Set Theory fifty years after Cohen

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Plan

- 1873–1963: The Continuum Problem up to Cohen
- 1963–1987: The first step in the post-Cohen theory
- 1987–present: Toward a solution of the Continuum Problem

1. 1873–1963: The Continuum Problem up to Cohen

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$\rightsquigarrow$ equivalently: Every uncountable set of reals has the cardinality of $\mathbb{R}$.

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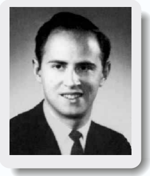

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Example: CH may be taken as an additional axiom, but not a good idea...

2. 1963–1987: The first step in the post-Cohen theory

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  (= no connection with ordinary objects).

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- Consensus: The base system for 21th century Set Theory is no longer ZF, but ZF+PD.

3. 1987–present: Toward a solution of the Continuum Problem

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- ($\approx$ Cohen) CH and $\neg$CH not provable from ZF+PD.

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Approach 1: Neutralizing forcing

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- Theorem (Foreman–Magidor–Shelah, 1988).— Under ZF+PD,
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- Question (*): Can one find (natural) axioms making the properties of sets that are $\underset{\overset{\uparrow}{H(\aleph_1)}}{\text{hereditarily of cardinality} \leqslant \aleph_1}$ generically absolute?

Freezing $H(\aleph_1)$

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- Limitation: Generic absoluteness impossible for $H(\aleph_2)$ and higher

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- Important for CH, because CH always encodable as a property of $H(\aleph_1)$: an axiom making the properties of $H(\aleph_1)$ generically absolute should decide CH.

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$\downarrow$ $\downarrow$

- Theorem (Woodin, 1999).— Under ZF+LC, if the strong Ω-conjecture is true, every axiom making the properties of $H(\aleph_1)$ generically absolute implies ¬CH.

- Meaning of the result (Woodin): Does not solve the Continuum Problem, but proves that a mathematical answer (a theorem) can eventually be given:

In spite of forcing, CH and ¬CH are not indiscernible.

- Limitation: Generic absoluteness impossible for $H(\aleph_2)$ and higher

$\rightsquigarrow$ a good test of what can be done, but cannot be the final answer.

Approach 2: Restricting to forcing-invariant properties

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- The complexity of larger and larger fragments should be higher and higher.

$\rightsquigarrow$ Impossible to stick to such a point of view...

Approach 3: Identifying one satisfactory universe

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- Question.— Can one find an L-like universe compatible with large cardinals?

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- A reference: W. Hugh Woodin, *Strong axioms of infinity and the search for V*, Proceedings ICM Hyderabad 2010, pp. 504–528