# Dual presentation of Thompson's group $F$ and flip distance between triangulations

Patrick Dehornoy

Laboratoire de Mathématiques
Nicolas Oresme, Université de Caen

- Main theme: Algebraic and combinatorial properties of $F$

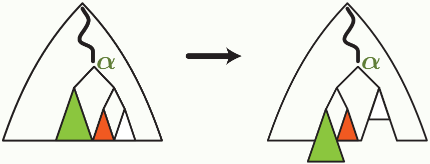
- An alternative—more natural?—presentation of $F$ that gives a lattice ("pre-Garside") structure;
- A connection with the rotation distance between finite trees and the flip distance between triangulations of a polygon.

**Plan :**

- 1. $F$ as the geometry group of associativity (as in Thompson–McKenzie, cf. Matt Brin's course 1)
- 2. The lattice structure of $F$
- 3. Associahedra and flip distance
- 4. Proving lower bounds for the distance in $F$

## 1. $F$ as the geometry group of associativity

- Starting idea: Associate with every algebraic law a certain monoid that captures its geometry—here: associativity $(\mathcal{A})\ x(yz) = (xy)z$.
- What means applying $\mathcal{A}$ to a term (= bracketted expression)?




$\rightsquigarrow$ Depends on the position and the orientation.

- Def.: $A_\alpha$:= (partial) operator "apply $\mathcal{A}$ at $\alpha$ in the $\rightarrow$ direction";

  \- $\mathcal{G}_\mathcal{A}$ ("geometry monoid of $\mathcal{A}$") := monoid generated by all $A_\alpha^{\pm 1}$.

**Fact**: Two terms $T, T'$ are $\mathcal{A}$-equivalent iff some element of $\mathcal{G}_\mathcal{A}$ maps $T$ to $T'$.

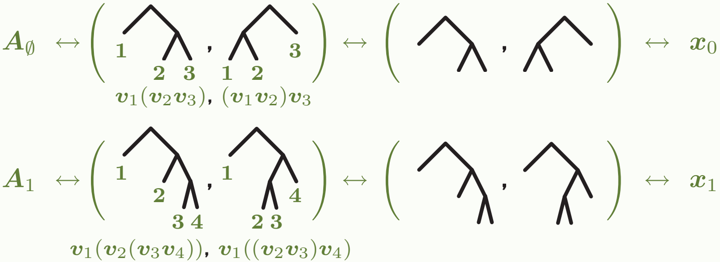
- The monoid $\mathcal{G}_\mathcal{A}$ is not a group ($A_\alpha A_\alpha^{-1} \neq \mathrm{id}$), but almost:

**Proposition:** The monoid $\mathcal{G}_\mathcal{A}/\!\approx$ is a group, isomorphic to $F$.

↑
to agree on at least one term

$\rightsquigarrow$ "$F$ is the geometry group of associativity"

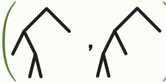
- Every element of $\mathcal{G}_\mathcal{A}$ consists of the instances of a pair of terms $\simeq$ a pair of binary trees (as variables are not permuted):



## The dual Thompson monoid

- Under the previous correspondence: $x_n \leftrightsquigarrow A_{11\dots1}$, $n$ times $1$.
- A (redundant) family of generators for $F$: all $A_\alpha$'s.

$\uparrow$ a binary address = a finite sequence of $0$'s and $1$'s

- Example: $A_0$ ("associativity at $0$") $=\Big($  $\Big) = x_0^{-1}x_1^{-1}x_0^2$.

$\rightsquigarrow$ a new presentation of $F$ + a new submonoid $F^{+*}$ that includes $F^+$.

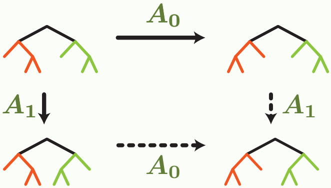
$\uparrow$ submonoid of $F$ generated by all $A_\alpha$

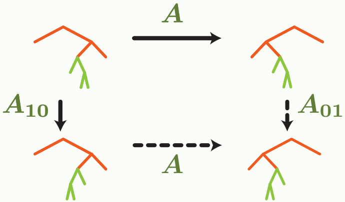
$\uparrow$ submonoid of $F$ generated by all $x_i$

- Reminiscent of the Birman–Ko–Lee generators of $B_n$ and the dual braid monoids $\rightsquigarrow$ the "dual Thompson monoid".

- Claim: The monoid $F^{+*}$ is a very natural object, and it has good properties, more symmetric than those of $F^+$.

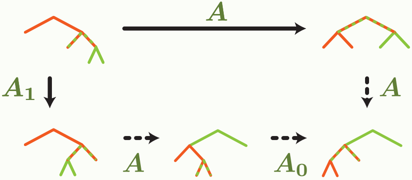
- Relations connecting the $A_\alpha$'s? Find common multiples in $F^{+*}$.
- **Commutation and twisted commutation relations:**




$$A_{\alpha 1\beta}\ A_{\alpha 0\gamma} = A_{\alpha 0\gamma}\ A_{\alpha 1\beta} \qquad A_{\alpha 10\beta}\ A_\alpha = A_\alpha\ A_{\alpha 01\beta}$$

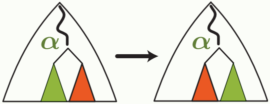
- **MacLane–Stasheff** pentagon relations**:**




$$A_{\alpha 1}\ A_\alpha\ A_{\alpha 0} = A_\alpha\ A_\alpha$$

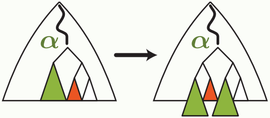
**Proposition**: The previous relations (commutation + pentagon) make a presentation of $F$—and of $F^{+*}$—in terms of the $A_\alpha$'s.

- Other laws:
Associativity + commutativity:
in addition to $A_\alpha$, define
$C_\alpha :=$ "apply commutativity at $\alpha$":



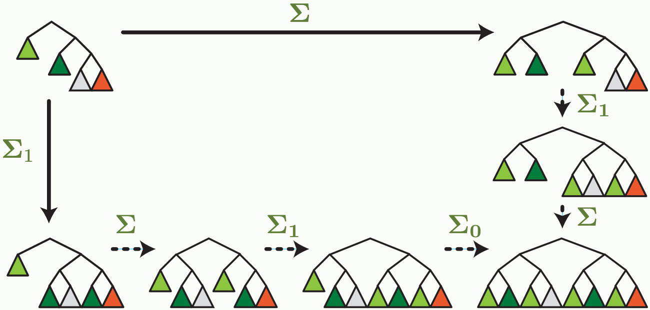

$\rightsquigarrow$ Thompson's group $V$, presented by pentagon + hexagon rel's.

- Other laws:
Self-distributivity $x(yz) = (xy)(xz)$:
define $\Sigma_\alpha :=$ "apply self-distributivity at $\alpha$":




$\rightsquigarrow$ Group $\mathcal{G}_{\mathcal{LD}}$—"Thompson's group for $\mathcal{LD}$"—presented by ...

- The heptagon relation of self-distributivity:




$$\Sigma_\alpha\ \Sigma_{\alpha 1}\ \Sigma_\alpha = \Sigma_{\alpha 1}\ \Sigma_\alpha\ \Sigma_{\alpha 1}\ \Sigma_{\alpha 0}$$

- Put $\sigma_i := \Sigma_{11\dots1}$, $n$ times $1$. The heptagon becomes

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \cdot \Sigma_{1^i0}:$$

$\rightsquigarrow$ The group $\mathcal{G}_{\mathcal{LD}}$ is an extension of Artin's braid group $B_\infty$.

## 2. The lattice structure of $F$

**Theorem**: The group $F$ is a group of (left and right) fractions for $F^{+*}$; the latter admits (left and right) lcm's and gcd's.

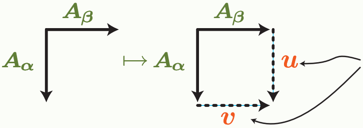
↑

least common multiples / greatest common divisors

**Corollary**: The relation $a^{-1}b \in F^{+*}$ defines a lattice ordering on $F$.

- Fact: For each pair of generators $A_\alpha, A_\beta$, the presentation contains exactly one relation of the form $A_\alpha ... = A_\beta ...$.
- Can one deduce anything?
  (typically that $A_\alpha ... = A_\beta ...$ is an lcm of $A_\alpha$ and $A_\beta$)
- In general: no; but yes, if some specific syntactic condition holds.

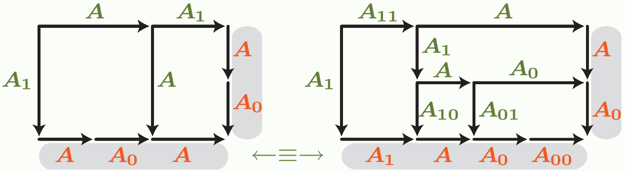
- Associate with every word in the letters $A_\alpha^{\pm 1}$ a stair by concatenating arrows $A_\alpha \mapsto \xrightarrow{A_\alpha}$, $A_\alpha^{-1} \mapsto \downarrow A_\alpha$.
- Close the ⌜ patterns until an ⌟ square is obtained using




where $A_\alpha v = A_\beta u$ is the unique relation of the presentation of the form $A_\alpha ... = A_\beta ...$

**Criterion**: Suppose that, for each relation $u = v$ of the presentation and each $\alpha$, reversing $A_\alpha^{-1} u$ and $A_\alpha^{-1} v$ leads to equivalent words. Then $F^{+*}$ admits (right) lcm's (and is left cancellative).

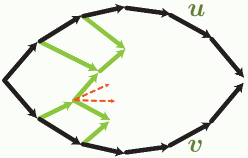
- Example: $u = AA_1$, $v = A_{11}A$, and $\alpha = 1$:




□

- Thus: $F^{+*}$ admits lcm's and gcd's—whence the lattice structure.
- Moreover, word reversing solves the word problem of $F^{+*}$:

**Proposition**: Two words $u, v$ in the letters $A_\alpha$ represent the same element of $F^{+*}$ iff reversing $u^{-1}v$ leads to the empty word.




- Extends to $F$—by combining left and right reversings.

**Corollary**: The Dehn function of $F$ w.r.t. the $A_\alpha$'s is quadratic.

- Gives a unique irreducible $F^{+*}$-fraction for each element of $F$, whence a unique normal form in $F$ once one is chosen in $F^{+*}$ (for instance, a "Polish normal form").

## 3. Associahedra and flip distance

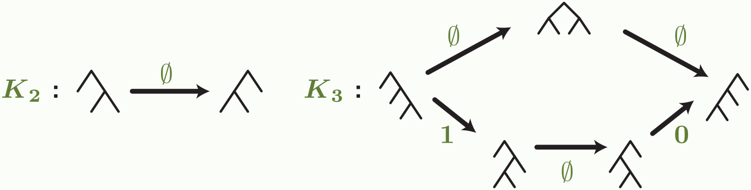
- $F$ has a partial action on finite trees (= bracketted expressions)

  ⇝ for every finite tree, a finite orbit:

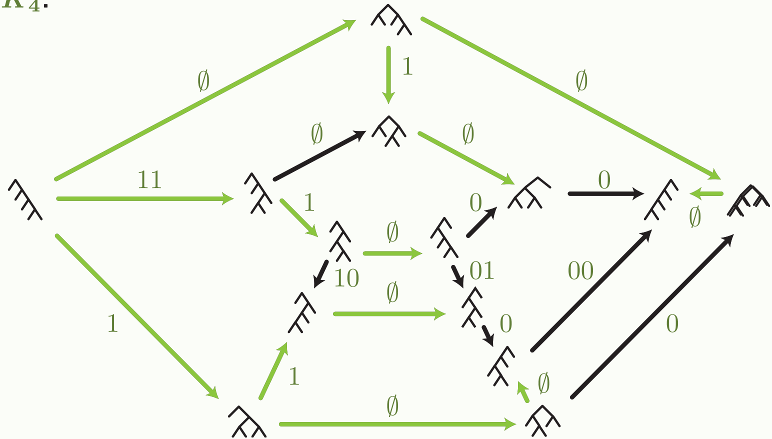
**Definition**: The associahedron $K_n$ is the oriented graph s.t.
- vertex set: the orbit of (any) size $n$ tree under $F$.
- edges: $\alpha$-labelled edge from $T$ to $T'$ iff $A_\alpha$ maps $T$ to $T'$.

⇝ a finite fragment of the Cayley graph of $F^{+*}$ w.r.t. the $A_\alpha$'s.

⇝ as $F^{+*}$ admits lcm's, a lattice: the Tamari lattice.



- ***K*$_4$:**




- Normal form in $F^{+*}$ provides a **spanning tree** on $K_n$.

**Problem**: Compute the distance between vertices in $K_n$, *i.e.*, between binary trees; in particular, compute the diameter of $K_n$.

↑
"rotation distance"

- Equivalently:
  Compute distances in (the Cayley graph of) $F$ w.r.t. the $A_\alpha$'s.

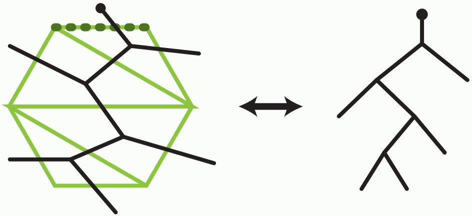
- Reminiscent of, but seemingly unrelated to,
  the similar problem with $x_0, x_1$—solved by B.Fordham.

**Proposition**: The embedding of $F^{+*}$ into $F$ is not a quasi-isometry:
For each $C$, there exist $f$ in $F^{+*}$ satisfying

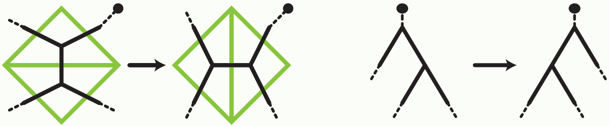
$$\mathrm{dist}_F(1, f) < \frac{1}{C}\,\mathrm{dist}_{F^{+*}}(1, f).$$

The distance in $(F, \{A_\alpha\}_\alpha)$ is not the distance in $(F^{+*}, \{A_\alpha\}_\alpha)$.

- Binary trees of size $n$ ⇝ triangulations of an $(n+2)$-gon



- Then flipping on edge in a triangulation ⇝ applying one $A_\alpha^{\pm 1}$



⇝ Diameter $d_n$ of $K_n$

= maximal rotation distance between two size $n$ trees.

= max. flip distance between two triangulations of an $(n+2)$-gon.

**Theorem (Sleator-Tarjan-Thurston, '88):** $d_n = 2n - 6$ for $n \gg 0$.

- The easy direction: $d_n \leqslant 2n - 6$ holds for $n > 10$.
  (Every size $n$ tree is at distance at most $n - 1$ of a right comb.)

- The problem: Proving lower bounds for $\mathrm{dist}(T, T')$.
- The solution: Use hyperbolic geometry.
  - Glue $\partial T$ and $\partial T'$ to get a triangulated polytope $\Pi$ in $S^2$.
  - Then $\mathrm{dist}(T, T') \approx \#$ tetrahedra in $\Pi \geqslant \dfrac{\mathrm{vol}(\Pi)}{\max \mathrm{vol}(\text{tetrahedron})}$.
  - In $\mathbb{E}^3$: $\mathrm{vol}(\Pi) \leqslant \frac{4}{3}\pi R^3 \leqslant 3 \max \mathrm{vol}(\text{tetrahedron})$: no hope!
  - In $\mathbb{H}^3$: $\mathrm{vol}(\Pi)$ may be large.
  - ⇝ Find polytopes with few vertices and large hyperbolic volume:
  - Use carefully chosen finite plane tesselations and close them. □

- Wonderful, but nothing for small $n$. ⇝ Is there another method?

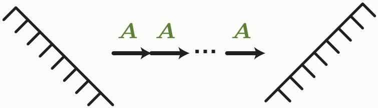
## 4. Proving lower bounds on $d_n$

↑ using combinatorial methods that work for each $n$.

- Recall: $d_n$ = max. # of $A_\alpha^{\pm 1}$ to transform a size $n$ tree into another.
- The problem: find size $n$ trees $T, T'$ s.t. many $A_\alpha^{\pm 1}$ (possibly $2n-6$) are provably needed to transform $T$ to $T'$.
- Use the algebraic properties of the monoid $F^{+*}$?
- A naive attempt: Invariants of the relations.

- All relations (including pentagon) preserve global # of $A_{11\ldots 1}$.

**Fact 1**: For each $n$, one has $d_n \geqslant n-1$.

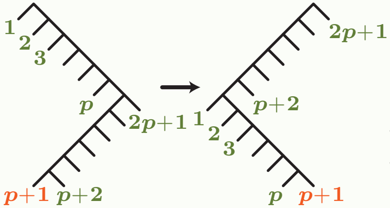


- A semi-naive attempt: Action on addresses.
- Number the leaves, and follow the address of a given leaf;

  ↑ finite sequence of 0's and 1's describing the path from the root

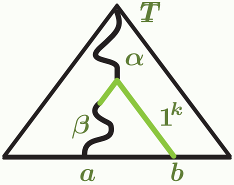
- Action of $A_\alpha$ on an address: $\beta \mapsto \begin{cases} \alpha 00\gamma & \text{if } \beta = \alpha 0\gamma, \\ \alpha 01\gamma & \text{if } \beta = \alpha 10\gamma, \\ \alpha 1\gamma & \text{if } \beta = \alpha 11\gamma, \\ \beta & \text{otherwise.} \end{cases}$

**Fact 2**: For each $n$, one has $d_n \geqslant \frac{3}{2}n - 2$.




address of $p+1$ in $T$: $1^p0^p$
address of $p+1$ in $T'$: $0^p1^p$
$\rightsquigarrow$ at least $3p-2$ steps. □

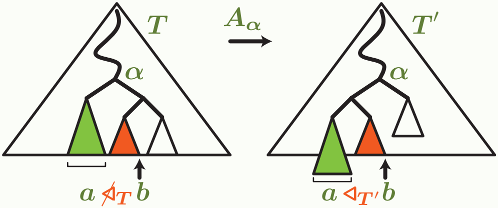
- Number the leaves again.
- Definition: For $a, b$ leaves of $T$, declare $a \lhd_T b$ true if, for some $\alpha, \beta, k$,




$a$ is covered by $b$ in $T$

⇝ a transitive relation attached with $T$.

**Lemma**: Assume that $A_\alpha$ maps $T$ to $T'$. Then
$a \lhd_{T'} b$ iff $a \lhd_T b$ or ( $a \in T_{\alpha 0}$ and $b = \mathrm{last}(T_{\alpha 10})$ ).




⇝ Applying associativity = adding covering.

**Principle 1**: Assume $a \not\triangleleft_T b$ and $a \triangleleft_{T'} b$.
Then any $A_\alpha^{\pm 1}$-sequence from $T$ to $T'$ contains at least one $b$-step.

↑

s.t. $b$ is the last leaf of the current $\alpha 10$-subtree

- Attaching one step with a leaf cannot give more than $n$.

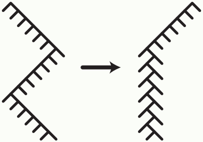
**Principle 2**: Assume $a \not\triangleleft_T b$, $a \triangleleft_{T'} b$, with $a \not\triangleleft_{T'} b{-}1$, $b{-}1 \triangleleft_{T'} b$.
Then any $A_\alpha^{\pm 1}$-sequence from $T$ to $T'$ contains at least two $(b)$-steps.

- Proof: Consider the first step where $b{-}1 \triangleleft b$ holds, a $b$-step.

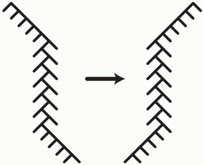
- Case 1: After this step, we still have $a \not\triangleleft b$:
then a second $b$-step will occur subsequently.

- Case 2: After this step, we have $a \triangleleft b$, which requires $a \triangleleft b{-}1$;
then a negative $b{-}1$-step will occur subsequently. □

**Fact 3**: For each $n$, one has $d_n \geqslant \frac{5}{3}n - 3$.



**Fact 4**: For each $n$, one has $d_n \geqslant \frac{7}{4}n - 4$.



- Conjecture: Such methods lead to $d_n = 2n - 6$ for $n > 10$.

• R. McKenzie & R.J. Thompson, An elementary construction of unsolvable word problems in group theory; *in* Word Problems, Boone & *al.* eds.,
Studies in Logic vol. 71, North Holland (1973)

• B. Fordham, Minimal length elements of Thompson's group $F$
Geom. Dedicata 99 (2003) 179–220

• D. Sleator, R. Tarjan, W. Thurston, Rotation distance, triangulations, and hyperbolic geometry J. Amer. Math. Soc. 1 (1988) 647–681

• P. Dehornoy, The structure group for the associativity identity,
J. Pure Appl. Algebra 111 (1996) 59–82

• P. Dehornoy, Geometric presentations of Thompson's groups,
J. Pure Appl. Algebra 203 (2005) 1–44

• P. Dehornoy, Braids and Self-Distributivity,
Progress in Math. vol. 192, Birkhäuser, (2000)

• D. Krammer, A class of Garside groupoid structures on the pure braid group,
Trans. Amer. Math. Soc. 360 (2008), 4029-4061