Unprovability results involving braids

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Patrick Dehornoy
Laboratoire de Mathématiques Nicolas Oresme
Université de Caen

joint work with A.Weiermann, L.Carlucci, A.Bovykin

- **Aim**: Describe combinatorial statements involving braids that are unprovable in weak subsystems of Peano arithmetic

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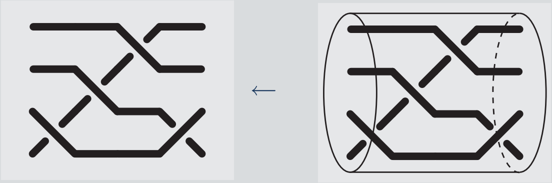

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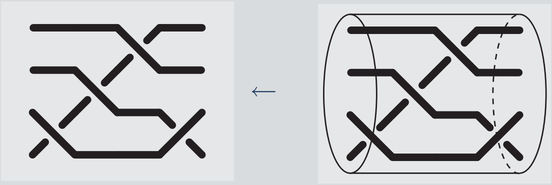

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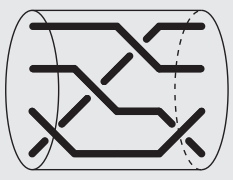

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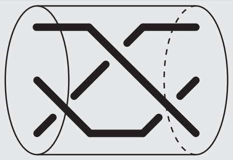

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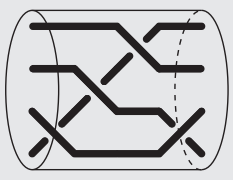

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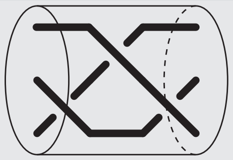

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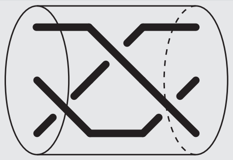

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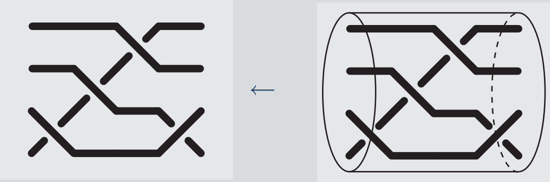

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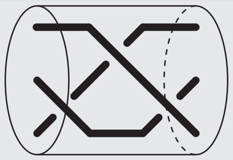

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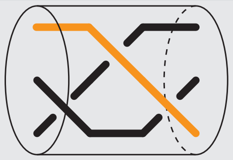

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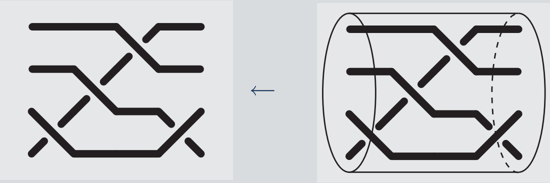

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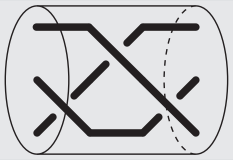

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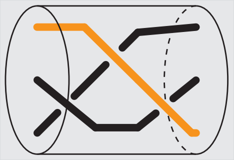

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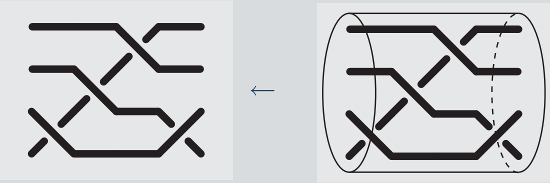

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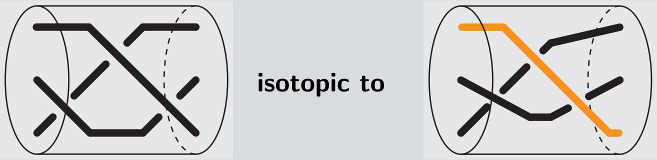

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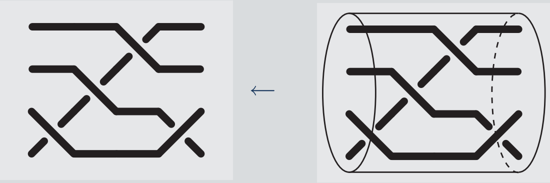

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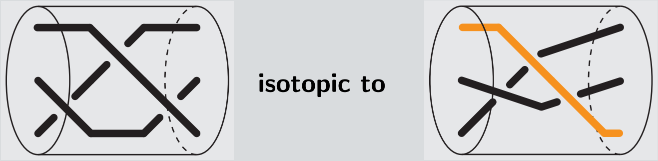

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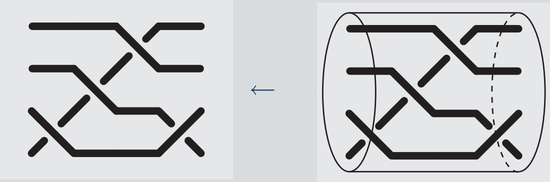

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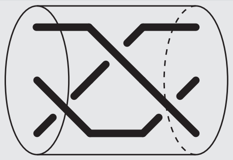

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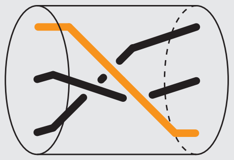

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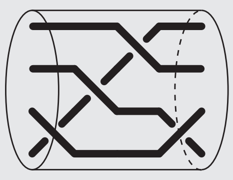

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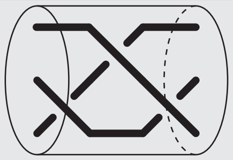

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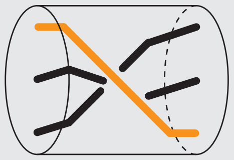

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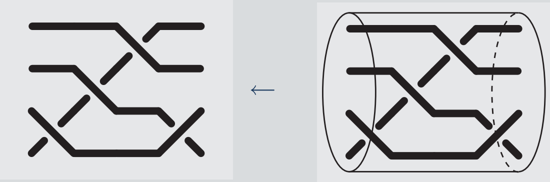

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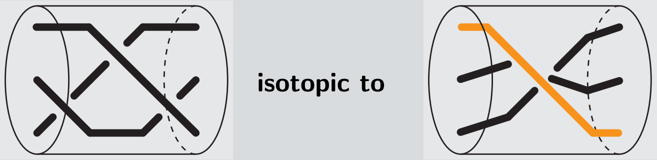

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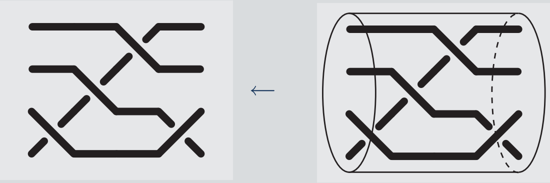

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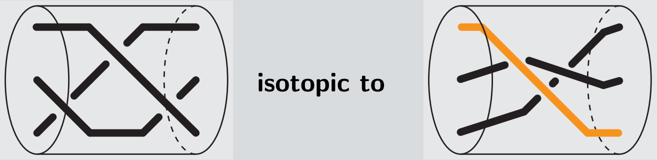

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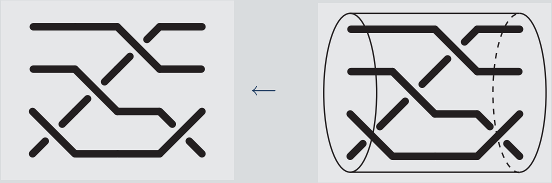

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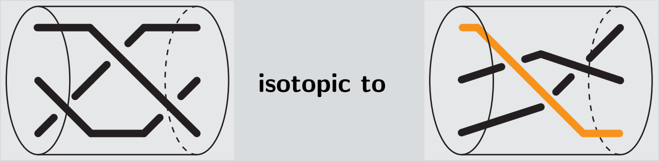

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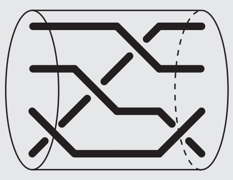

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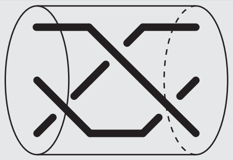

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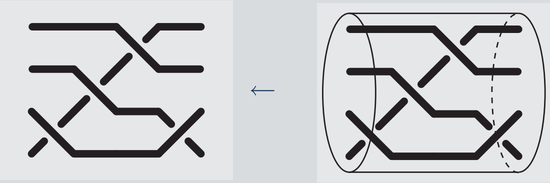

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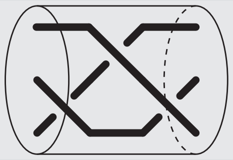

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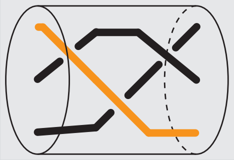

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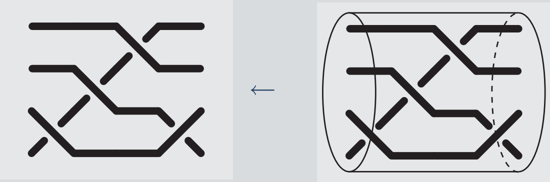

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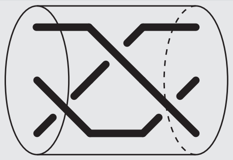

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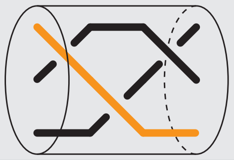

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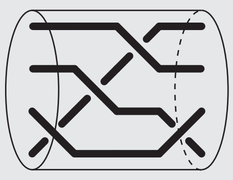

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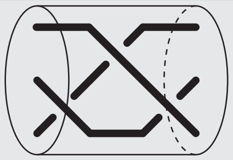

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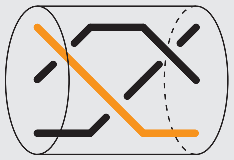

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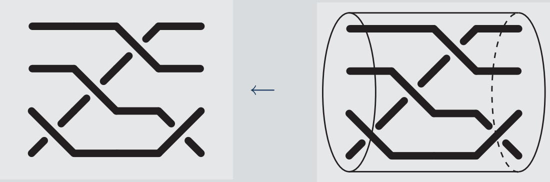

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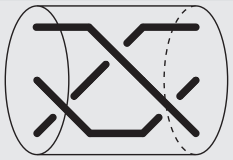

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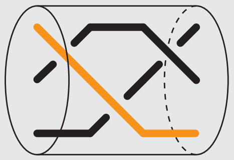

- a braid := an isotopy class $\rightsquigarrow$ represented by 2D-diagram,
  but different 2D-diagrams may give rise to the same braid.

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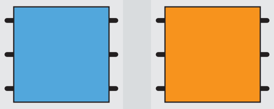

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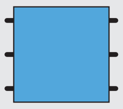
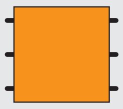
 $*$  $:=$

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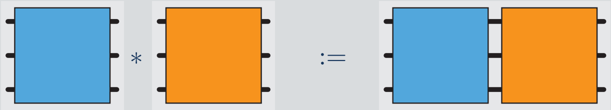

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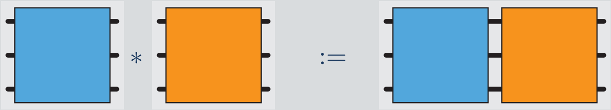
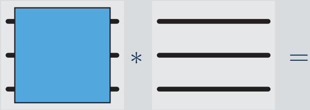


- **Then well-defined (w.r.t. isotopy), associative, admits a unit:**



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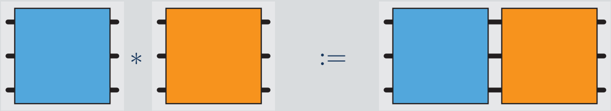

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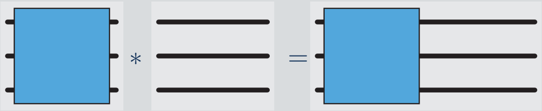

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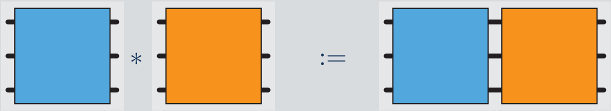

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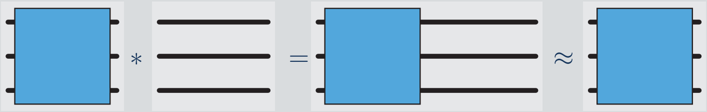

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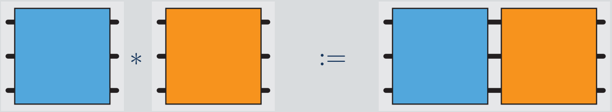

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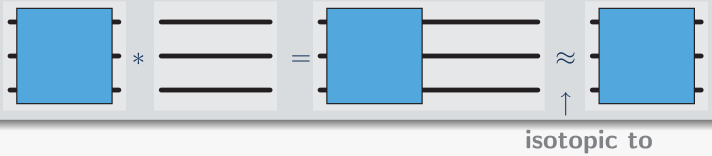

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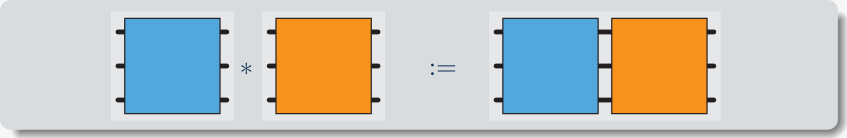

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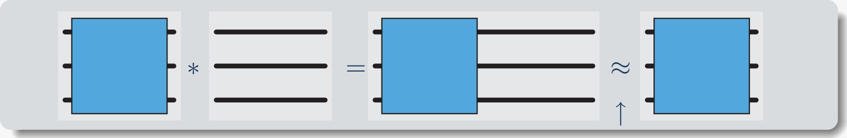

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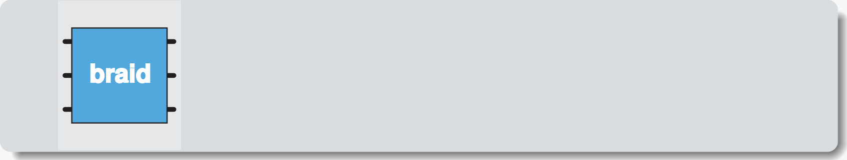

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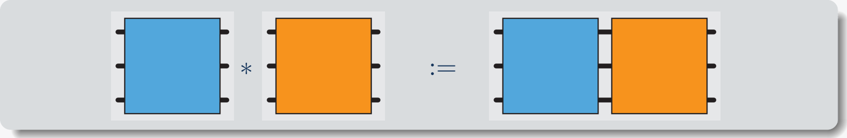

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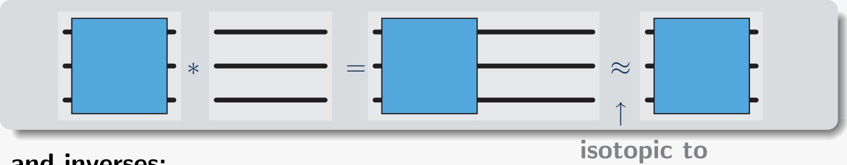

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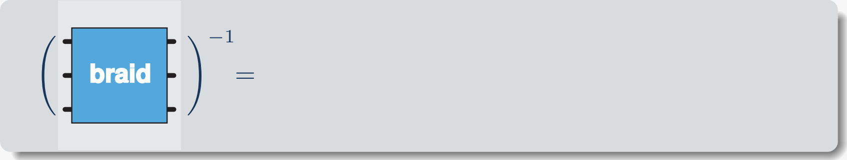

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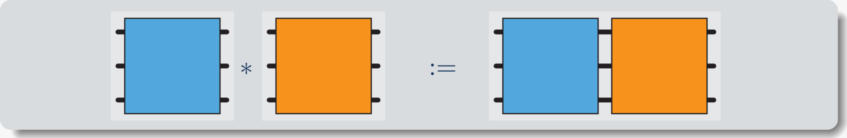

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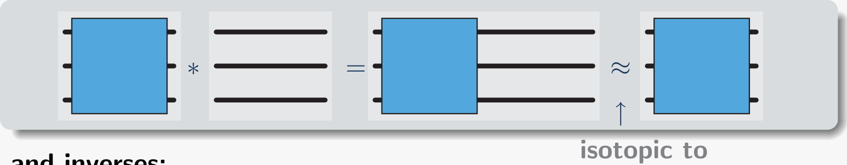

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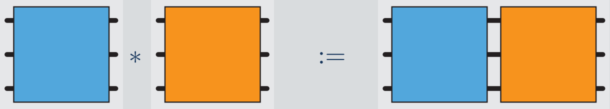

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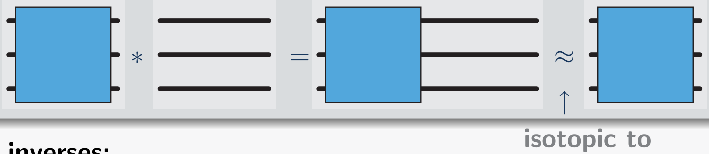

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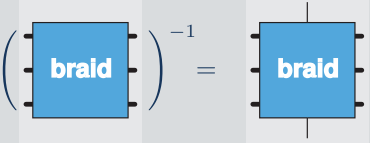

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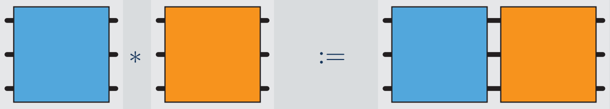

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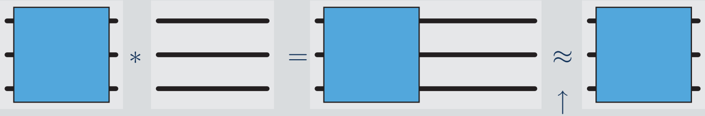

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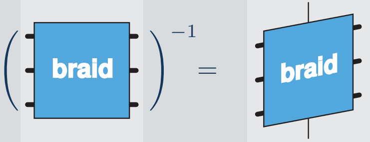

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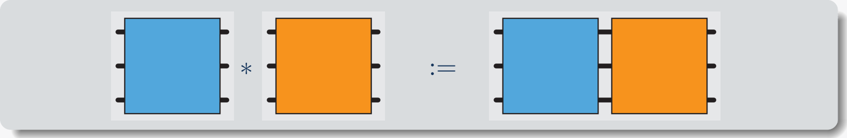

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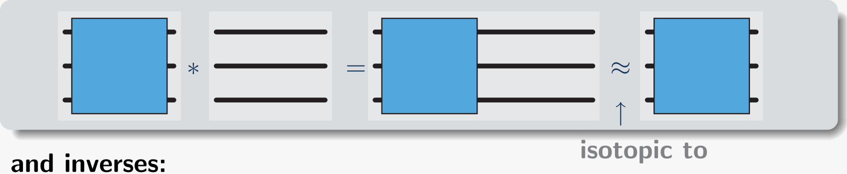

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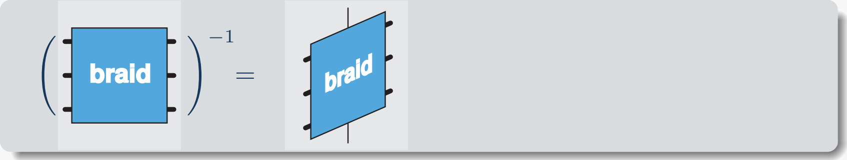

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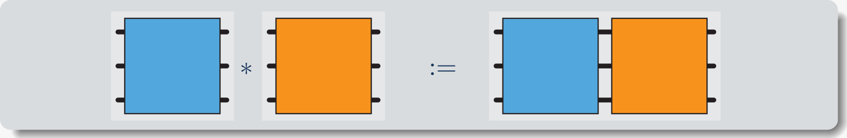

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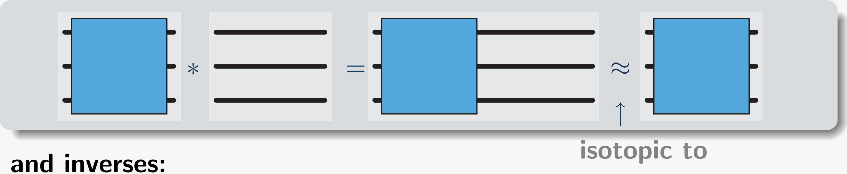

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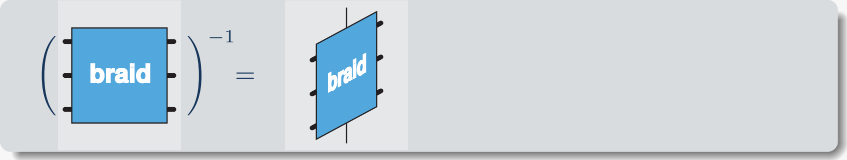

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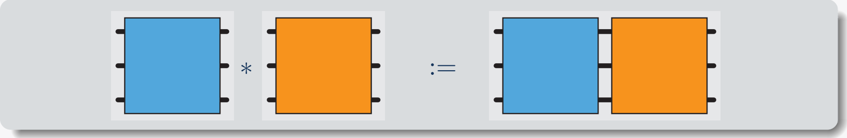

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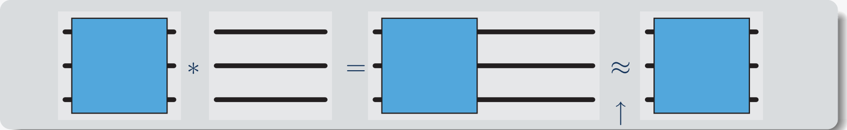

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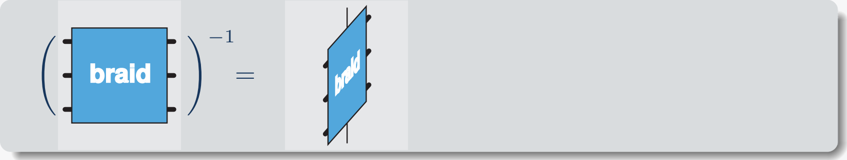

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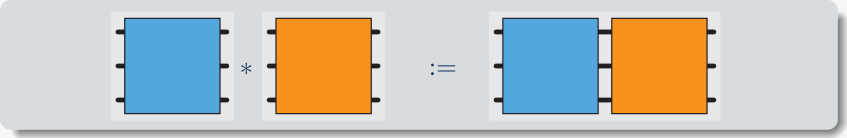

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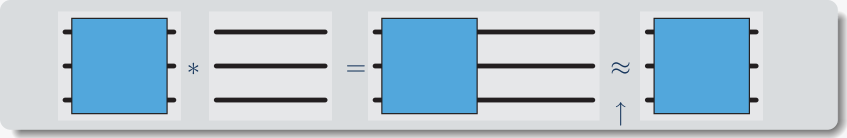

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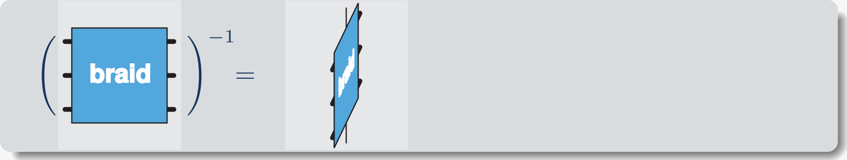

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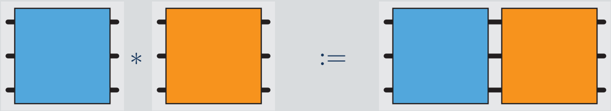

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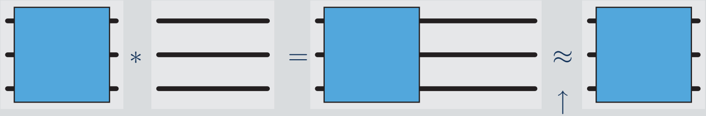

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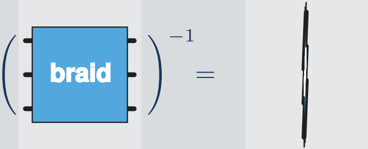

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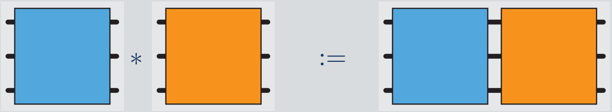

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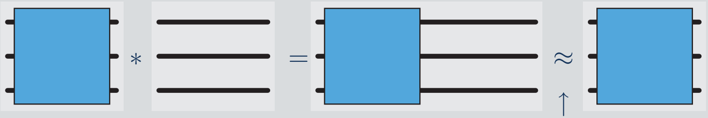

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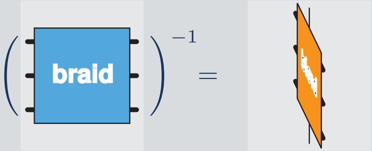

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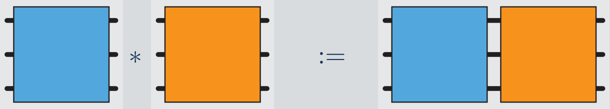

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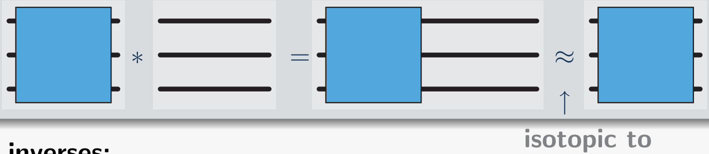

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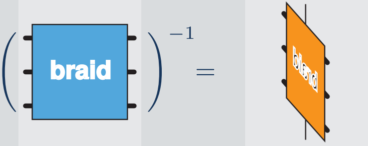

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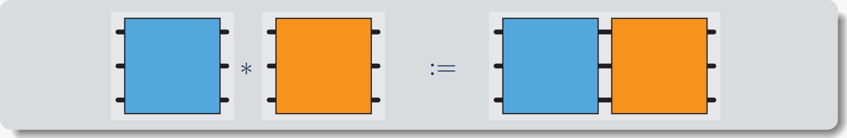

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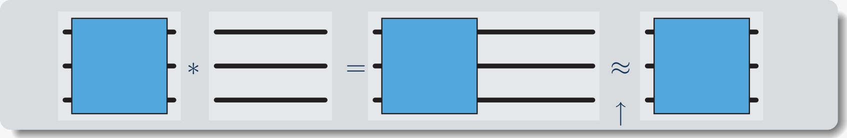

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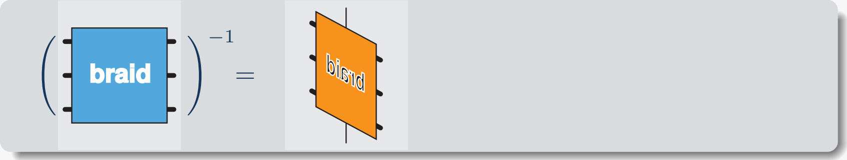

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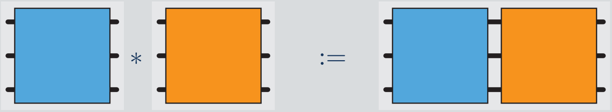

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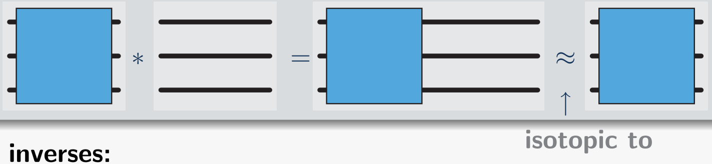

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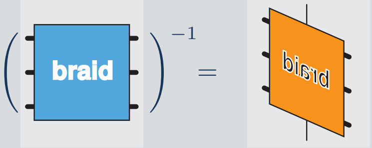

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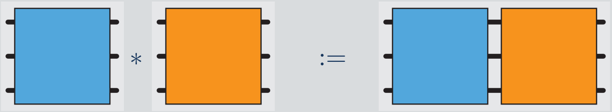

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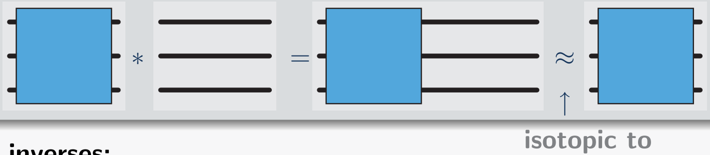

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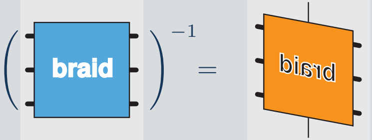

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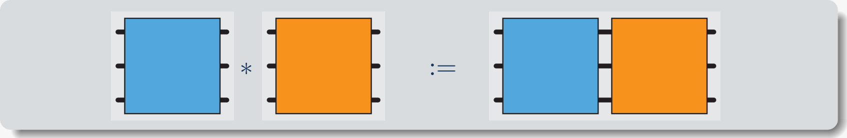

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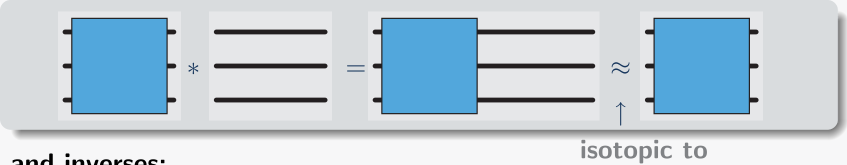

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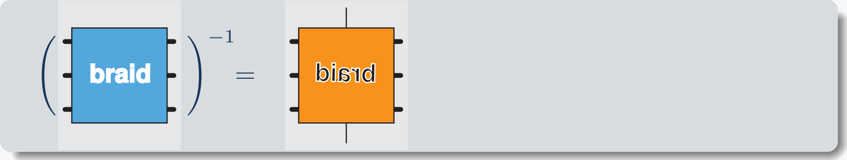

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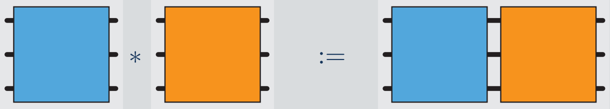

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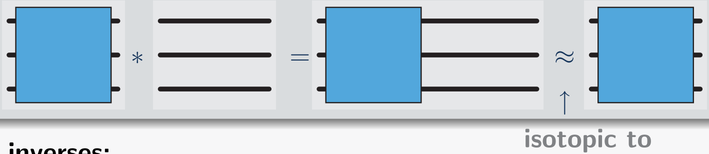

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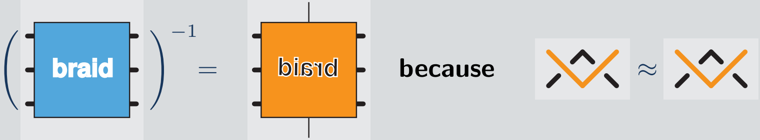

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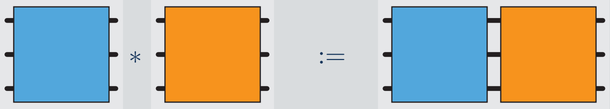

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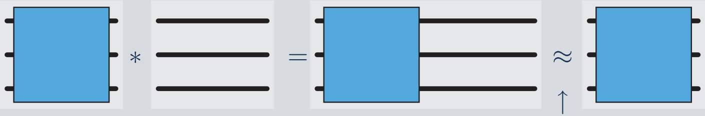

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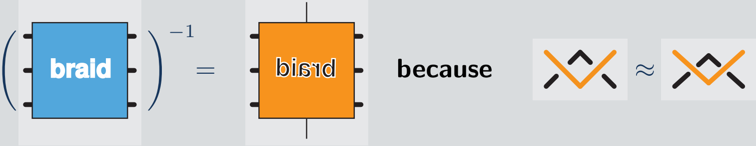

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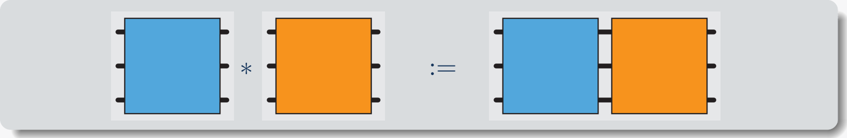

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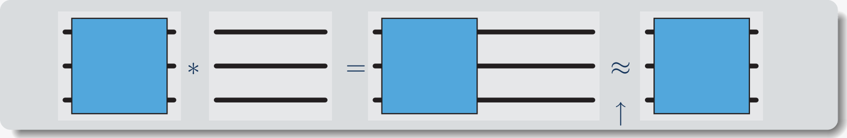

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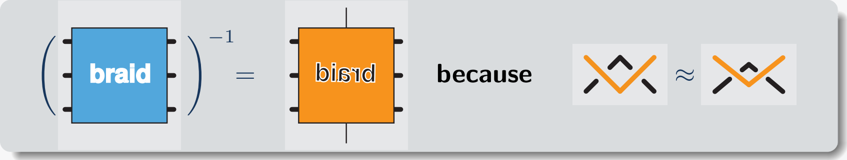

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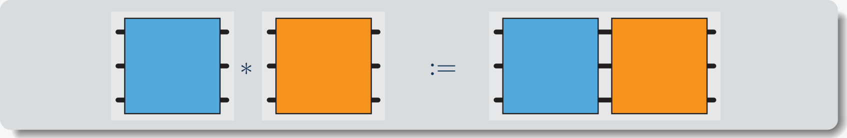

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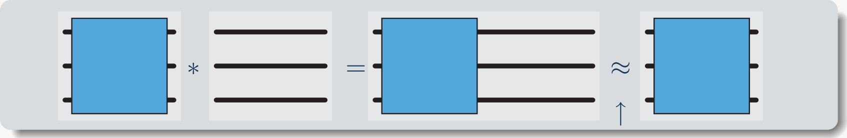

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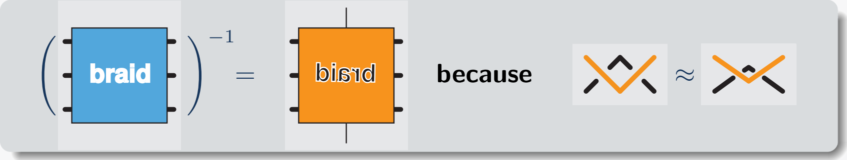

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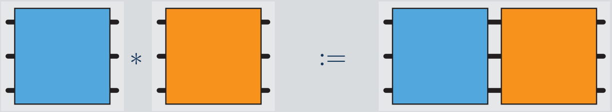

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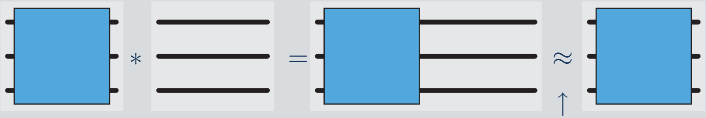

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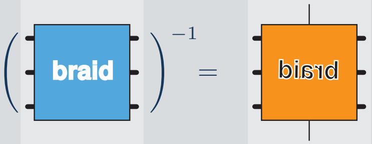

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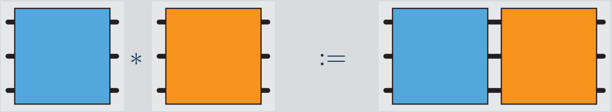

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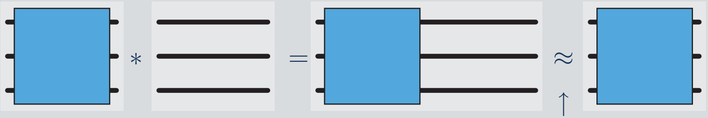

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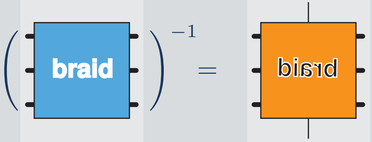

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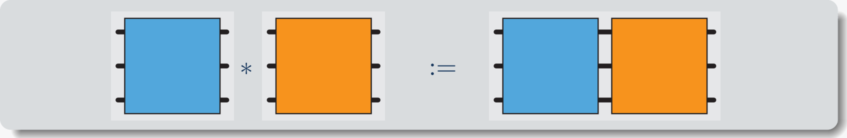

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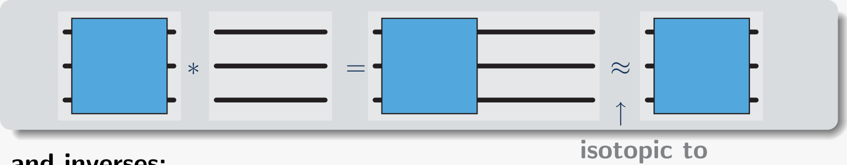

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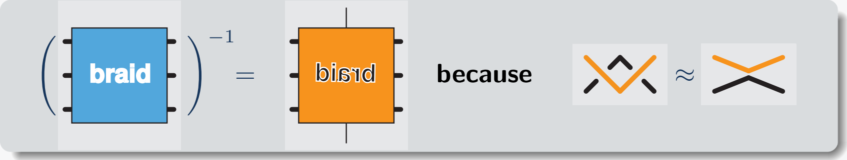

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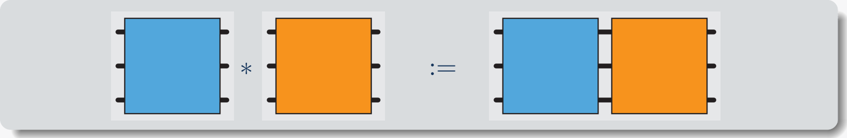

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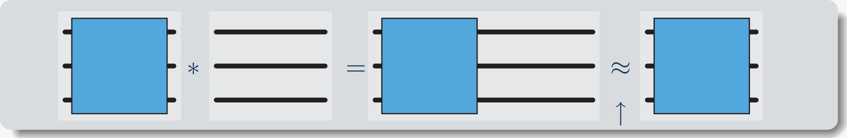

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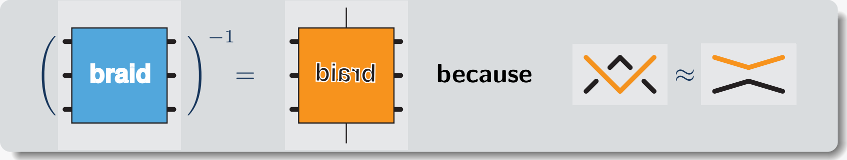

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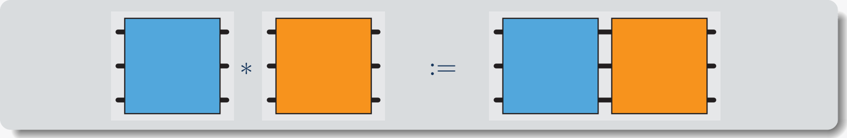

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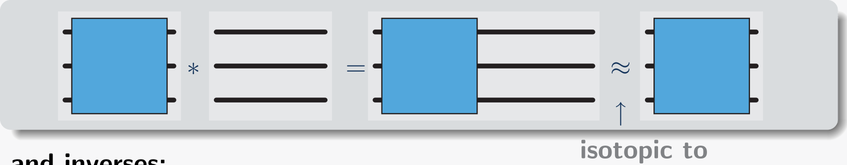

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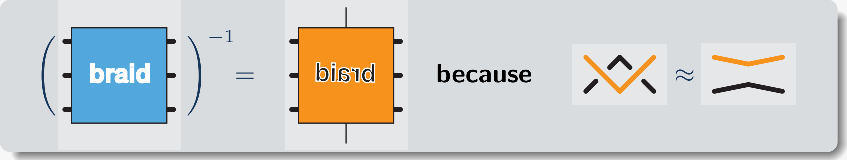

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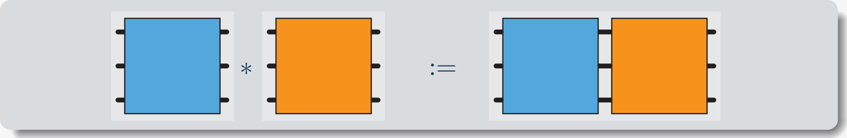

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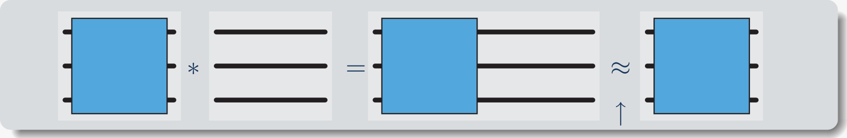

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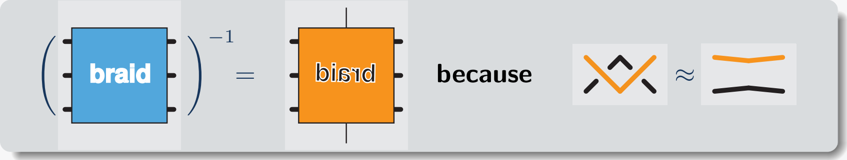

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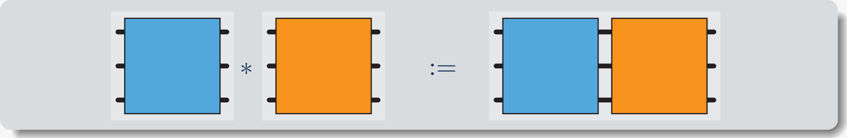

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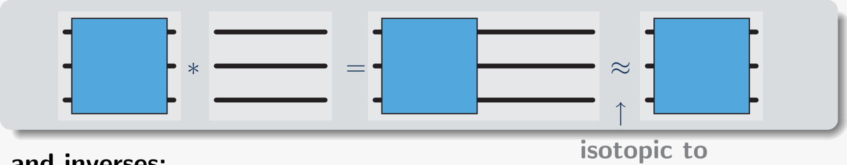

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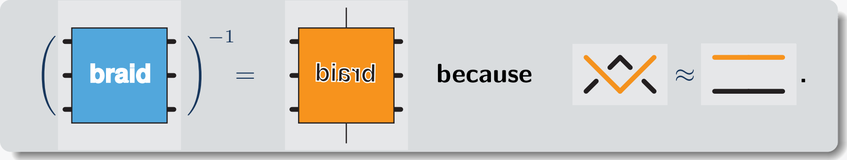

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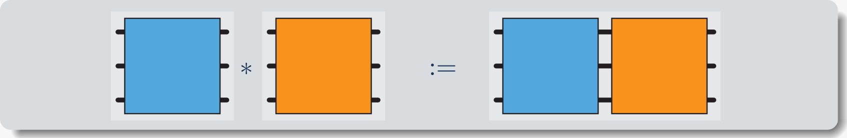

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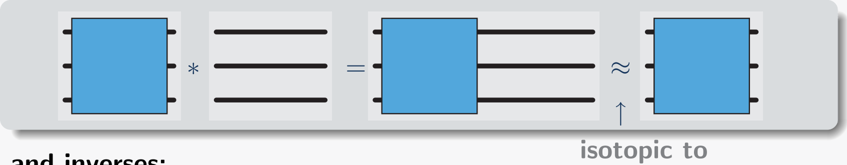

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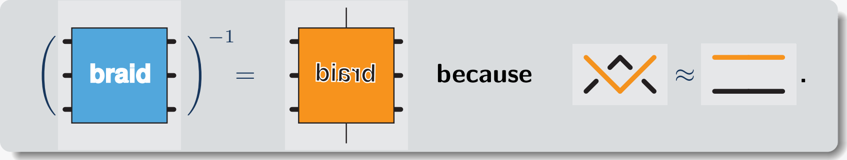

⇝ For each n, the group B_n of n strand braids (E.Artin, ∼1925).

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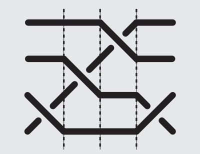

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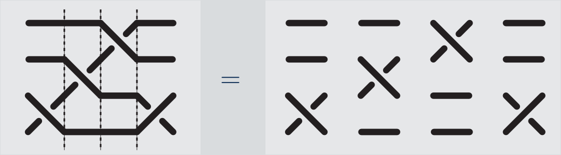

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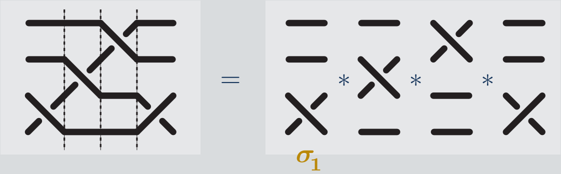

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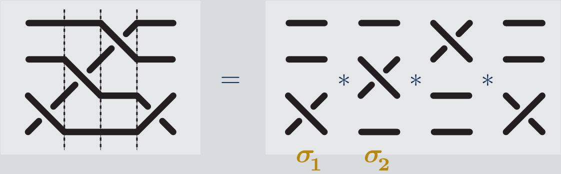

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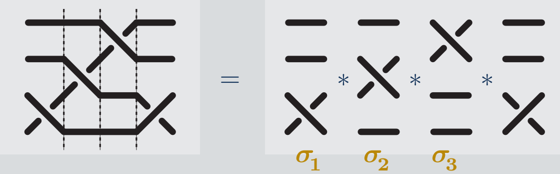

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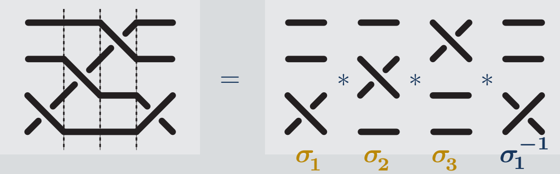

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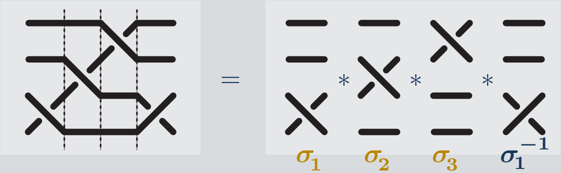

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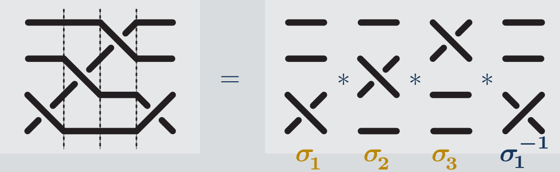

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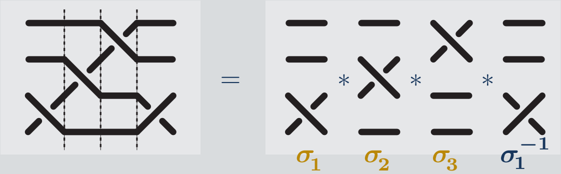

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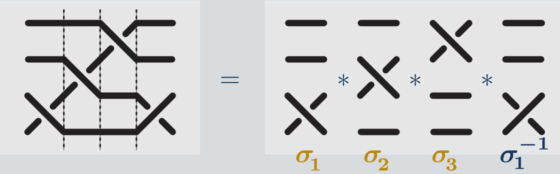

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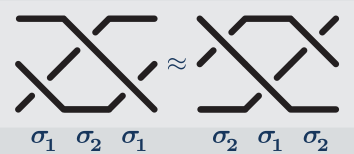

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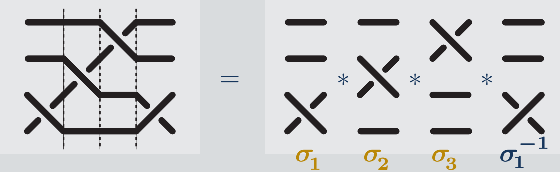

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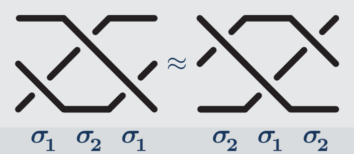

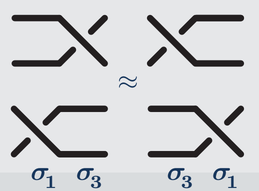

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- Here in the 3 strand version—but exists for each n.

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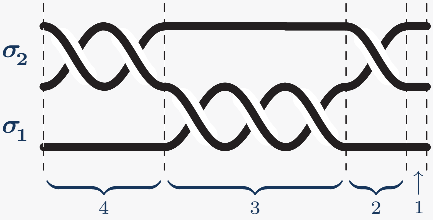

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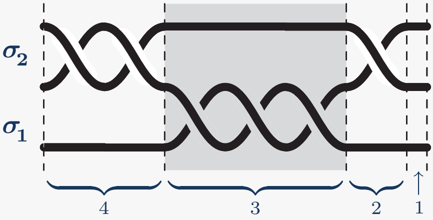

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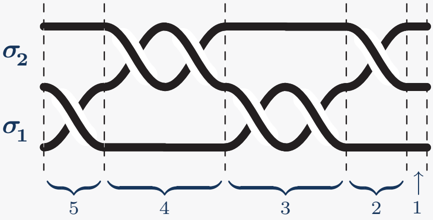

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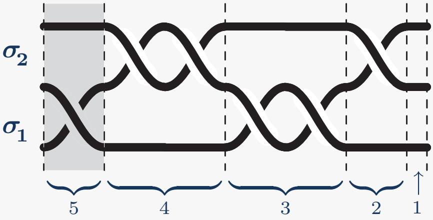

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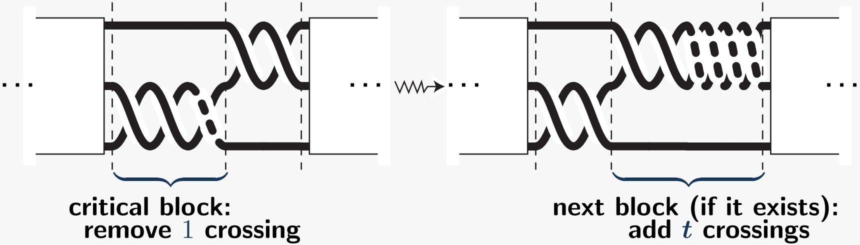

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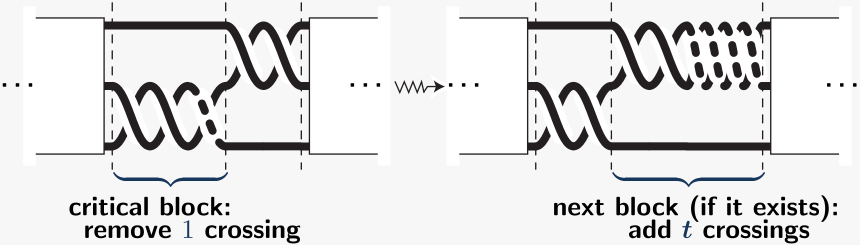

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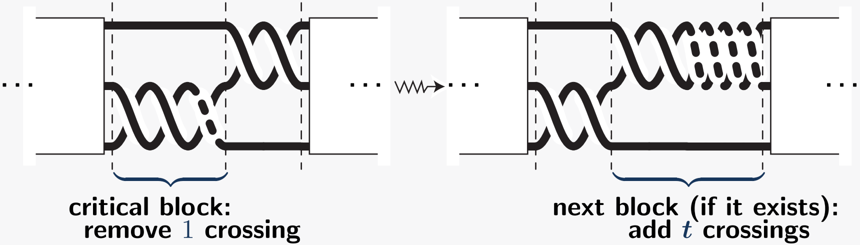

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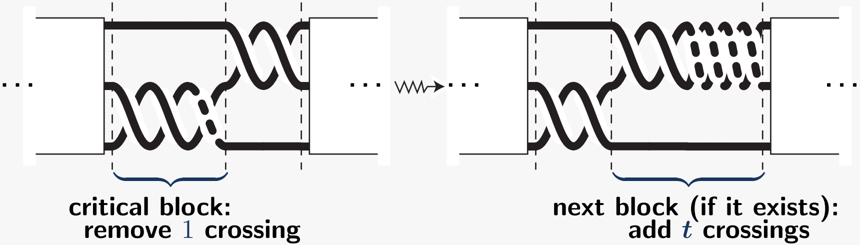

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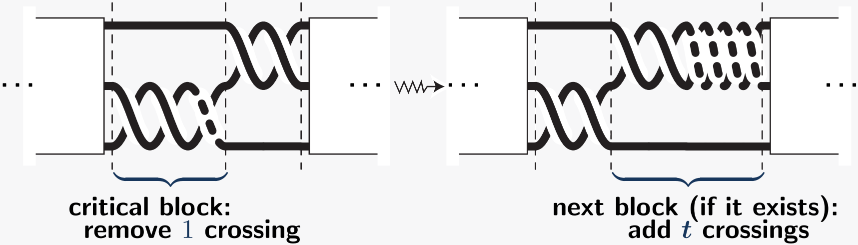

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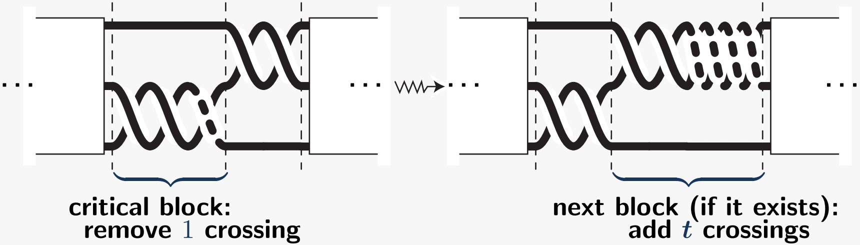

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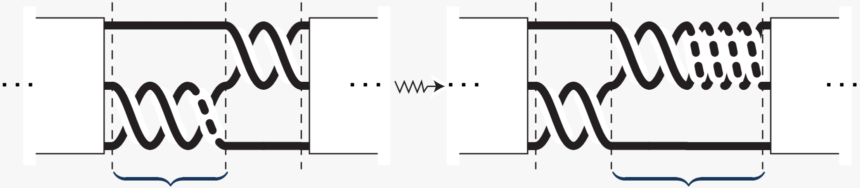

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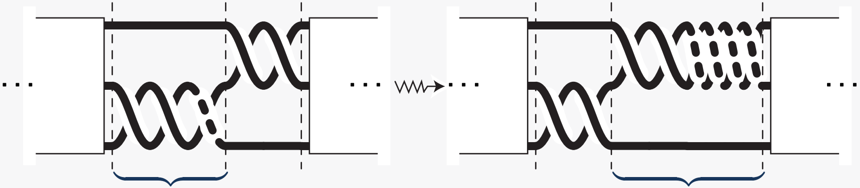

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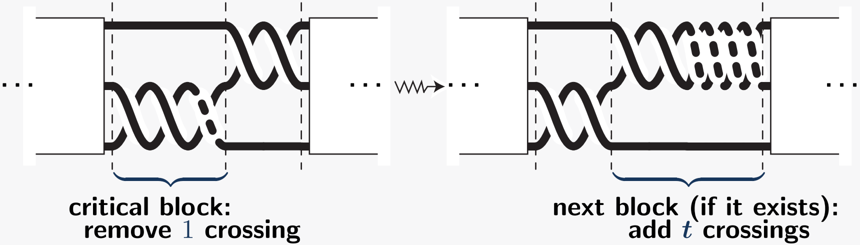

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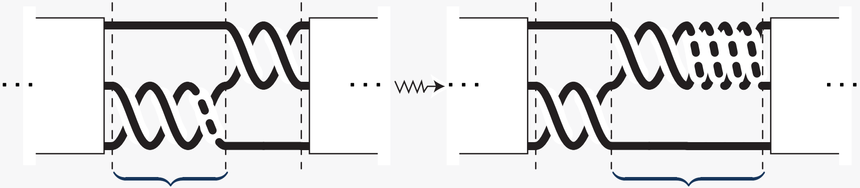

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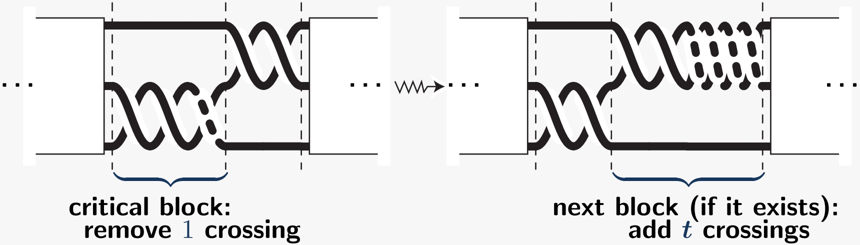

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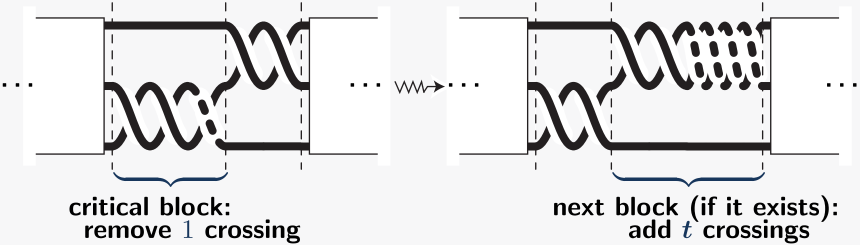

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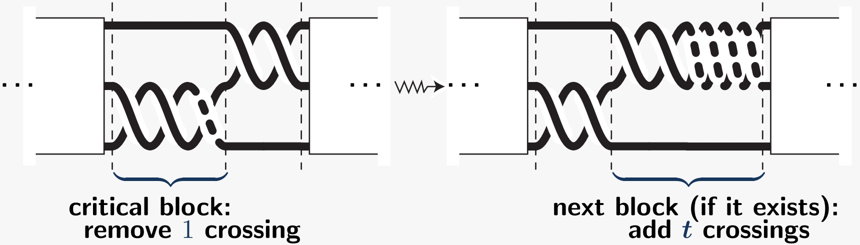

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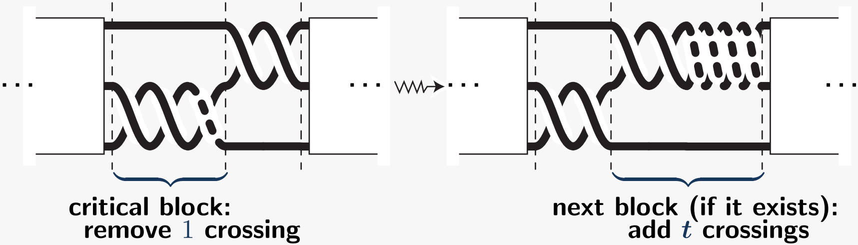

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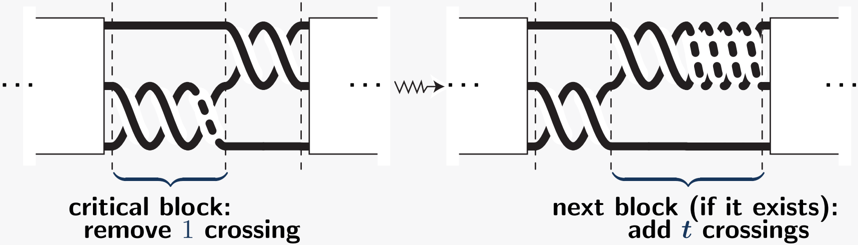

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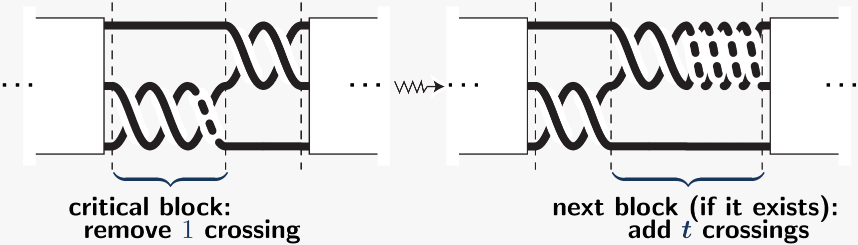

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in contrast with the folklore result:

- **All usual (algebraic) properties of braids can be proved in $I\Sigma_1$.**

- **Unprovability of the finiteness of $\mathcal{G}_3$-sequences in $I\Sigma_1$:**

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$\rightsquigarrow$ $I\Sigma_1$ does not prove the totality of the Ackermann function, hence it cannot prove the finiteness of $\mathcal{G}_3$-sequences of braids.

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- So far: $\mathcal{G}_3$-sequences = particular descending sequences of braids.

$\rightsquigarrow$ Now: arbitrary descending sequences of braids.

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$\rightsquigarrow$ Question: Where is the transition between $I\Sigma_1$-provability and $I\Sigma_1$-unprovability?

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- Key point: Fine counting arguments in B_3, namely evaluating

$$\mathrm{card}\{b \in B_3 \mid \|b\| \leqslant \ell \;\;\&\;\; b < \Delta_3^k\}.$$

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⇝ A notion of $\mathcal{G}_\infty$-sequence similar to $\mathcal{G}_3$-sequence, but involving arbitrary braids instead of 3-braids.

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