Garside families and germs

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- General principle:

  “Garside families give all results about Garside groups at no extra cost.”

- Here: two ways of characterizing Garside families:
  - (extrinsic) recognizing that a subfamily of a category is a Garside family,
  - (intrinsic) recognizing that a family generates a category in which it embeds as a Garside family.

- Text in progress, joint with F.Digne, E.Godelle, D.Krammer, J.Michel:

  www.math.unicaen/∼dehornoy/Books/Garside.pdf

- **Definition.**— A **category** is a monoid with a partial product, namely
  two families $\mathcal{C}$ and $Obj(\mathcal{C})$,
  plus two maps, **source** and **target**, of $\mathcal{C}$ to $Obj(\mathcal{C})$,
  plus a partial associative product: fg exists iff $\text{target}(f) = \text{source}(g)$,
  plus a identity-element 1_x for each object x.

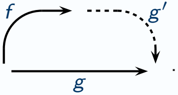
- Viewing elements as morphisms : $x \xrightarrow{f} y \xrightarrow{g} z$.

- **Definition.**— A category is **left-cancellative** if $fg = fg'$ implies $g = g'$.

P1 • **Lemma.**— If $\mathcal{C}$ is left-cancellative, an element g of $\mathcal{C}$
has a left-inverse $(\exists f (fg = 1_y))$ iff it has a right-inverse $(\exists f (gf = 1_x))$.

- **Notation.**— $\mathcal{C}^\times$:= all invertible elements of $\mathcal{C}$ ($= 1_\mathcal{C}$ if no nontrivial invertible elements),
  For $\mathcal{S} \subseteq \mathcal{C}$, $\mathcal{S}^\sharp := \mathcal{S}\mathcal{C}^\times \cup \mathcal{C}^\times$ = closure of $\mathcal{S}$ under right-multiplication
  by invertible elements ($= \mathcal{S} \cup 1_\mathcal{C}$ if no nontrivial invertible elements).

• **Definition.**— For f, g in a category $\mathcal{C}$, say that f left-divides g, or g is a right-multiple of f, denoted $f \preccurlyeq g$, if $fg' = g$ for some g'.

• Viewing elements as morphisms : 

• **Notation.**— $g =^{\times} g'$ if $\exists e \in \mathcal{C}^{\times}\ (g' = ge)$.

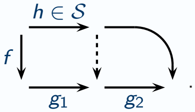
P2 • **Lemma.**— If $\mathcal{C}$ is left-cancellative, $\preccurlyeq$ is a partial preordering on $\mathcal{C}$, and the conjunction of $g \preccurlyeq g'$ and $g' \preccurlyeq g$ is equivalent to $g =^{\times} g'$.

• Definition.— Assume $\mathcal{C}$ left-cancellative and $\mathcal{S} \subseteq \mathcal{C}$.
For g_1, g_2 in $\mathcal{C}$, say that (g_1, g_2) is $\mathcal{S}$-greedy if $g_1 g_2$ is defined and

$$\forall h{\in}\mathcal{S}\ \forall f{\in}\mathcal{C}\ (h \preccurlyeq f g_1 g_2 \Rightarrow h \preccurlyeq f g_1).$$

Say that $(g_1, \dots, g_p)$ is $\mathcal{S}$-greedy if (g_i, g_{i+1}) is $\mathcal{S}$-greedy for every i.

• Viewing elements as morphisms :



• In particular, (g_1, g_2) $\mathcal{S}$-greedy implies $\forall h{\in}\mathcal{S}\ (h \preccurlyeq g_1 g_2 \Rightarrow h \preccurlyeq g_1)$.

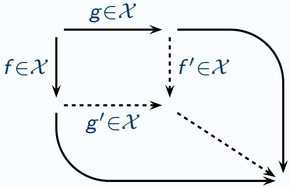
• Example.— Monoid $(\mathbb{N}^n, +)$. Then $f \preccurlyeq g$ iff $\forall i{\leqslant}n\,(f(i) \leqslant g(i))$.
- Let $S = \{f \mid \forall i\,(f(i) \leqslant 1)\}$.
- Then (g_1, g_2) is S-greedy iff $\forall i\,(h(i) \leqslant f(i) + g_1(i) + g_2(i) \Rightarrow h(i) \leqslant f(i) + g_1(i))$
  iff $\forall i\,(g_1(i) = 0 \Rightarrow g_2(i) = 0)$.

P3 • Lemma.— $\mathcal{S}$-greedy $\Leftrightarrow$ $\mathcal{S}^\sharp$-greedy.

• Definition.— $\mathcal{X}$ closed under right-complement if

$$\forall f, g{\in}\mathcal{X}\ \forall h{\in}\mathcal{C}\ (f, g \preccurlyeq h \Rightarrow \exists f', g'{\in}\mathcal{X}\ (fg' = gf' \preccurlyeq h)).$$

• Viewing elements as morphisms :



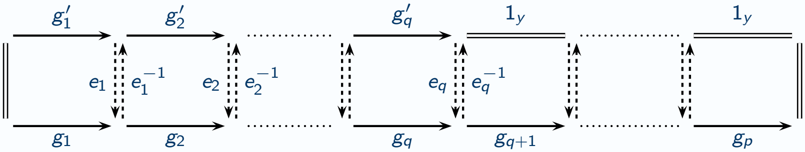
P4 • Lemma.— Assume $\mathcal{C}$ is left-cancellative, and $\mathcal{S}^\sharp$ generates $\mathcal{C}$ and is closed under right-complement. Then (g_1, g_2) is $\mathcal{S}$-greedy iff $\forall h{\in}\mathcal{S}\ (h \preccurlyeq g_1 g_2 \Rightarrow h \preccurlyeq g_1)$.

• **Definition.**— A sequence is $\mathcal{S}$-*normal* if it is $\mathcal{S}$-greedy and its entries lie in $\mathcal{S}^\sharp$.

P6 • **Proposition.**— Assume that $\mathcal{C}$ is left-cancellative and $\mathcal{S} \subseteq \mathcal{C}$. Then any two $\mathcal{S}$-normal decompositions of an element g are $\mathcal{C}^\times$-deformations of one another.

↑ a sequence $(g_1, \ldots, g_p)$ satisfying $g = g_1 \ldots g_p$

↑ $\exists$ commutative diagram like that ↓



P5 • **Claim.**— $(g_1, \ldots, g_p)$ $\mathcal{S}$-greedy $\Rightarrow$ $(g_1, g_2 \ldots g_p)$ $\mathcal{S}$-greedy.

P7 • Corollary.— Assume $\mathcal{C}$ left-cancellative and $\mathcal{S} \subseteq \mathcal{C}$.
The number of non-invertible entries in an $\mathcal{S}$-normal decomposition of an element g does not depend on the decomposition: the $\mathcal{S}$-length $\|g\|_{\mathcal{S}}$ of g.

P9 • Proposition.— Assume $\mathcal{C}$ left-cancellative and $\mathcal{S} \subseteq \mathcal{C}$.
Then $\|g\|_{\mathcal{S}} \leqslant r$ holds for every g in $(\mathcal{S}^\sharp)^r$ that admits an $\mathcal{S}$-normal decomposition.

P8 • Claim.— (g_1, g_2, g_3) $\mathcal{S}$-greedy $\Rightarrow$ $(g_1 g_2, g_3)$ $\mathcal{S}^2$-greedy.

• Definition.— Assume $\mathcal{C}$ left-cancellative. A subfamily $\mathcal{S}$ of $\mathcal{C}$ is a Garside family in $\mathcal{C}$ if every element of $\mathcal{C}$ admits an $\mathcal{S}$-normal decomposition.

• Example.— $\mathcal{C}$ is always Garside in $\mathcal{C}$.

- $S = \{f \mid \forall i\ (f(i) \leqslant 1)\}$ is Garside in $(\mathbb{N}^n, +)$:
  the S-normal decomposition of g is $(g_1, g_2, ...)$ with $g_k(i) = \begin{cases} 1 & \text{if } g(i) \geqslant k, \\ 0 & \text{otherwise.} \end{cases}$

P10- If (M, Δ) is a Garside monoid, then $Div(\Delta)$ is a Garside family in M.

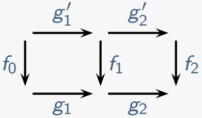
↑

- M is cancellative,
- M admits least common multiples and greatest common divisors,
- there exists $N : M \to \mathbb{N}$ s.t. $g \neq 1 \Rightarrow N(g) \geqslant 1$ and $N(fg) \geqslant N(f) + N(g)$,
- $Div(\Delta) = \widetilde{Div}(\Delta)$, and generates M.

P14 • Proposition.— A subfamily $\mathcal{S}$ of a left-cancellative category $\mathcal{C}$ is a Garside family iff (*) $\mathcal{S}^\sharp$ generates $\mathcal{C}$ and every element of $(\mathcal{S}^\sharp)^2$ admits an $\mathcal{S}$-normal decomposition.

P11 • Claim 1.— (*) $\Rightarrow$ every element of $(\mathcal{S}^\sharp)^2$ admits an $\mathcal{S}$-normal decomposition of length 2.

P12 • Lemma (first domino rule).— Assume $\mathcal{C}$ left-cancellative and



is commutative. If (g_1, g_2) and (g'_1, f_1) are $\mathcal{S}$-greedy, then so is (g'_1, g'_2).

P13 • Claim 2.— (*) $\Rightarrow$ if g admits an $\mathcal{S}$-normal decomposition of length p and $f \in \mathcal{S}^\sharp$, then fg admits an $\mathcal{S}$-normal decomposition of length $p+1$ when defined.

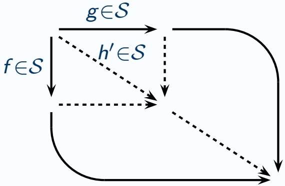
P15 • Scholium.— If $\mathcal{S}$ is a Garside family and f right-divides of g, then $\|f\|_{\mathcal{S}} \leqslant \|g\|_{\mathcal{S}}$. In particular, $\mathcal{S}^\sharp$ is closed under right-divisor.

• Definition.— Say that g_1 is an $\mathcal{S}$-head of g if $g_1 \in \mathcal{S}$, $g_1 \preccurlyeq g$, and $\forall h{\in}\mathcal{S}\ (h \preccurlyeq g \Rightarrow h \preccurlyeq g_1)$.

• Definition.— $\mathcal{S}$ closed under right-comultiple if

$$\forall f, g{\in}\mathcal{S}\ \forall h{\in}\mathcal{C}\ (f, g \preccurlyeq h \Rightarrow \exists h'{\in}\mathcal{S}\ (f, g \preccurlyeq h' \preccurlyeq h)).$$

• Viewing elements as morphisms :



• Proposition.— A subfamily $\mathcal{S}$ of a left-cancellative category $\mathcal{C}$ is a Garside family iff

(**) $\mathcal{S}^\sharp$ generates $\mathcal{C}$, is closed under right-complement, and every non-invertible element of $(\mathcal{S}^\sharp)^2$ admits an $\mathcal{S}$-head, or

(**bis) $\mathcal{S}^\sharp$ generates $\mathcal{C}$, is closed under right-divisor and right-comultiple, and every element of $(\mathcal{S}^\sharp)^2$ admits a maximal left-divisor lying in $\mathcal{S}$.

- Want: $\mathcal{S}$ Garside $\Rightarrow$ - $\mathcal{S}^\sharp$ generates $\mathcal{C}$: OK
  - $\mathcal{S}^\sharp$ closed under right-divisor: OK
  - every element of $(\mathcal{S}^\sharp)^2$ admits an $\mathcal{S}$-head.
  - $\mathcal{S}^\sharp$ closed under right-comultiple,
  - $\mathcal{S}^\sharp$ closed under right-complement,

P16 • Claim 1.— $g_1 \in \mathcal{S}$ and (g_1, g_2) $\mathcal{S}$-greedy $\Rightarrow$ g_1 is an $\mathcal{S}$-head of $g_1 g_2$.

P17 • Claim 2.— $\mathcal{S}$ Garside $\Rightarrow \mathcal{C}^\times \mathcal{S}^\sharp \subseteq \mathcal{S}^\sharp$.

P18 • Claim 3.— Every element admitting an $\mathcal{S}$-decomposition admits one where all entries except maybe the last one belong to $\mathcal{S}$.

P19 • Claim 4.— $\mathcal{S}$ Garside $\Rightarrow$ $\mathcal{S}^\sharp$ and $\mathcal{S}$ closed under right-comultiple.

P20 • Claim 5.— ($\mathcal{S}^\sharp$ closed under right-comultiple + under right-divisor) $\Rightarrow$ $\mathcal{S}^\sharp$ closed under right-complement;
($\mathcal{S}^\sharp$ closed under right-complement + $\mathcal{S}^\sharp$ generates $\mathcal{C}$) $\Rightarrow$ $\mathcal{S}^\sharp$ closed under right-divisor.

• Want: - $\mathcal{S}^\sharp$ generates $\mathcal{C}$ + closed under right-complement
+ every element of $(\mathcal{S}^\sharp)^2$ admits an $\mathcal{S}$-head $\Rightarrow$ $\mathcal{S}$ Garside.

and: - $\mathcal{S}^\sharp$ generates $\mathcal{C}$ + closed under right-comultiple + closed under right-divisor
+ every element of $(\mathcal{S}^\sharp)^2$ admits a maximal left-divisor in $\mathcal{S}$ $\Rightarrow$ $\mathcal{S}$ Garside.

P21 • Claim 1.— ($\mathcal{S}^\sharp$ closed under right-complement + $\mathcal{S}^\sharp$ generates $\mathcal{C}$) $\Rightarrow$
every element of $(\mathcal{S}^\sharp)^2$ admitting an $\mathcal{S}$-head admits an $\mathcal{S}$-normal decomposition.

P22 • Claim 2.— $\mathcal{S}^\sharp$ closed under right-comultiple $\Rightarrow$
a maximal left-divisor in $\mathcal{S}$ is an $\mathcal{S}$-head.

P23

• **Definition.**— A category $\mathcal{C}$ is right-Noetherian if right-divisibility is a well-founded relation.

↑ every nonempty subfamily has a least element

P23 • **Lemma.**— A left-cancellative category $\mathcal{C}$ is right-Noetherian iff there is no infinite bounded $\prec$-increasing sequence.

↑ $g_1 \prec g_2 \prec \dots \prec g$

P24 • **Proposition.**— A subfamily $\mathcal{S}$ of a left-cancellative right-Noetherian category $\mathcal{C}$ is a Garside family iff $\mathcal{S}^\sharp$ generates $\mathcal{C}$ and is closed under right-divisor and right-comultiple.

• **Definition.**— A category $\mathcal{C}$ admits right-lcm's if any two elements of $\mathcal{C}$ admit a least upper bound for $\preccurlyeq$.

P25 • **Corollary.**— Assume that $\mathcal{C}$ is a left-cancellative category $\mathcal{C}$ that is right-Noetherian and admits right-lcm's. A subfamily $\mathcal{S}$ of $\mathcal{C}$ is a Garside family iff $\mathcal{S}^\sharp$ generates $\mathcal{C}$ and is closed under right-divisor and right-lcm.

• **Definition.**— A map $H : \mathcal{X} \subseteq\!\!\to \mathcal{C}$ satisfies the $\mathcal{H}$-law if

$$H(fg) =^{\times} H(fH(g)).$$

• **Proposition.**— A subfamily $\mathcal{S}$ of a left-cancellative category $\mathcal{C}$ is a Garside family iff (***) $\mathcal{S}^\sharp$ generates $\mathcal{C}$, and there exists $H : \mathcal{C} \setminus \mathcal{C}^\times \to \mathcal{S}$ such that

- $H(g) \preccurlyeq g$ always holds,
- $f \preccurlyeq g$ implies $H(f) \preccurlyeq H(g)$,
- $g \in \mathcal{S}$ implies $H(g) =^{\times} g$,
- H satisfies the $\mathcal{H}$-law.

P26 • Proof ($\Rightarrow$): $\mathcal{S}$ Garside $\Rightarrow$ exists H satisfying...:
Define $H(g) =$ 1st entry in an $\mathcal{S}$-normal decomposition of $\mathcal{S}$ with 1st entry in $\mathcal{S}$.

• Proof ($\Leftarrow$): H satisfying... $\Rightarrow$ $\mathcal{S}$ Garside:

P27 • Claim 1.— $H(g)$ is an $\mathcal{S}$-head of g.

P28 • Claim 2.— $\mathcal{S}^\sharp$ is closed under right-comultiple.

P29• Claim 3.— $\mathcal{S}^\sharp$ is closed under right-divisor.

- So far: $\mathcal{C}$ (the ambient category) is given, and $\mathcal{S} \subseteq \mathcal{C}$.
- Now: $\mathcal{S}$ is given (no $\mathcal{C}$), and look for conditions ensuring that there exists a category $\mathcal{C}$ s.t. $\mathcal{S} \hookrightarrow \mathcal{C}$ and $\mathcal{S}$ is a Garside family in $\mathcal{C}$.

- Analysis: If $\mathcal{S} \subseteq \mathcal{C}$, the operation of $\mathcal{C}$ induces a partial operation on $\mathcal{S}$:

$$f \bullet g = h \text{ whenever } fg = h \text{ and } f, g, h \in \mathcal{S}.$$

$\rightsquigarrow$ Investigate the structure $(\mathcal{S}, \bullet)$. ← "a germ"

- **Lemma**.— If $\mathcal{S}$ is a subfamily of a left-cancellative category and includes $1_{\mathcal{S}}$, then $\bullet$ obeys the rules

  (1) If $f \bullet g$ exists, the target of f is the source of g;
  (2) If $f \in \mathcal{S}(x, y)$, then $1_x \bullet f = f = f \bullet 1_y$;
  (3) If $f \bullet g$ and $g \bullet h$ exist, then $f \bullet (g \bullet h)$ exists iff $(f \bullet g) \bullet h$ does, and then equal.

Moreover, if $\mathcal{S}$ is closed under right-divisor in $\mathcal{C}$, then
P30 (4) If $(f \bullet g) \bullet h$ exists, then $g \bullet h$ does.

- **Definition**.— A germ is a triple $(\mathcal{S}, 1_{\mathcal{S}}, \bullet)$ where $\mathcal{S}$ is a precategory and $\bullet$ is a partial binary operation on $\mathcal{S}$ satisfying (1), (2), (3).
  - It is called left-associative if, in addition, (4) is satisfied.

- For $\mathcal{S}$ a precategory, $\mathcal{S}^*$ = the free category generated by $\mathcal{S}$
  = the category of all $\mathcal{S}$-paths.

- **Definition.**— For $\mathcal{S}$ a germ, the category $Cat(\mathcal{S})$ associated with $\mathcal{S}$ is $\mathcal{S}^*/\equiv$
  where $\equiv$ is the congruence generated by all relations $f \mid g = h$ for $f \bullet g = h$ in $\mathcal{S}$.

P32 • **Proposition.**— If $\mathcal{S}$ is a Garside family in a left-cancellative category $\mathcal{C}$
and $\mathcal{S}^\sharp = \mathcal{S}$ holds, then $\mathcal{C} \cong Cat(\mathcal{S})$.

P31 • **Claim.**— $\mathcal{S}$ Garside in $\mathcal{C} \Rightarrow \mathcal{C}$ presented by the relations $fg = h$ with $f, g, h \in \mathcal{S}^\sharp$.

P34 • **Proposition.**— If $\mathcal{S}$ is a left-associative germ,
then $\mathcal{S}$ embeds in $Cat(\mathcal{S})$ as a subfamily that is closed under right-divisor.

P33 • **Claim.**— Define $\Pi : \mathcal{S}^* \to \mathcal{S}$ (partial) by $\Pi(\varepsilon_x) = 1_x$ and $\Pi(g \mid w) = g \bullet \Pi(w)$
whenever defined. Then $w \equiv w' \Rightarrow \Pi(w)$ defined iff $\Pi(w')$ defined, and then equal.

• Definition.— A germ $\mathcal{S}$ is a Garside germ if there exists a left-cancellative category $\mathcal{C}$ such that $\mathcal{S}$ embeds in $\mathcal{C}$ and $\mathcal{S}$ is a dense Garside family in $\mathcal{C}$.

↑ generates the category and is closed under right-divisor

• Definition.— A $\mathcal{J}$-function for a germ $\mathcal{S}$ is a map $J : \mathcal{S}^{[2]} \to \mathcal{S}$ s.t., for every (g_1, g_2),

$$J(g_1, g_2) \in \{h \in \mathcal{S} \mid g_1 \bullet h \text{ is defined and } h \preccurlyeq_{\mathcal{S}} g_2\}.$$

↑ $\mathcal{J}(g_1, g_2)$

• Proposition.— A germ is a Garside germ iff it is left-associative, left-cancellative, and there exists a $\mathcal{J}$-function J that satisfies the $\mathcal{J}$-law:

$$g = g_1 \bullet g_2 \text{ implies } J(g_1, g_2 \bullet J(g_2, g_3)) = g_2 \bullet J(g, g_3).$$

P36 • Proof ($\Rightarrow$): Assume $\mathcal{S}$ is a Garside germ.

P35 • Claim.— Exists a sharp $\mathcal{S}$-head function: $g =^{\times} g' \Rightarrow H(g) = H(g')$.

• Define $J(g_1, g_2)$ by $g_1 \bullet J(g_1, g_2) = H(g_1 g_2)$. Then $g = g_1 \bullet g_2$ implies $H(g_1(g_2 g_3)) = H(g_1 H(g_2 g_3))$, which translates to $J(g_1, g_2 \bullet J(g_2, g_3)) = g_2 \bullet J(g, g_3)$.

the $\mathcal{H}$-law — the $\mathcal{J}$-law

- Proof ($\Leftarrow$): Assume $\mathcal{S}$ is a left-associative, left-cancellative germ that admits a $\mathcal{J}$-function J satisfying the $\mathcal{J}$-law.

- Aim: Show that $\mathit{Cat}(\mathcal{S})$ is left-cancellative and $\mathcal{S}$ is a Garside family in $\mathit{Cat}(\mathcal{S})$.

- Claim 1.— Define $K : \mathcal{S}^{[2]} \to \mathcal{S}$, $\Theta : \mathcal{S}^* \to \mathcal{S}$ and $\Omega : \mathcal{S}^* \to \mathcal{S}^*$ by

$$g_2 = J(g_1, g_2) \bullet K(g_1, g_2),$$
$$\Theta(\varepsilon_x) = 1_x \text{ and } \Theta(g \mid w) = g \bullet J(g, \Theta(w)),$$
$$\Omega(\varepsilon_x) = \varepsilon_x \text{ and } \Omega(g \mid w) = K(g, \Theta(w)) \mid \Omega(w).$$

Then Θ and Ω are compatible with $\equiv$, and $w \equiv \Theta(w) \mid \Omega(w)$ always holds.

$\uparrow$ the congruence that defines $\mathit{Cat}(\mathcal{S})$ from $\mathcal{S}^*$ = the congruence generated by the relations $f \mid g = h$ with $f \bullet g = h$

- Claim 2.— $w \equiv J(g, \Theta(w)) \bullet \Omega(g \mid w)$.

- Claim 3.— $\mathit{Cat}(\mathcal{S})$ is left-cancellative.

- Claim 4.— Θ induces an $\mathcal{S}$-head function on $\mathcal{S}$.

- Definition.— A $\mathcal{J}$-function J for a germ $\mathcal{S}$ is called maximum if
$$\forall h \in \mathcal{J}(g_1, g_2)\ (h \preccurlyeq J(g_1, g_2)).$$

- Proposition.— A germ is a Garside germ iff it is left-associative, left-cancellative, and admits a maximum $\mathcal{J}$-function.

- Proof: $(\Rightarrow)$ The J-function deduced from an $\mathcal{S}$-head function is maximum.
$(\Leftarrow)$ A maximum J almost satisfies the $\mathcal{J}$-law.

- Claim 1.— A maximum $\mathcal{J}$-function J satisfies the weak $\mathcal{J}$-law
$$g = g_1 \bullet g_2 \text{ implies } J(g_1, g_2 \bullet J(g_2, g_3)) =^{\times}_{\mathcal{S}} g_2 \bullet J(g, g_3).$$

- Definition.— An $\mathcal{I}$-function is ... $\in \{g \in \mathcal{S} \mid \exists h\ (g = g_1 \bullet h \text{ and } h \preccurlyeq_{\mathcal{S}} g_2)\}$.

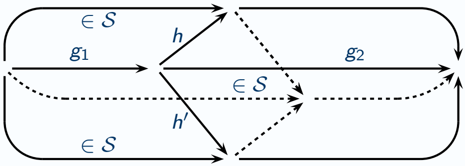
- Claim 2.— A maximum $\mathcal{I}$-function I satisfies the weak $\mathcal{I}$-law
$$g = g_1 \bullet g_2 \text{ implies } I(g_1, I(g_2, g_3)) =^{\times}_{\mathcal{S}} I(g, g_3).$$

- Claim 3.— $\exists$ function satisfying the weak $\mathcal{I}$-law $\Rightarrow$ $\exists$ function satisfying the $\mathcal{I}$-law.

• Definition.— A germ $\mathcal{S}$ is right-Noetherian
if right-divisibility is a well-founded relation on $\mathcal{S}$.

↑
every nonempty subfamily has a least element

• Proposition.— A right-Noetherian germ $\mathcal{S}$ is a Garside germ iff it is left-associative, left-cancellative, and, for every (g_1, g_2) in $\mathcal{S}^{[2]}$, any two elements of $\mathcal{J}(g_1, g_2)$ have a common right-multiple in $\mathcal{J}(g_1, g_2)$.



• Corollary.— For a germ $\mathcal{S}$ to be a Garside germ, it is sufficient that $\mathcal{S}$ is left-associative, left-cancellative, right-Noetherian, admits right-lcm's, and satisfies

(*) If $g_1 \bullet h$ and $g_1 \bullet h'$ are defined, then $g_1 \bullet \mathrm{lcm}(h, h')$ is defined.

If $\mathcal{S}$ is right-associative, (*) is necessarily satisfied.

- **Definition.**— For G a group and Σ positively generating G:
  - $(g_1, \ldots, g_r)$ is Σ-tight if $\|g_1 \ldots g_r\|_\Sigma = \|g_1\|_\Sigma + \ldots + \|g_r\|_\Sigma$;
  - f is a Σ-prefix of g if $(f, f^{-1}g)$ is Σ-tight (resp. Σ-suffix ... (gf^{-1}, f) ...).

- **Definition.**— For G a group, Σ positively generating G, and $H \subseteq G$, the derived germ $H_{/\Sigma}$ is $(H, 1, \bullet)$ where $f \bullet g = h$ is $fg = h$ and (f, g) is Σ-tight.

- **Lemma.**— $H_{/\Sigma}$ is a cancellative and Noetherian germ; it is left-associative (resp. right-associative) whenever H is closed under Σ-suffix (resp. prefix) in G.

- **Proposition.**— Assume G is a group, Σ positively generates G, and $H \subseteq G$ is closed under Σ-suffix and Σ-prefix in G. If any two elements of H admits a least common Σ-prefix, then $H_{/\Sigma}$ is a Garside germ.

- **Example.**— $G = \mathfrak{S}_n$ with $\Sigma = \{(i, i+1) \mid i < n\}$.
  Then $G_{/\Sigma} = Div(\Delta_n)$ is a Garside germ, and $Cat(H_{/\Sigma}) = B_n^+$.