The Subword Reversing Method

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Plan :

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- A motivating example
- Van Kampen diagrams
- Reversing : geometric description
- Reversing : syntactic description

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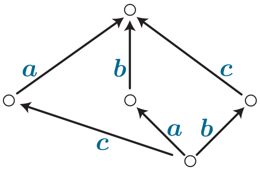

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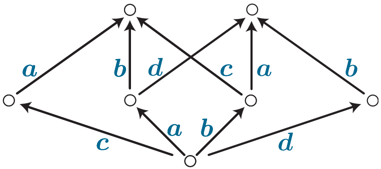

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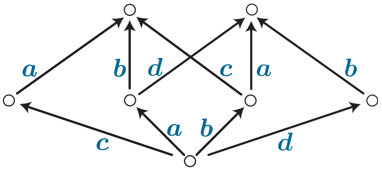

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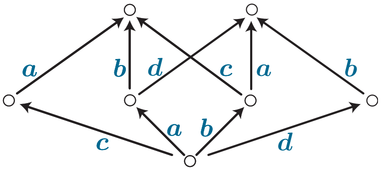

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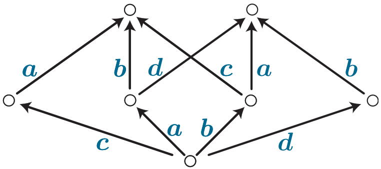

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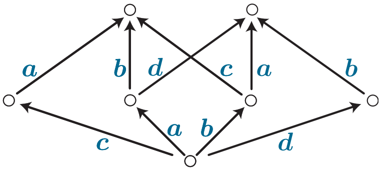

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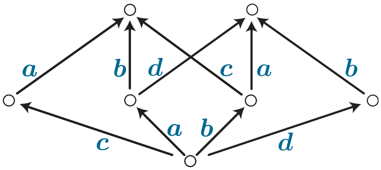

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- Note: M is **not** eligible for Adjan's cancellativity criterion.

all relations of the form $u = v$ with u, v nonempty words on S

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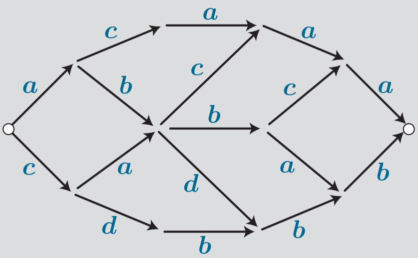

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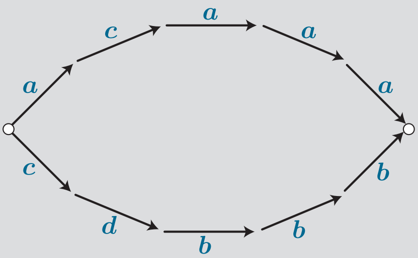

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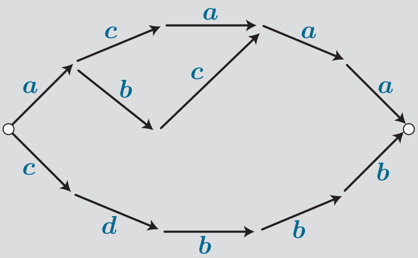

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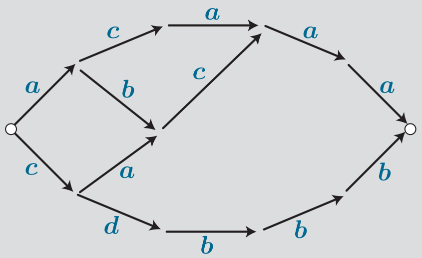

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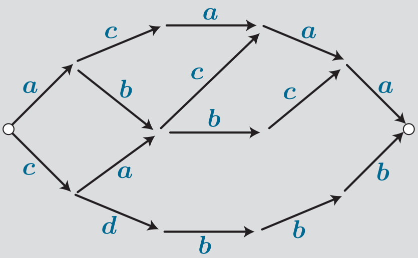

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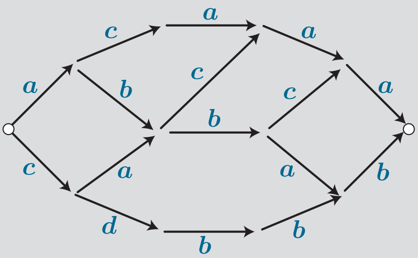

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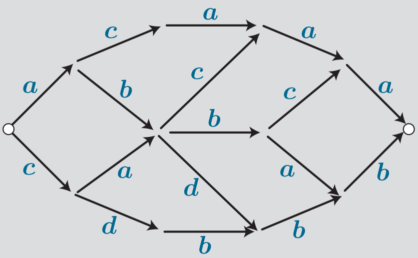

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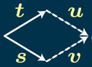
  - choose a relation $sv = tu$ of R to close this pattern into 

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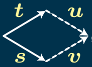
  - choose a relation $sv = tu$ of R to close this pattern into 

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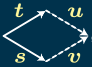
  - choose a relation $sv = tu$ of R to close this pattern into 

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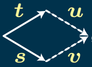
  - choose a relation $sv = tu$ of R to close this pattern into 

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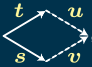
  - choose a relation $sv = tu$ of R to close this pattern into 
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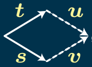
  - choose a relation $sv = tu$ of R to close this pattern into 

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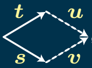
  - choose a relation $sv = tu$ of R to close this pattern into 

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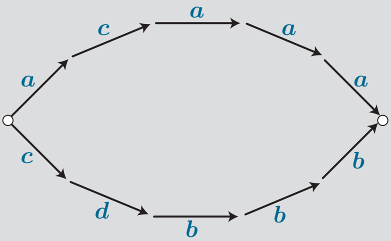

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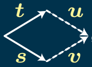
  - choose a relation $sv = tu$ of R to close this pattern into 

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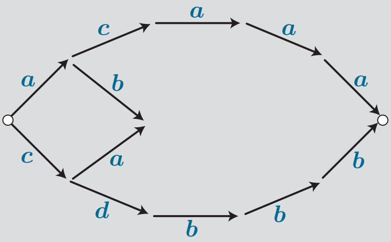

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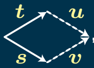
  - choose a relation $sv = tu$ of R to close this pattern into 

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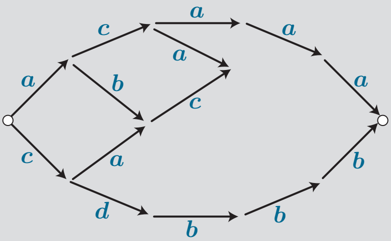

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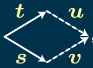
  - choose a relation $sv = tu$ of R to close this pattern into 
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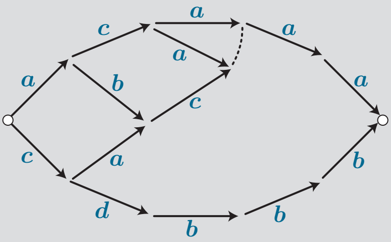

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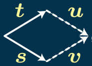
  - choose a relation $sv = tu$ of R to close this pattern into 

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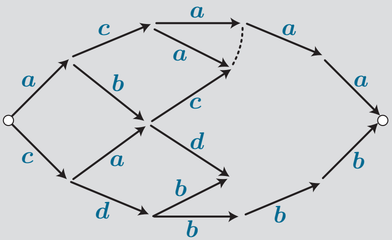

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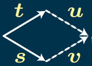
  - choose a relation $sv = tu$ of R to close this pattern into , and repeat.

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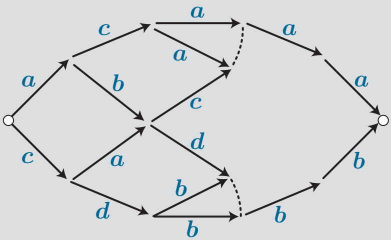

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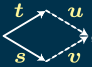
  - choose a relation $sv = tu$ of R to close this pattern into 

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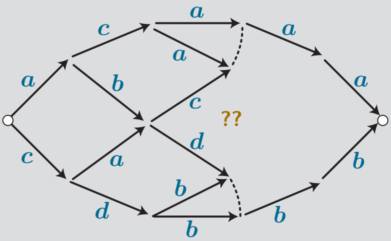

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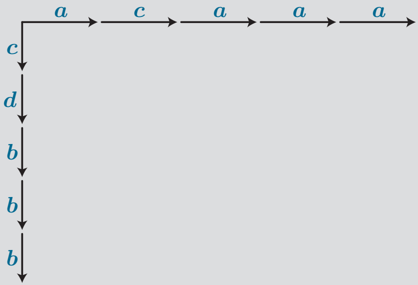

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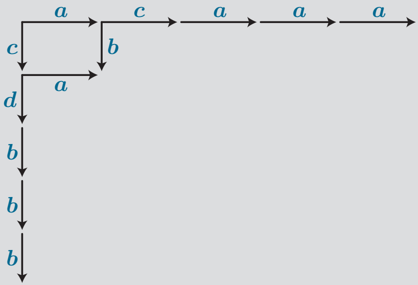

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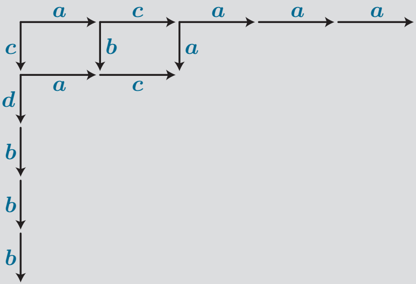

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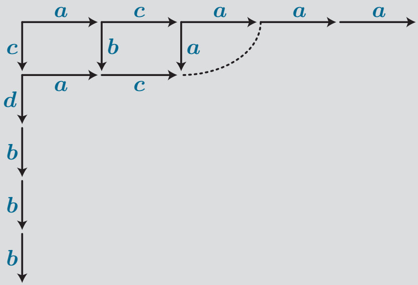

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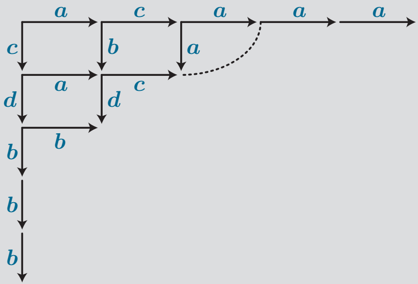

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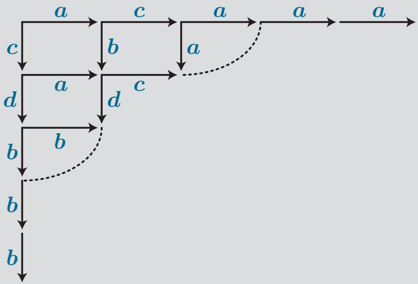

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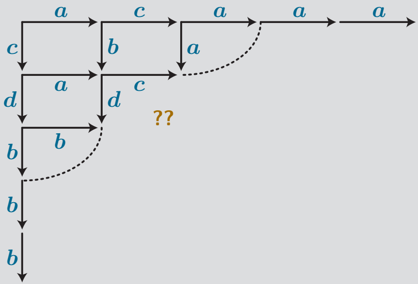

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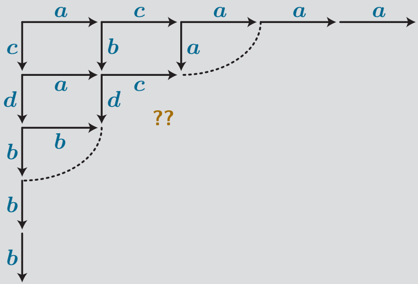

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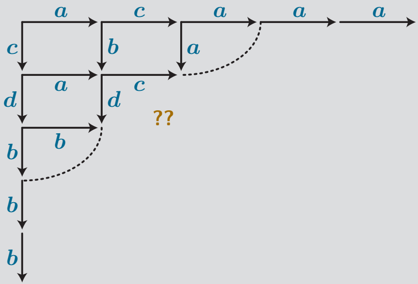

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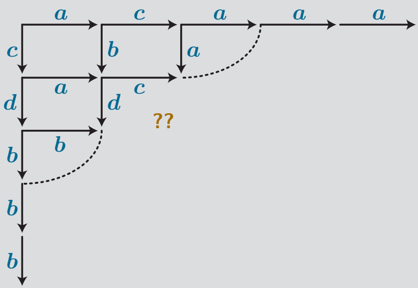

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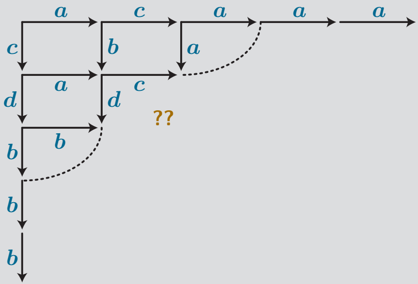

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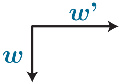

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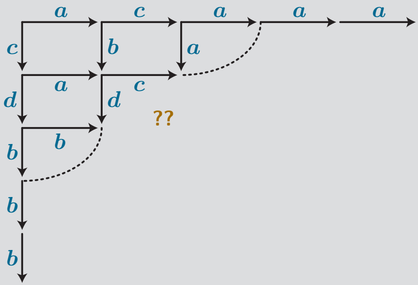

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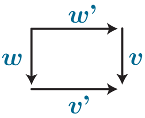

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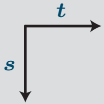

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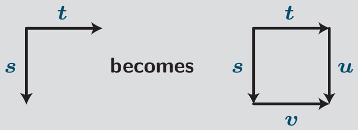

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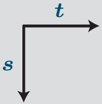

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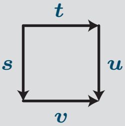

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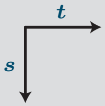

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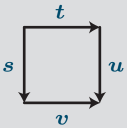

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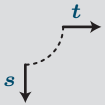

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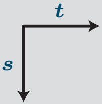

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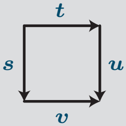

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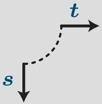

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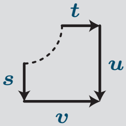

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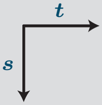

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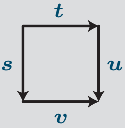

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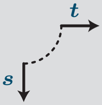

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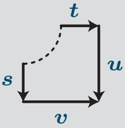

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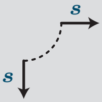

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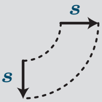

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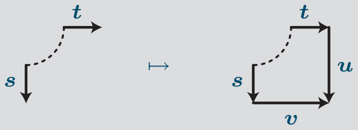

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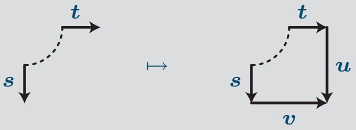

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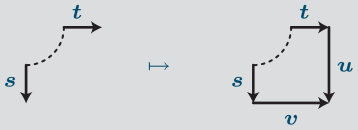

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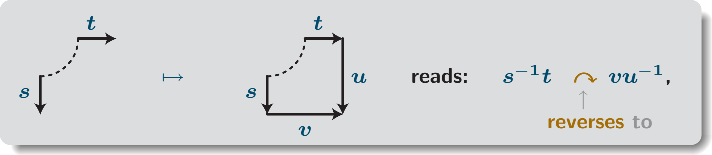

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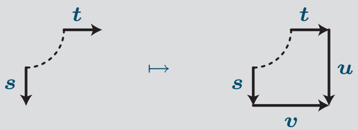

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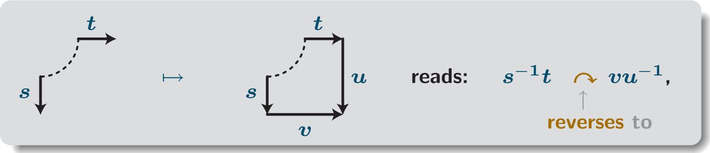

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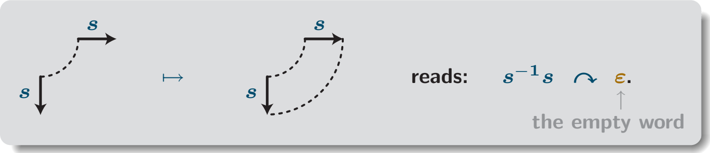

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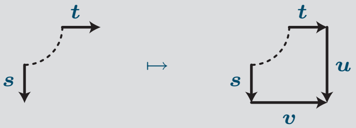

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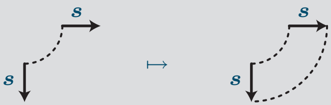

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↑
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- In this setting, "subword reversing" means replacing $-+$ with $+-$, whence the terminology.

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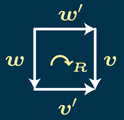
  then $w^{-1}w' \curvearrowright_R v'v^{-1}$, i.e.,  ,

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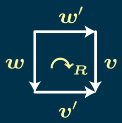

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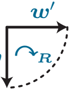
- In particular, $w^{-1}w' \curvearrowright_R \varepsilon$, i.e., w 

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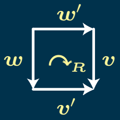
  then $w^{-1}w' \curvearrowright_R v'v^{-1}$, i.e.,  , implies $wv' \equiv_R^+ w'v$.

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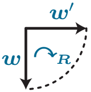
- In particular, $w^{-1}w' \curvearrowright_R \varepsilon$, i.e.,  , implies $w \equiv_R^+ w'$.

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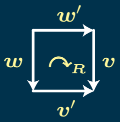
  then $w^{-1}w' \curvearrowright_R v'v^{-1}$, i.e.,  , implies $wv' \equiv^+_R w'v$.

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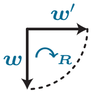
- In particular, $w^{-1}w' \curvearrowright_R \varepsilon$, i.e.,  , implies $w \equiv^+_R w'$.

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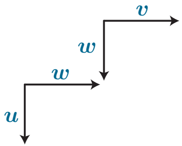

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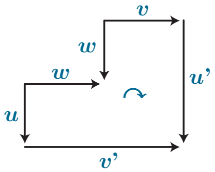

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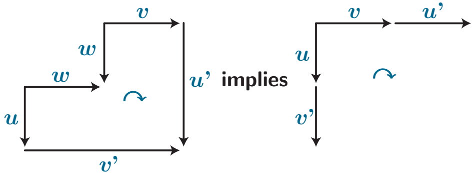

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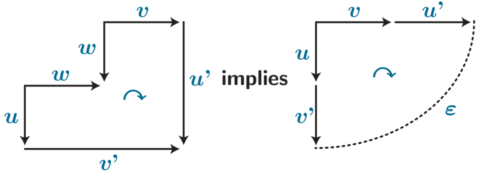

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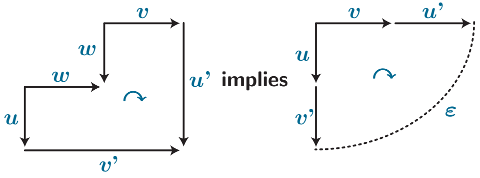

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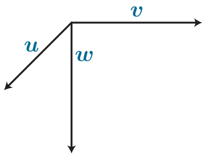

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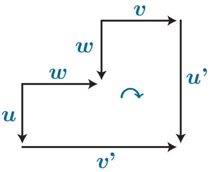

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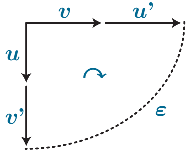

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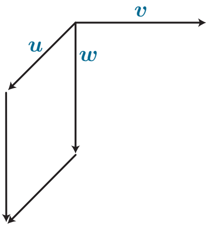

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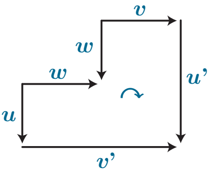

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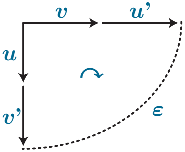

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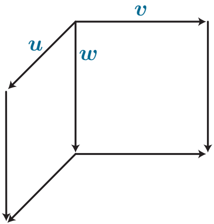

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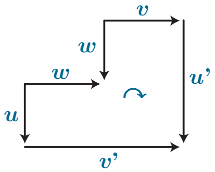

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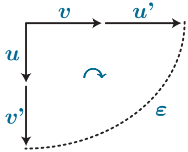

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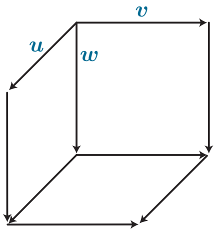

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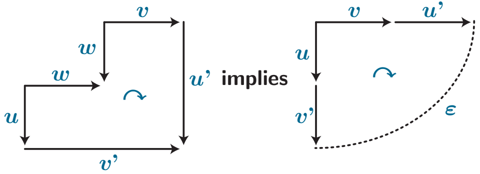

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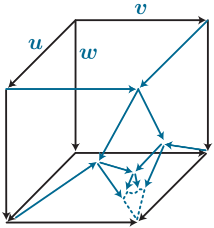

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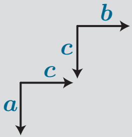

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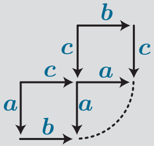

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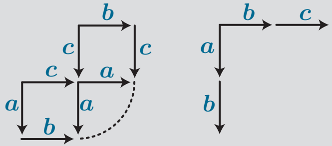

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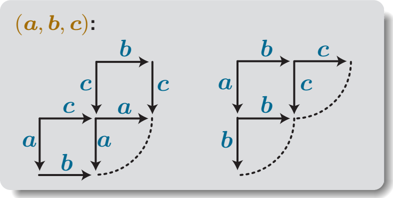

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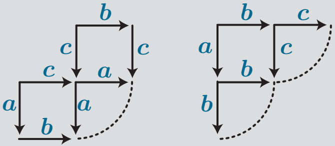
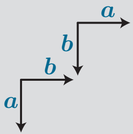


(a, a, b):



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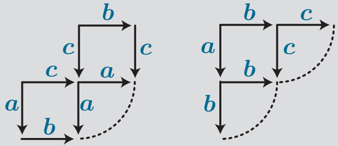
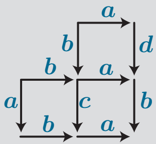


(a, a, b):



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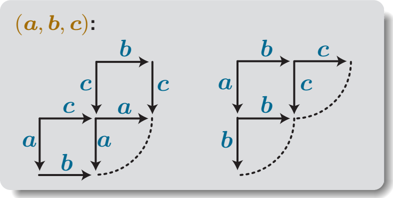

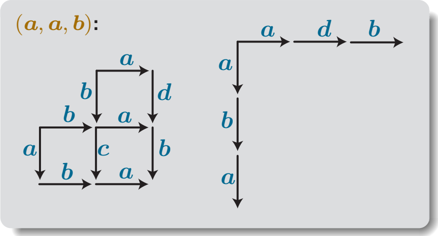

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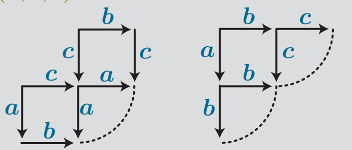

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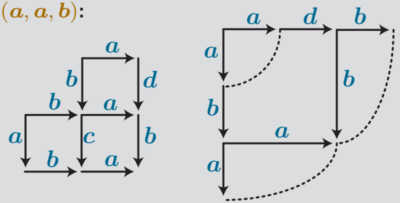

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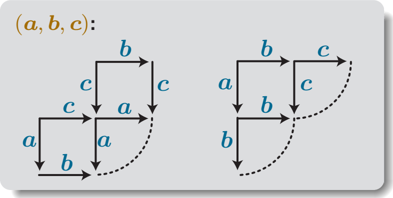

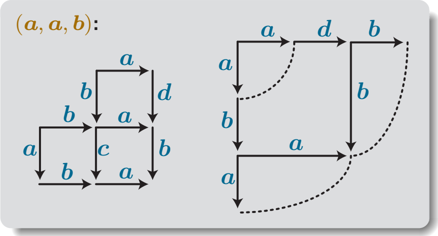

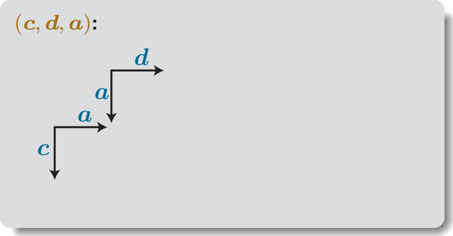

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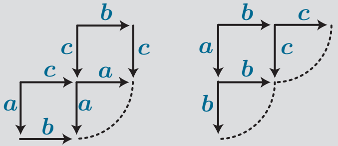
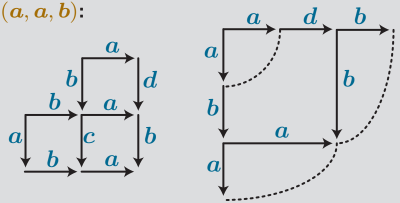
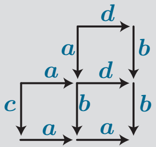


(a, a, b):



(c, d, a):



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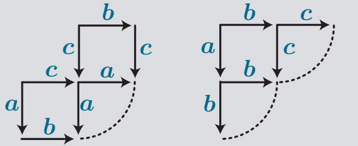
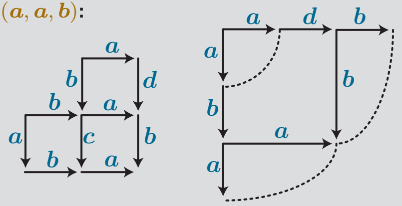
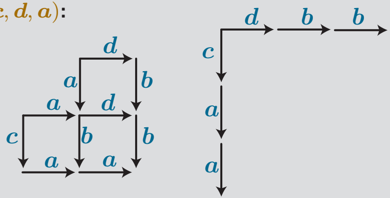


(a, a, b):



(c, d, a):



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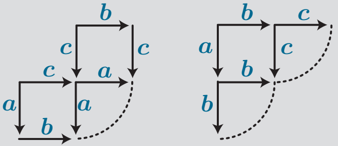
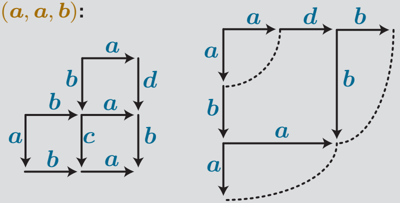


(a, a, b):



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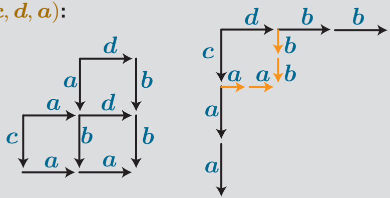

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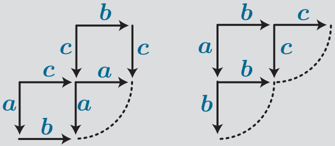

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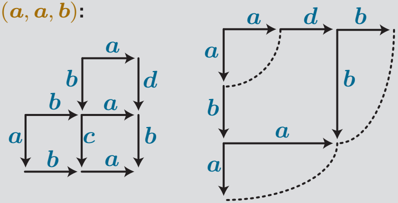
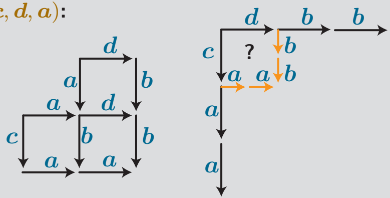


(c, d, a):



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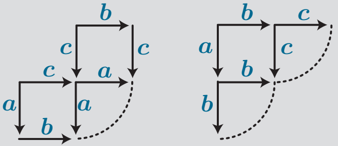

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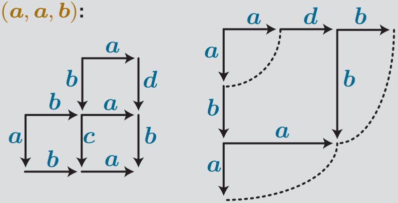
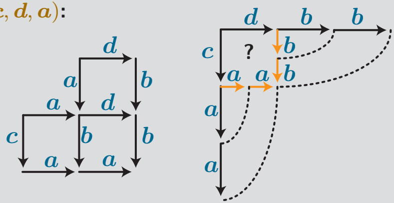


(c, d, a):



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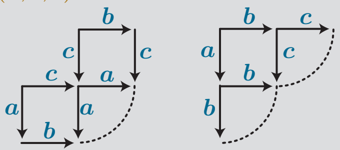
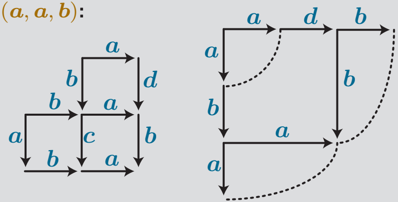
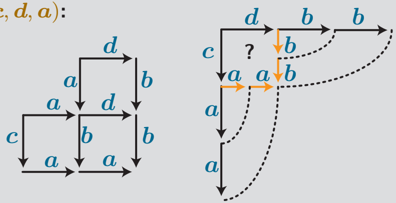


(a, a, b):

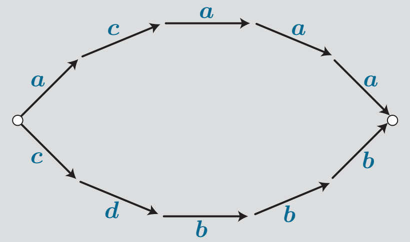


(c, d, a):



$\rightsquigarrow$ A completion procedure: if the cube fails, add the (redundant) missing relation.

here: add $caa = dbb$.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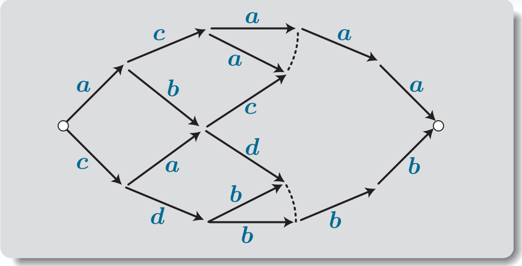
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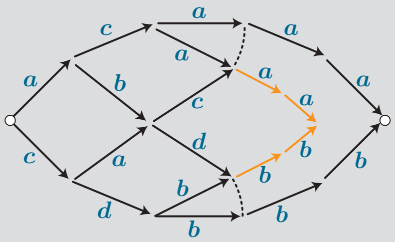
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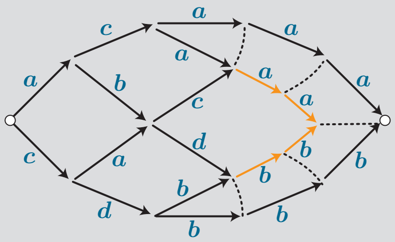
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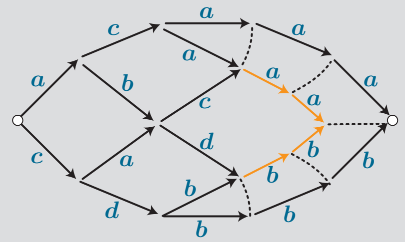
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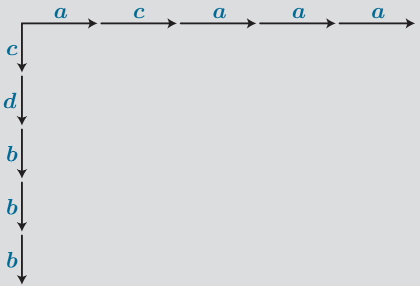
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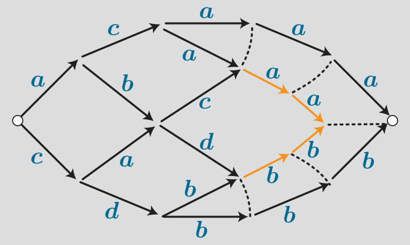
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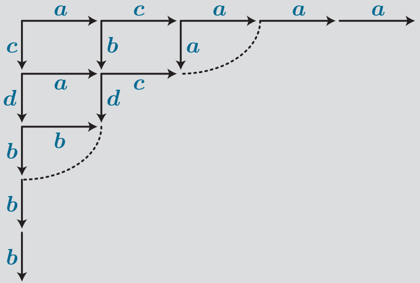
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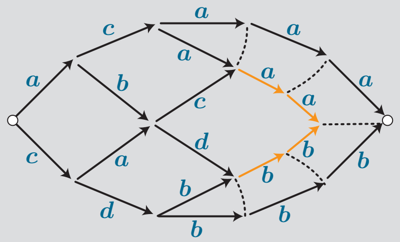
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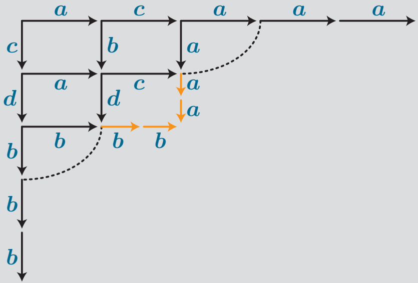
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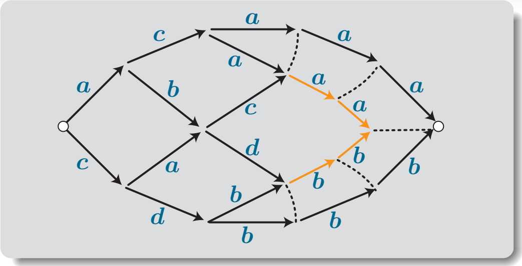
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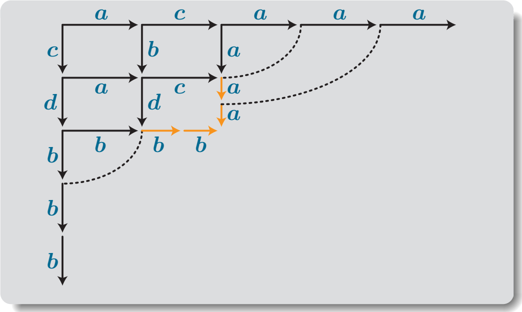
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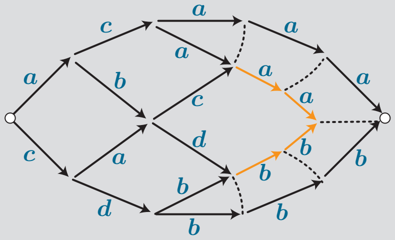
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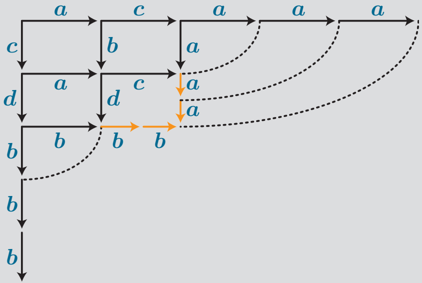
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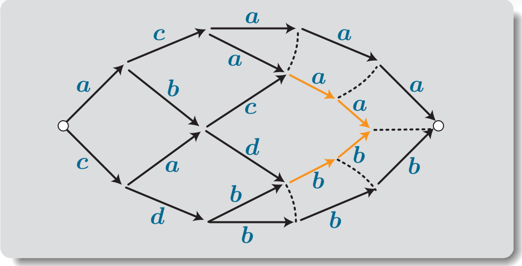
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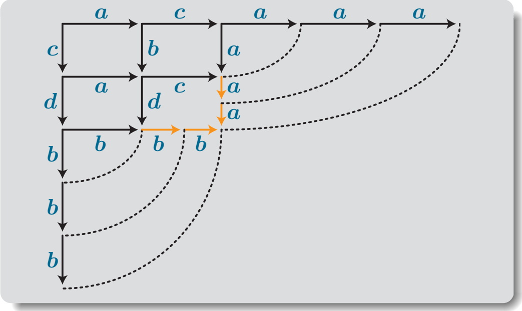
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- **Three possible cases:**

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- **A particular framework:**

- **Definition : A semigroup presentation (S, R) is called complemented if,**

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  (= only one reversing diagram for all initial words u, v)

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- For a complemented presentation, reversing is deterministic:
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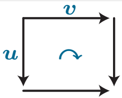

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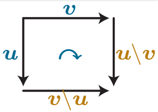

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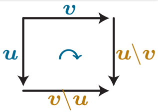

- **Proposition** : If (S, R) is complemented, the cube condition for u, v, w holds iff

$$(u\backslash v)\backslash(u\backslash w) \equiv_R^+ (v\backslash u)\backslash(v\backslash w).$$

(the cube law)

3. Subword Reversing : Uses

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- Cancellativity
- Word problems
- Recognizing Garside structures
- Computing in Garside structures

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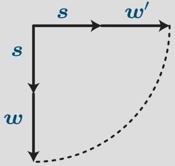

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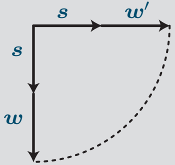

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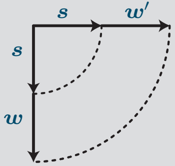

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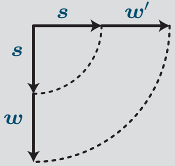

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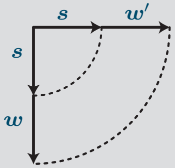

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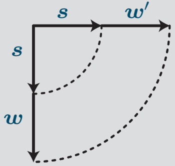

- **Example** : M is left-cancellative

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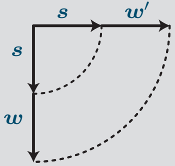

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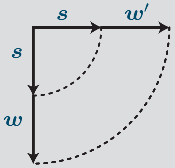

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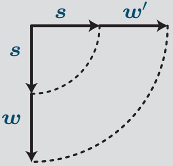

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- Remark: Applies in particular to every complete complemented presentation.

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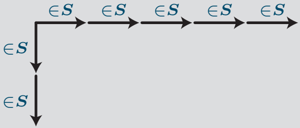

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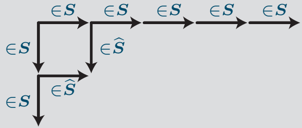

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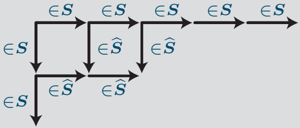

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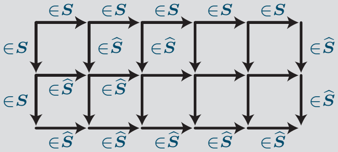

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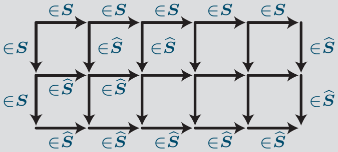

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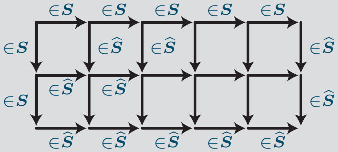

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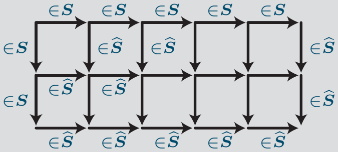

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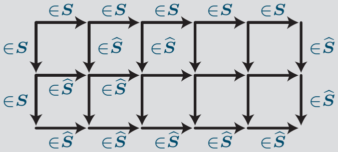

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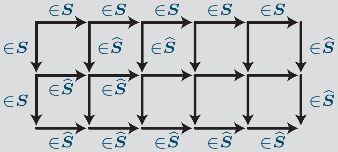

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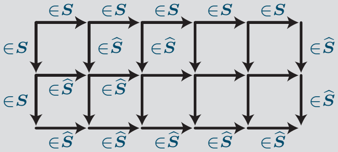

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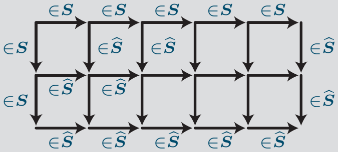

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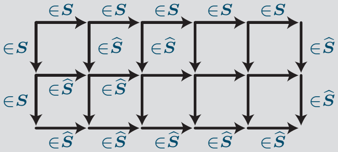

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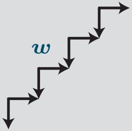

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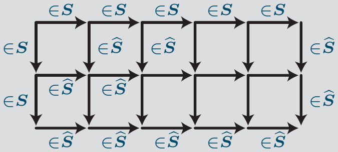

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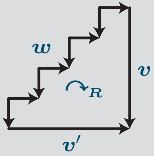

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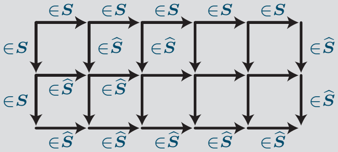

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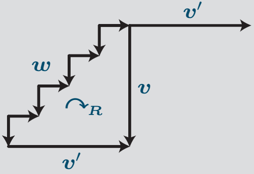

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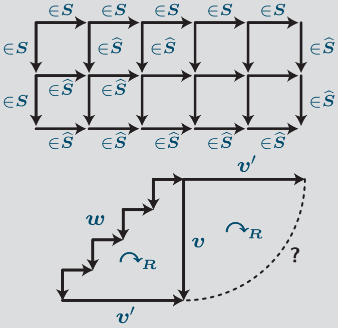

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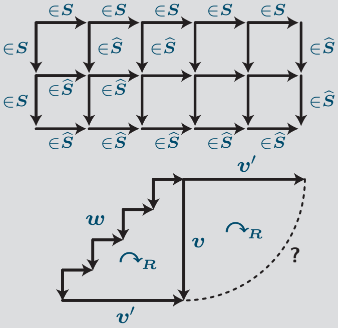

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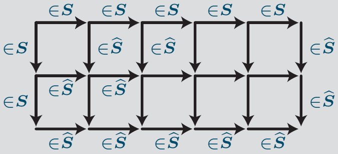

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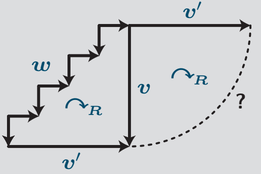

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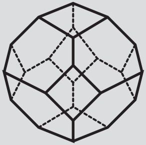
Δ_4



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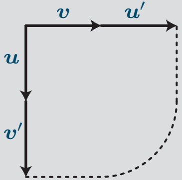

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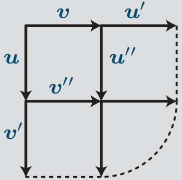

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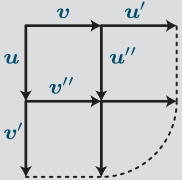

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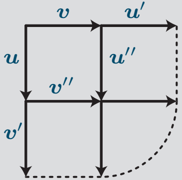

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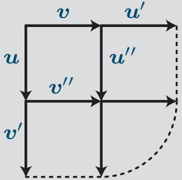

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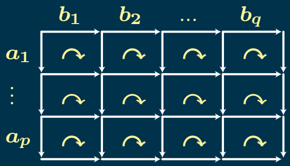

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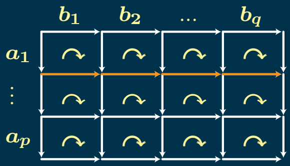

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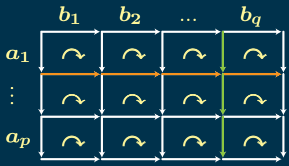

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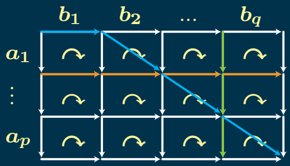

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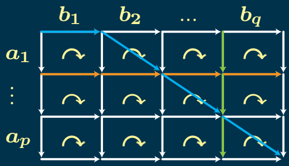

- Leads to the so-called grid property in Garside groups (≈ CAT(0) geometry).

4. Subword Reversing : Efficiency

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- **Proposition** : There exist positive constants C, C' s.t.
  - dist$(u, v) \leqslant Cn^4$ for all f in $\mathfrak{S}_n$ and all reduced expressions u, v of f,
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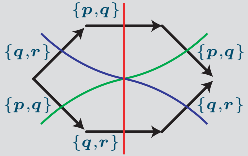

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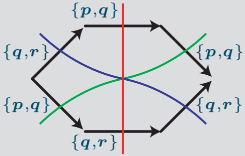

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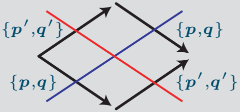

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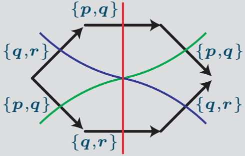

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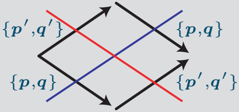

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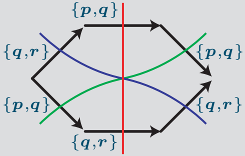

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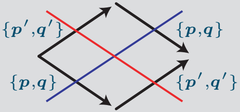

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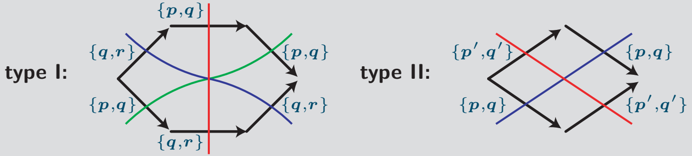

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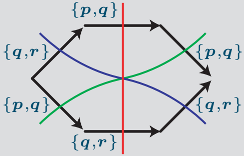

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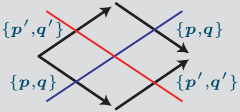

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- **Lemma** : A sufficient condition for a van Kampen diagram $\mathcal{D}$ to be optimal is that any two separatrices cross at most once in $\mathcal{D}$.

- **Example** : $w = \sigma_3\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3$, $w' = \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1$.

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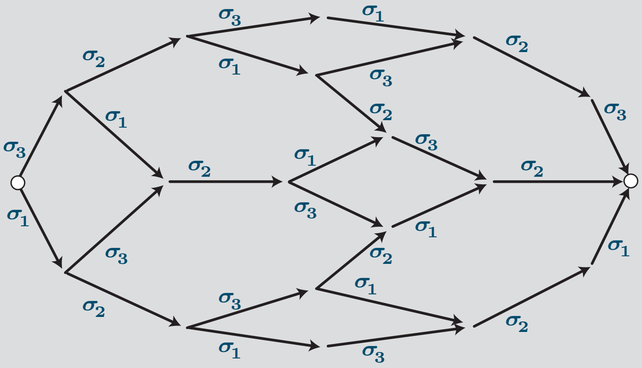

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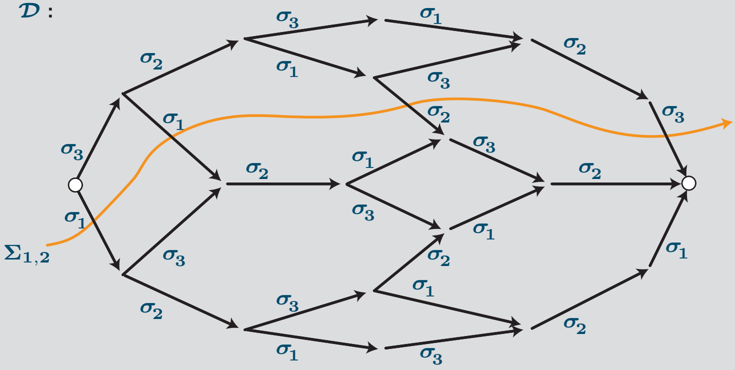

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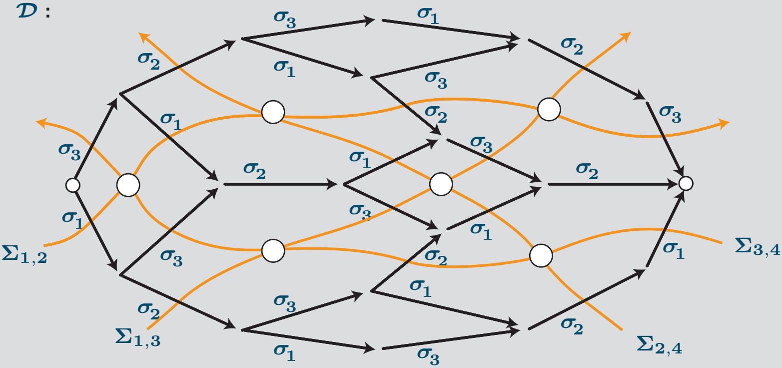

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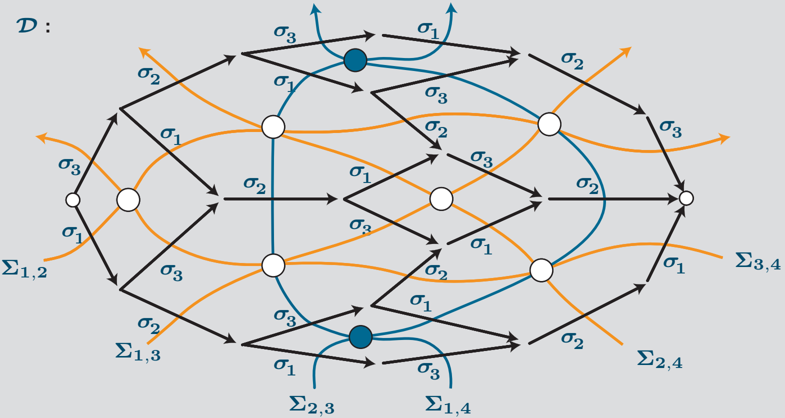

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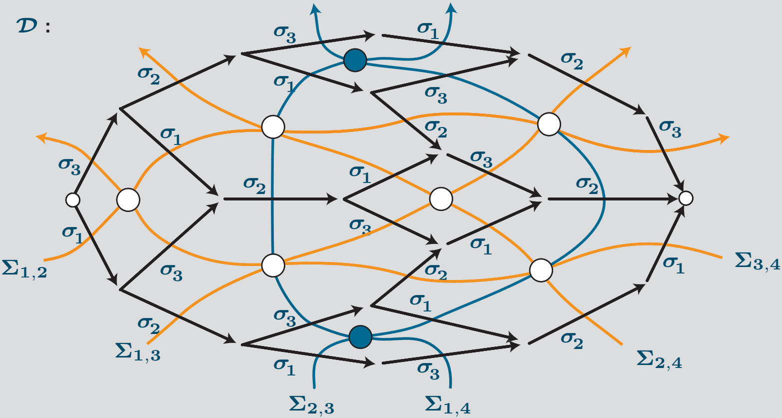

⇝ The separatrices $\Sigma_{2,3}$ and $\Sigma_{1,4}$ cross twice, hence $\mathcal{D}$ is not optimal.

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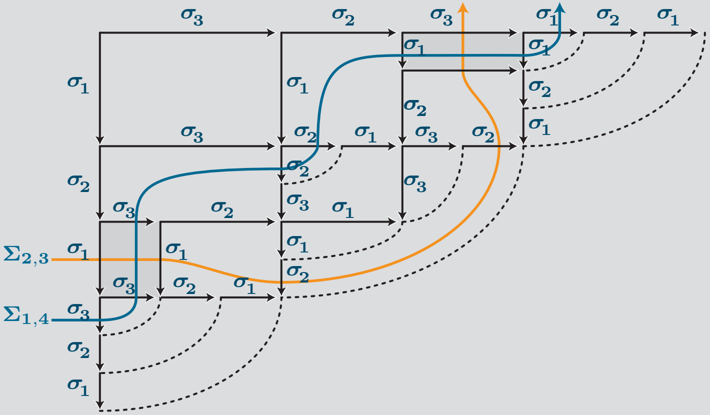

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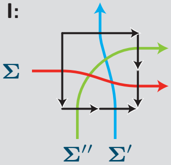

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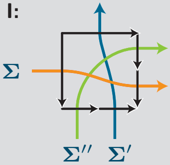

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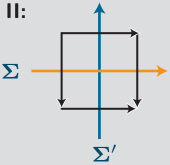

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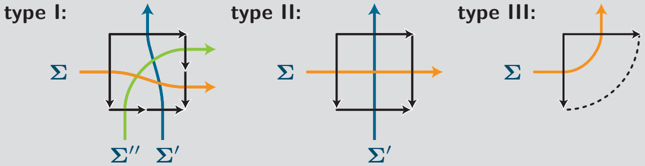

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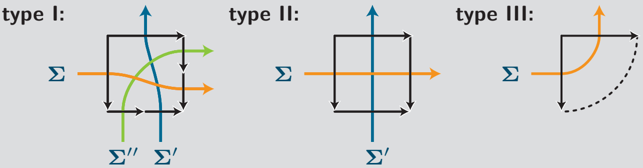

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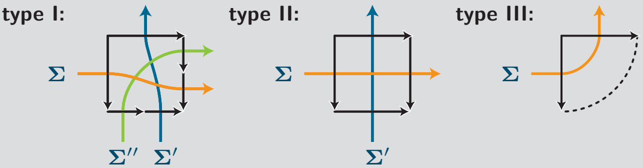

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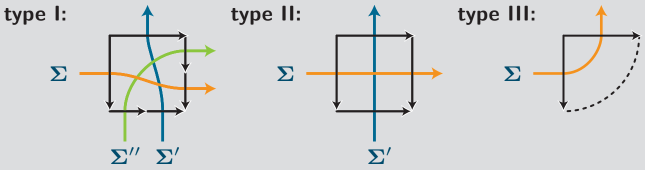

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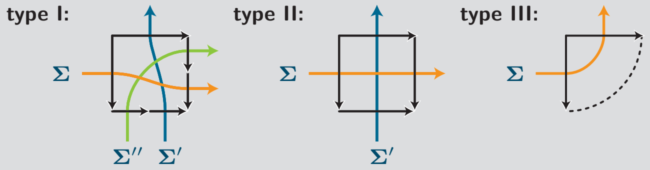

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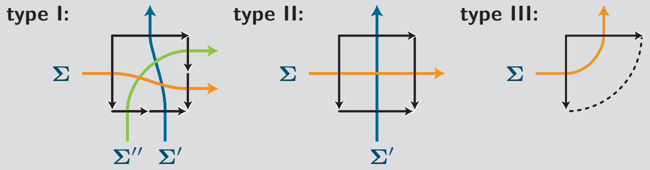

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