Subword Reversing and Ordered Groups

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- Use subword reversing to constructing examples of ordered groups.

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(and a new proof of the orderability of the braid group B_3)

– Plan –

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I. What is subword reversing?

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Appendix. The μ function on positive braids

– Part I –

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- A strategy for constructing Kampen diagrams

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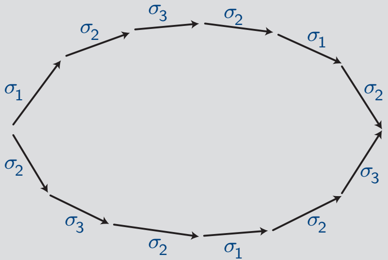

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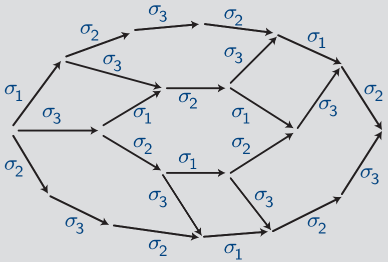

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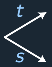
  - look at the (a) leftmost pending pattern 

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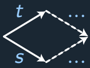
  - choose a relation $s\ldots = t\ldots$ of R to close this pattern into 

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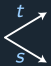
  - look at the (a) leftmost pending pattern 

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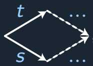
  - choose a relation $s ... = t ...$ of R to close this pattern into 

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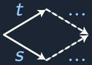
  - choose a relation $s... = t...$ of R to close this pattern into , and repeat.

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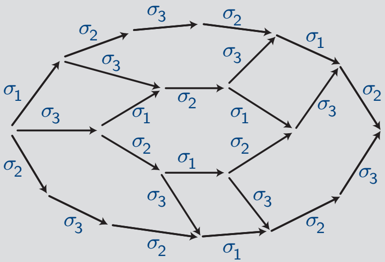

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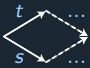
  - choose a relation $s... = t...$ of R to close this pattern into 

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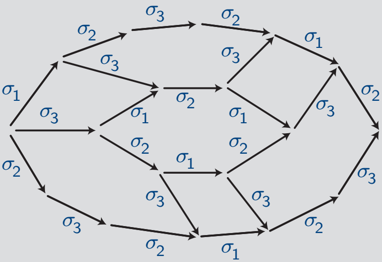

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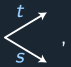
  - look at the (a) leftmost pending pattern 
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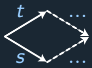
  - choose a relation $s\ldots = t\ldots$ of R to close this pattern into 
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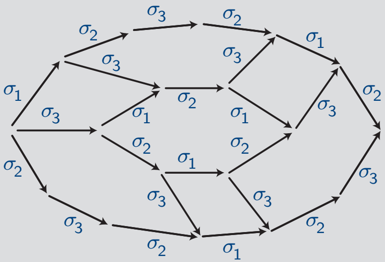

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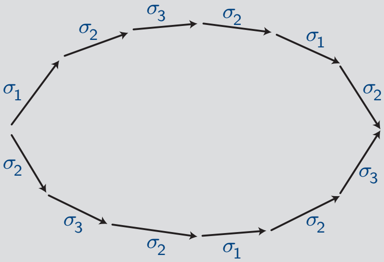

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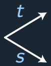
  - look at the (a) leftmost pending pattern 
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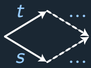
  - choose a relation $s... = t...$ of R to close this pattern into 
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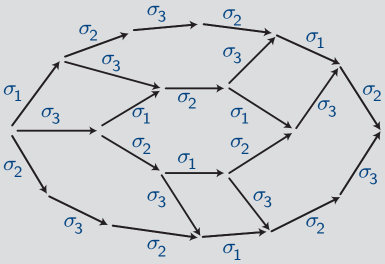

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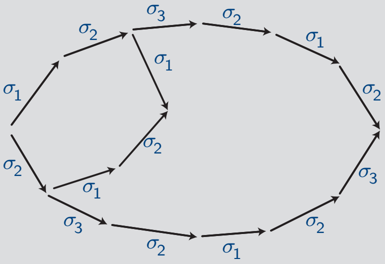

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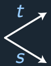
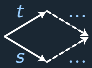
  - look at the (a) leftmost pending pattern  ,

  - choose a relation $s\ldots = t\ldots$ of R to close this pattern into , and repeat.

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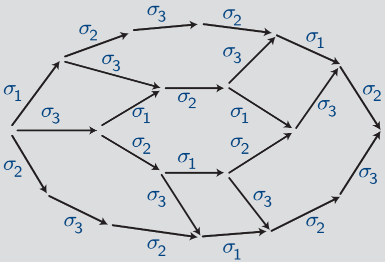

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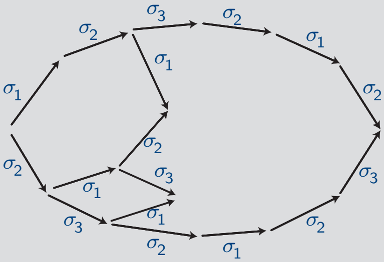

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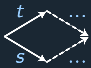
  - choose a relation $s\ldots = t\ldots$ of R to close this pattern into 
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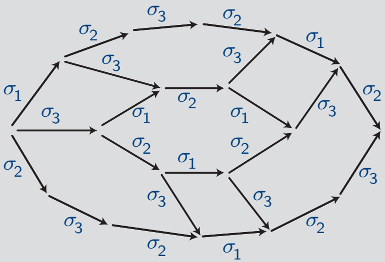

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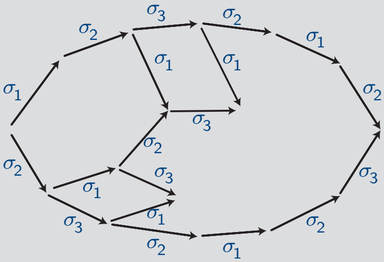

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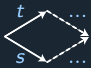
  - choose a relation $s\dots = t\dots$ of R to close this pattern into  , and repeat.

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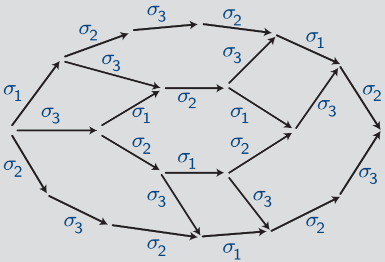

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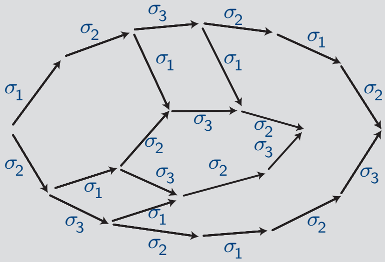

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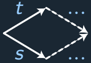
  - choose a relation $s\ldots = t\ldots$ of R to close this pattern into 

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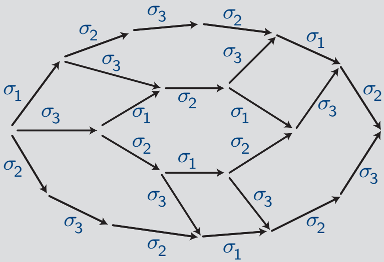

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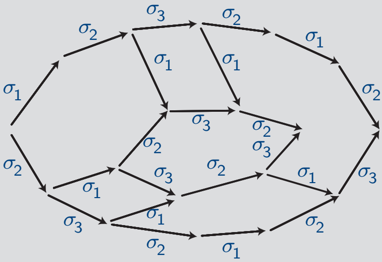

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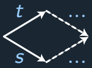
  - choose a relation $s... = t...$ of R to close this pattern into , and repeat.

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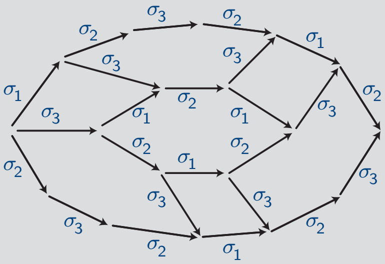

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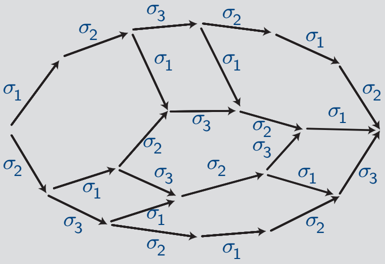

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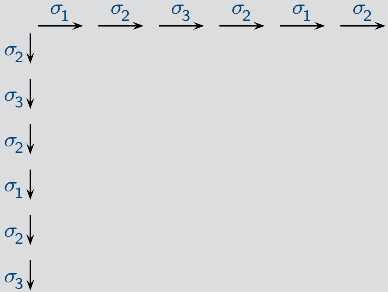

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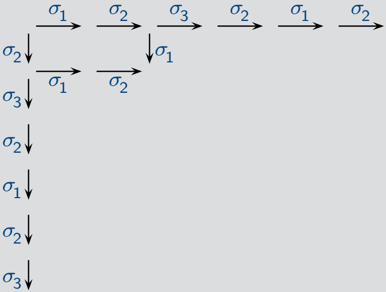

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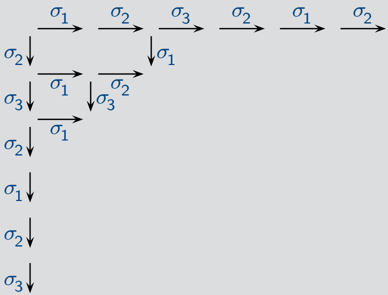

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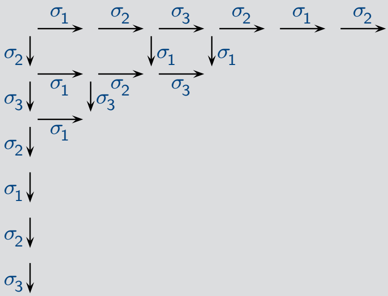

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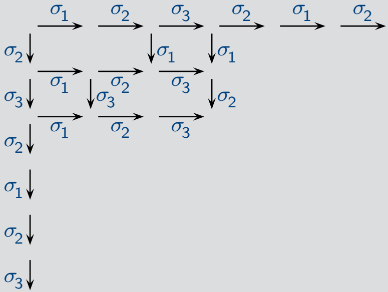

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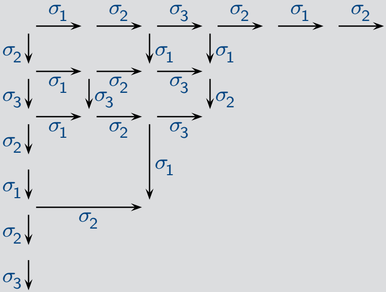

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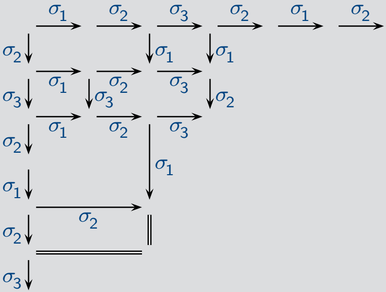

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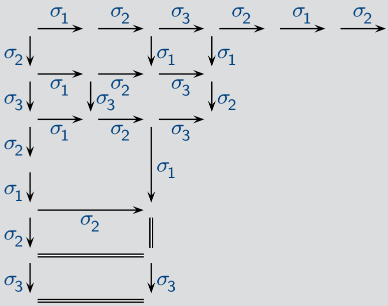

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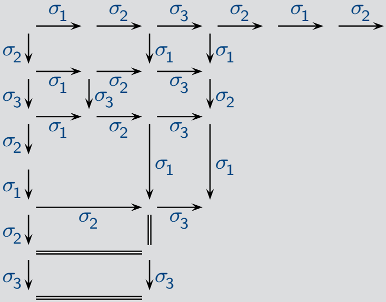

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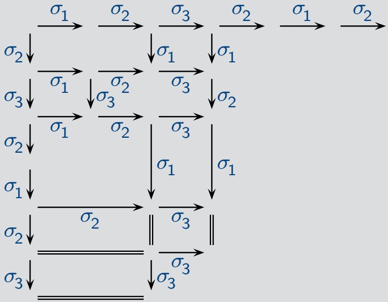

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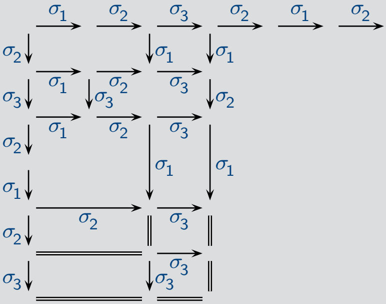

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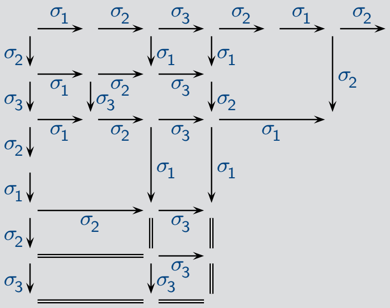

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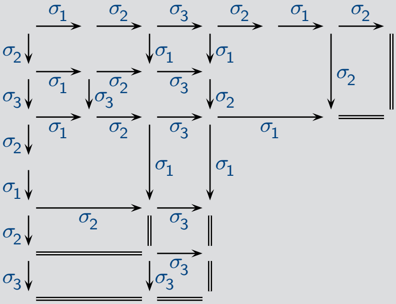

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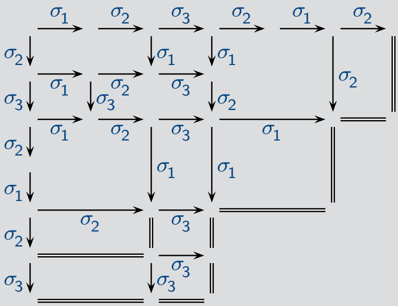

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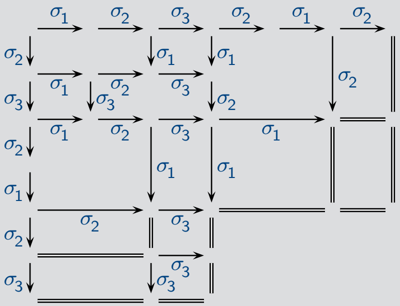

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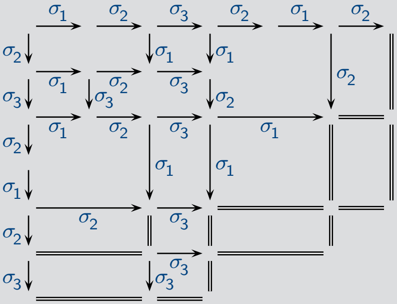

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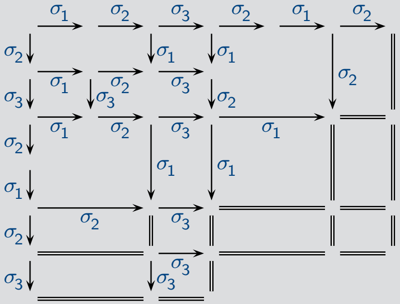

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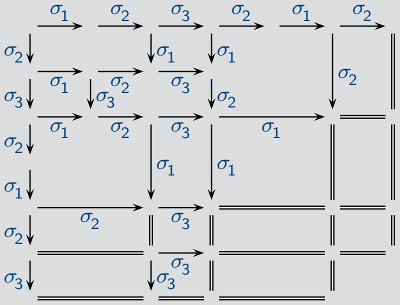

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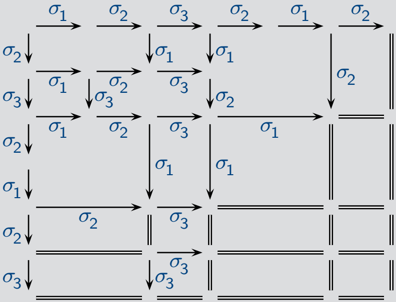

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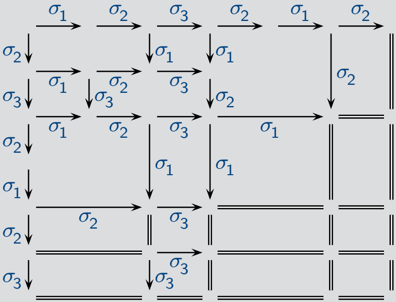

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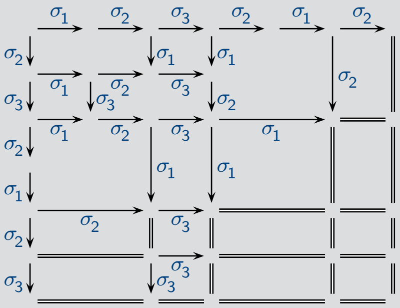

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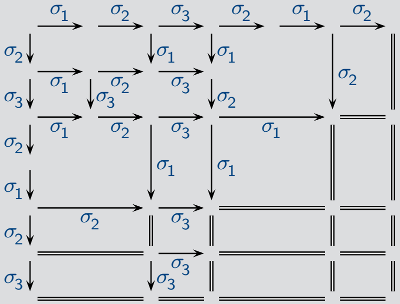

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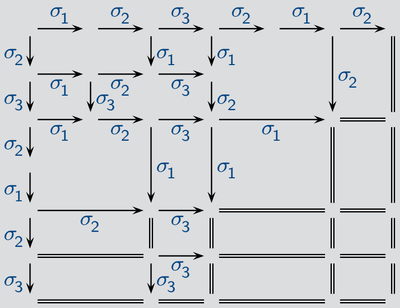

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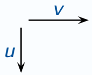

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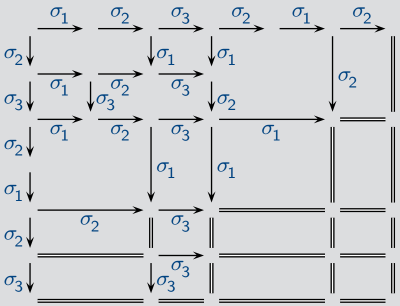

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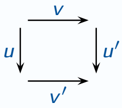

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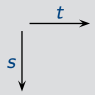

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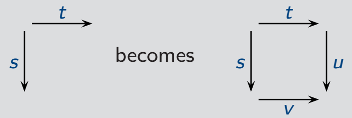

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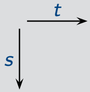

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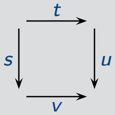

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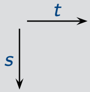

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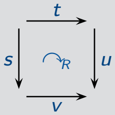

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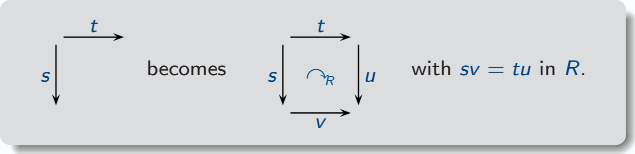

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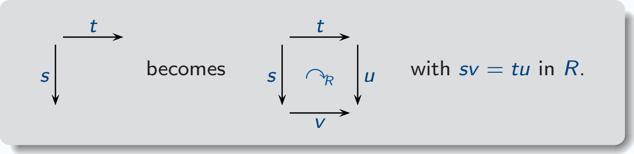

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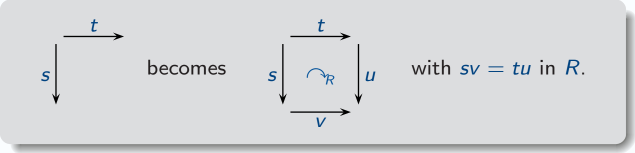

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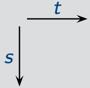

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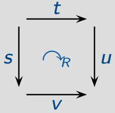

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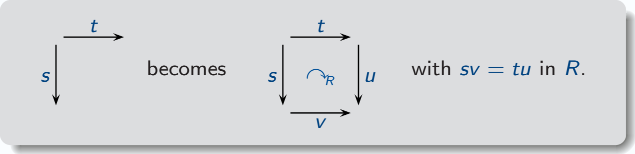

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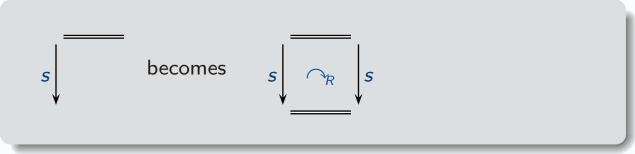

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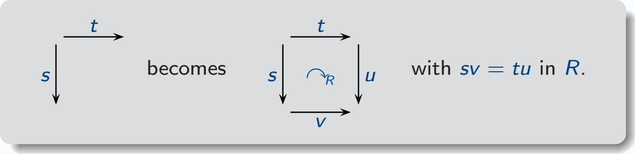

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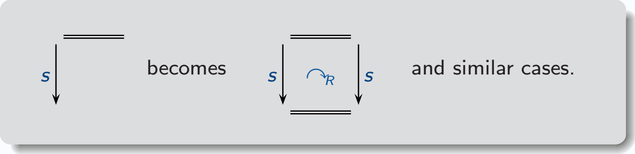
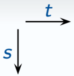
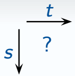


- Reversing may be stuck: 

- Reversing may be stuck: 

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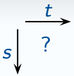
- Reversing may be stuck: 

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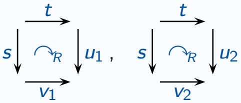
- Result may not be unique: 

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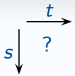
- Reversing may be stuck: 

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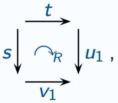
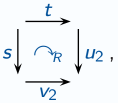
- Result may not be unique: 

, 

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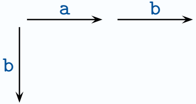
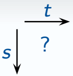


- Reversing may be stuck: 

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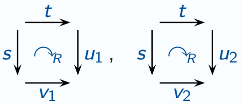
- Result may not be unique: 

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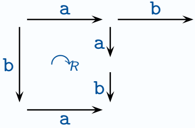
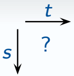


- Reversing may be stuck: 

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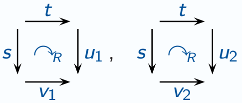
- Result may not be unique: 

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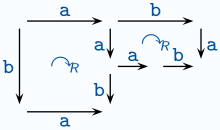
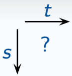


- Reversing may be stuck: 

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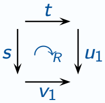
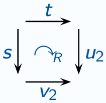
- Result may not be unique: 

, 

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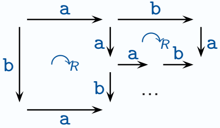
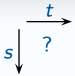


- Reversing may be stuck: 

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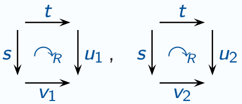
- Result may not be unique: 

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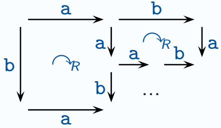

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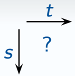
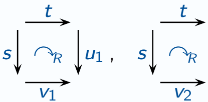
• Reversing may be stuck:  if no relation $s\ldots = t\ldots$;

• Result may not be unique:  if several relations $s\ldots = t\ldots$;

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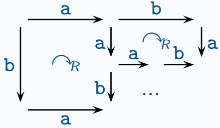
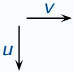
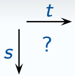


• Reversing may terminate with nonempty output words: 

- Reversing may be stuck: 

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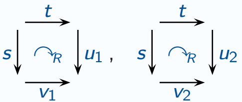
- Result may not be unique: 

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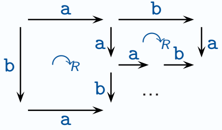

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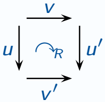
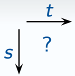


- Reversing may be stuck: 

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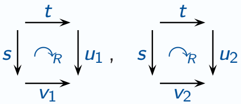
- Result may not be unique: 

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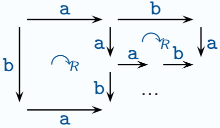

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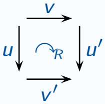
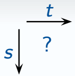


• Reversing may be stuck: 

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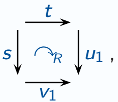
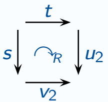
• Result may not be unique: 

, 

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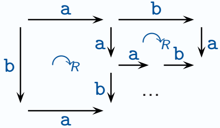

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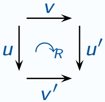
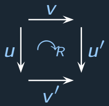
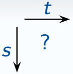


• **Lemma.**— 

- Reversing may be stuck: 

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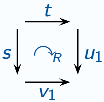
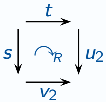
- Result may not be unique: 

, 

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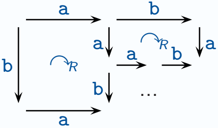

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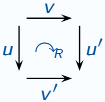
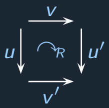


- **Lemma.**— 

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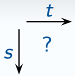
- Reversing may be stuck: 

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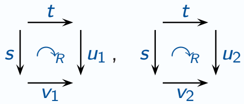
- Result may not be unique: 

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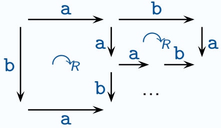

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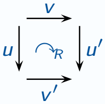
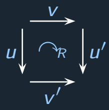
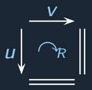
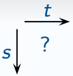


- Lemma.— 

implies $uv' \equiv_R^+ vu'$. In particular, 

- Reversing may be stuck: 

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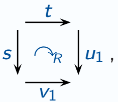
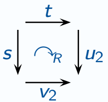
- Result may not be unique: 

, 

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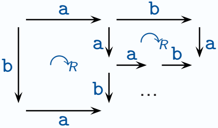

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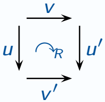
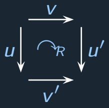
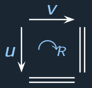


- **Lemma.**— 

implies $uv' \equiv_R^+ vu'$. In particular, 

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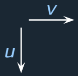

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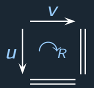
  $u \equiv_R^+ v$ implies  .

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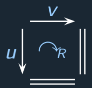
  $u \equiv_R^+ v$ implies 

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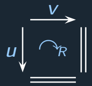
  $u \equiv_R^+ v$ implies  .

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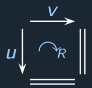
  $u \equiv_R^+ v$ implies  .

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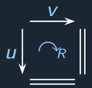
  $u \equiv_R^+ v$ implies 

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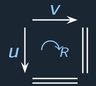
- **Definition.**— A presentation (S, R) is called complete for $\curvearrowright$ if $u \equiv_R^+ v$ implies 

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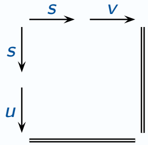
By completeness, 

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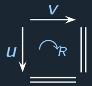
  $u \equiv_R^+ v$ implies  .

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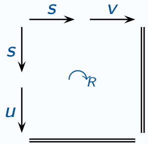
By completeness,  .

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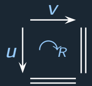
- **Definition.**— A presentation (S, R) is called complete for $\curvearrowright$ if $u \equiv_R^+ v$ implies  .

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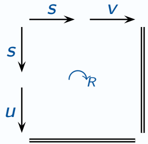
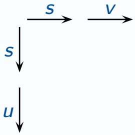
By completeness, 

. Must be 

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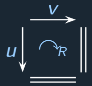

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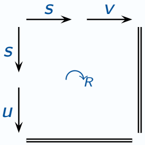

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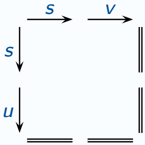

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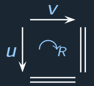

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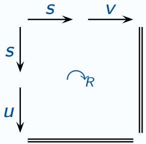

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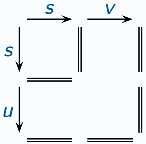

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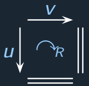
- **Definition.**— A presentation (S, R) is called complete for $\curvearrowright$ if $u \equiv_R^+ v$ implies 

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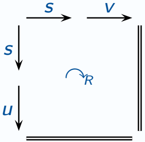
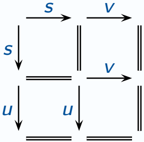
By completeness, 

. Must be 

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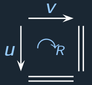
- **Definition.**— A presentation (S, R) is called complete for $\curvearrowright$ if $u \equiv_R^+ v$ implies  .

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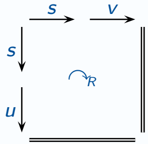
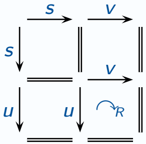
By completeness,  . Must be  .

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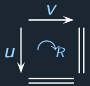
- **Definition.**— A presentation (S, R) is called complete for $\curvearrowright$ if $u \equiv_R^+ v$ implies 

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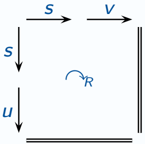
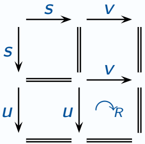
By completeness, 

. Must be 

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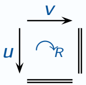
  $u \equiv_R^{+(k)} v$ ($\exists$ a length k derivation from u to v) implies  .

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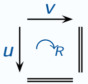
  $u \equiv_R^{+(k)} v$ ($\exists$ a length k derivation from u to v) implies .

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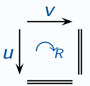
  $u \equiv_R^{+(k)} v$ ($\exists$ a length k derivation from u to v) implies  .

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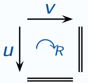
  $u \equiv_R^{+(k)} v$ ($\exists$ a length k derivation from u to v) implies  .

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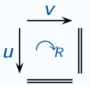
  $u \equiv_R^{+(k)} v$ ($\exists$ a length k derivation from u to v) implies 

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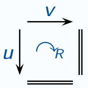

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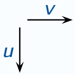
Step 1: Reversing  must terminate in finitely many steps.

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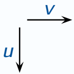
Step 1: Reversing 

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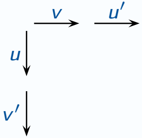
Indeed: by completeness, we must have 

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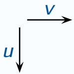
Step 1: Reversing 

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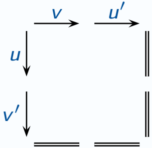
Indeed: by completeness, we must have 

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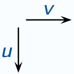
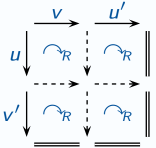
Step 1: Reversing  must terminate in finitely many steps.

Indeed: by completeness, we must have  .

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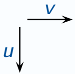
Step 1: Reversing 

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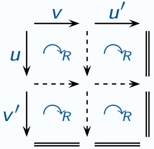
Indeed: by completeness, we must have 

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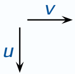
Step 1: Reversing 

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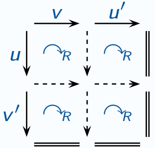
Indeed: by completeness, we must have 

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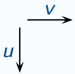
Step 1: Reversing 

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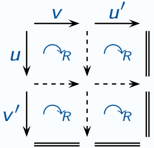
Indeed: by completeness, we must have 

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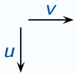
Step 1: Reversing 

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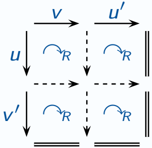
Indeed: by completeness, we must have 

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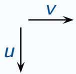
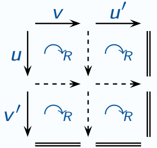
Step 1: Reversing  must terminate in finitely many steps.

Indeed: by completeness, we must have  .

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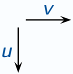
Step 1: Reversing 

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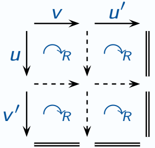
Indeed: by completeness, we must have 

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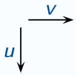
Step 1: Reversing 

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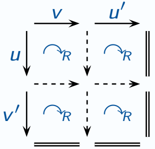
Indeed: by completeness, we must have 

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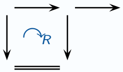

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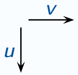
Step 1: Reversing 

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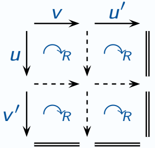
Indeed: by completeness, we must have 

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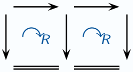

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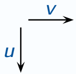
Step 1: Reversing 

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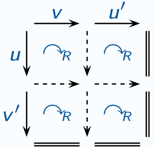
Indeed: by completeness, we must have 

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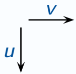
Step 1: Reversing 

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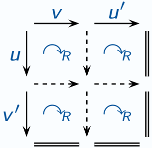
Indeed: by completeness, we must have 

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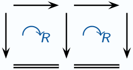

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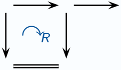

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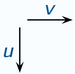
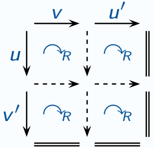
Step 1: Reversing  must terminate in finitely many steps.

Indeed: by completeness, we must have  .

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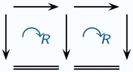

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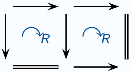

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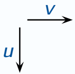
Step 1: Reversing 

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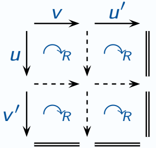
Indeed: by completeness, we must have 

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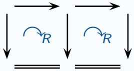
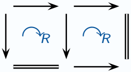
 or  or 

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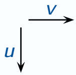
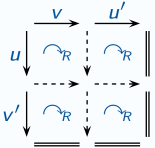
Step 1: Reversing  must terminate in finitely many steps.

Indeed: by completeness, we must have  .

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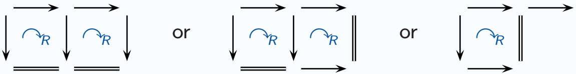
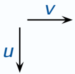


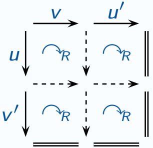
• Recall: Interested in the case when the left-divisibility relation $\preccurlyeq$ — which in general is a partial (pre)-order — is a linear order: any two elements are comparable.

• **Lemma**.— Assume (S, R) is a triangular presentation. Then any two elements f, g of $\langle S \mid R\rangle^+$ that admit a common right-multiple are comparable for $\preccurlyeq$.

• Proof. Choose words u, v in S^* that represent f and g. The assumption is that $uv' \equiv_R^+ vu'$ holds for some u', v'.

Step 1: Reversing 

must terminate in finitely many steps.

Indeed: by completeness, we must have 

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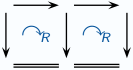

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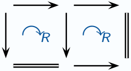

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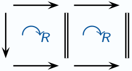

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• ...(under the same hypotheses) any two elements of M admit a common right-multiple.

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$$\mathtt{a}\delta = \mathtt{a}\mathtt{b}^p\mathtt{b} = \mathtt{a}\mathtt{b}^p(\mathtt{a}\mathtt{b}^p)^q\mathtt{a} = (\mathtt{a}\mathtt{b}^p)^q\mathtt{a}\mathtt{b}^p\mathtt{a} = \mathtt{b}\mathtt{b}^p\mathtt{a} = \delta\mathtt{a}.$$

So δ lies in the center of $G^+_{p,q}$. Hence any two elements of $G^+_{p,q}$ have a common right-multiple, and the left-divisibility relation $\preccurlyeq$ is a linear order on $G^+_{p,q}$.

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• Remark.—

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• Particular cases:

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• Particular cases:
- $p = q = 1$: $\mathtt{aba} = \mathtt{b}$ (or $x^2 = y^2$), the Klein bottle group K;

• **Proposition.**— For $p, q \geqslant 1$, let $G_{p,q} = \langle \mathtt{a}, \mathtt{b} \mid (\mathtt{ab}^p)^q \mathtt{a} = \mathtt{b} \rangle$. Then $G_{p,q}$ is left-orderable, and $LO(G_{p,q})$ has isolated points.

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• Particular cases:
- $p = q = 1$: $\mathtt{aba} = \mathtt{b}$ (or $x^2 = y^2$), the Klein bottle group K;
- $p = 2, q = 1$:

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• Particular cases:
- $p = q = 1$: $\mathtt{aba} = \mathtt{b}$ (or $x^2 = y^2$), the Klein bottle group K;
- $p = 2, q = 1$: $\mathtt{ab}^2\mathtt{a} = \mathtt{b}$

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• Particular cases:
- $p = q = 1$: $\mathtt{aba} = \mathtt{b}$ (or $x^2 = y^2$), the Klein bottle group K;
- $p = 2, q = 1$: $\mathtt{ab}^2\mathtt{a} = \mathtt{b}$ (or $x^2 = y^3$),

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• Particular cases:
- $p = q = 1$: $\mathtt{a}\mathtt{b}\mathtt{a} = \mathtt{b}$ (or $x^2 = y^2$), the Klein bottle group K;
- $p = 2, q = 1$: $\mathtt{a}\mathtt{b}^2\mathtt{a} = \mathtt{b}$ (or $x^2 = y^3$), the 3-strand braid group B_3;

• **Proposition.**— For $p, q \geqslant 1$, let $G_{p,q} = \langle \mathtt{a}, \mathtt{b} \mid (\mathtt{a}\mathtt{b}^p)^q\mathtt{a} = \mathtt{b} \rangle$. Then $G_{p,q}$ is left-orderable, and $LO(G_{p,q})$ has isolated points.

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• Remark.— Put $x = \mathtt{a}\mathtt{b}^p$ and $y = \mathtt{b}$. Then $(\mathtt{a}\mathtt{b}^p)^q\mathtt{a} = \mathtt{b}$ (iff) $x^{q+1} = y^{p+1}$.

• Particular cases:
- $p = q = 1$: $\mathtt{a}\mathtt{b}\mathtt{a} = \mathtt{b}$ (or $x^2 = y^2$), the Klein bottle group K;
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– Appendix –

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The μ function on positive braids

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…The orderability of braid groups is 20 years old this week.

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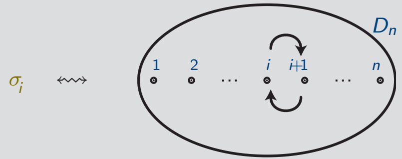

- Associate with every braid β in B_n^+
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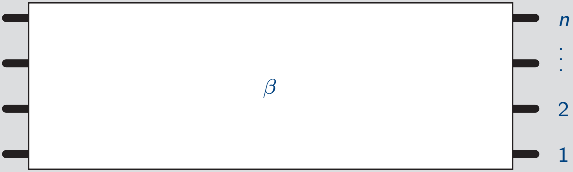

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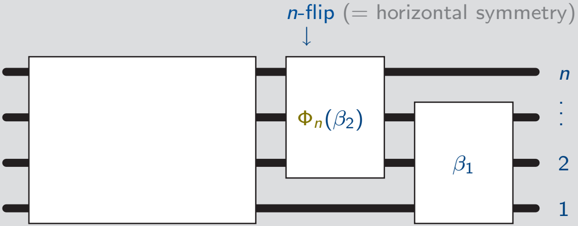

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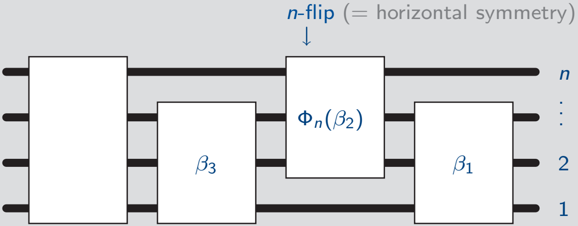

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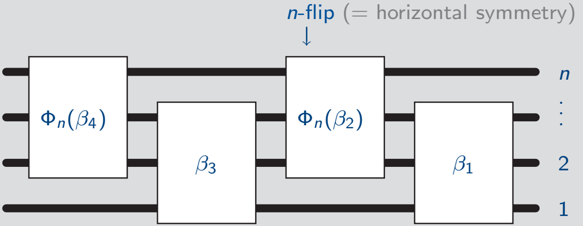

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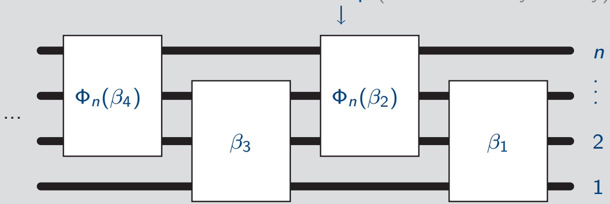

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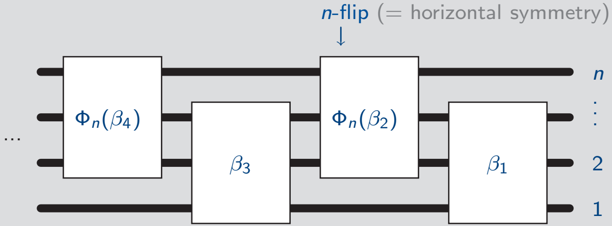

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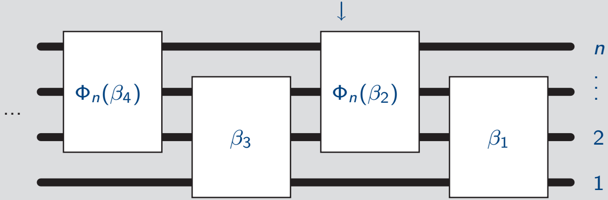

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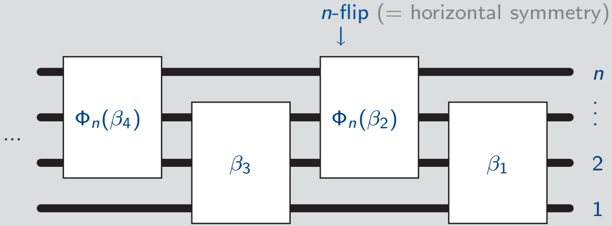

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- **Corollary**.— The D-ordering on B_n^+ is a well-ordering of type $\omega^{\omega^{n-2}}$.

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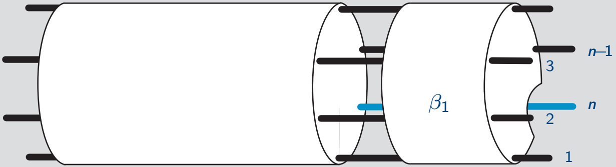

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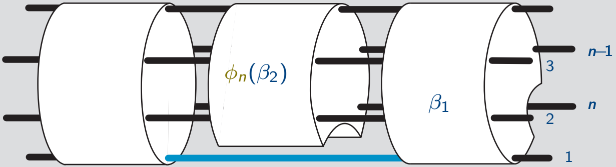

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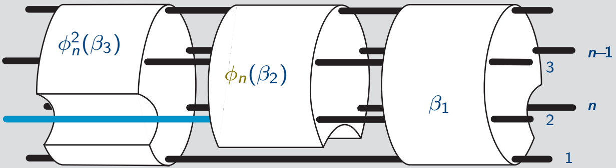

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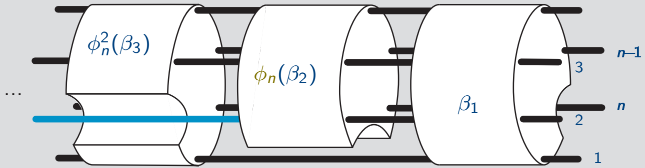

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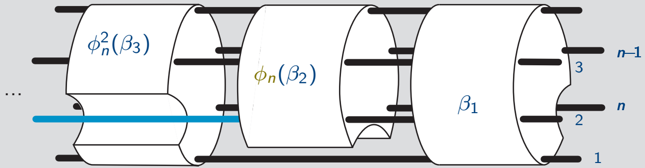

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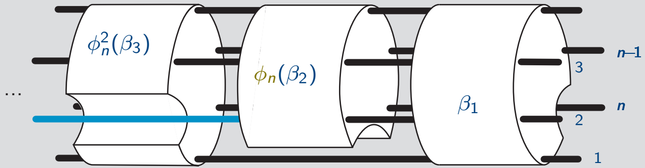

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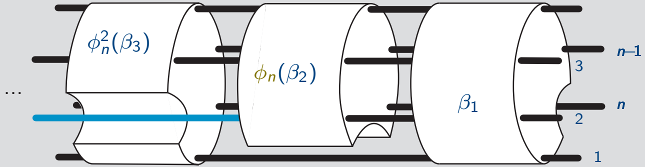

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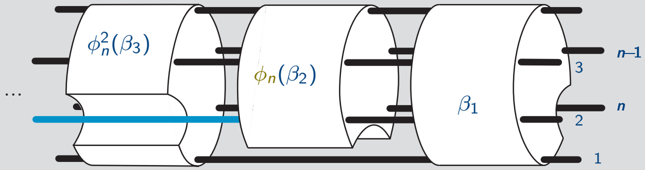
• Then similar splitting of braids in B_n^{+*} into sequences of braids in B_{n-1}^{+*}.



• Iterating the dual splitting operation gives for every β in B_n^{+*} a finite tree $T_n^*(\beta)$.

• Theorem (Fromentin, 2008): For β, β' in B_n^{+*},

$$\beta <_D \beta' \quad \text{is equivalent to} \quad T_n^*(\beta) <^{\text{ShortLex}} T_n^*(\beta').$$

• Corollary.— The D-ordering on B_n^{+*} is a well-ordering of type $\omega^{\omega^{n-2}}$.

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- Conjecture (D., Fromentin, Gebhardt, 2009).— For β in B_3^+,

$$\mu(\beta\Delta_3^2) = \sigma_1\sigma_2^2\sigma_1 \cdot \mu(\beta) \cdot \sigma_2^2.$$

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