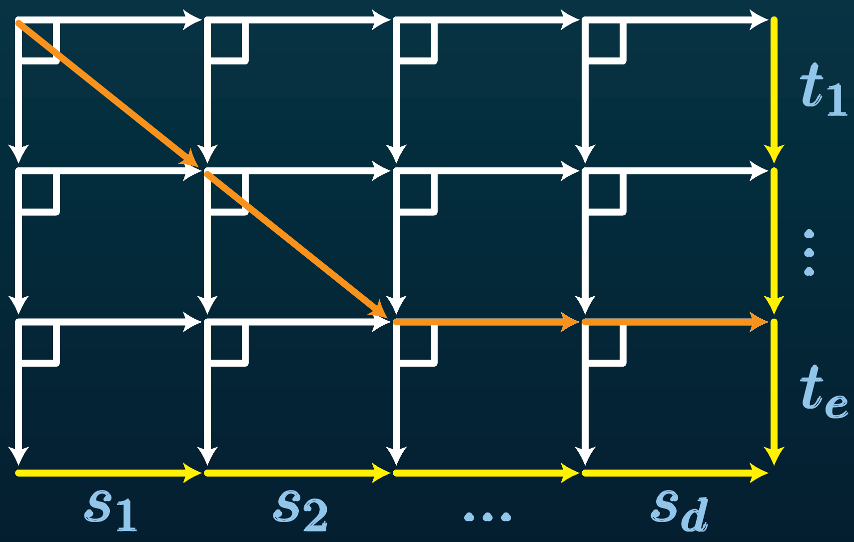

is normal as well.

⇝ Computation of the normal form of a product, or of an lcm.

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- Remark: $N_{n,d} = \sum_{s \text{ simple}} N_{n,d}(s) = N_{n,d+1}(1)$.

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$$M_1 = (1), \quad M_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \ldots$$

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- Example: $M'_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ ($\rightsquigarrow$ size 4 instead of 6)

• Lemma: Let $a_{I,J} := \#\ f$ in $\mathfrak{S}_n$ s.t. $D(f) \supseteq I$ and $D(f^{-1}) \supseteq J$.
Then $a_{I,J} = \#\ k \times \ell$ integral matrices s.t. the sum of the ith row is p_i and the sum of the j column is q_j, where $(p_1, ..., p_k)$ is the composition of I, and $(q_1, ..., q_\ell)$ is the composition of J.

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- Example: $M''_3 = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ ($\rightsquigarrow$ size 3 instead of 4)

• Going from $\boldsymbol{M_n}$ to $\boldsymbol{M''_n}$ = reducing the size of the automata involved in the automatic structure of $\boldsymbol{B_n}$ from $\boldsymbol{n!}$ to $\boldsymbol{p(n)}$.

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• (Hivert–Novelli) M''_n interprets in the context of quasi-symmetric functions (Malvenuto–Reutenauer).

⇝ LU decomposition of M''_n

- Back to the counting problem: all numbers $N_{n,d}(s)$ express in terms of the powers of the eigenvalues of M_n, hence of M''_n.

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$\rightsquigarrow$ $N_{4,d} = c_1(3+\sqrt{6})^d + c_2(3-\sqrt{6})^d + c_3 2^d + c_4 d + c_5$ with $c_1 = \ldots$

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- A few more experiments:

n	2	3	4	5	6	7	8
$\lambda_{max}(M_n)$	1	2	5.5	18.7	77.4	373.9	2066.6
$\frac{\lambda_{max}(M_n)}{n \cdot \lambda_{max}(M_{n-1})}$	0.5	0.667	0.681	0.687	0.689	0.690	0.691

- A few experimental data:

$$\begin{aligned}
\mathsf{CharPol}(M''_1) &= x-1\\
\mathsf{CharPol}(M''_2) &= \mathsf{CharPol}(M''_1)\cdot(x-1)\\
\mathsf{CharPol}(M''_3) &= \mathsf{CharPol}(M''_2)\cdot(x-2)\\
\mathsf{CharPol}(M''_4) &= \mathsf{CharPol}(M''_3)\cdot(x^2-6x+3)\\
\mathsf{CharPol}(M''_5) &= \mathsf{CharPol}(M''_4)\cdot(x^2-20x+24)\\
\mathsf{CharPol}(M''_6) &= \mathsf{CharPol}(M''_5)\cdot(x^4-82x^3+359x^2-260x+60)\ldots
\end{aligned}$$

- Conjecture: The characteristic polynomial of M_{n-1} divides that of M_n.

- A few more experiments:

n	2	3	4	5	6	7	8
$\lambda_{max}(M_n)$	1	2	5.5	18.7	77.4	373.9	2066.6
$\frac{\lambda_{max}(M_n)}{n\cdot\lambda_{max}(M_{n-1})}$	0.5	0.667	0.681	0.687	0.689	0.690	0.691

- Question: What is the asymptotic behaviour of $\lambda_{max}(M_n)$?

• Recall: $N_{n,d}$=#pos.n-braids of degree$\leqslant d$; $N_{n,d}(s)$=***id.***with last factor s.
Instead of leaving n fixed and letting d vary, leave d fixed and let n vary:

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- $N_{n,4}(\Delta_{n-1}) = \lfloor n!e \rfloor - 1$, corresponding to $x_n = nx_{n-1} + 2n - 1$.

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