Set Theory fifty years after Cohen

Patrick Dehornoy

Laboratoire de Mathématiques
Nicolas Oresme, Université de Caen

Clermont Ferrand, February 2014

Set Theory fifty years after Cohen

Abstract

- Cohen's work is not the end of History.
- Today much more is known about sets and infinities.
- There is a reasonable hope that the Continuum Problem will be solved.

Plan

- 1873–1963: The Continuum Problem up to Cohen
- 1963–1987: The first step in the post-Cohen theory
- 1987–present: Toward a solution of the Continuum Problem

1. 1873–1963: The Continuum Problem up to Cohen

- Theorem (Cantor, 1873).— There exist
  at least two non-equivalent infinities.

- Theorem (Cantor, 1880's).— There exist
  infinitely many non-equivalent infinities,
  which organize in a well-ordered sequence

$$\aleph_0 < \aleph_1 < \aleph_2 < \ldots < \aleph_\omega < \ldots$$

- Facts. - $\mathrm{card}(\mathbb{N}) = \aleph_0$,
  - $\mathrm{card}(\mathbb{R}) = \mathrm{card}(\mathfrak{P}(\mathbb{N})) = 2^{\aleph_0} > \mathrm{card}(\mathbb{N})$.

- Question (Continuum Problem).— For which α does $\mathrm{card}(\mathbb{R}) = \aleph_\alpha$ hold?

- Conjecture (Continuum Hypothesis, Cantor, 1879).— $\mathrm{card}(\mathbb{R}) = \aleph_1$.

$\rightsquigarrow$ equivalently: Every uncountable set of reals has the cardinality of $\mathbb{R}$.

- Theorem (Cantor–Bendixson, 1883).— Closed sets satisfy CH.

↑
Every uncountable closed set of reals has the cardinality of $\mathbb{R}$.

- Theorem (Alexandroff, 1916).— Borel sets satisfy CH.

... and then no progress for 70 years.

- In the meanwhile: Formalization of First Order logic (Frege, Russell, ...) and axiomatization of Set Theory (Zermelo, then Fraenkel, ZF)

Consensus: "*We agree that these properties express our current intuition of sets (but this may change in the future)*".

- First question.— Is CH or ¬CH (negation of CH) provable from ZF?

- Theorem (Gödel, 1938).— Unless ZF is contradictory, $\neg$CH cannot be proved from ZF.

- Theorem (Cohen, 1963).— Unless ZF is contradictory, CH cannot be proved from ZF.

- Conclusion.— The system ZF is incomplete.
  $\leadsto$ Discover further properties of sets, and adopt an extended list of axioms!

- Question.— How to recognize that an axiom is true? (?)

Example: CH may be taken as an additional axiom, but not a good idea...

2. 1963–1987: The first step in the post-Cohen theory

- Which new axioms?

- From 1930's, axioms of large cardinal :

  various solutions to the equation

$$\frac{\text{super-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}.$$

- Examples: inaccessible cardinals, measurable cardinals, etc.

- X infinite: $\exists j : X \to X$ (j injective not bijective)
- X super-infinite: $\exists j : X \to X$ (j injective not biject. preserving definable notions)

  ↑

  $\mathbb{N}$ not super-infinite, as no $j : \mathbb{N} \to \mathbb{N}$ can preserve $<, +, \ldots$

- Quite natural axioms (= iteration of the postulate that infinite sets exist),

  but no evidence that they are true or, rather, useful

  (= no connection with ordinary objects).

- Definition.— For $A \subseteq \mathbb{R}$, consider the two player $\{0,1\}$-game G_A:

$$\begin{array}{lcccccc} \text{I} & a_1 & & a_3 & & \dots & \\ \text{II} & & a_2 & & a_4 & & \dots \end{array}$$

where I wins if the real $[0, a_1 a_2 ...]_2$ belongs to A. Then A is called determined if one of the players has a winning strategy in G_A.

- An infinitary statement of a special type:

$$\exists a_1 \forall a_2 \exists a_3 ...([0, a_1 a_2 ...]_2 \in A) \text{ or } \forall a_1 \exists a_2 \forall a_3 ...([0, a_1 a_2 ...]_2 \notin A).$$

- A model for many properties: there exist codings $C_{\mathcal{L}}, C_{\mathcal{B}} : \mathfrak{P}(\mathbb{R}) \to \mathfrak{P}(\mathbb{R})$ s.t.
  A is Lebesgue measurable iff $C_{\mathcal{L}}(A)$ is determined,
  A has the Baire property iff $C_{\mathcal{B}}(A)$ is determined, etc.

- Always true for simple sets:
  All closed sets are determined (Gale–Stewart, 1962),
  All Borel sets are determined (Martin, 1975).

- Always (false) for complicated sets:
  "All sets are determined" contradicts AC (Mycielski–Steinhaus, 1962),
  "All projective sets are determined" unprovable from ZF ($\approx$ Gödel, 1938).
  ↑
  closure of Borel sets under continuous image and complement

- Definition.— The Axiom of Projective Determinacy (PD) is the statement "Every projective set of reals is determined".

- Propositions (Moschovakis, Kechris, ..., 1970's).— When added to ZF, PD provides a complete and satisfactory description of projective sets of reals.

↑ heuristically complete

↑ no pathologies: Lebesgue measurable, etc.

- Example.— Under ZF+PD, projective sets satisfy CH.

- So PD is a useful axiom, but not a natural one (why consider this axiom?), contrary to large cardinal axioms, which are natural but (a priori) not useful.

- Theorem (Martin–Steel 1985, Woodin, 1987).— PD is a large cardinal axiom.

  ↑ infinitely many Woodin cardinals imply PD — ↑ PD (implies) infinitely many Woodin cardinals

- "Corollary" (Woodin).— PD is true.

  "Proof": PD is both natural (as a large cardinal axiom), and useful (as a determinacy property). □

- Why "true"? (Woodin) *true* = *"validated on the basis of accepted and compelling principles of infinity"*.

  ⇝ Think of the axiom of infinity: Is it true? (Yes) Why?

- Consensus: The base system for 21th century Set Theory is no longer ZF, but ZF+PD.

3. 1987–present: Toward a solution of the Continuum Problem

- ($\approx$ Cohen) CH and $\neg$CH not provable from ZF+PD.

  $\rightsquigarrow$ Adding PD to ZF is only the first (second) step.

- So far three approaches (with three theorems of Woodin):
  - Neutralizing forcing: "generic absoluteness" (1990's)

    test-approach, but limited
  - Restricting to forcing-invariant properties: "generic multiverse" († 2005)

    a dead end
  - Identifying one satisfactory universe: "ultimate-L" (1938-2006-present)

    currently most promising

Approach 1: Neutralizing forcing

- Cohen's method of generic extensions: analogous to algebraic extensions
  for K a field, a larger field $K[\alpha]$ controlled from within K;
  for M a universe, a larger universe $M[G]$ controlled from within M.

any structure that satisfies the axioms of ZF (↑ universe) — "forcing" (↑ controlled from within)

- Example (Cohen, '63): From M satisfying CH, extension $M[G]$ satisfying $\neg$CH.

- Many properties can be changed using forcing,
  but not the properties of (hereditarily) finite sets: cannot change $2 + 2 = 4$.

- Theorem (folklore, 1960's).— Under ZF,
  properties of hereditarily finite sets are generically absolute.

↑ invariant under forcing

(explains why ZF heuristically complete for properties of hereditarily finite sets)

- Theorem (Foreman–Magidor–Shelah, 1988).— Under ZF+PD,
  properties of hereditarily countable sets are generically absolute.

- Question (*): Can one find (natural) axioms making the properties of sets that are hereditarily of cardinality $\leqslant \aleph_1$ generically absolute?

$\uparrow$

$H(\aleph_1)$

- Important for CH, because CH always encodable as a property of $H(\aleph_1)$: an axiom making the properties of $H(\aleph_1)$ generically absolute should decide CH.

large cardinals exist $\approx$there is no exotic large cardinal

$\downarrow$ $\downarrow$

- Theorem (Woodin, 1999).— Under ZF+LC, if the strong Ω-conjecture is true, every axiom making the properties of $H(\aleph_1)$ generically absolute implies $\neg$CH.

- Meaning of the result (Woodin): Does not solve the Continuum Problem, but proves that a mathematical answer (a theorem) can eventually be given:

In spite of forcing, CH and $\neg$CH are not indiscernible.

- Limitation: Generic absoluteness impossible for $H(\aleph_2)$ and higher

$\rightsquigarrow$ a good test of what can be done, but cannot be the final answer.

Approach 2: Restricting to forcing-invariant properties

- Possible (approach) viewpoint: There is no way to prefer one universe or another one, in particular a generic extension.

- Hence introduce the generic multiverse,
  ↑ smallest family of universes that is closed under generic extension

  and consider as valid only those properties that are satisfied in all universes of the generic multiverse (and consider the others, e.g., CH, as meaningless).

- Theorem (Woodin, 2005).— Under ZF+LC, if the strong Ω-conjecture is true, the family of all statements that are valid in the sense above has the same algorithmic complexity as the family of all true statements of third-order arithmetic.

  ↑ Turing reducibility

  ↑ involving $\mathbb{N}$, $\mathfrak{P}(\mathbb{N})$, and $\mathfrak{P}(\mathfrak{P}(\mathbb{N}))$

- The complexity of larger and larger fragments should be higher and higher.

  $\rightsquigarrow$ Impossible to stick to such a point of view...

Approach 3: Identifying one satisfactory universe

- As the multiverse approach is impossible, try to identify
  one distinguished universe that could be adopted as a satisfactory reference.

- Typical candidate: Gödel's universe L of constructible sets (1938).
  ↑
  the minimal universe (cf. prime subfield): only definable sets

- Fully understood: "fine structure" theory (Jensen, Silver, ..., 1970's)
  ... but impossible as a reference universe:
  - incompatible with large cardinals: does not satisfy PD,
  - implies pathologies: existence of a non-measurable projective subset of $\mathbb{R}$.

- Question.— Can one find an L-like universe compatible with large cardinals?

- (Kunen, 1971) Universe $L[U]$: compatible with large cardinals
  up to the level of one measurable cardinal
- (Mitchell–Steel, 1980-90's) Universe $L[E]$: compatible with large cardinals
  up to the level of PD (infinitely many Woodin cardinals)
- But: how to hope completing the program,
  as there is an endless hierarchy of increasingly complex large cardinals?

- Theorem (Woodin, 2006).— There exists an explicit level (one supercompact cardinal) such that the (possible) L-like universe that is compatible with large cardinals up to that level is automatically compatible with all large cardinals.

↑
"ultimate-L"

- Now, a realistic hope to complete the program.
- Still to do (2014): Give an explicit construction of ultimate-L, and complete the proof that it is L-like (= as canonical and well understood as L, $L[U]$, $L[E]$).

- Conjecture (Woodin, 2010).— ZF+PD+V=ultimate-L is true.

↑

an explicit axiom expressing that the universe is (intrinsically) ultimate-L.

$\rightsquigarrow$ proving that the axiom V=ultimate-L has the same quality as PD,

(Woodin again) *true* = *"validated on the basis of accepted and compelling principles of infinity"*.

- Proposition.— ZF+PD+V=ultimate-L implies GCH (and the Ω-conjecture).

$\rightsquigarrow$ If ZF+PD+V=ultimate-L becomes accepted as the base of Set Theory, then the Continuum Problem will have been solved (at last).

- In any case: possibility of a coherent theory beyond ZF and of a solution of the Continuum Problem.

- (R.Solovay): "*Though I am an enthusiastic platonist, I don't think there is anything magical about ZFC. It's just one waystation along a long long road.*"

- A reference: W. Hugh Woodin, *Strong axioms of infinity and the search for V*, Proceedings ICM Hyderabad 2010, pp. 504–528