(P1). Assume $fg = 1_X$.
Then $fgf = f$, hence $gf = 1_Y$
by cancelling f! □

(P2). $f = f1_X$ hence $f \preceq f$

$f \preceq g \preceq h$: $fg' = g$, $gh' = h$
$\Rightarrow (fg')h' = f(g'h') = h$
Hence $f \preceq h$.

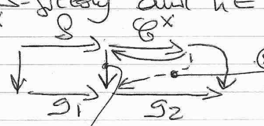
Assume $f \preceq g$ and $g \preceq f$:
$fg' = g$ and $gf' = f$
Hence $f(g'f') = f$
Hence (left-cancell.) $g'f' = 1_Y$
Hence f' and g' invertible
So $f =^{\times} g$. □

(P3) $\mathcal{S}^\#$-greedy $\Rightarrow$ $\mathcal{S}$-greedy because $\mathcal{S} \subseteq \mathcal{S}^\#$.

$\mathcal{S}$-greedy $\overset{?}{\Rightarrow}$ $\mathcal{S}^\#$-greedy

Assume (g_1, g_2) $\mathcal{S}$-greedy and $h \in \mathcal{S}^\#$

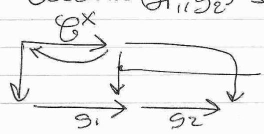
Case 1. $h \in \mathcal{S}\mathcal{C}^\times$



(2) by composing

(1) because (g_1, g_2) $\mathcal{S}$-greedy

Case 2 $h \in \mathcal{C}^\times$



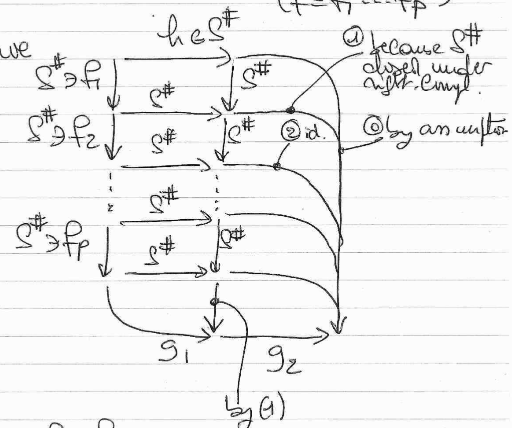
by composing □

(P4) Assume (1) $\forall h \in S^{(\sharp)} \,(h \preccurlyeq g_1 g_2 \Rightarrow h \preccurlyeq g_1)$

Assume $h \in S^\sharp$ and $h \preccurlyeq f g_1 g_2$
(want: $h \preccurlyeq f g_1$)

1) $f \in \mathcal{C}$ and $S^\sharp$ generates $\mathcal{C}$ $\Rightarrow$ $\exists f_1 \dots f_p \in S^\sharp$
($f = f_1 \cdots f_p$)

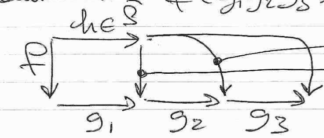
2) So we have



3) So $h \preccurlyeq \underbrace{f_1 \cdots f_p}_{f} g_1$. □

Remark: (P5) really requires full notion of S-greedy

(P5) Assume (g_1, g_2, g_3) S-greedy
and $h \preccurlyeq f(g_1 g_2 g_3)$



① $h \preccurlyeq (f g_1) g_2 g_3$ and (g_2, g_3) greedy
② $h \preccurlyeq f g_1 g_2$ and (g_1, g_2) greedy □

(P6) Assume $(g_1, \ldots, g_p)$ and $(g'_1, \ldots, g'_q)$ $\mathcal{S}$-normal decompositions of g. Then we have

$$\begin{tikzcd}
\cdot \arrow[r, "g'_1"] \arrow[d, equal] & \cdot \arrow[r, "g'_2"] \arrow[d, shift left, "f_1"'] & \cdot \arrow[r, phantom, "\cdots"] \arrow[d, shift left] & \cdot \arrow[r, "g'_q"] \arrow[d, shift left] & \cdot \arrow[r, equal] \arrow[d, shift left] & \cdot \arrow[r, phantom, "\cdots"] & \cdot \arrow[r, equal] \arrow[d, shift left] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, "g_1"'] & \cdot \arrow[r, "g_2"'] \arrow[u, shift left, "f'_1"'] & \cdot \arrow[r, phantom, "\cdots"] \arrow[u, shift left] & \cdot \arrow[r, "g_q"'] \arrow[u, shift left] & \cdot \arrow[r, "g_{q+1}"'] \arrow[u, shift left] & \cdot \arrow[r, phantom, "\cdots"] & \cdot \arrow[r, "g_p"'] \arrow[u, shift left] & \cdot
\end{tikzcd}$$

① because $g'_1 \in \mathcal{S}^\#$ and $(g_1, g_2, \ldots, g_q)$ $\mathcal{S}^\#$-greedy

② because $g'_2 \in \mathcal{S}^\#$ and $(g_2, g_3, \ldots, g_p)$ greedy, etc...

- Then $\uparrow$ similarly
- Finally $g_1 f_1 f'_1 = g_1 \Rightarrow f_1$ and f'_1 invertible etc. inductively.

So $\mathcal{C}^\times$-deformation $\square$

(P7) If
$$\begin{tikzcd}
\cdot \arrow[r, "g"] \arrow[d, shift left] & \cdot \arrow[d, shift left] \\
\cdot \arrow[r, "g'"'] \arrow[u, shift left, "\mathcal{C}^\times"] & \cdot \arrow[u, shift left, "\mathcal{C}^\times"']
\end{tikzcd}$$
, then

g invertible $\Leftrightarrow$ g' invertible. $\square$

(P8)

$$\begin{tikzcd}
\cdot \arrow[r, "h_1 \in \mathcal{S}^\#"] \arrow[d] & \cdot \arrow[r, "h_2 \in \mathcal{S}^\#"] \arrow[d] & \cdot \arrow[d] \arrow[rr] & & \cdot \arrow[d] \\
\cdot \arrow[r, "g_1"'] & \cdot \arrow[r, "g_2"'] & \cdot \arrow[r, "g_3"'] & \cdots \arrow[r, "g_p"'] & \cdot
\end{tikzcd}$$

① because $(g_1, g_2 \ldots g_p)$ $\mathcal{S}^\#$-greedy

② because $(g_2, g_3 \ldots g_p)$ $\mathcal{S}^\#$-greedy

etc. $\square$

Remark. Requires full $\mathcal{S}^\#$-greediness

(P9)

$$\begin{tikzcd}
\cdot \arrow[rrr, "g \in (S^{\#})^r"] \arrow[d, equal] & & & \cdot \arrow[d] \arrow[rrr, equal, bend left] & & & {} \\
\cdot \arrow[r, "g_1"'] & \cdots \arrow[rr, "g_r"'] & & \cdot \arrow[r, "g_{r+1}"'] & \cdots \arrow[rr, "g_p"'] & & \cdot
\end{tikzcd}$$

Assume $(g_1, \dots, g_p)$ S-normal.

1) By the Claim, $g \preccurlyeq g_1 \cdots g_r$
2) Hence $g_{r+1}, \dots, g_p$ are invertible

~~3)~~ [Hence $(g_1, \dots, \underbrace{g_r g_{r+1} \cdots g_p}_{\in S^{\#}\mathcal{C}^{\times} \subseteq S^{\#}})$ is S-normal]

Hence all non-invertible entries are among $g_1, \dots, g_r$. So $\|g\|_S \leqslant r$ $\square$

(P10) "Usual" construction of the NF:

If $g \neq 1$, let $g_1 = \underline{\gcd}(g, \Delta)$ and $g = g_1 g'$

Then $g_1 \in \underline{\mathrm{Div}}(\Delta)$, and $\lambda(g') < \lambda(g)$

Then iterate: in finitely many steps
$g = g_1 \cdots g_p$

For $h \preccurlyeq \Delta$, $h \preccurlyeq g \Rightarrow h \preccurlyeq g_1$ so
$(g_1, g_2 \cdots g_p)$ is $\underline{\mathrm{Div}}(\Delta)$-greedy

Then induction: $(g_2, g_3 \cdots g_p)$ greedy, $\square$

So every element of M has a ete $\underline{\mathrm{Div}}(\Delta)$-normal dec.

(P11) Have to prove: $\exists$ normal dec. of length (not more than) $\underline{\underline{2}}$.

Assume $g_1, g_2 \in S^\#$. Then $g_1 g_2 \in (S^\#)^2$

So, by previous, if it admits an S-normal dec. it admits one with at most 2 non-invertible entries, hence it admits one with length 2:

if shorter, add 1's
if longer, incorporate the invertible entries in g_2 $\square$

(P12) Domino:

Assume

$$\begin{tikzcd}
\cdot \arrow[r, "h \in S"] \arrow[d] & \cdot \arrow[d] \arrow[dr, bend left] \arrow[dd, bend left] & \\
\cdot \arrow[r, "g_1'"] \arrow[d] & \cdot \arrow[r, "g_2'"] \arrow[d, "f_1"] & \cdot \arrow[d] \\
\cdot \arrow[r, "g_1"'] & \cdot \arrow[r, "g_2"'] & \cdot
\end{tikzcd}$$

① because (g_1, g_2) S-greedy

② because (g_1', f_1) S-greedy

So (g_1', g_2') S-greedy

---

① $f, g_1 \in S^\#$ $\overset{\text{assumption}}{\Longrightarrow}$ (g_1', f_1) S-normal $\square$

(P13)

$$\begin{tikzcd}
\cdot \arrow[r, "g_1'"] \arrow[d, "f \in S^\#"'] & \cdot \arrow[r, "g_2'"] \arrow[d, "f_1"] & \cdot \arrow[r, phantom, "\cdots"] \arrow[d, "f_2"] & \cdot \arrow[r, "g_p'"] \arrow[d] & \cdot \arrow[d, "f_p"] \\
\cdot \arrow[r, "g_1"'] & \cdot \arrow[r, "g_2"'] & \cdot \arrow[r, dashed] & \cdot \arrow[r, "g_p"'] & \cdot
\end{tikzcd}$$

g_2' (similar)

$\underbrace{g_1 \quad g_2 \quad \cdots \quad g_p}_{\text{"}S\text{-normal"}}$

Then Domino $\Downarrow$ $(g_1', \dots, g_p', f_p)$ S-normal $\square$

(P14) ($\Leftarrow$) Induction on q s.t. $g \in (S^{\#})^q$: for adding one factor, use Claim 2.

($\Rightarrow$) obvious. $\square$

(P15) The proof shows:

left-multiplying by one element of $S^{\#}$ increases the length by at most 1,

(*) and does not diminish: if g_1' is invertible then g_1 is invertible

So, if f right-divides g, i.e., $g = (h_p \dots h_1) f$, then

$\|f\|_S \leqslant \|h_1 f\|_S \leqslant \dots \leqslant \|g\|_S$.

Because: If (g_1, g_2) S-normal + g_1 invertible, then g_2 invertible.

Pf:

$$\begin{tikzcd}
\cdot \arrow[r, "g_2"] \arrow[d, "g_1^{-1}"'] & \cdot \arrow[d] \arrow[dr, equal, bend left] & \\
\cdot \arrow[r, "g_1"'] & \cdot \arrow[r, "g_2"'] & \cdot
\end{tikzcd}$$

because S-greedy

Hence $g_2 \preccurlyeq 1_y$

Hence invertible

P16

$$\begin{tikzcd} \bullet \arrow[r, "h"] \arrow[d, equal] & \bullet \arrow[d] \arrow[dr, bend left] & \\ \bullet \arrow[r, "g_1"'] & \bullet \arrow[r, "g_2"'] & \bullet \end{tikzcd}$$

— def. of $\mathcal{S}$-greedy

$\square$

P17 Assume $e \in \mathcal{C}^\times$ $g \in \mathcal{S}^\#$
Want $eg \in \mathcal{S}^\#$

$$\begin{tikzcd} \bullet \arrow[r, "e"] \arrow[d, equal] & \bullet \arrow[l, bend left] \arrow[r, "g"] & \bullet \arrow[dll, bend left] \arrow[rrr, equal] & & & \bullet \arrow[d, equal] \\ \bullet \arrow[r, "g_1"'] & \bullet \arrow[r, "g_2"'] & \bullet \arrow[r, dashed] & \cdots \arrow[rr, "g_p"'] & & \bullet \end{tikzcd}$$

$\underbrace{g_1 \; g_2 \cdots g_p}_{\mathcal{S}\text{-normal}}$

— because e^{-1} exist and $(g_1, g_2 \cdots g_p)$ is $\mathcal{S}$-greedy

Then $g_2 \cdots g_p$ invertible. Hence $eg = g_1 \cdots g_p \in \underbrace{\mathcal{S}^\# \mathcal{C}^\times \subseteq \mathcal{S}^\#}$.

$\square$

P18 Want entries in $\mathcal{S}$, not only in $\mathcal{S}^\#$.

$$\begin{tikzcd} \bullet \arrow[r, "s"] \arrow[d, equal] & \bullet \arrow[r, "s"] \arrow[d, "\mathcal{C}^\times"] & \bullet \arrow[r, dashed] \arrow[d, "\mathcal{C}^\times"] & \bullet \arrow[r, "s"] \arrow[d] & \bullet \arrow[d, "\mathcal{C}^\times"] \\ \bullet \arrow[r, "g_1 \in \mathcal{S}^\#"'] & \bullet \arrow[r, "g_2 \in \mathcal{S}^\#"'] & \bullet \arrow[r, dashed] & \bullet \arrow[r, "g_p \in \mathcal{S}^\#"'] & \bullet \end{tikzcd}$$

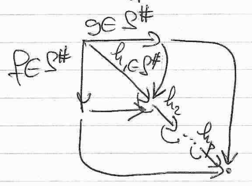
by Claim 2

① by $\mathcal{S}^\# = \mathcal{S}\mathcal{C}^\times (\cancel{\cup \mathcal{C}^\times})$

$\underbrace{g_1 \in \mathcal{S}^\# \; g_2 \in \mathcal{S}^\# \cdots g_p \in \mathcal{S}^\#}_{\mathcal{S}\text{-normal}}$

$\square$

(P19) Hyp. S Garside



① Take $(h_1, \dots, h_p)$ S-normal decomp. of the common multiple
② Because $(h_1, h_2, \dots, h_p)$ is $S^\#$-greedy and $f \in S^\#$, $f \preceq h_1$ and, similarly, $g \preceq h_1$ ∎

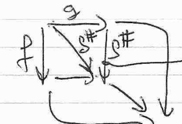
So $S^\#$ closed under right-comultiple

For S: idem by require $h_1 \in S$ by Claim 3.

□

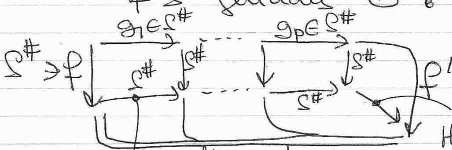
(P20) Hyp. $S^\#$ closed under right-comultiple.

① Hyp + $S^\#$ closed under right-divisor: Then



because right-divisor of an element of $S^\#$

② Hyp + $S^\#$ closed under right-complement. Then + $S^\#$ generate $\mathcal{C}$:



Assume $f \in S^\#$ and f' right-divides f -
② must be invertible
Hence $f' \in S^\# \mathcal{C}^\times = S^\#$ □

① by right-complement

(P21) Assume $g_1 g_2 \in (S^\#)^2$ $(g_1, g_2 \in S^\#)$
+ g_1' is an S-head of $g_1 g_2$.

Then $g_1 \preceq g_1'$: say $g_1' = g_1 h$.

$g_1 h = g_1' \preceq g_1 g_2 \Rightarrow h \preceq g_2$, say
$g_2 = h g_2'$.

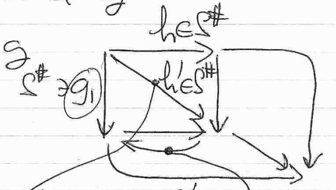
Then $g_1 g_2 = g_1' g_2'$
$g_1' \hookrightarrow \in S^\#$; $g_2' \hookrightarrow$ right-divides g_2, hence (by Claim 5 of previous page)

It remains to see: (g_1', g_2') is S-greedy.
By Lemma (P4), it is enough to show
$\forall h \in S\ (h \preceq g_1' g_2' \Rightarrow h \preceq g_1')$,
which is true by defi. of the S-head. $\square$

(P22) Assume $g_1 \in S^\#$ is a maximal left-divisor of g

Take $h \in S$, $h \preceq g$

$g = g_1 g'$



① because $S^\#$ closed under right co-multiple

② because $g_1 \preceq h'$ and g_1 maximal $\square$

P22 A00 generates
closed complement
$\exists$ head
Then (Claim 1) every elt of $(S^{\#})^2$ has an
S-normal decomposition
Hence (by ~~[illegible]~~ "Recognizing Garside I")
S is Garside $\square$

A01 generates
closed comultiple + divisor
$\exists$ maximal
By Claim 2 every elt of $(S^{\#})^2$ has a head
By Claim 5 of P20, $S^{\#}$ closed under
right-complement. Then apply $\square$

P23 Assume $g_1 \prec g_2 \prec \ldots \prec g$

Say $g_1 f_1 = g_2 f_2 = \ldots = g$

Then $f_1, f_2, \ldots$ is strictly decreasing for
right-divisibility
$\square$

P24 Noetherian $\Rightarrow$ $\exists$ maximal
left-divisor in S.
$\square$

P25 If lcm exist, then closed under right-
comultiple $\Leftrightarrow$ closed under right-lcm.
$\square$

P26 Only nontrivial: H satisfies $\mathcal{H}$-law.

① $H(g) \preceq g \Rightarrow fH(g) \preceq fg \Rightarrow H(fH(g)) \preceq H(fg)$

② Conversely: $h \in \mathcal{S} \preceq fg \Rightarrow h \preceq fH(g)$

because $(H(g), g')$ is $\mathcal{S}$-greedy (assumption)

$\Rightarrow h \preceq H(fH(g))$

In particular (for $h = H(fg)$):

$H(fg) \preceq H(fH(g))$

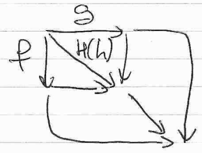
$\mathcal{H}$-law $\square$

P27 Ass. $h \in \mathcal{S}$ and $h \preceq g$.

Then $h =^{\times} H(h) \preceq H(g)$

Hence $h \preceq H(g)$. $\square$

P28



$\in \mathcal{S}^{\#} \in \mathcal{S}^{\#}$

Assume $f, g \preceq h$

then

$f, g \preceq \underset{\in \mathcal{S}}{H(h)} \preceq h$

$\square$

P29 Assume $g \in \mathcal{S}^{\#}$, $g = g'f$ Want: $f \in \mathcal{S}^{\#}$?

Assume $ge \in \mathcal{S}$, $e \in \mathcal{C}^{\times}$

$g = g'f =^{\times} ge' =^{\times} H(ge) =^{\times} H(g) = H(g'f)$

$=^{\times} H(g' H(f)) \preceq g' H(f)$

$\Rightarrow f \preceq H(f) \Rightarrow f =^{\times} H(f) \in \mathcal{S}$

$\Rightarrow f \in \mathcal{S}^{\#}$. $\square$

(P30) $(f\circ g)\circ h$ exists

$\Rightarrow (fg)h \in S$

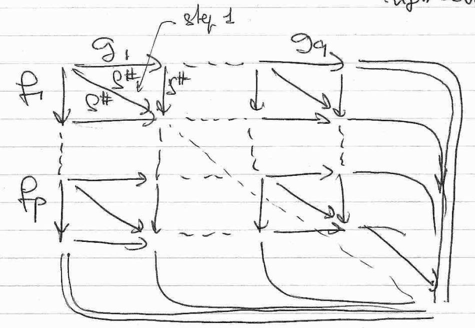
$\Rightarrow f(gh) \in S$

$\Rightarrow gh \in S \Rightarrow g\circ h$ exists. □

(P31) Assume $f_1 \cdots f_p = g_1 \cdots g_q$ with $f_1, \dots, g_q$ in $S^\#$.

Want to decompose into finitely many relations "$fg = h$".

Apply "$S^\#$ closed under right-comultiple and right-complement"



Construct the grid

Then each small piece is a relation of the form $fg = h$ with f, g, h in $S^\#$. ∎

(P32) By definition $Cat(S)$ is the category presented by the relations of the Claim. $\square$

(P33) The critical case is proving that if $g_1 \circ g_2$ exists, then

$\Pi(g_1|g_2|g_3)$ exists iff $\Pi(g_1 \circ g_2 | g_3)$ exists and then equal.

($\Longrightarrow$) $\Pi(g_1|g_2|g_3)$ exists implies that

$$\left.\begin{array}{l} g_1 \circ (g_2 \circ g_3) \text{ exists} \\ + \; g_1 \circ g_2 \text{ exists (assumption)} \end{array}\right\} \Rightarrow (g_1 \circ g_2) \circ g_3 \text{ exists},$$

i.e., $\Pi(g_1 \circ g_2 | g_3)$ exists and $=$.

($\Longleftarrow$) $\Pi(g_1 \circ g_2 | g_3)$ exists means

$(g_1 \circ g_2) \circ g_3$ exists $\underset{\text{left-associativity}}{\Longrightarrow}$ $g_2 \circ g_3$ exists

and $g_1 \circ (g_2 \circ g_3)$ exists and $=$ $\square$

(P34) Assume $(g) \equiv (g')$. Then $\Pi((g)) = \Pi((g'))$ (by Claim)

$\Pi((g)) = g$, $\Pi((g')) = g'$ $\square$

(P35) Use the Axiom of Choice to select one element in each $=^{\times}$-class $\square$

(P36) Define J starting with a sharp H-function.

$$H(g_1(g_2g_3)) = g_1 J(g_1, H(g_2g_3))$$

$$H((g_1g_2)g_3) = (g_1 \circ g_2) \circ J(g_1 \circ g_2, g_3)$$

"sharp" H-law $\rightarrow$ $\shortparallel$

$$= g_1 \circ (g_2 \circ J(g_1 \circ g_2, g_3))$$

$$H(g_1, H(g_2g_3)) = g_1 \circ J(g_1, g_2 \circ J(g_2, g_3))$$

By left-cancellativity

$$J(g_1, g_2 \circ J(g_2, g_3)) = g_2 \circ J(g_1 \circ g_2, g_3)$$

J-law

$\square$
