The Subword Reversing Method

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- A strategy for constructing van Kampen diagrams for semigroups,

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- A strategy for constructing van Kampen diagrams for semigroups, with an application to the combinatorial distance between the reduced expressions of a permutation.

Plan :

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- The general case:
  - Subword reversing as a strategy for constructing van Kampen diagrams

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  - recognizing the optimality of a van Kampen diagram

all relations of the form $\mathrm{u} = \mathrm{v}$ with u, v nonempty words on S

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a tesselated disk with (oriented) edges labeled by elements of S and faces labelled by relations of R, with boundary paths labelled w and w'.

- **Example: Let** $B_n^+ = \left\langle s_1, \ldots, s_{n-1} \;\middle|\; \begin{array}{ll} s_i s_j s_i = s_j s_i s_j & \text{for } |i-j| = 1 \\ s_i s_j = s_j s_i & \text{for } |i-j| \geqslant 2 \end{array} \right\rangle^+$

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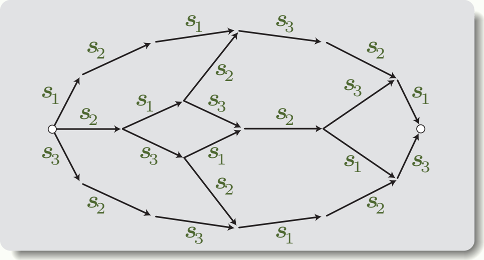
Then



is a van Kampen diagram for $(s_1 s_2 s_1 s_3 s_2 s_1, s_3 s_2 s_3 s_1 s_2 s_3)$.

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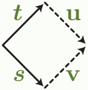
  - choose a relation $s\mathrm{v} = t\mathrm{u}$ of R to close it into 

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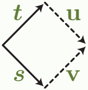
  - **choose a relation $s\mathrm{v} = t\mathrm{u}$ of R to close it into** 

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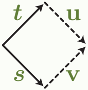
  - choose a relation $s\mathrm{v} = t\mathrm{u}$ of R to close it into 

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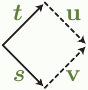
  - choose a relation $s\mathrm{v} = t\mathrm{u}$ of R to close it into 

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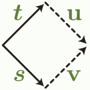
  - choose a relation $s\mathrm{v} = t\mathrm{u}$ of R to close it into 

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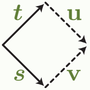
  - choose a relation $s\mathrm{v} = t\mathrm{u}$ of R to close it into 

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  - May not be unique (several relations $s... = t...$);
  - May never terminate (when u, v have length more than 1);
  - May terminate but boundary words are longer than w, w'
    (certainly happens if w, w' are not R-equivalent).

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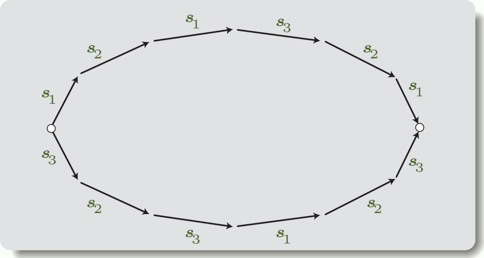

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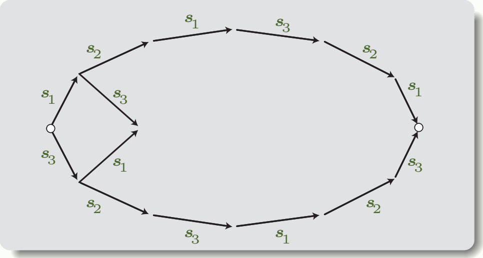

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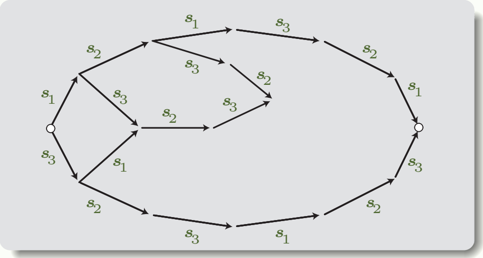

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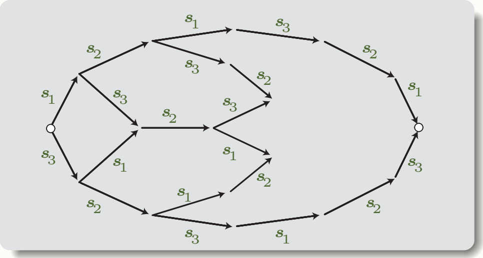

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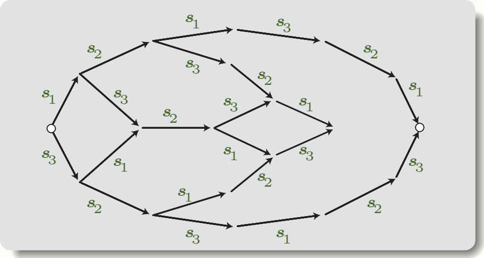

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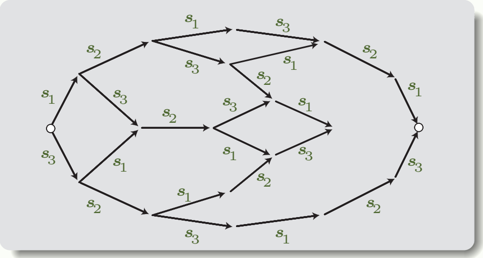

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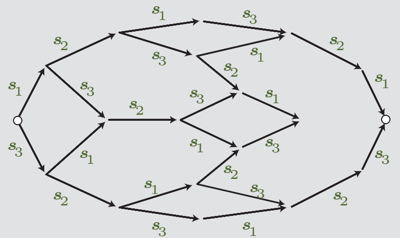

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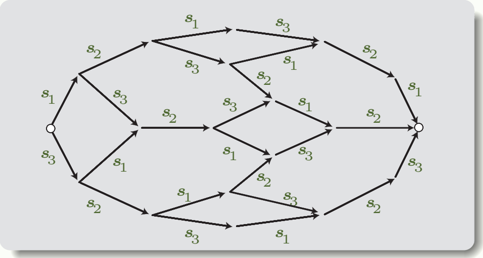

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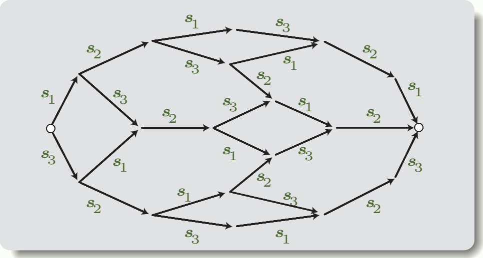

So, on this particular example, the reversing strategy works.

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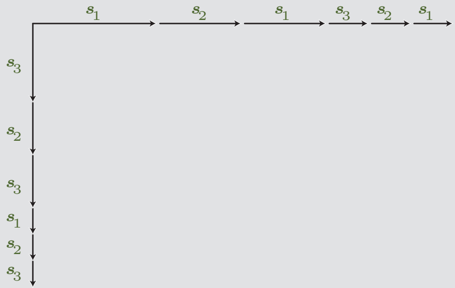

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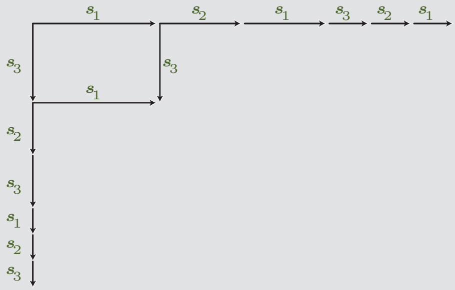

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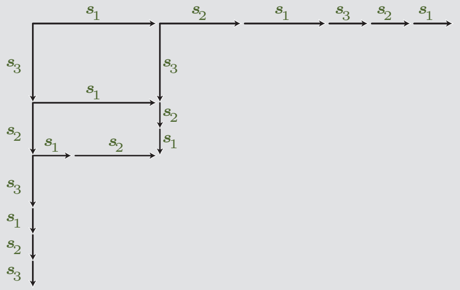

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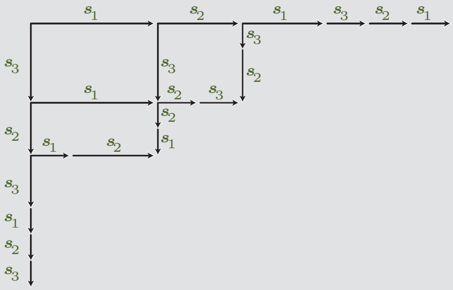

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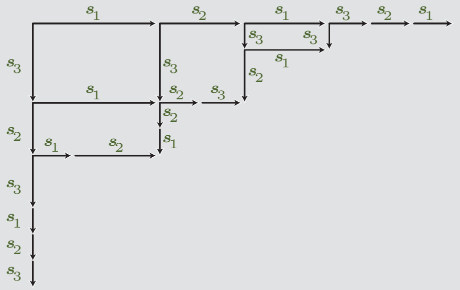

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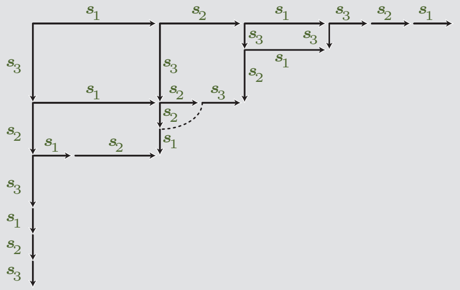

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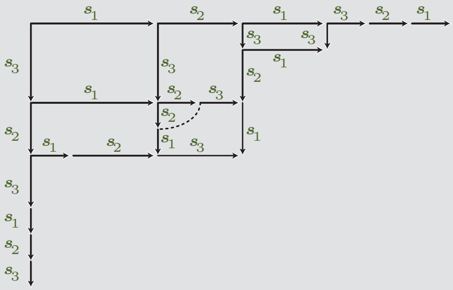

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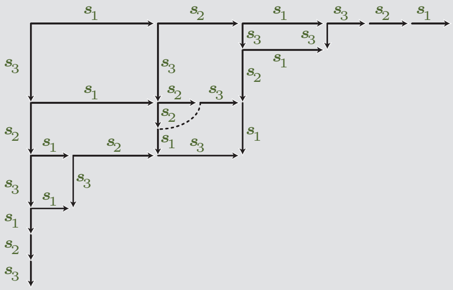

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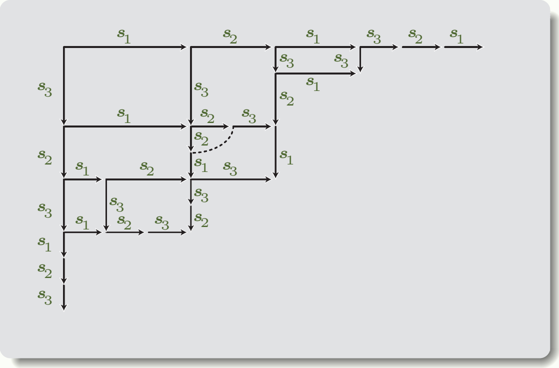

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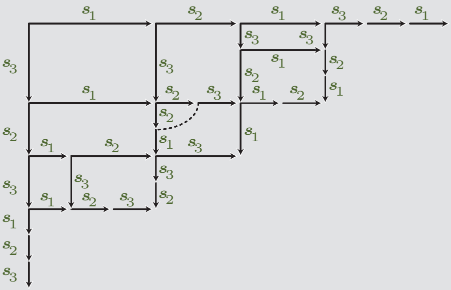

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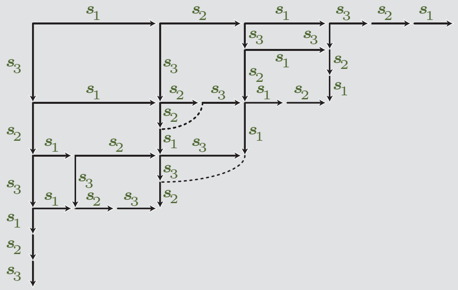

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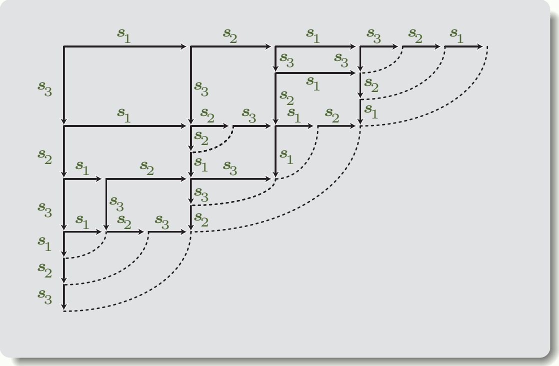

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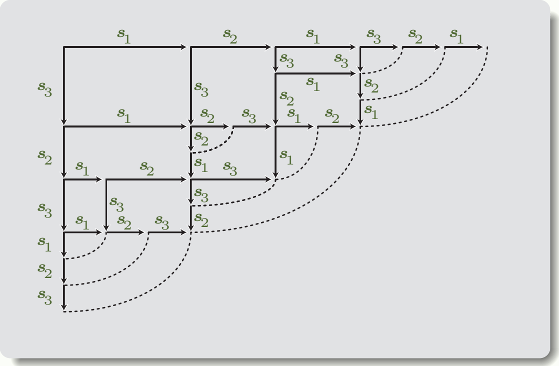

⇝ **only vertical and horizontal edges,**
plus dotted arcs connecting vertices that are to be identified
in order to get an actual van Kampen diagram.

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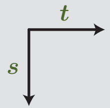

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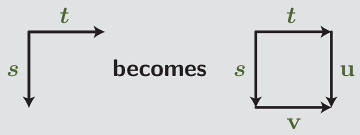

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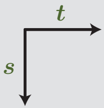
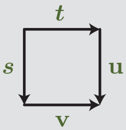


becomes



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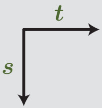

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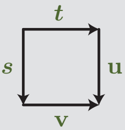

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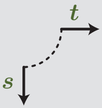

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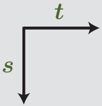

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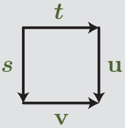

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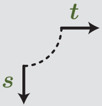

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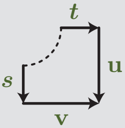

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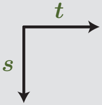

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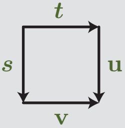

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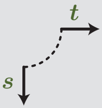

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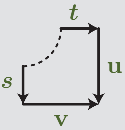

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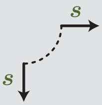

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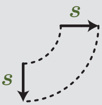

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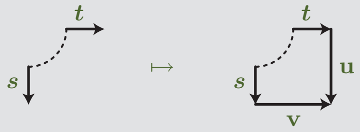

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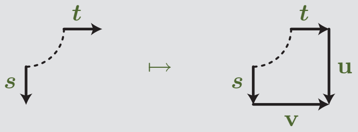

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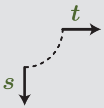

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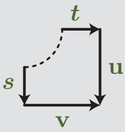

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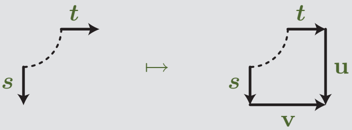

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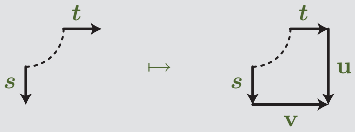

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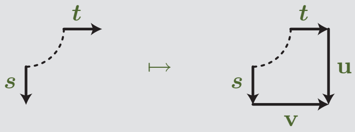

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- Syntactically, "subword reversing": replacing $-+$ with $+-$.

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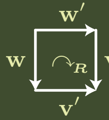
i.e., 

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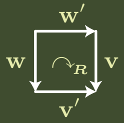
i.e., 

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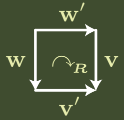
  i.e., 

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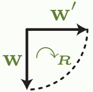
- In particular, if $\mathrm{w}^{-1}\mathrm{w}' \curvearrowright_R \varepsilon$, *i.e.*, if 

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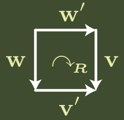
  i.e., 

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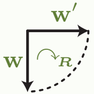
- In particular, if $\mathrm{w}^{-1}\mathrm{w}' \curvearrowright_R \varepsilon$, *i.e.*, if 

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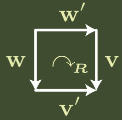
  i.e., 

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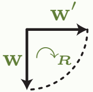
- In particular, if $\mathrm{w}^{-1}\mathrm{w}' \curvearrowright_R \varepsilon$, *i.e.*, if 

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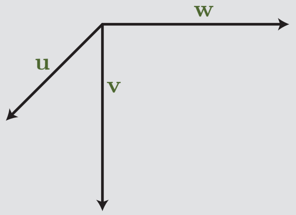

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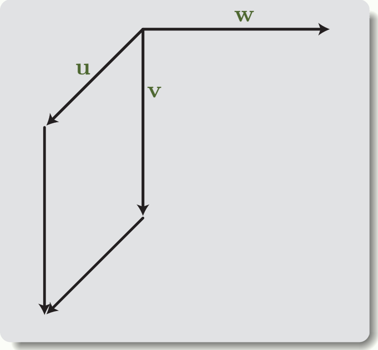

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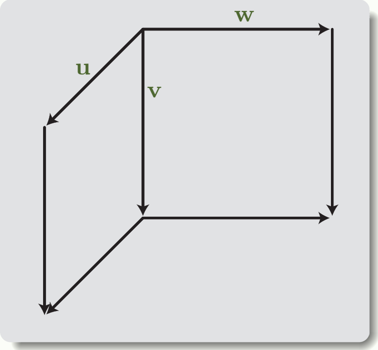

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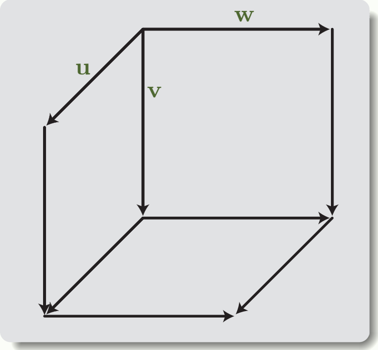

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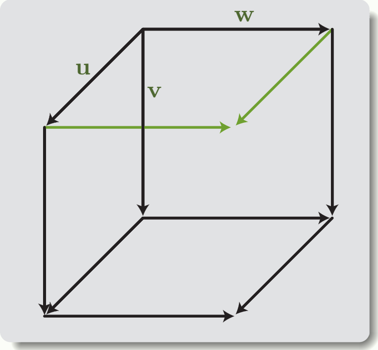

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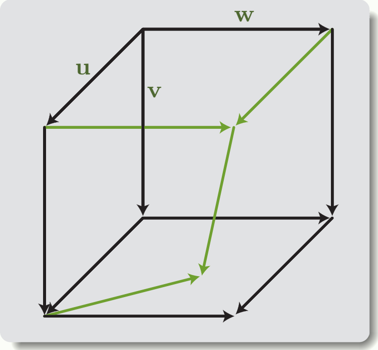

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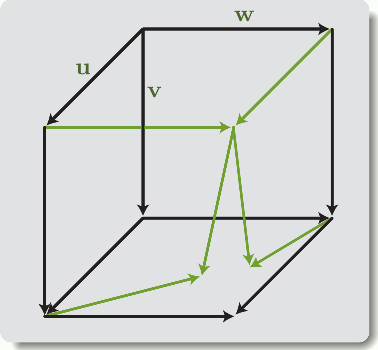

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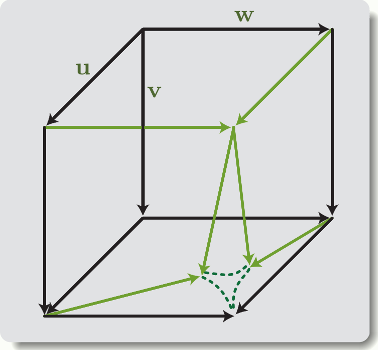

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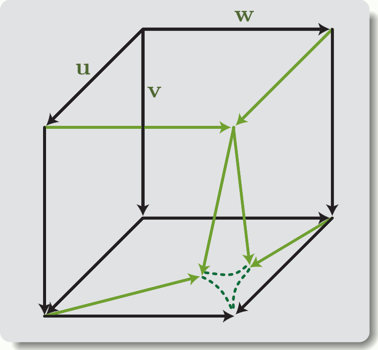

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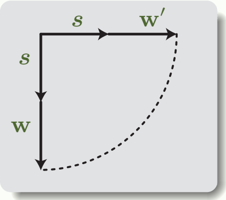

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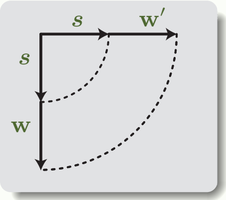

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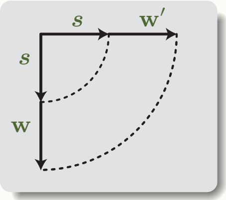

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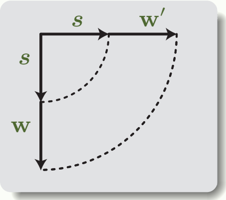

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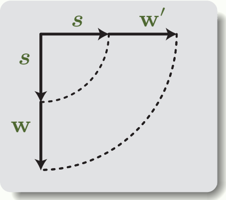

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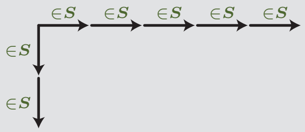

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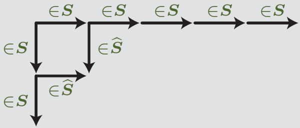

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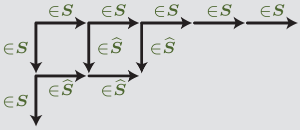

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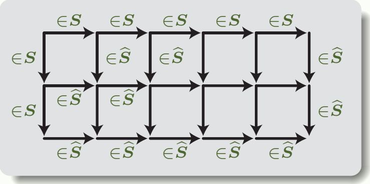

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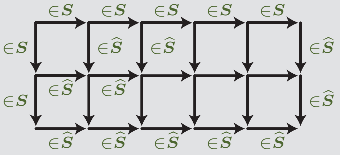

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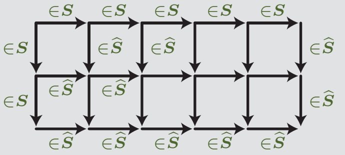

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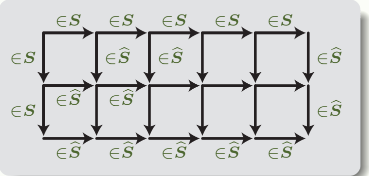

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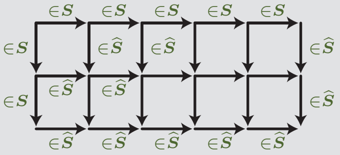

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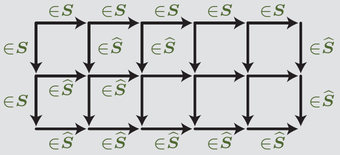

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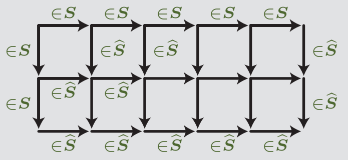

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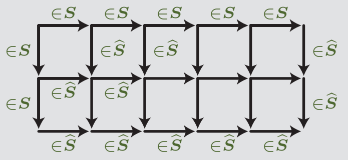
- Proof: Reversing terminates in quadratic time: construct an $\widehat{S}$-labeled grid:



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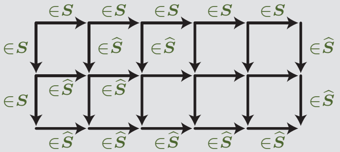

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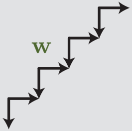

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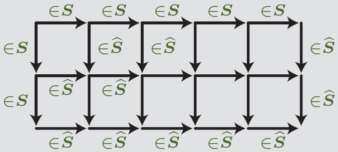

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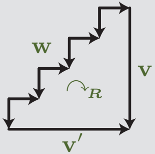

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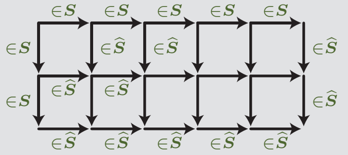

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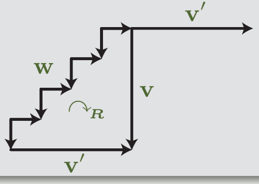

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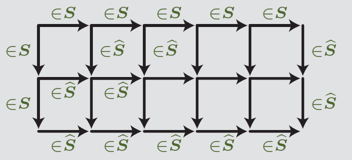

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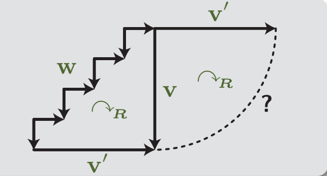

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- (with M. Autord) Combinatorial distance between the reduced expressions of a permutation.

Reduced expressions of a permutation

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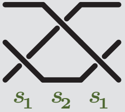

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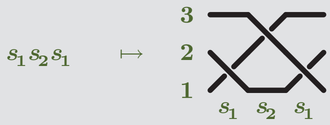

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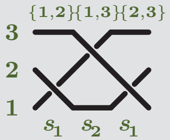

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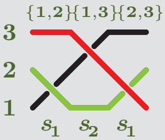

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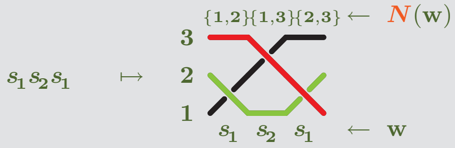

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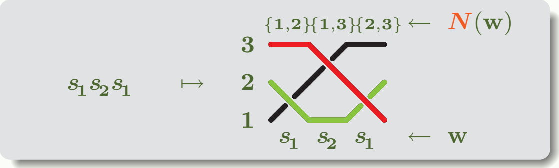

$\rightsquigarrow$ a sequence $N(\mathrm{w})$ of pairs of integers in $\{1, ..., n\}$.

- **For S, S' sequences of pairs of integers in $\{1, ..., n\}$:**
  - $I_3(S, S')$

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  - $I_3(S, S') = \#$ **triples** $\{p, q, r\}$ **s.t.**
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  Hence $d(\mathrm{w}, \mathrm{w}') \geqslant 4 + 2 = 6$.

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- **Question** (Conjecture?): Is the above inequality an equality?

- **Back to van Kampen diagrams with the aim of recognizing optimality.**

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  ↑

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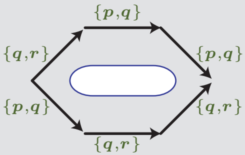

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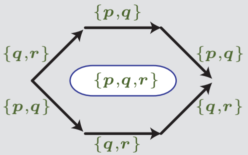

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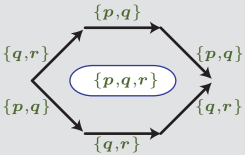

type II:

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↑
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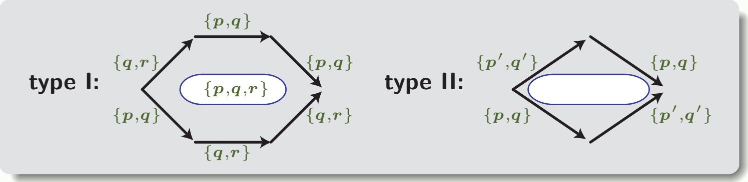

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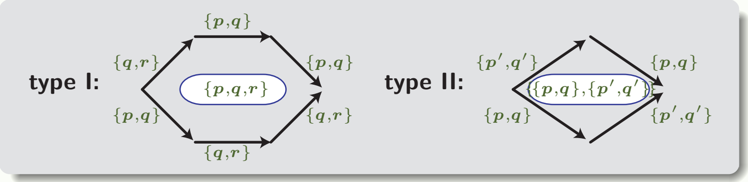

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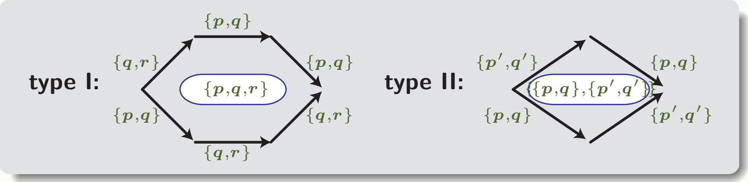
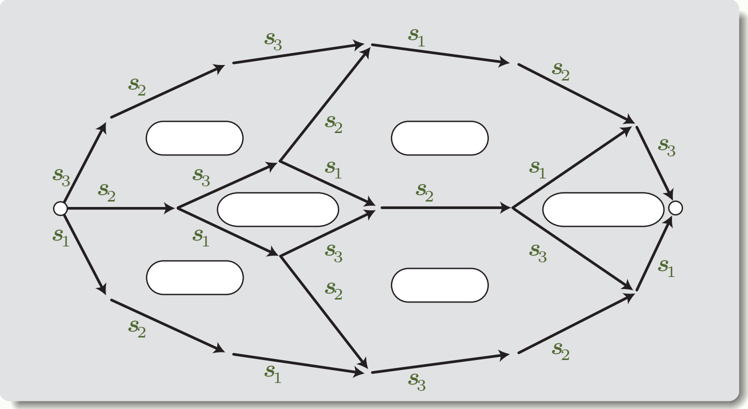


• **Criterion 1: A van Kampen diagram in which different faces have different names is optimal.**

- **Example:**



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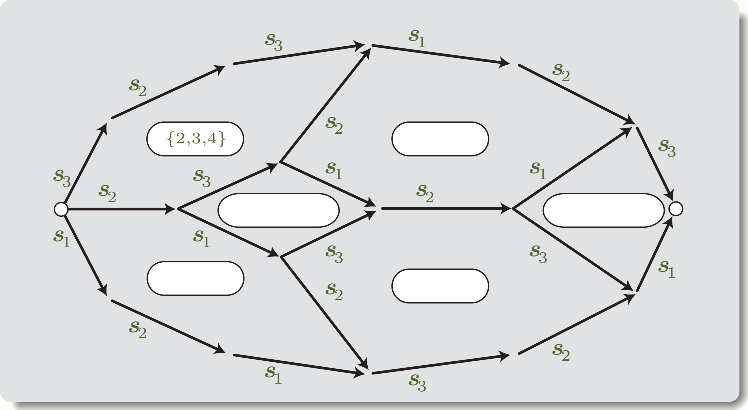

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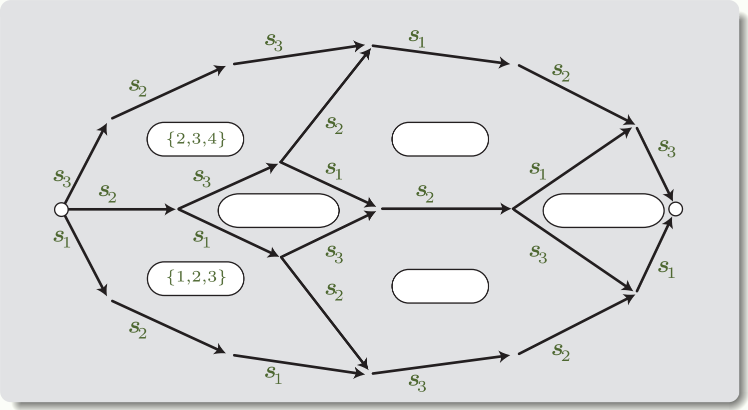

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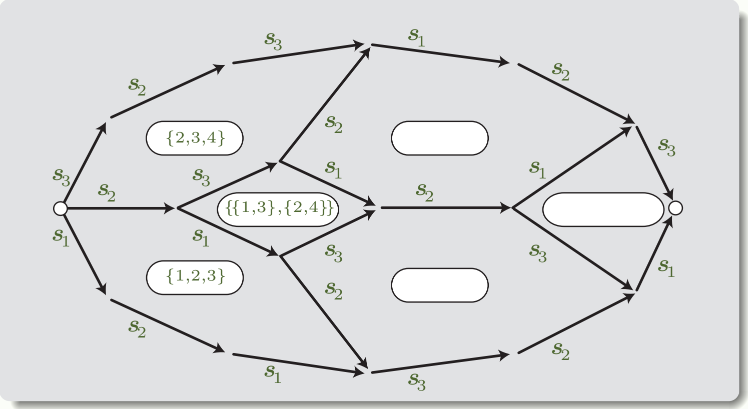

s_3
s_1
s_2
s_2
s_2
{2,3,4}
s_3
s_3
s_1
s_1
s_2
s_2
{{1,3},{2,4}}
s_1
s_1
s_3
s_3
s_1
{1,2,3}
s_2
s_2
s_2
s_1
s_3

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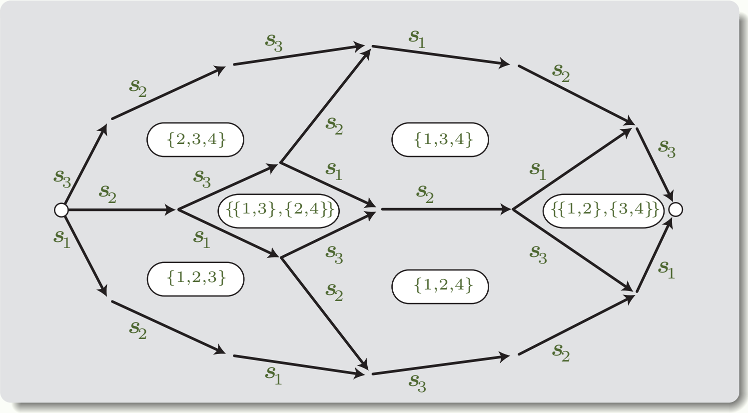

s_3
s_1
s_2
s_2
{2,3,4}
s_2
{1,3,4}
s_3
s_3
s_1
s_1
s_3
s_2
s_2
{{1,3},{2,4}}
{{1,2},{3,4}}
s_1
s_1
s_3
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{1,2,4}
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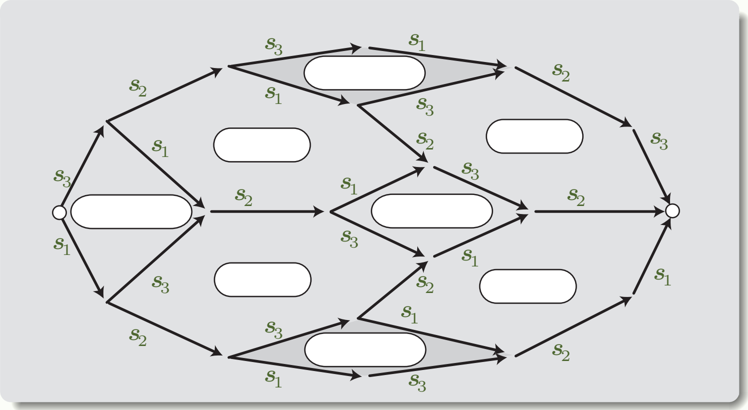
s_3
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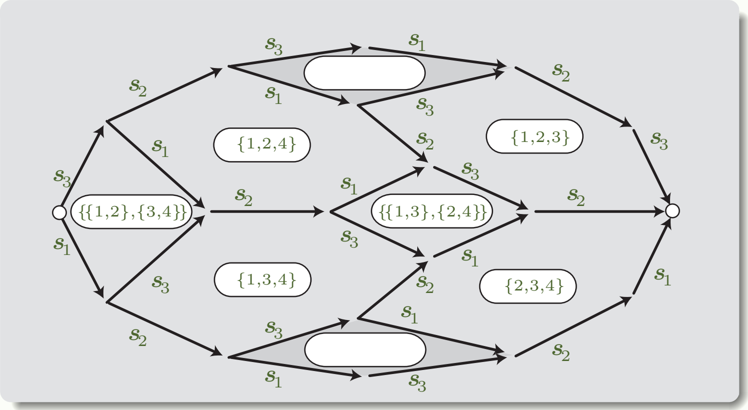
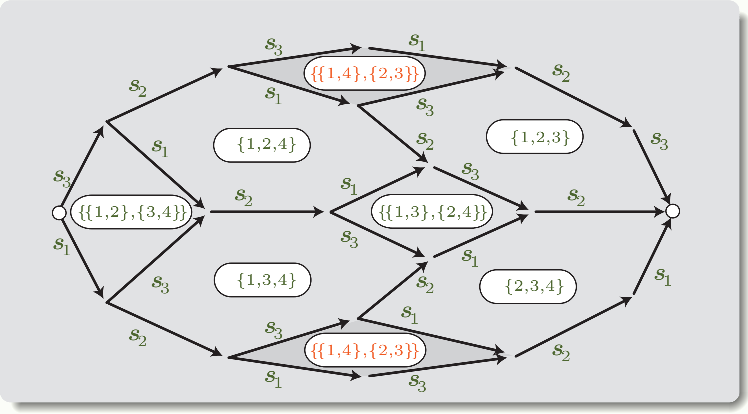


Naming faces (2)

- **Example:**



- (Again in a van Kampen diagram) connect the edges with the same name:

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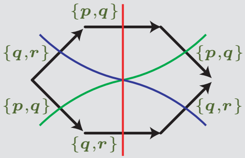

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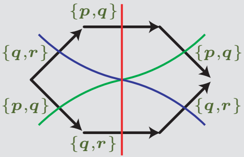
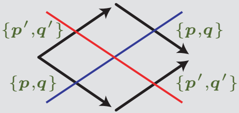


type II:



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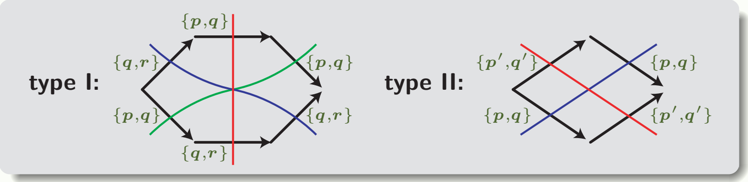

⇝ for each pair $\{p, q\}$, an (oriented) curve that connect all $\{p, q\}$-edges:

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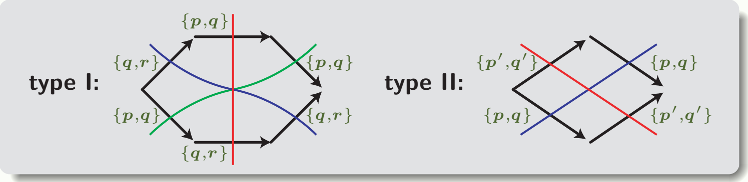

⇝ for each pair $\{p, q\}$, an (oriented) curve that connect all $\{p, q\}$-edges: the $\{p, q\}$-separatrix $\Sigma_{p,q}$.

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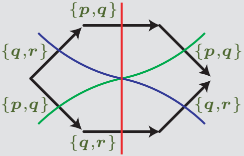
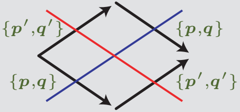


type II:



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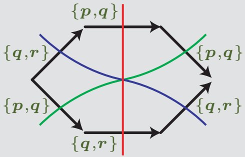
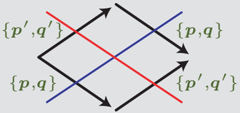


type II:



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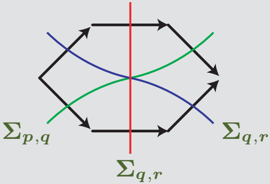

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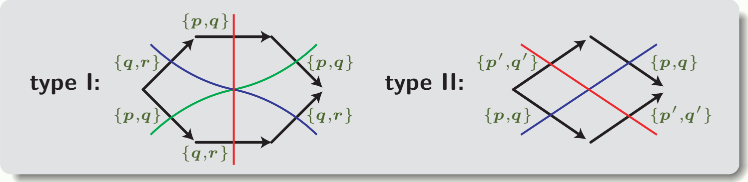

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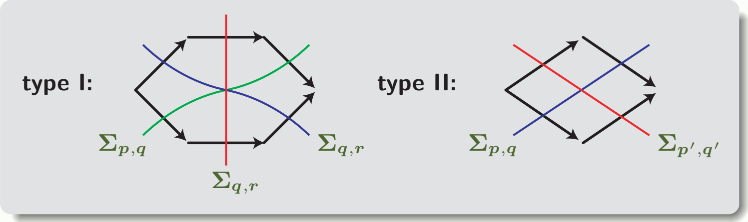

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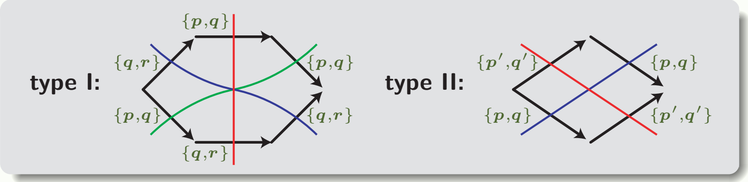

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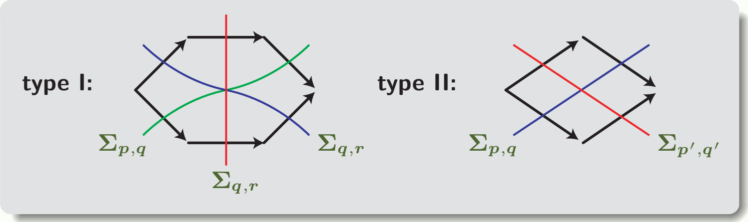

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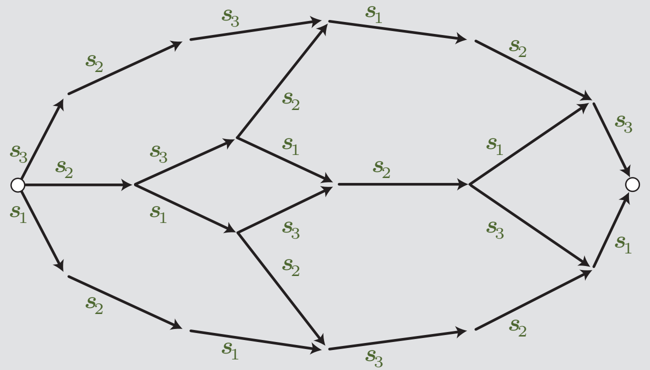
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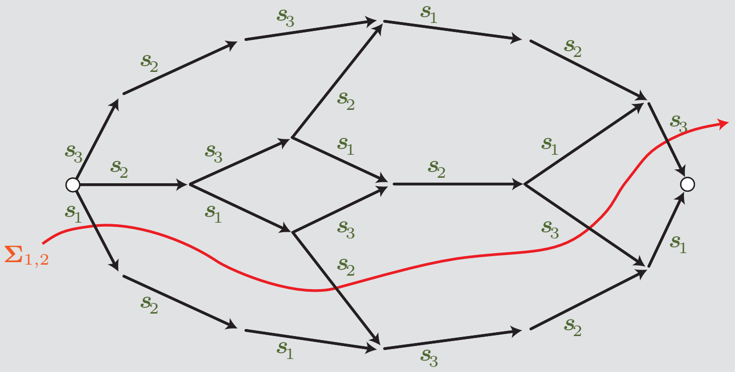
- **Example:**



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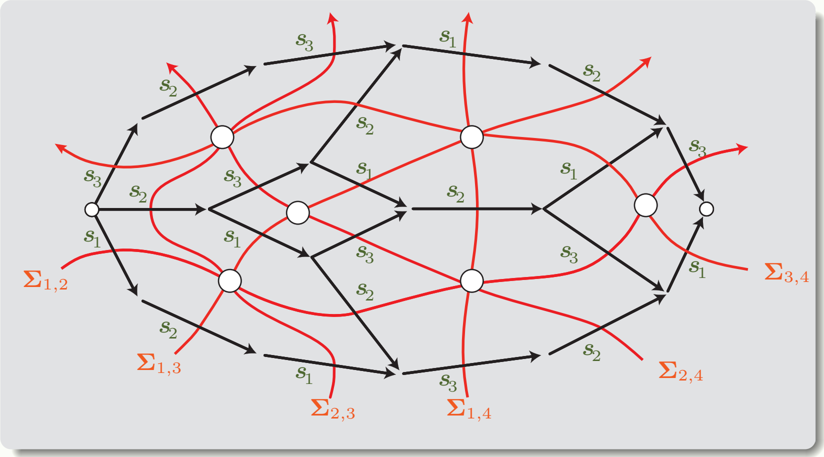
- **Example:**



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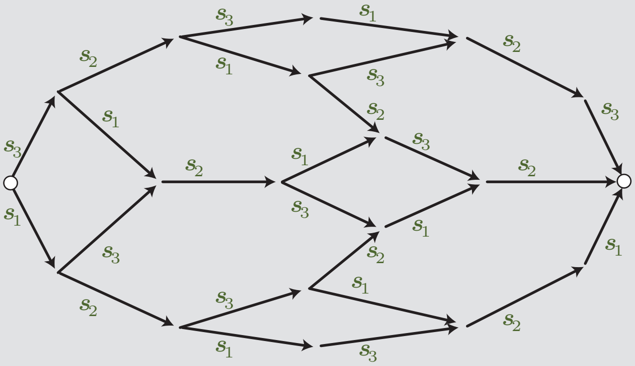
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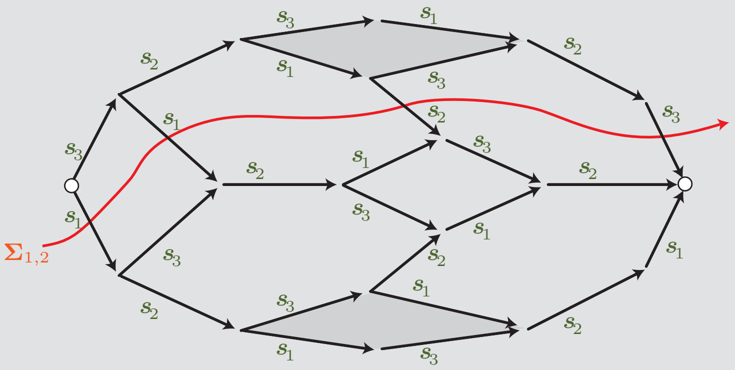

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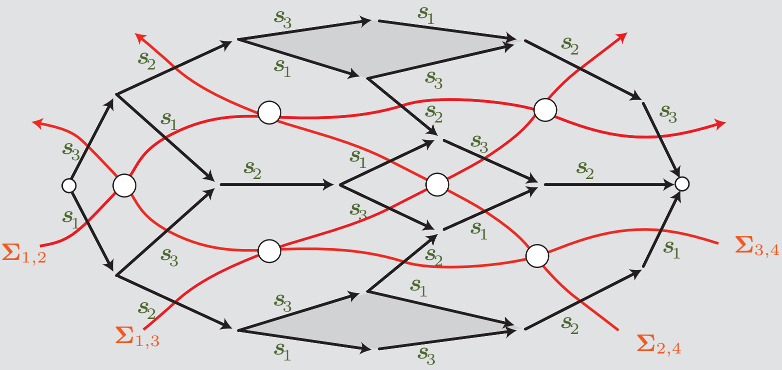
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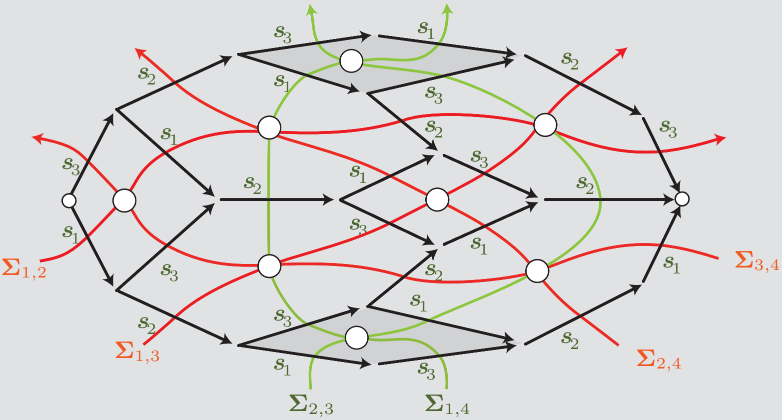

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- Criterion 2: A van Kampen diagram in which any two separatrices cross at most once is optimal.

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- Question: Is the condition necessary, *i.e.*, do any two separatrices cross at most once in an optimal van Kampen diagram?

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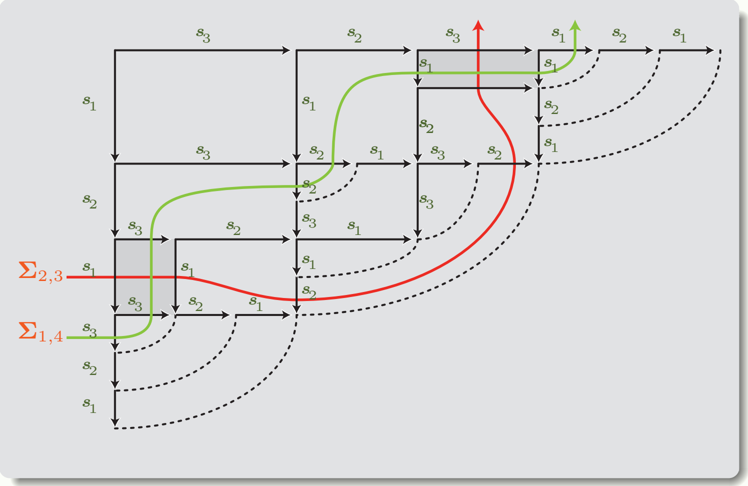
- Remark: Compare with "a s-word is reduced iff any two strands in the associated braid diagram cross at most one".

- **Applies in particular to reversing diagrams**
  (viewed as particular van Kampen diagrams):



- **How are separatrices in a reversing diagram?**

- **How are separatrices in a reversing diagram? Three types of faces:**

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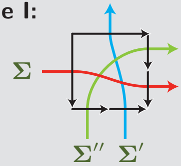
type I:



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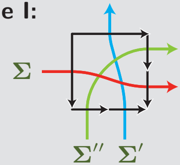
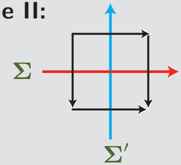


type II:



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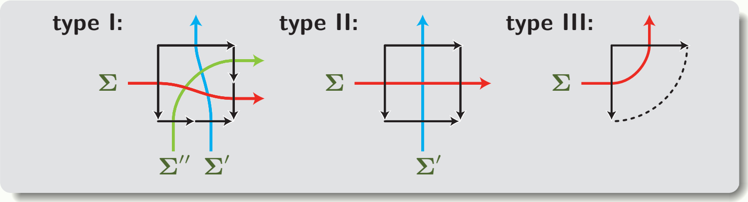

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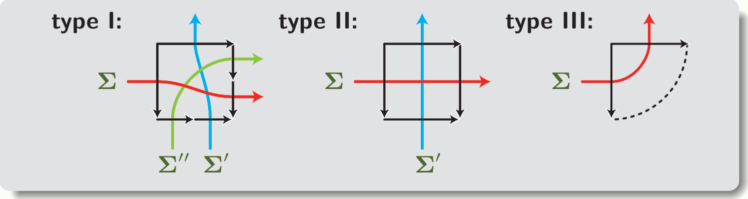

- Criterion 3: A reversing diagram containing no type III face is optimal.

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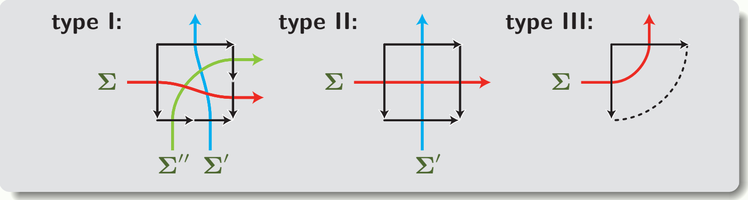

- Criterion 3: A reversing diagram containing no type III face is optimal.

- Proof: In order that two separatrices cross twice, one has to go from horizontal to vertical. □

- **An improvement: Same argument when reversing steps are grouped:**

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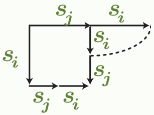
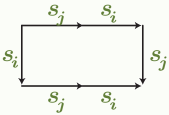
replace 

with 

for $|i - j| = 1$**,**

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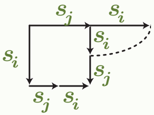
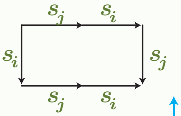
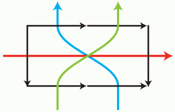
replace 

with 

for $|i - j| = 1$**,**

corresponding to 

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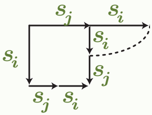
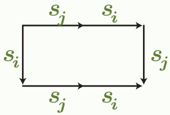
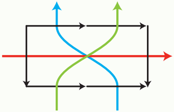
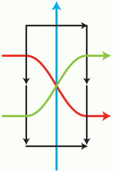
replace 

with 

for $|i-j|=1$,

corresponding to  **and**  .

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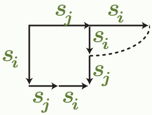
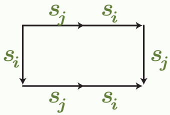
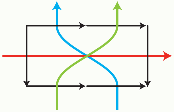
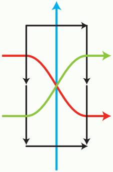
replace 

with 

for $|i - j| = 1$,

corresponding to  **and**  .

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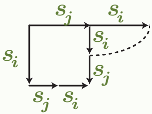
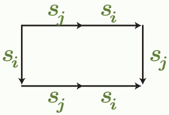
replace 

with 

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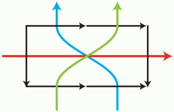
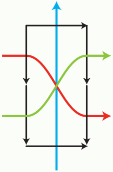
corresponding to  and  .

- **An application:**

> • **Proposition**: For each ℓ, there exist length ℓ reduced s-words w, w' satisfying $\mathrm{w}^{-1}\mathrm{w}' \curvearrowright_R \mathrm{v}'\mathrm{v}^{-1}$ and $d(\mathrm{wv}', \mathrm{w}'\mathrm{v}) \geqslant \ell^4/8$.

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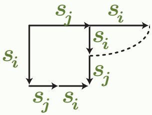
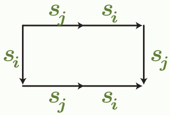
replace 

with 

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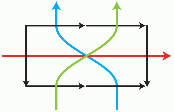
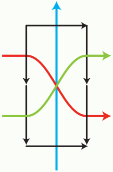
corresponding to  and  .

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By contrast: for **fixed** n, Garside's theory gives an upper bound in $O(\ell^2)$.

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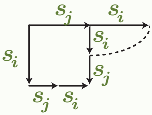
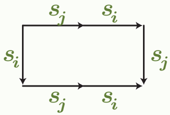
replace 

with 

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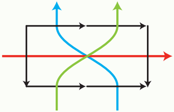
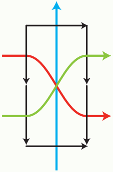
corresponding to  and  .

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- **Two conclusions:**

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- Even in the simple(?) case of braids and permutations, many open questions.

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- Importance of having van Kampen diagrams included in a grid.

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