News from Garside theory

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- Extending Garside's algebraic approach to braid groups to other structures, an (old) ongoing program in two steps.
- A text in progress, joint with F.Digne (Amiens), E.Godelle (Caen), D.Krammer (Warwick), J.Michel (Paris) www.math.unicaen/∼dehornoy/Books/Garside.pdf.

• **Theorem:** (F.A. Garside, PhD, 1967) Artin's n-strand braid group B_n is a group of fractions for the monoid

$$B_n^+ = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle^+ .$$

• **Application.**— Solution to the Conjugacy Problem of B_n (not mentioned in MR!)

• Two main ingredients:

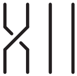
- The monoid B_n^+ is cancellative (argument credited to G.Higman);

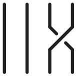
  $fg' = fg \Rightarrow g' = g$ and $g'f = gf \Rightarrow g' = g$

- Any two elements of B_n^+ admit a common right-multiple, and even a right-lcm.

  $f \preccurlyeq g$ iff $\exists g' \, (fg' = g)$

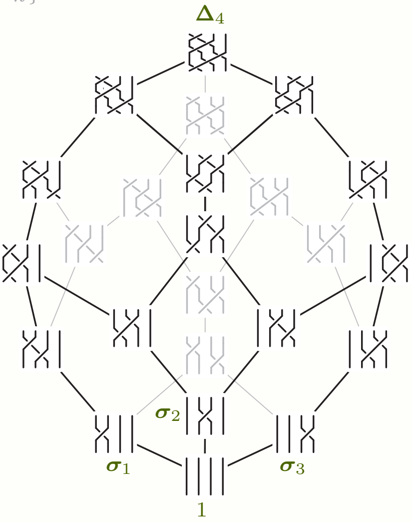
• Key fact: Δ_n is a (least) common right-multiple of $\sigma_1, \ldots, \sigma_{n-1}$ in B_n^+.





- Precisely: $(\mathbf{Div}(\Delta_n), \preccurlyeq)$ is a lattice with $n!$ elements, the simple n-strand braids $\cong (\mathfrak{S}_n,$ weak order$)$.

  $\{g \in B_n^+ \mid g \preccurlyeq \Delta_n\}$



- (Deligne, 1971, Brieskorn–Saito, 1971)
  Similar results for all Artin–Tits groups of spherical type.
  ↑
  s.t. the associated Coxeter group is finite

- Theorem: (Adjan,1984, Thurston, ∼1988, El Rifai–Morton, ∼1988)
Every braid of B_n admits a unique decomposition ("greedy normal form")

$$\Delta_n^d \, g_1 \dots g_\ell$$

with $d \in \mathbb{Z}$ and $g_1, \dots, g_\ell$ in $\mathrm{Div}(\Delta_n)$ s.t. $g_1 \neq \Delta_n$, $g_\ell \neq 1$, and, for every k,

$$\forall h \preccurlyeq \Delta_n \ (h \preccurlyeq g_k g_{k+1} \ \Rightarrow h \preccurlyeq g_k).$$

- Corollary (Epstein & *al.*, Chapter IX).— Braid groups are biautomatic.

- (Charney, 1992) All Artin–Tits groups of spherical type are biautomatic.

- Theorem (D., 1991).— The monoid of Self-Distributivity M_{LD} is cancellative and admits common right-multiples.

- Key ingredients:
  - Word reversing ($\approx$ Garside's Theorem H),
  - For every term t, there exists a maximal simple LD-expansion of t ($\approx$ there exists a maximal simple n-strand braid).

- Application 1.— The Word Problem of the Self-Distributivity Law is decidable without any set-theoretical assumption.

... and, because M_{LD} projects onto B_∞^+,

- Application 2.— Braid groups are orderable.

- Revelation (L.Paris, 1997).— "*Mmhh... ça devrait marcher pour d'autres groupes...*"

$\rightsquigarrow$ list the needed properties of B_n^+ (and M_{LD})

$\rightsquigarrow$ the notion of a Garside monoid.

- Several definitions proposed, then

- Definition (Berder, 2001).— A Garside monoid is a pair (M, Δ) such that
  - M is a cancellative monoid,
  - there exists $\lambda : M \to \mathbb{N}$ s.t. $g \neq 1 \Rightarrow \lambda(g) \geqslant 1$ and $\lambda(fg) \geqslant \lambda(f) + \lambda(g)$,
  - M admits lcm's and gcd's (least common multiples, greatest common divisors),
  - Δ is a Garside element in M, this meaning:
    - the left and right divisors of Δ coincide,
    - they generate M,
    - they are finite in number.

- Definition: A quasi-Garside monoid is a pair (M, Δ) such that ...
  ... *(the same except)* : no restriction on # of divisors of Δ.

- Definition: A (quasi)-Garside group is a group that can be expressed
  (in at least one way) as a group of fractions of a (quasi)-Garside monoid.

- Example 1 (Garside, 1967).— (B_n^+, Δ_n) is a Garside monoid.

- Example 2 (Birman, Ko, Lee, 1998).— (B_n^{+*}, Δ_n^*) is a Garside monoid.

dual braid monoid
assocciated with band generators

$\sigma_1\sigma_2 \dots \sigma_{n-1}$

- Example 3.— Torus knot groups are Garside groups.

$\langle \mathrm{a}, \mathrm{b} \mid \mathrm{a}^p = \mathrm{b}^q \rangle$

- Example 4 (Bessis, 2006).— Free groups are quasi-Garside groups.

- Example 5 (Digne, 2010).— Artin–Tits groups of type $\widetilde{A}_n$ and $\widetilde{C}_n$ are quasi-Garside groups.

• Proposition: Assume (M, Δ) is a Garside monoid. Then every element of M admits a unique decomposition

$$\Delta^d g_1 \dots g_\ell$$

with $p \in \mathbb{Z}$ and $g_1, \dots, g_\ell$ in $\mathbf{Div}(\Delta)$ s.t. $g_1 \neq \Delta$, $g_\ell \neq 1$, and, for every k,

$$\forall h \preccurlyeq \Delta \ (h \preccurlyeq g_k g_{k+1} \ \Rightarrow h \preccurlyeq g_k).$$

• Corollary.— Garside groups are biautomatic.

Principle: Everything works, but requires new (better) proofs.

Is that the end of the story?

… No, because similar results (e.g. existence of greedy decompositions) hold in structures that are not Garside monoids.

- Example 1.— M_{LD} is not a Garside monoid: not finitely generated, not (known to be) right-cancellative, no global Δ.

- Example 2.— B_∞^+ is not a Garside monoid: not finitely generated.

- Example 3.— $\widetilde{\mathbb{N}}^n := \mathbb{N}^n \ltimes \mathfrak{S}_n$ is not a Garside monoid: nontrivial invertible elements.

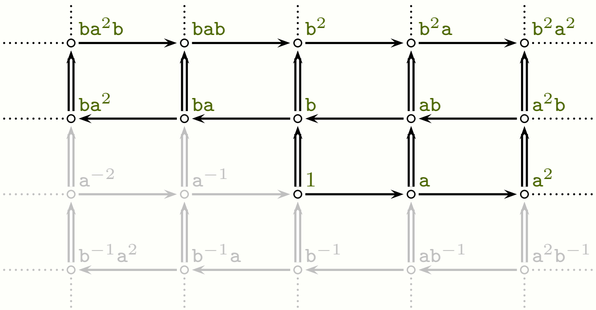
$\uparrow$

$\widetilde{\mathbb{N}}^2 = \langle \mathrm{a}, \mathrm{b}, \mathrm{s} \mid \mathrm{ab} = \mathrm{ba}, \mathrm{s}^2 = 1, \mathrm{sa} = \mathrm{bs} \rangle^+$

- Example 4.— $K^+ = \langle \mathtt{a}, \mathtt{b} \mid \mathtt{b} = \mathtt{aba} \rangle^+$ is not a Garside monoid: no λ function.

↑ the Klein bottle monoid

- An interesting example: K^+ defines a linear ordering on its group of fractions:
  ⇝ the left-divisibility lattice is a chain

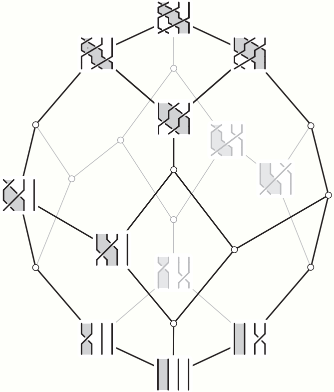


• Example 5.— The category $\mathcal{BR}_n$ is not a Garside monoid: not a monoid.

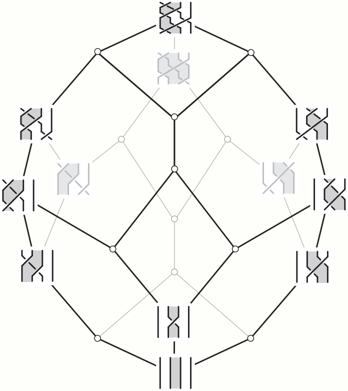
↑

the n-strand braid ribbon category:

- objets: integers from 1 to $n-1$;
- elements (morphisms): $\mathcal{BR}_n(i,j)$:= all n-strand braids that admit an (i,j)-ribbon.



the lattice $\mathbf{Div}(\Delta_4)$ in $\mathcal{BR}_4(1,-)$



the lattice $\mathbf{Div}(\Delta_4)$ in $\mathcal{BR}_4(2,-)$

Can one extend the notion of a Garside monoid
so as to cover the previous examples (and more) ? ... Yes.

- The category framework:
  - a category = two families $\mathcal{C}$ and $\mathcal{O}bj(\mathcal{C})$,
    plus two maps, source and target, of $\mathcal{C}$ to $\mathcal{O}bj(\mathcal{C})$,
    plus a partial associative product: fg exists iff target(f) = source(g).
  - $\mathcal{C}^{\times}$:= all invertible elements of $\mathcal{C}$ $(=1_{\mathcal{C}}$ if no nontrivial invertible elements),
  - For $\mathcal{S} \subseteq \mathcal{C}$, $\mathcal{S}^{\sharp} := \mathcal{S}\mathcal{C}^{\times} \cup \mathcal{C}^{\times}$ = closure of $\mathcal{S}$ under right-multiplication
    by invertible elements $(= \mathcal{S} \cup 1_{\mathcal{C}}$ if no nontrivial invertible elements).

• Definition.— Assume $\mathcal{C}$ is a left-cancellative category. A Garside family in $\mathcal{C}$ is a subfamily $\mathcal{S}$ of $\mathcal{C}$ s.t.
- $\mathcal{S} \cup \mathcal{C}^\times$ generates $\mathcal{C}$,
- $\mathcal{S}^\sharp$ is closed under right-divisor (every right-divisor of an el't of $\mathcal{S}^\sharp$ belongs to $\mathcal{S}^\sharp$),
- every element of $\mathcal{C}$ has a maximum left-divisor lying in $\mathcal{S}$.

↑

$\forall g \in \mathcal{C}\ \exists g_1 \in \mathcal{S}\ \forall h \in \mathcal{S}\ (h \preccurlyeq g \Leftrightarrow h \preccurlyeq g_1)$

• Example 1.— If (M, Δ) is a Garside monoid, $\mathbf{Div}(\Delta)$ is a Garside family in M.

• Proof: - $\mathbf{Div}(\Delta)$ generates M: hypothesis;
- $\mathbf{Div}(\Delta)$ closed under right-divisor: because $\mathbf{Div}(\Delta) = \widetilde{\mathbf{Div}}(\Delta)$;
- Every element g has a maximum left-divisor in $\mathbf{Div}(\Delta)$: left-gcd(g, Δ). $\square$

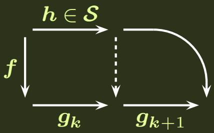
• Example 2.— If $\mathcal{C}$ is any left-cancellative category, $\mathcal{C}$ is a Garside family in $\mathcal{C}$.

• Example 3.— $\mathbf{Div}(\mathtt{b}^2)$ is a Garside family in K^+.

• **Definition.**— Assume $\mathcal{C}$ is a left-cancellative category, $\mathcal{S} \subseteq \mathcal{C}$, and g in $\mathcal{C}$.
An $\mathcal{S}$-normal decomposition of g is a sequence $(g_1, \dots, g_\ell)$ s.t.

- $g = g_1 \dots g_\ell$ holds,
- $g_1, \dots, g_\ell$ lie in $\mathcal{S}^\sharp$,
- for every k, the pair (g_k, g_{k+1}) is $\mathcal{S}$-greedy:

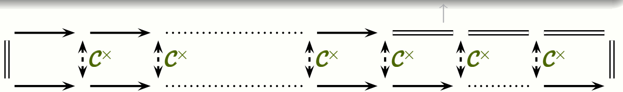
$$\forall h \in \mathcal{S}\ \forall f \in \mathcal{C}\ (h \preccurlyeq f g_k g_{k+1} \Rightarrow h \preccurlyeq f g_k)$$



$\rightsquigarrow$ the standard definition, except for the additional f.

• **Proposition.**— Assume $\mathcal{C}$ is a left-cancellative category and $\mathcal{S}$ is included in $\mathcal{C}$. TFAE:
- $\mathcal{S}$ is a Garside family in $\mathcal{C}$;
- Every element of $\mathcal{C}$ admits an $\mathcal{S}$-normal decomposition.

When it exists, an $\mathcal{S}$-normal decomposition is unique up to $\mathcal{C}^\times$-deformation.

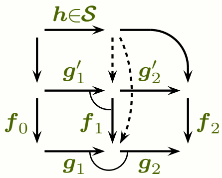


- **Lemma** (first domino rule).— Assume $\mathcal{C}$ is a left-cancellative category, and

$$\begin{array}{ccccc} \cdot & \xrightarrow{g'_1} & \cdot & \xrightarrow{g'_2} & \cdot \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ \cdot & \xrightarrow[g_1]{} & \cdot & \xrightarrow[g_2]{} & \cdot \end{array}$$

is commutative. If (g_1, g_2) and (g'_1, f_1) are $\mathcal{S}$-greedy, then so is (g'_1, g'_2).

- Proof.—



□

... *and everything goes smoothly*...

• If $\mathcal{S}$ is a Garside family, the current notion of $\mathcal{S}$-greediness is the classic one.

• There exist many equivalent definitions of Garside families (head functions, closure properties, etc.), including intrinsic definitions (germs of Digne–Michel).

↑ referring to no pre-existing category

• There exist specific (more simple) definitions of Garside families when the ambient category satisfies additional hypotheses.

no infinite descending sequence for right-divisibility ↓

• Proposition.— Assume $\mathcal{C}$ is a left-Noetherian left-cancellative category that admits local right-lcm's, and $\mathcal{S} \subseteq \mathcal{C}$. Then $\mathcal{S}$ is a Garside family in $\mathcal{C}$ iff $\mathcal{S} \cup \mathcal{C}^{\times}$ generates $\mathcal{C}$, and $\mathcal{S}^{\sharp}$ is closed under right-divisor and right-lcm.

↑ any two elements with a common right-multiple have a right-lcm

• Corollary.— (...) every generating family is included in a smallest Garside family.

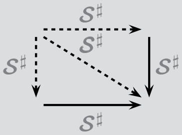
• **Proposition.**— Assume (M, Δ) is a Garside monoid and G is the group of fractions of M. Every element of G admits a unique decomposition $(f_p^{-1}, \dots, f_1^{-1}, g_1, \dots, g_q)$ with $(f_1, \dots, f_p)$, $(g_1, \dots, g_q)$ normal and left-gcd$(f_1, g_1) = 1$.

cancellative + any two elements have a common left-multiple
↓

• **Definition.**— Assume $\mathcal{C}$ is a left-Ore category. A Garside family $\mathcal{S}$ of $\mathcal{C}$ is called strong (*resp.* full) if

$$\forall f, g \in \mathcal{S}^\sharp \ \exists f', g' \in \mathcal{S}^\sharp \ (f'g = g'f = \text{left-lcm}(f, g))$$

$$(\textit{resp. } \forall f, g \in \mathcal{S}^\sharp \ \exists f', g' \in \mathcal{S}^\sharp \ (f'g = g'f = \text{left-lcm}(f, g) \in \mathcal{S}^\sharp)).$$



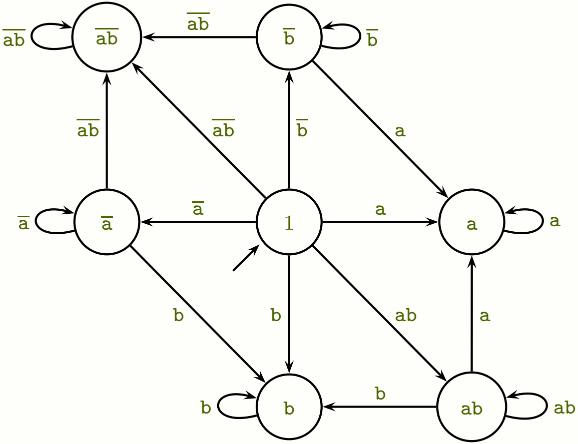
• **Proposition.**— Assume $\mathcal{C}$ is a left-Ore category and $\mathcal{S} \subseteq \mathcal{C}$. TFAE:
- $\mathcal{S}$ is a strong Garside family in $\mathcal{C}$;
- Every element of $\mathcal{C}$ admits a geodesic symmetric $\mathcal{S}$-normal decomposition.

When it exists, a symmetric $\mathcal{S}$-normal decomposition is unique up to $\mathcal{C}^\times$-deformation.

• **Proposition.**— Assume $\mathcal{C}$ is a Ore category that admits a finite strong Garside family. Then the groupoid of fractions of $\mathcal{C}$ is automatic.

• Proof: Being symmetric $\mathcal{S}$-normal is a local property,
and the Fellow Traveler Property is satisfied. □

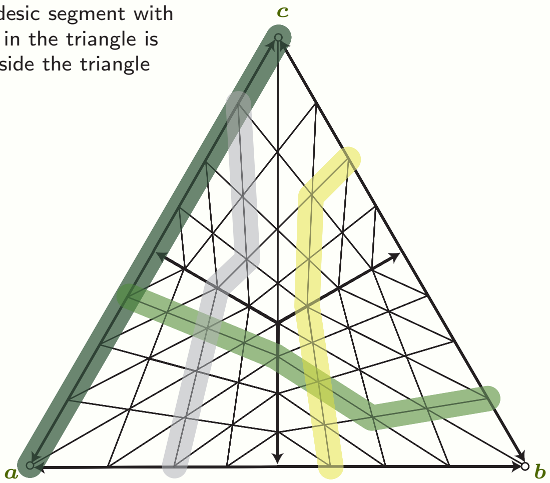
• Example:
Automaton for the symmetric normal form on $\mathbb{Z}^2$



• Proposition.— Assume $\mathcal{C}$ is a Ore category with no nontrivial invertible element and $\mathcal{S}$ is a full Garside family in $\mathcal{C}$. Then, for all a, b, c in the groupoid of fractions of $\mathcal{C}$, there exists a convex planar triangular diagram with endpoints a, b, c.

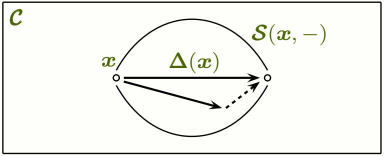
↑

every geodesic segment with endpoints in the triangle is entirely inside the triangle



Where is Δ ?

• **Definition.**— Assume $\mathcal{C}$ is a left-cancellative category, $\mathcal{S} \subseteq \mathcal{C}$, and $\Delta : \mathcal{O}bj(\mathcal{C}) \to \mathcal{C}$. Then $\mathcal{S}$ is right-bounded by Δ if $\forall x{\in}\mathcal{O}bj(\mathcal{C})\ \forall g{\in}\mathcal{S}(x,-)\ (g \preccurlyeq \Delta(x))$.



• If $\mathcal{S}$ is Garside, $\mathcal{S}$ right-bounded by Δ means

(*) $$\forall g{\in}\mathcal{S}^\sharp\ \exists g'{\in}\mathcal{S}^\sharp\ (gg' = \Delta(\text{source}(g))).$$

• **Definition.**— Assume... Then $\mathcal{S}$ is bounded by Δ if (*) and, symmetrically,

$$\forall g{\in}\mathcal{S}^\sharp\ \exists g''{\in}\mathcal{S}^\sharp\ (g''g = \Delta(\text{target}(g))).$$

• Example: If (M, Δ) is a Garside monoid, then $\mathbf{Div}(\Delta)$ is bounded by Δ.

- **Definition.**— Assume $\mathcal{C}$ is a left-cancellative category.

A **Garside map** in $\mathcal{C}$ is a map $\Delta : \mathcal{O}bj(\mathcal{C}) \to \mathcal{C}$ s.t.
  - for every x in $\mathcal{O}bj(\mathcal{C})$, source$(\Delta(x)) = x$,
  - $\mathbf{Div}(\Delta)$ generates $\mathcal{C}$,
  - $\mathbf{Div}(\Delta) = \widetilde{\mathbf{Div}}(\Delta)$ (left- and right-divisors of some $\Delta(x)$),
  - for every g in $\mathcal{C}(x, -)$, the elements g and $\Delta(x)$ have a left-gcd.

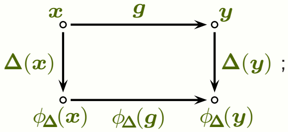
- Example: If (M, Δ) is a Garside monoid, Δ is a Garside (map) element in M.

- **Proposition.**— Assume $\mathcal{C}$ is a cancellative category.
  - If Δ is a Garside map in $\mathcal{C}$, then $\mathbf{Div}(\Delta)$ is a Garside family that is bounded by Δ.
  - Conversely, if $\mathcal{S}$ is a Garside family in $\mathcal{C}$ that is bounded by a map Δ, then Δ is a Garside map and $\mathcal{S}^\sharp = \mathbf{Div}(\Delta)$.

So: Bounded Garside families $\Leftrightarrow$ Garside maps

- Claim 1.— All properties of Garside monoids (groups) extend to categories (groupoids) with a Garside map.

- In particular:
  - There is an automorphic functor ϕ_Δ such that



  - There exists a (unique) Δ-normal form;
  - The Conjugacy Problem is decidable, ...

- Claim 2.— The extended theory applies to really more cases (Klein bottle monoid, ribbon categories, monoid M_{LD}, ...) ...and it gives better proofs and new results.

- Recap.— Two main notions:
  - Garside family: what is needed to get normal decompositions;
  - Garside map (= bounded Garside family): to get the complete theory.

⇝ Follow www.math.unicaen/∼dehornoy/Books/Garside.pdf or www.math.unicaen/∼DDKM/Garside.pdf