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Two half-talks:

1. A conjecture about Artin–Tits groups
2. News from Garside theory

1. A conjecture about Artin–Tits groups

- **A** (very vague) **claim.**— Some elementary facts about the word problem of Artin-Tits presentations might have not yet been discovered.

- $G = \langle S \mid R \rangle$ means

$$G = (S \underset{\uparrow}{\cup} S^{-1})^* / \underset{\uparrow}{\equiv}_R$$

the free monoid generated by S and a copy S^{-1} of S

the smallest congruence that includes R plus the free group relations $ss^{-1} = s^{-1}s = 1$

- Special case (positive presentation) : relations of the form $u = v$ with $u, v \in S^*$

- **Fact.**— Two words w, w' of $(S \cup S^{-1})^*$ represent the same element of $\langle S \mid R \rangle$ iff one can go from w to w' using transformations of
  - type 0 : Erasing some subword $s^{-1}s$ or ss^{-1} with s in S;
  - type 1 : Replacing some subword u by v with $u = v$ in R;
  - type ∞ : Inserting some subword $s^{-1}s$ or ss^{-1} with s in S.

Avoiding type ∞

- **Summary :** $w \equiv w'$ iff $w \overset{0,1,\infty}{\rightsquigarrow} w'$.

Question: Can one avoid type ∞ ?

- Stupid : $\equiv$ is symmetric, $\overset{0,1}{\rightsquigarrow}$ is not.

- **Special case :** $w \equiv \varepsilon$ iff $w \overset{0,1,\infty}{\rightsquigarrow} \varepsilon$ (ε = empty word, representing 1)

Question: Does $w \equiv \varepsilon$ imply $w \overset{0,1}{\rightsquigarrow} \varepsilon$?

- YES for a free group.
- The monoid $\langle S \mid R \rangle^{+}$ embeds in the group $\langle S \mid R \rangle$ iff YES for every word of the form $u^{-1}v$ with u, v in S^{*}.
- But NO in general: $\langle \mathbf{a}, \mathbf{b} \mid \mathbf{ab} = \mathbf{ba} \rangle$ and $w = \mathbf{aBAb}$ ($\mathtt{A} = \mathtt{a}^{-1}$, $\mathtt{B} = \mathtt{b}^{-1}$, ...)

$\Rightarrow$ Complete definition with :

\- **type 1:** Replacing u by v, or u^{-1} by v^{-1}, with $u = v$ in R.

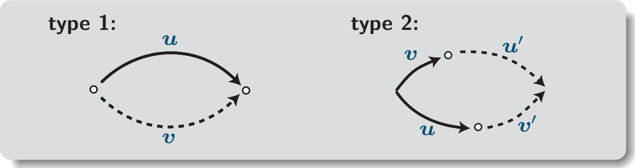
- Then $\mathbf{aBAb} \overset{1}{\rightsquigarrow} \mathbf{aABb} \overset{0}{\rightsquigarrow} \mathbf{Bb} \overset{0}{\rightsquigarrow} \varepsilon$

Type 2 transformations

- Still NO : $\langle \mathtt{a}, \mathtt{b}, \mathtt{c} \mid \mathtt{ab} = \mathtt{ba}, \mathtt{bc} = \mathtt{cb}, \mathtt{ac} = \mathtt{ca} \rangle$ and $w = \mathtt{aBcAbC}$.

$\Rightarrow$ Introduce

- **type 2**: Replacing $u^{-1}v$ by $v'u'^{-1}$ s.t. $u, v \neq \varepsilon$ and $uv' = vu'$ lies in R, or vice versa.

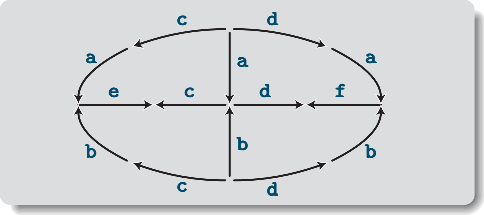


- Then : $\mathtt{aBcAbC} \overset{2}{\rightsquigarrow} \mathtt{aBAcbC} \overset{1}{\rightsquigarrow} \mathtt{aABcbC} \overset{0}{\rightsquigarrow} \mathtt{BcbC} \overset{1}{\rightsquigarrow} \mathtt{BbcC} \overset{0}{\rightsquigarrow} \mathtt{cC} \overset{0}{\rightsquigarrow} \varepsilon$.

Definition.— A positive presentation (S, R) satisfies $(\#)$ if $w \equiv \varepsilon$ implies $w \overset{0,1,2}{\rightsquigarrow} \varepsilon$.

Property $(\#)$

- **Fact.— Some presentations do not satisfy $(\#)$.**



Conjecture.— All Artin–Tits presentations satisfy $(\#)$.

↑

all relations are of the form $sts... = tst...$ with same length on both sides

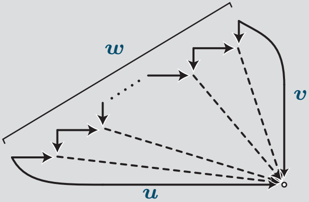
- **Possible interest of $(\#)$?**

a technical hypothesis satisfied by all Artin–Tits presentations

↓

Proposition.— Assume that (S, R) is complete with respect to right-reversing and satisfies $(\#)$. Then the monoid $\langle S \mid R\rangle^+$ embeds in the group $\langle S \mid R\rangle$.

- Principle of proof : Say that w is a bridge from u to v if there exists a commutative positive equivalence diagram



If w is a bridge from u to v and $w \overset{0,1,2}{\rightsquigarrow} w'$ holds, then w' is a bridge from u to v.

Now assume $u \equiv v$. Then uv^{-1} is a bridge from u to v.
Hence, if $(\#)$ is true, ε is also a bridge from u to v : this means $u \equiv^+ v$. ☐

Proposition.— Artin–Tits presentations of spherical type satisfy $(\#)$.

$\uparrow$

the associated Coxeter group is finite

- Principle of proof : $\langle S \mid R\rangle$ is a group of fractions of $\langle S \mid R\rangle^+$, and type 2 transformations compute lcm's :

$$w \overset{0^+,2^+}{\rightsquigarrow} uv^{-1} \overset{0^-,2^-}{\rightsquigarrow} v'^{-1}u',$$

with $u, v, u', v' \in S^*$ and $v'^{-1}u'$ shortest fractionary word equivalent to w. Then $w \equiv \varepsilon$ implies $u' = v' = \varepsilon$, hence $w \overset{0,2}{\rightsquigarrow} \varepsilon$. □

Special case 2

Proposition.— Right-angled Artin–Tits presentations satisfy $(\#)$.

all relations of the fom $st = ts$

- Principle of proof : Start with a derivation $w \overset{0,1,2,\infty}{\rightsquigarrow} \varepsilon$ and [project] it to another derivation $w \overset{0,1,2}{\rightsquigarrow} \varepsilon$ by following the pairs $s^{-1}s$ and ss^{-1} created in type ∞ steps.

Point : All such pairs become $s^e v s^{-e}$ with s commuting with $\pi(v)$, the word obtained by erasing all later pairs $t^d t^{-d}$ (hence induction). ☐

- Example :

$$\begin{tikzcd}
\mathtt{aBcAbC} \arrow[r, squiggly, "\infty"] \arrow[d, "\pi"] & \mathtt{aBAacAbC} \arrow[r, squiggly, "1"] \arrow[d, "\pi"] & \mathtt{aABacAbC} \arrow[r, squiggly, "0"] \arrow[d, "\pi"] & \mathtt{BacAbC} \arrow[r, squiggly, "1"] \arrow[d, "\pi"] & \mathtt{BcaAbC} \arrow[r, squiggly, "0"] \arrow[d, "\pi"] & \mathtt{BcbC} \arrow[r, squiggly] \arrow[d, "\pi"] & \dots \\
\mathtt{aBcAbC} \arrow[r, phantom, "="] & \mathtt{aBcAbC} \arrow[r, phantom, "="] & \mathtt{aBcAbC} \arrow[r, squiggly, "2"] & \mathtt{BacAbC} \arrow[r, squiggly, "1"] & \mathtt{BcaAbC} \arrow[r, squiggly, "0"] & \mathtt{BcbC} \arrow[r, squiggly] & \dots
\end{tikzcd}$$

- Connection with the word problem? NO (at least, not directly)

Proposition.— (D.–Wiest) For type A_n with $n \geqslant 3$ (braids with at least 4 strands) there exist words w such that $\{w' \mid w \overset{0,1,2}{\rightsquigarrow} w'\}$ is infinite.

So $w \overset{0,1,2}{\rightsquigarrow} \varepsilon$ need not be decidable.

- On the other hand, there may exist strategies for $\overset{0,1,2}{\rightsquigarrow}$:
  Handle reduction (type A) is such a strategy, for which termination is provable.

Is the existence of such a strategy really specific to type A ?

2. News from Garside theory

(ongoing work with F.Digne, D.Krammer, and J.Michel)

Definition.— Assume that $\mathcal{C}$ is a left-cancellative category. A subfamily $\mathcal{S}$ of $\mathcal{C}$ $(= \mathcal{H}om(\mathcal{C}))$ is said to be a **Garside family** in $\mathcal{C}$ if $1_{\mathcal{C}}$ is included in $\mathcal{S}$, and

(i) $\mathcal{S} \cup \mathcal{C}^{\times}$ generates $\mathcal{C}$,
(ii) $\mathcal{S}\mathcal{C}^{\times}$ is closed under right-divisor,
(iii) each element of $\mathcal{C}$ admits a maximum left-divisor lying in $\mathcal{S}$.

↑ $\mathcal{C}^{\times}$:= invertible elements of $\mathcal{C}$

↑ $\forall g\ \exists g_1{\in}\mathcal{S}\ \forall h{\in}\mathcal{S}\ (h \preccurlyeq g \Leftrightarrow h \preccurlyeq g_1)$

Proposition.— Assume that $\mathcal{C}$ is a left-cancellative category and $\mathcal{S}$ is a Garside family in $\mathcal{C}$. Then every element of $\mathcal{C}$ admits an $\mathcal{S}$-normal decomposition, which is unique up to $\mathcal{C}^{\times}$-deformation.

An $\mathcal{S}$-normal decomposition : $(g_1, ..., g_\ell)$ s.t.
$g_1, ..., g_{\ell-1}$ lie in $\mathcal{S}$, g_ℓ lies in $\mathcal{S}\mathcal{C}^{\times}$, and (g_i, g_{i+1}) is $\mathcal{S}$-greedy for each i.

↑ $\forall h \in \mathcal{S}\ \forall f\ (h \preccurlyeq f g_i g_{i+1} \Rightarrow h \preccurlyeq f g_i)$

A $\mathcal{C}^{\times}$-deformation : left- and right-multiplication, by invertible elements

Garside map

- For $\Delta : \mathcal{O}bj(\mathcal{C}) \to \mathcal{C}$, put

$$\mathcal{D}iv(\Delta) = \{g \mid \exists x\ (g \underset{\substack{\uparrow \\ \text{left-divides}}}{\preccurlyeq} \Delta(x))\}, \qquad \widetilde{\mathcal{D}iv}(\Delta) = \{g \mid \exists y\ (\Delta(y) \underset{\substack{\uparrow \\ \text{right-divides}}}{\widetilde{\succcurlyeq}} g)\}.$$

Definition.— Assume that $\mathcal{C}$ is a left-cancellative category. A map $\Delta : \mathcal{O}bj(\mathcal{C}) \to \mathcal{C}$ is said to be a **Garside map** in $\mathcal{C}$ if

(i) $x = \mathrm{source}(\Delta(x))$ for each object x,

(ii) $\mathcal{D}iv(\Delta)$ generates $\mathcal{C}$,

(iii) $\widetilde{\mathcal{D}iv}(\Delta) = \mathcal{D}iv(\Delta)$,

(iv) g and $\Delta(\mathrm{source}(g))$ admit a left-gcd for each element g of $\mathcal{C}$.

Proposition.— Assume that $\mathcal{C}$ is a left-cancellative category.

(i) If Δ is a Garside map in $\mathcal{C}$, then $\mathcal{D}iv(\Delta)$ is a Garside family that is closed under left-divisor and is **bounded** by Δ.

(ii) Conversely, if $\mathcal{S}$ is a Garside family in $\mathcal{C}$ that is closed under left-divisor and is **bounded** by a map Δ, then Δ (nearly) is a Garside map in $\mathcal{C}$.

$\mathcal{S}$ bounded by Δ : $\forall g \in \mathcal{S}\ (g \preccurlyeq \Delta(\mathrm{source}(g)))$

- If (M, Δ) is a Garside monoid, then Δ is a Garside (map) element in M, and $\mathcal{D}iv(\Delta)$ is a Garside family in M.

Definition.— A left-cancellative category $\mathcal{C}$ is called left-Noetherian if proper right-divisibility in $\mathcal{C}$ has no infinite descending sequence.

Proposition.— Assume that $\mathcal{C}$ is a left-cancellative category that is left-Noetherian. For $\mathcal{S}$ included in $\mathcal{C}$ such that $\mathcal{S} \cup \mathcal{C}^\times$ generates $\mathcal{C}$, TFAE
(i) $\mathcal{S}$ is Garside in $\mathcal{C}$,
(ii) $\mathcal{S}\mathcal{C}^\times$ is closed under right-divisor and $\mathcal{S}$ is closed under right-comultiple.

$\mathcal{S}$ closed under right-comultiple : for all f, g in $\mathcal{S}$, every right-comultiple of f and g is a right-multiple of some right-comultiple that lies in $\mathcal{S}$.

- If the ambient category admits right-lcm's, then (ii) is equivalent to $\mathcal{S}\mathcal{C}^\times$ being closed under right-divisor and right-lcm
(whence the existence of a smallest Garside family including a given family).

- Reference : `http://www.math.unicaen.fr/~DDKM/DDKM.pdf`