Three termination problems

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- Three unrelated termination problems :

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- Three unrelated termination problems : partial specific answers known,

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- Three unrelated termination problems : partial specific answers known,
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- Three unrelated termination problems : partial specific answers known,
  but no global understanding: can some general tools be useful?

- Plan :

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  1. The Polish Algorithm for Left-Selfdistributivity

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- The associativity law

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(In particular, it terminates.)

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- Example: $\boldsymbol{t} = x*((x*x)*(x*x))$, $\boldsymbol{t}' = (x*x)*(x*(x*x))$, i.e., in Polish,

$$\begin{aligned}
\boldsymbol{t}_0 &= x\,x\,x*x\,x*\,*\,* \\
\boldsymbol{t}'_0 &= x\,x*x\,x\,x*\,*\,* \\[1ex]
\boldsymbol{t}_1 &= x\,x*x\,x*\,*x\,x\,x*\,*\,* \\
\boldsymbol{t}'_1 &= x\,x*x\,x\,x*\,*\,* && (= \boldsymbol{t}'_0) \\[1ex]
\boldsymbol{t}_2 &= x\,x*x\,x*\,*x\,x\,x*\,*\,* && (= \boldsymbol{t}_1) \\
\boldsymbol{t}'_2 &= x\,x*x\,x*x\,x*\,* \\[1ex]
\boldsymbol{t}_3 &= x\,x*x\,x*\,*x\,x\,x*\,*\,* && (= \boldsymbol{t}_2) \\
\boldsymbol{t}'_3 &= x\,x*x\,x*\,*x\,x*x\,x*\,*\,*
\end{aligned}$$

- Left-selfdistributivity (LD) : $x*(y*z) = (x*y)*(x*z)$,

  i.e., in Polish, $\begin{cases} x\,y\,z**\\ x\,y*x\,z** \end{cases}$ compare with associativity $\begin{cases} x\,y\,z**\\ x\,y*x* \end{cases}$

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- Example: $\boldsymbol{t} = x*((x*x)*(x*x))$, $\boldsymbol{t}' = (x*x)*(x*(x*x))$, i.e., in Polish,

$$\begin{aligned}
\boldsymbol{t}_0 &= xxx*xx*** \\
\boldsymbol{t}'_0 &= xx*xxx*** \\[1ex]
\boldsymbol{t}_1 &= xx*xx**xxx*** \\
\boldsymbol{t}'_1 &= xx*xxx*** && (= \boldsymbol{t}'_0) \\[1ex]
\boldsymbol{t}_2 &= xx*xx**xxx*** && (= \boldsymbol{t}_1) \\
\boldsymbol{t}'_2 &= xx*xx*xx** \\[1ex]
\boldsymbol{t}_3 &= xx*xx**xxx*** && (= \boldsymbol{t}_2) \\
\boldsymbol{t}'_3 &= xx*xx**xx*xx***
\end{aligned}$$

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$$
\begin{aligned}
\boldsymbol{t}_0 &= x\,x\,x*x\,x***\\
\boldsymbol{t}'_0 &= x\,x*x\,x\,x***\\[1ex]
\boldsymbol{t}_1 &= x\,x*x\,x**x\,x\,x***\\
\boldsymbol{t}'_1 &= x\,x*x\,x\,x*** && (= \boldsymbol{t}'_0)\\[1ex]
\boldsymbol{t}_2 &= x\,x*x\,x**x\,x\,x*** && (= \boldsymbol{t}_1)\\
\boldsymbol{t}'_2 &= x\,x*x\,x*x\,x**\\[1ex]
\boldsymbol{t}_3 &= x\,x*x\,x**x\,x\,x*** && (= \boldsymbol{t}_2)\\
\boldsymbol{t}'_3 &= x\,x*x\,x**x\,x*x\,x***\\[1ex]
\boldsymbol{t}_4 &= x\,x*x\,x**x\,x*x\,x***\\
\boldsymbol{t}'_4 &= x\,x*x\,x**x\,x*x\,x*** && (= \boldsymbol{t}'_3)
\end{aligned}
$$

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  i.e., in Polish, $\begin{cases} x\,y\,z**\\ x\,y*x\,z** \end{cases}$ compare with associativity $\begin{cases} x\,y\,z**\\ x\,y*x* \end{cases}$

- Polish Algorithm: the same as for associativity.

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\boldsymbol{t}_0 &= xxx*xx***\\
\boldsymbol{t}'_0 &= xx*xxx***\\[1ex]
\boldsymbol{t}_1 &= xx*xx**xxx***\\
\boldsymbol{t}'_1 &= xx*xxx*** && (= \boldsymbol{t}'_0)\\[1ex]
\boldsymbol{t}_2 &= xx*xx**xxx*** && (= \boldsymbol{t}_1)\\
\boldsymbol{t}'_2 &= xx*xx*xx**\\[1ex]
\boldsymbol{t}_3 &= xx*xx**xxx*** && (= \boldsymbol{t}_2)\\
\boldsymbol{t}'_3 &= xx*xx**xx*xx***\\[1ex]
\boldsymbol{t}_4 &= xx*xx**xx*xx***\\
\boldsymbol{t}'_4 &= xx*xx**xx*xx*** && (= \boldsymbol{t}'_3)
\end{aligned}$$

So $\boldsymbol{t}_4 = \boldsymbol{t}'_4$, hence $\boldsymbol{t}_0$ and $\boldsymbol{t}'_0$ are LD-equivalent.

- Conjecture.— The Polish Algorithm works for left-selfdistributivity.

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- Known.— (i) If it terminates, the Polish Algorithm works for left-selfdistributivity.

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- Known.— (i) If it terminates, the Polish Algorithm works for left-selfdistributivity.
  (ii) The smallest counter-example to termination (if any) is huge.

1. The Polish Algorithm for Left-Selfdistributivity
2. Handle reduction of braids
3. Subword reversing for positively presented groups

- A true (but infinite) rewrite system.

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- Alphabet: `a`, `b`, `A`, `B`

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and, more generally,

  - $\mathtt{ab}^i\mathtt{A} \longrightarrow \mathtt{Ba}^i\mathtt{b}$,

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$$\begin{aligned} \boldsymbol{w}_0 &= \mathtt{a}\underline{\mathtt{abA}}\mathtt{bbAA} \\ \boldsymbol{w}_1 &= \mathtt{aBabbbAA} \end{aligned}$$

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\boldsymbol{w}_2 &= \mathtt{aBBaa}\underline{\mathtt{abA}}\\
\boldsymbol{w}_3 &= \mathtt{aBBaaBab},
\end{aligned}$$

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- **Theorem.**— The process terminates in quadratic time.

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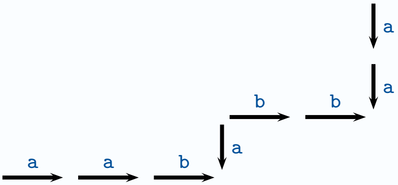

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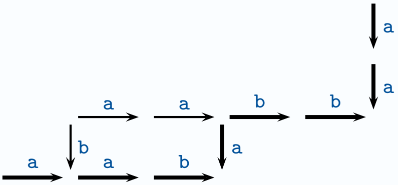

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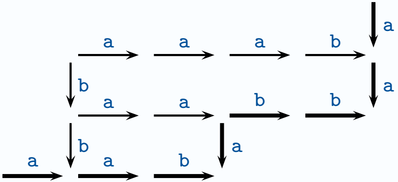

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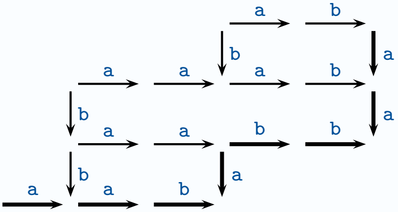

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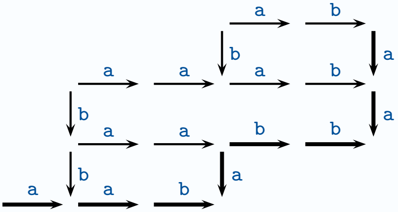
\boldsymbol{w}_3 &= \texttt{aBBaaBab}
\end{aligned}
$$

draw the grid:



□

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  so far: case of "3-strand" braids; now: case of "4-strand" braids

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  (case of "n strand" braids entirely similar for every n).

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  - for w in $\{\mathtt{b}, \mathtt{c}, \mathtt{C}\}^*$ or $\{\mathtt{B}, \mathtt{c}, \mathtt{C}\}^*$:

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  - for w in $\{\mathtt{b}, \mathtt{c}, \mathtt{C}\}^*$ or $\{\mathtt{B}, \mathtt{c}, \mathtt{C}\}^*$: $\mathtt{a}w\mathtt{A} \to \phi_{\mathtt{a}}(w)$, $\mathtt{A}w\mathtt{a} \to \phi_{\mathtt{A}}(w)$,
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  - for w in $\{\mathtt{b}, \mathtt{c}, \mathtt{C}\}^*$ or $\{\mathtt{B}, \mathtt{c}, \mathtt{C}\}^*$: $\mathtt{a}w\mathtt{A} \longrightarrow \phi_{\mathtt{a}}(w)$, $\mathtt{A}w\mathtt{a} \longrightarrow \phi_{\mathtt{A}}(w)$,
    with $\phi_{\mathtt{a}}(w)$ obtained from w by $\mathtt{b} \longrightarrow \mathtt{Bab}$ and $\mathtt{B} \longrightarrow \mathtt{BAb}$,
    and $\phi_{\mathtt{A}}(w)$ obtained from w by $\mathtt{b} \longrightarrow \mathtt{baB}$ and $\mathtt{B} \longrightarrow \mathtt{bAB}$,

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  - for w in $\{\mathtt{b}, \mathtt{c}, \mathtt{C}\}^*$ or $\{\mathtt{B}, \mathtt{c}, \mathtt{C}\}^*$: $\mathtt{a}w\mathtt{A} \longrightarrow \phi_{\mathtt{a}}(w)$, $\mathtt{A}w\mathtt{a} \longrightarrow \phi_{\mathtt{A}}(w)$,
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    and $\phi_{\mathtt{A}}(w)$ obtained from w by $\mathtt{b} \longrightarrow \mathtt{baB}$ and $\mathtt{B} \longrightarrow \mathtt{bAB}$,
  - for w in $\{\mathtt{c}\}^*$ or $\{\mathtt{C}\}^*$: $\mathtt{b}w\mathtt{B} \longrightarrow \phi_{\mathtt{b}}(w)$, $\mathtt{B}w\mathtt{b} \longrightarrow \phi_{\mathtt{B}}(w)$,

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    with $\phi_{\mathtt{a}}(w)$ obtained from w by $\mathtt{b} \longrightarrow \mathtt{Bab}$ and $\mathtt{B} \longrightarrow \mathtt{BAb}$,
    and $\phi_{\mathtt{A}}(w)$ obtained from w by $\mathtt{b} \longrightarrow \mathtt{baB}$ and $\mathtt{B} \longrightarrow \mathtt{bAB}$,
  - for w in $\{\mathtt{c}\}^*$ or $\{\mathtt{C}\}^*$: $\mathtt{b}w\mathtt{B} \longrightarrow \phi_{\mathtt{b}}(w)$, $\mathtt{B}w\mathtt{b} \longrightarrow \phi_{\mathtt{B}}(w)$,
    with $\phi_{\mathtt{b}}(w)$ obtained from w by $\mathtt{c} \longrightarrow \mathtt{Cbc}$ and $\mathtt{C} \longrightarrow \mathtt{CBc}$,
    and $\phi_{\mathtt{B}}(w)$ obtained from w by $\mathtt{c} \longrightarrow \mathtt{cbC}$ and $\mathtt{C} \longrightarrow \mathtt{cBC}$.

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    with $\phi_{\mathtt{a}}(w)$ obtained from w by $\mathtt{b} \longrightarrow \mathtt{Bab}$ and $\mathtt{B} \longrightarrow \mathtt{BAb}$,
    and $\phi_{\mathtt{A}}(w)$ obtained from w by $\mathtt{b} \longrightarrow \mathtt{baB}$ and $\mathtt{B} \longrightarrow \mathtt{bAB}$,
  - for w in $\{\mathtt{c}\}^*$ or $\{\mathtt{C}\}^*$: $\mathtt{b}w\mathtt{B} \longrightarrow \phi_{\mathtt{b}}(w)$, $\mathtt{B}w\mathtt{b} \longrightarrow \phi_{\mathtt{B}}(w)$,
    with $\phi_{\mathtt{b}}(w)$ obtained from w by $\mathtt{c} \longrightarrow \mathtt{Cbc}$ and $\mathtt{C} \longrightarrow \mathtt{CBc}$,
    and $\phi_{\mathtt{B}}(w)$ obtained from w by $\mathtt{c} \longrightarrow \mathtt{cbC}$ and $\mathtt{C} \longrightarrow \mathtt{cBC}$.

- Remark.— $\mathtt{ab}^i\mathtt{A} \longrightarrow (\mathtt{Bab})^i \longrightarrow \mathtt{Ba}^i\mathtt{b}$: extends the 3-strand case.

- Example:

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  `abcbABABCBA`

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    `abcbABABCBA`

- Example:

  ```
  abcbABABCBA
  BabcBabBABCBA
  ```

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  `abcbABABCBA`
  `BabcBabBABCBA`
  `BabcBaABCBA`
  `BabcBBCBA`
  `BaCbcBCBA`

- Example:

  ```
  abcbABABCBA
  BabcBabBABCBA
  BabcBaABCBA
  BabcBBCBA
  BaCbcBCBA
  BaCCbcCBA
  ```

- Example:

  abcbABABCBA
  BabcBabBABCBA
  BabcBaABCBA
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  BaCbcBCBA
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  BaCCbBA

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  abcbABABCBA
  BabcBabBABCBA
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⇝ Terminates: the final word does not contain both `a` and `A`

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$\rightsquigarrow$ Terminates: the final word does not contain both `a` and `A`
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- Theorem.— Handle reduction always terminates in exponential time

- Example:

  $\underline{\mathtt{abcbA}}\mathtt{BABCBA}$
  $\mathtt{BabcBa}\underline{\mathtt{bB}}\mathtt{ABCBA}$
  $\mathtt{BabcB}\underline{\mathtt{aA}}\mathtt{BCBA}$
  $\mathtt{Ba}\underline{\mathtt{bcB}}\mathtt{BCBA}$
  $\mathtt{BaC}\underline{\mathtt{bcB}}\mathtt{CBA}$
  $\mathtt{BaCCb}\underline{\mathtt{cC}}\mathtt{BA}$
  $\mathtt{BaCC}\underline{\mathtt{bB}}\mathtt{A}$
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- **Theorem.**— Handle reduction always terminates in exponential time
(and *id.* for n-strand version).

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- Theorem.— Handle reduction always terminates in exponential time
(and *id.* for n-strand version).

- Experimental evidence.— It terminates in quadratic time (for every n).

- A 4-strand braid diagram

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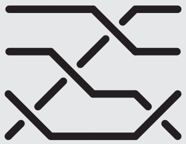
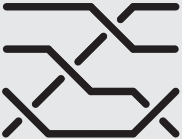


- A 4-strand braid diagram = 2D-projection of a 3D-figure:



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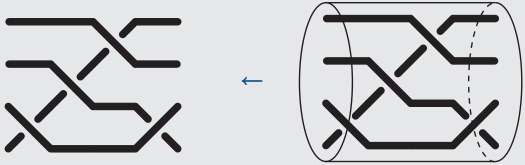

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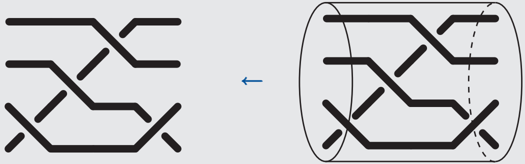

- isotopy = move the strands but keep the ends fixed:

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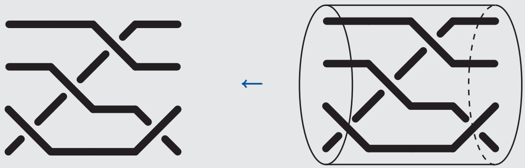

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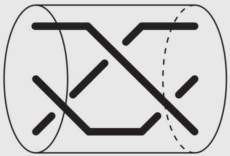
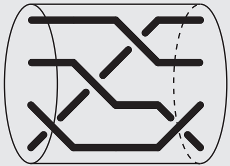
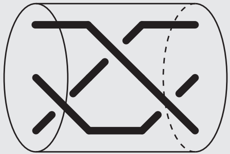


- A $\mathbf{4}$-strand braid diagram = 2D-projection of a 3D-figure:

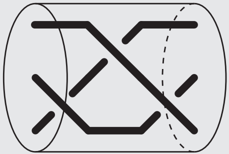
$\leftarrow$



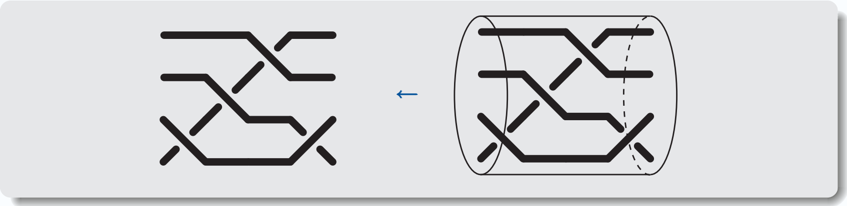
- isotopy = move the strands but keep the ends fixed:



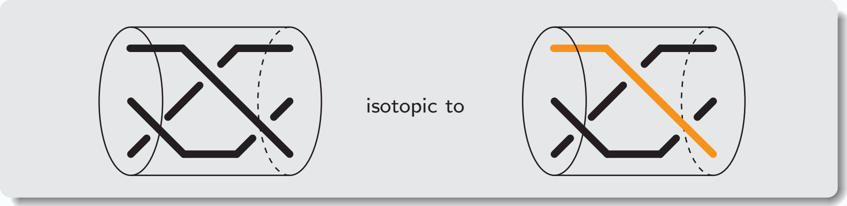
isotopic to



- A 4-strand braid diagram = 2D-projection of a 3D-figure:



- isotopy = move the strands but keep the ends fixed:



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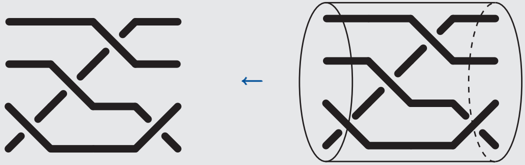

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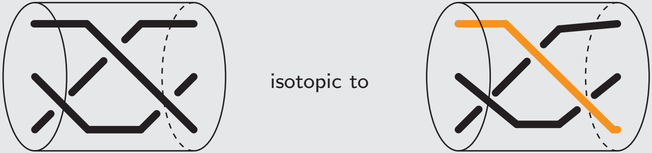

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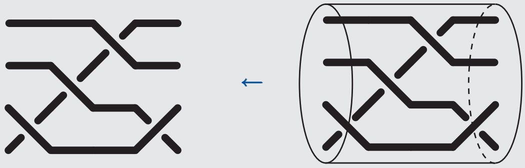

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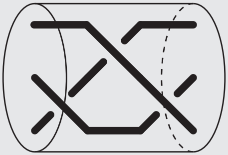
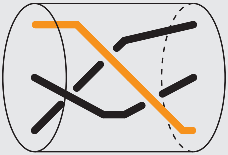


isotopic to



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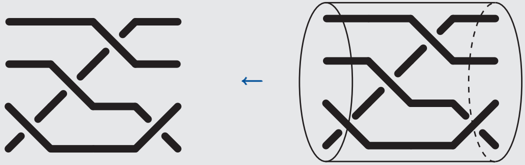

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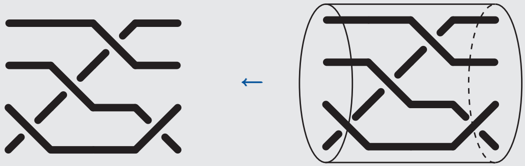

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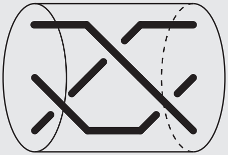
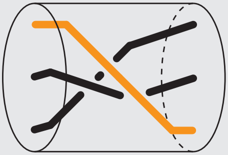


isotopic to



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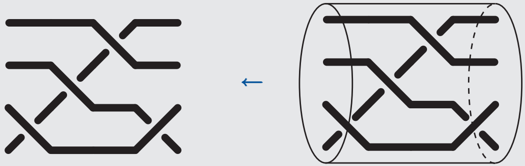

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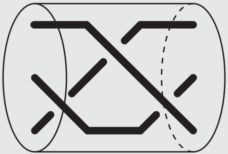
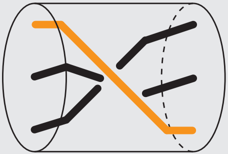


isotopic to



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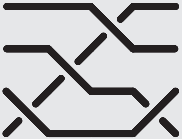
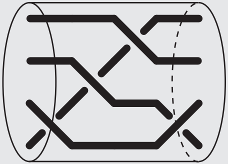

←


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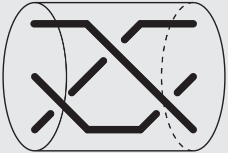
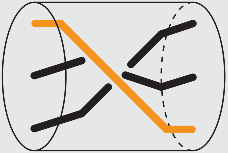
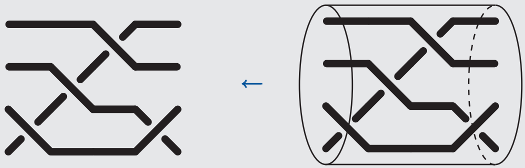
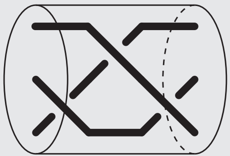

isotopic to


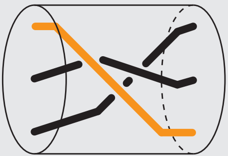
- A $\mathbf{4}$-strand braid diagram = 2D-projection of a 3D-figure:



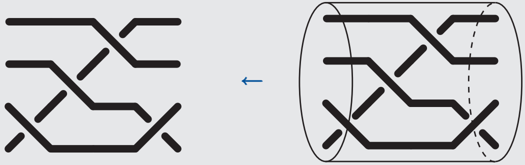
- isotopy = move the strands but keep the ends fixed:



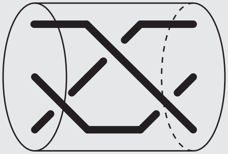
isotopic to



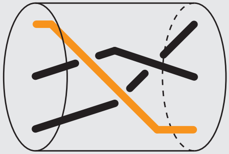
- A 4-strand braid diagram = 2D-projection of a 3D-figure:



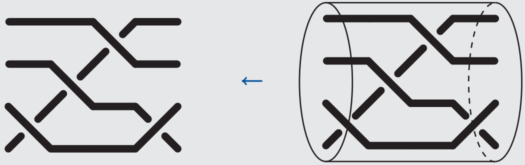
- isotopy = move the strands but keep the ends fixed:



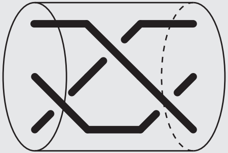
isotopic to



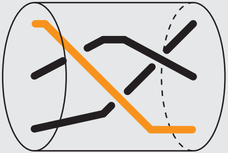
- A 4-strand braid diagram = 2D-projection of a 3D-figure:



- isotopy = move the strands but keep the ends fixed:



isotopic to



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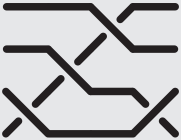

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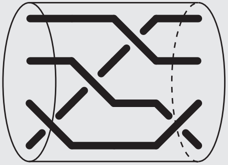
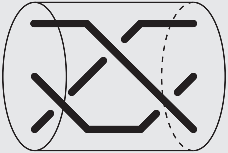
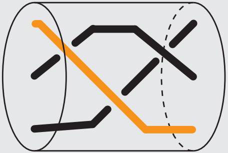


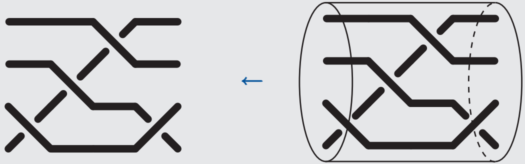
- isotopy = move the strands but keep the ends fixed:



isotopic to

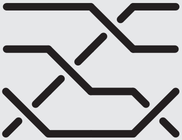


- A $\mathbf{4}$-strand braid diagram $=$ 2D-projection of a 3D-figure:

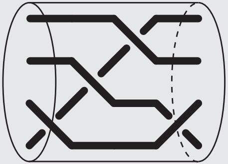


- isotopy $=$ move the strands but keep the ends fixed:

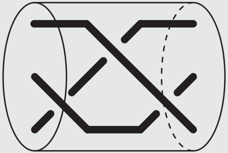
- A $\mathbf{4}$-strand braid diagram = 2D-projection of a 3D-figure:



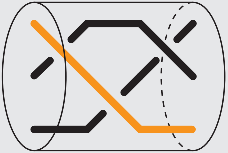
$\longleftarrow$



- isotopy = move the strands but keep the ends fixed:

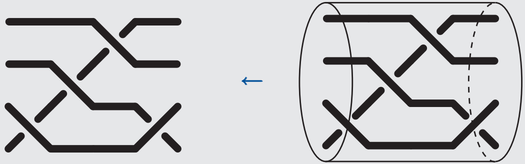


isotopic to



- a braid := an isotopy class $\rightsquigarrow$ represented by 2D-diagram,

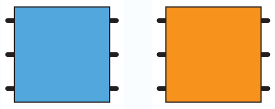
- A 4-strand braid diagram = 2D-projection of a 3D-figure:



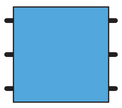
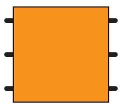
- isotopy = move the strands but keep the ends fixed:

- a braid := an isotopy class $\rightsquigarrow$ represented by 2D-diagram,
  but different 2D-diagrams may give rise to the same braid.

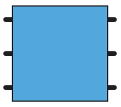
- Product of two braids:



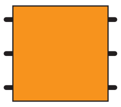
- Product of two braids:

 $*$  $:=$

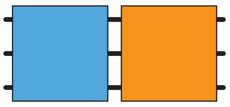
- Product of two braids:



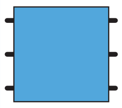
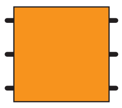
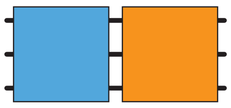
$*$



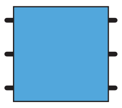
$:=$



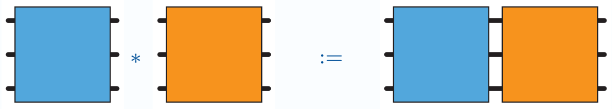
- Product of two braids:

 $*$  $:=$ 

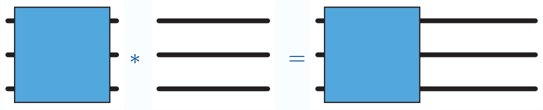
- Then well-defined with respect to isotopy), associative, admits a unit:

 $*$  $=$

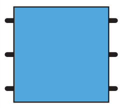
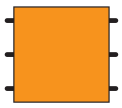
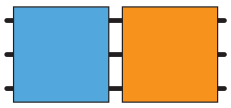
- Product of two braids:



- Then well-defined with respect to isotopy), associative, admits a unit:



- Product of two braids:

 $*$  $:=$ 

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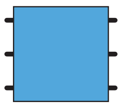
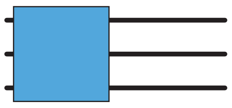
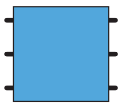
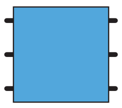
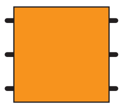
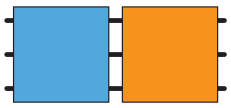
 $*$  $=$  $\approx$ 

- Product of two braids:

 $*$  $:=$ 

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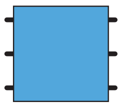
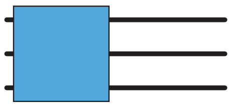
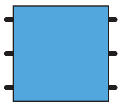
 $*$  $=$  $\underset{\substack{\uparrow \\ \text{isotopic to}}}{\approx}$ 

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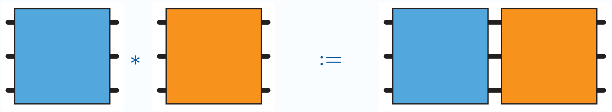

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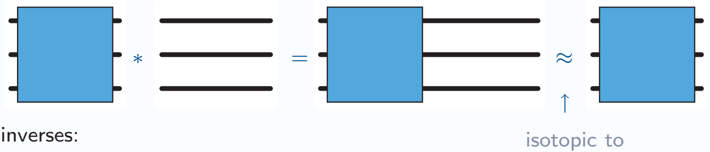
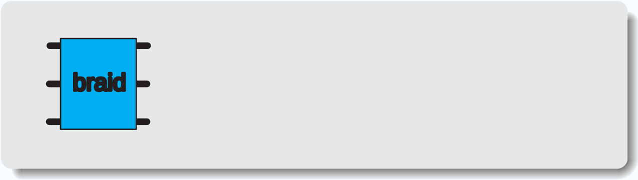


and inverses:



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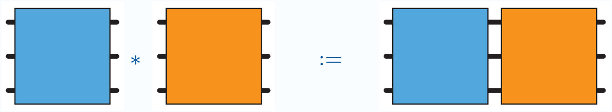

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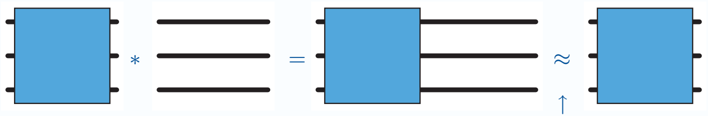
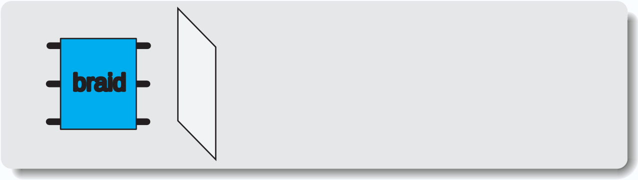
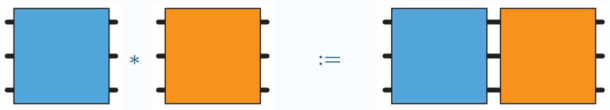


isotopic to

and inverses:



- Product of two braids:



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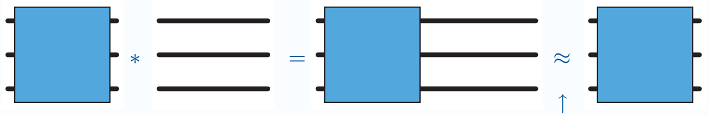
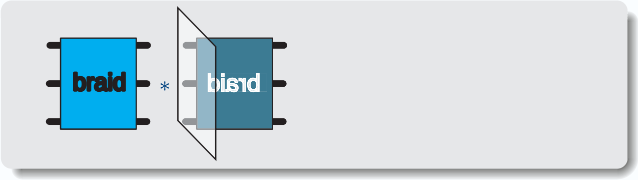
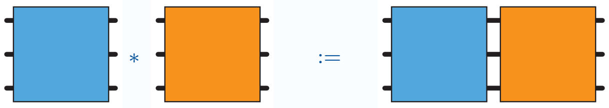


and inverses:

isotopic to



- Product of two braids:



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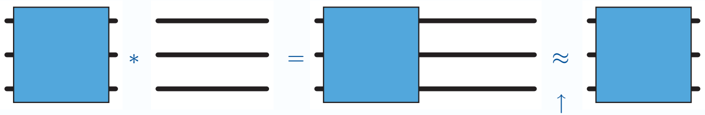
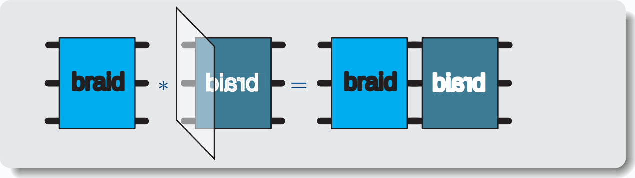


isotopic to

and inverses:



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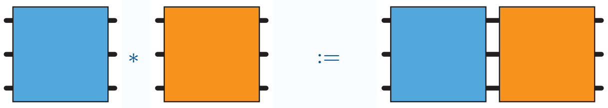

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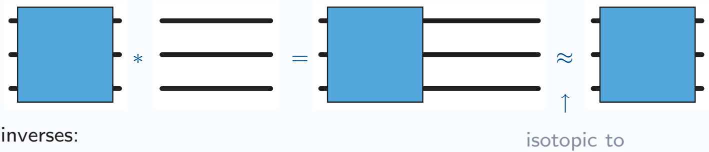
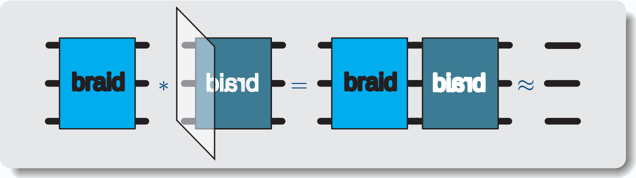


and inverses:



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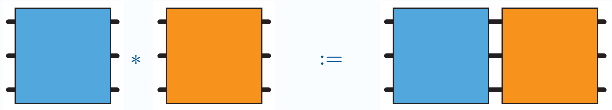

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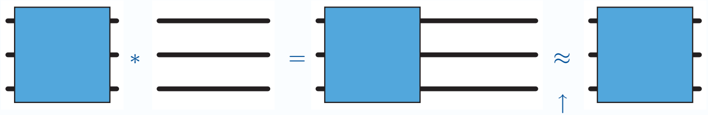

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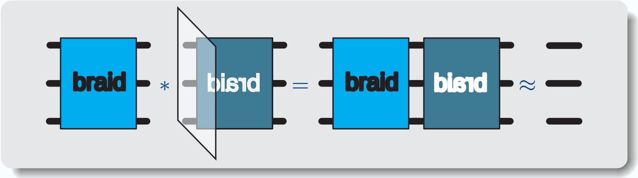
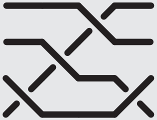


⇝ For each n, the group B_n of n-strand braids (E.Artin, 1925).

- Artin generators of B_n:



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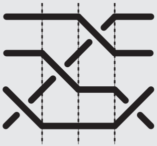
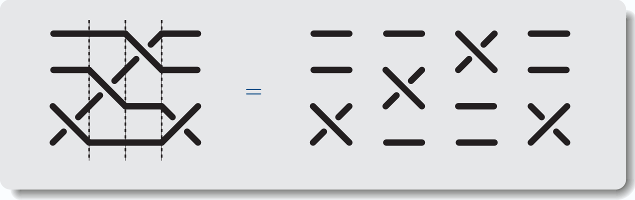


- Artin generators of B_n:



- Artin generators of B_n:

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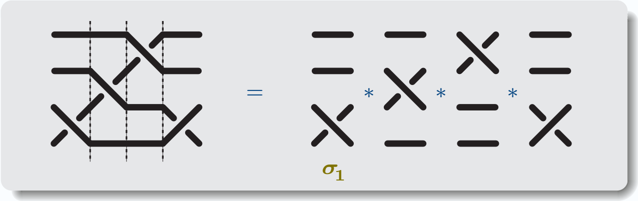
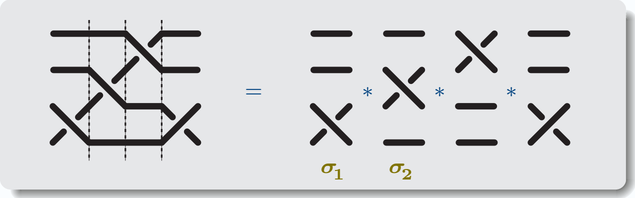
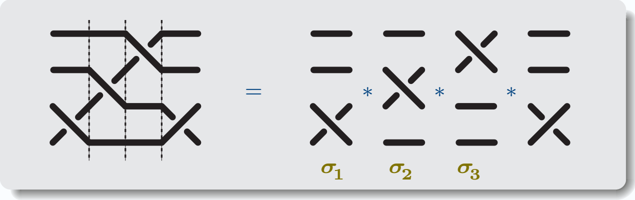


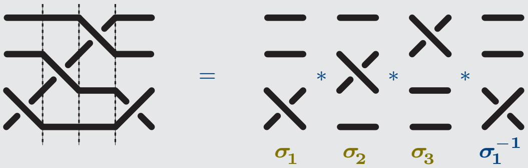
- Artin generators of B_n:



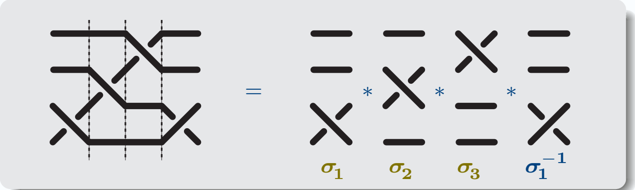
- Artin generators of B_n:



- Artin generators of B_n:

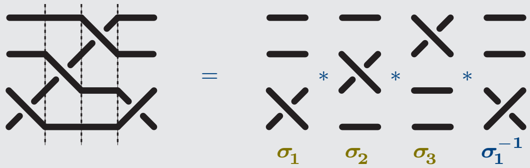


- Artin generators of B_n:



- Theorem (Artin): The group B_n is generated by $\sigma_1, \ldots, \sigma_{n-1}$,

- Artin generators of B_n:



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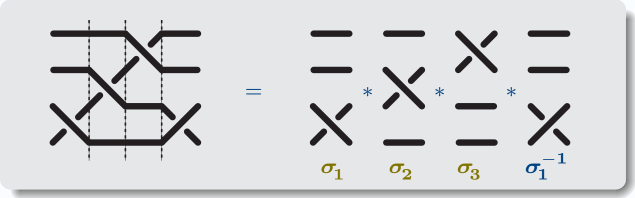

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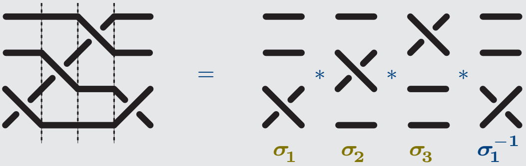

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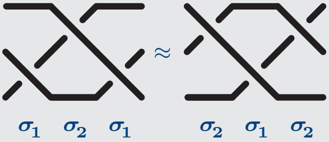

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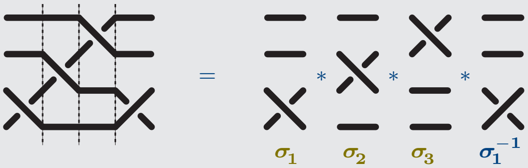

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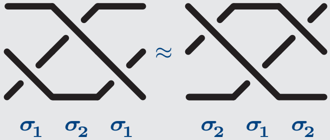

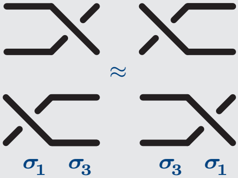

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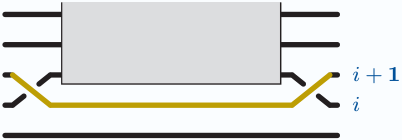

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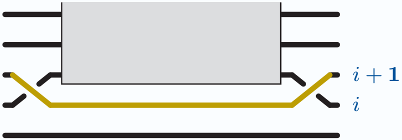

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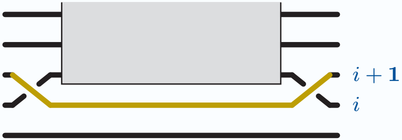

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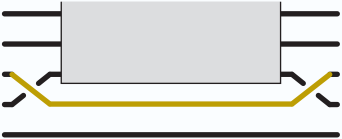

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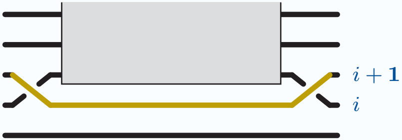

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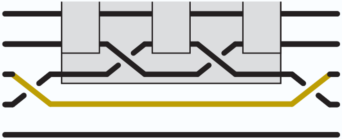

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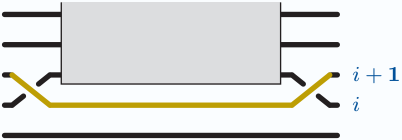

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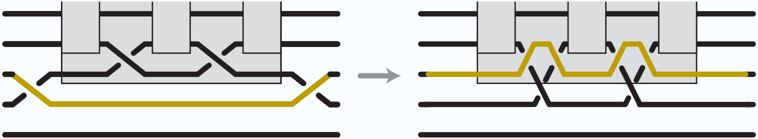

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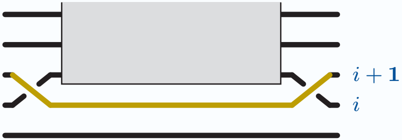

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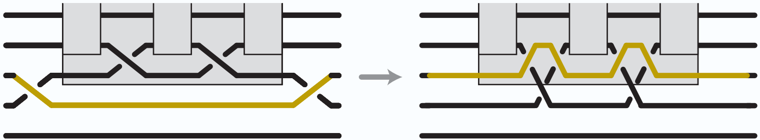

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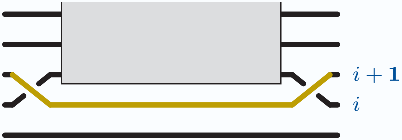

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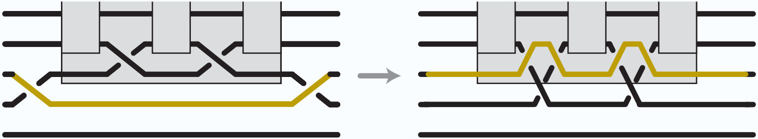

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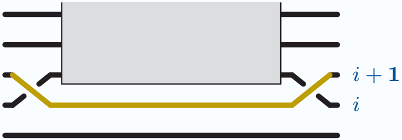

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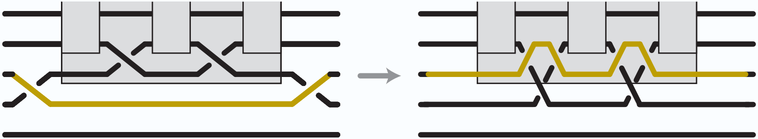

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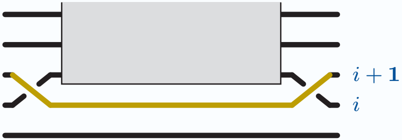

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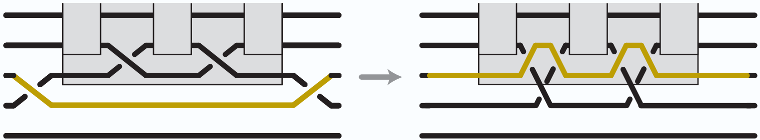

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- Theorem.— Every sequence of handle reductions terminates.

1. The Polish Algorithm for Left-Selfdistributivity
2. Handle reduction of braids
3. Subword reversing for positively presented groups

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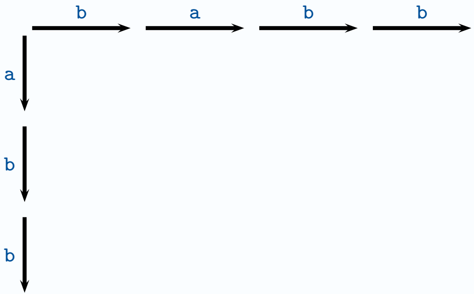

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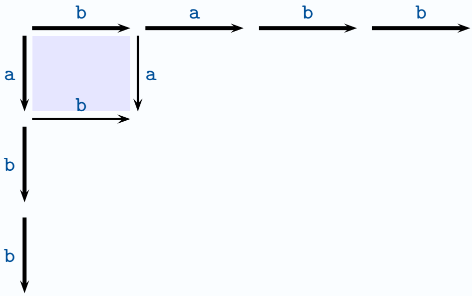

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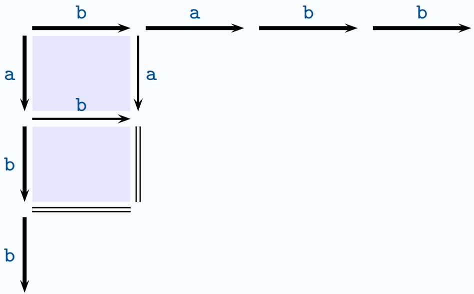

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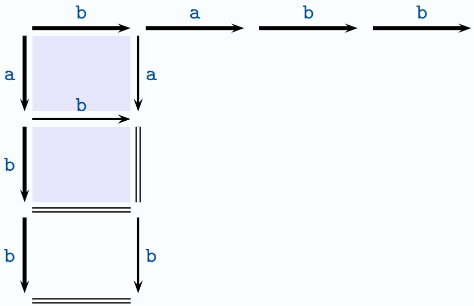

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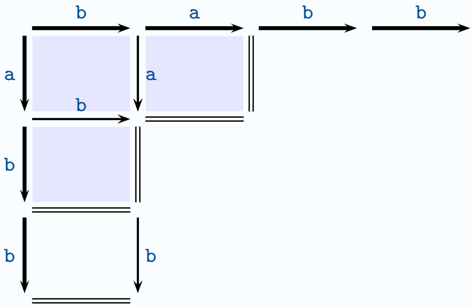

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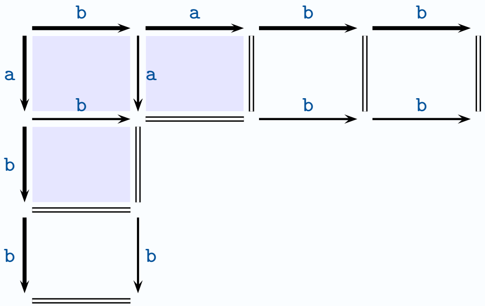

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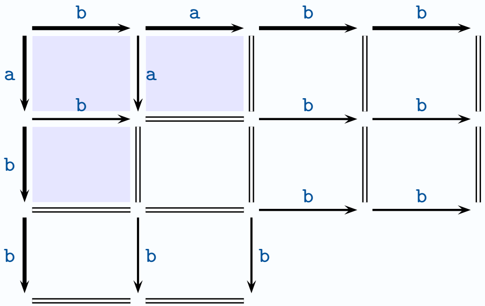

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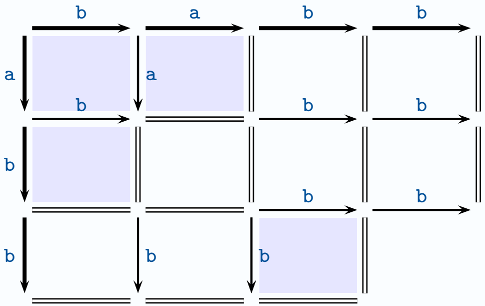

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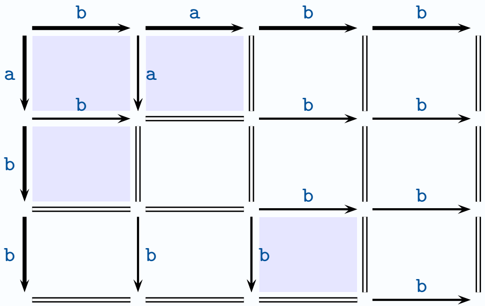

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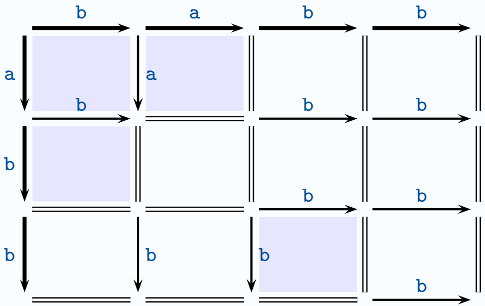

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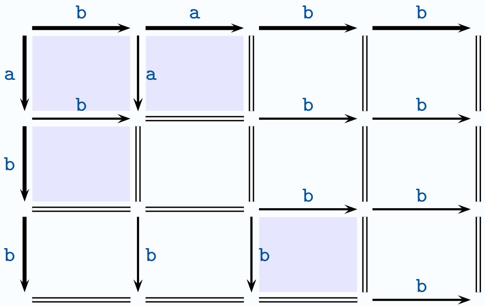

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  - $\mathtt{Ab} \longrightarrow \mathtt{baBA}$, $\mathtt{Ba} \longrightarrow \mathtt{abAB}$. ("reverse $-+$ into $+-$", but different rule)

    $\rightsquigarrow$ Again: transforms an arbitrary signed word into a positive–negative word.

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- Example: $\mathtt{BB\underline{Ab}abb} \longrightarrow \mathtt{B\underline{Bb}aBAabb} \longrightarrow \mathtt{\underline{Ba}BAabb}$

- Example 2:
- Same alphabet: $\mathtt{a}, \mathtt{b}, \mathtt{A}, \mathtt{B}$

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  $\longrightarrow$ $\mathtt{abAB}\underline{\mathtt{Aa}}\mathtt{bb}$ $\longrightarrow$ $\mathtt{abA}\underline{\mathtt{Bb}}\mathtt{b}$ $\longrightarrow$ $\mathtt{abA}\underline{\mathtt{Bb}}$

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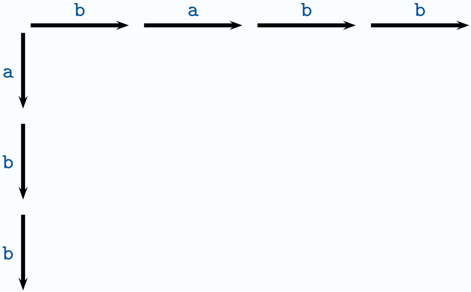
- Reversing grid:

- Reversing grid: same, but possibly smaller and smaller arrows.



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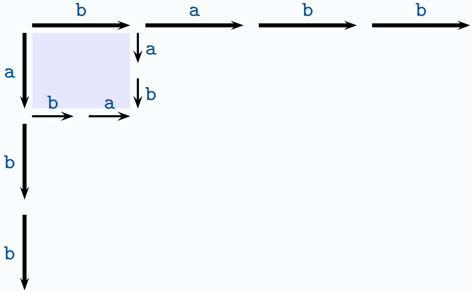

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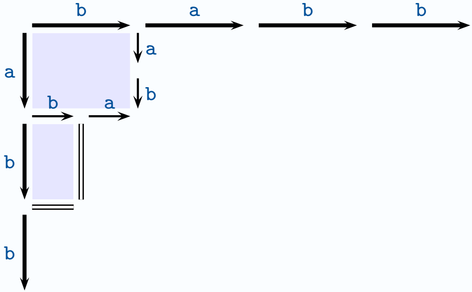

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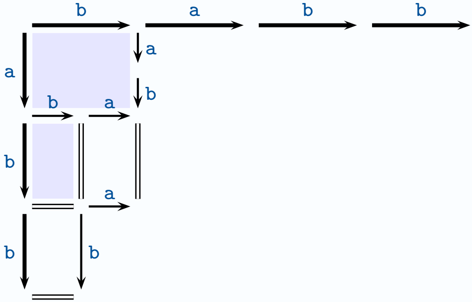

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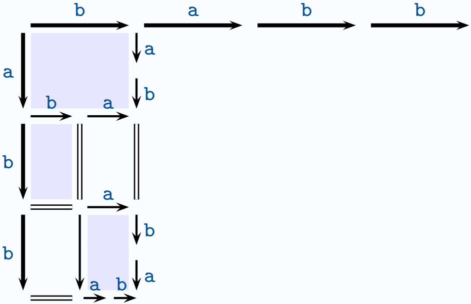

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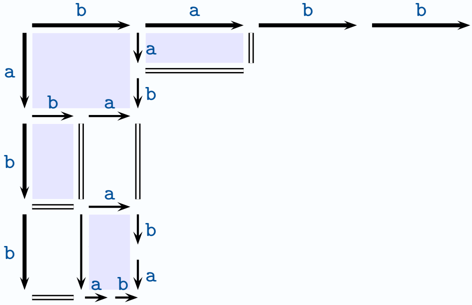

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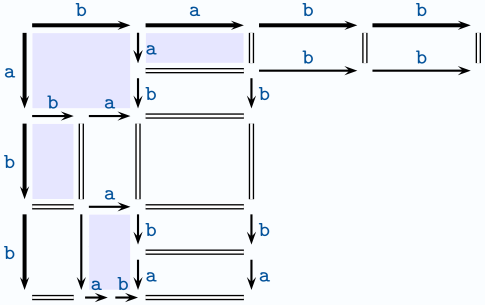

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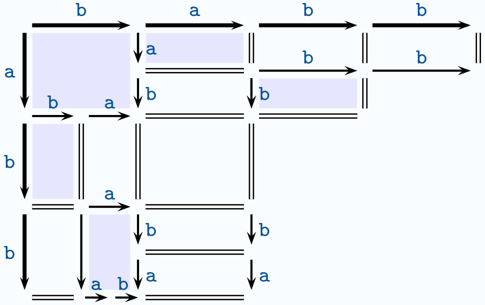

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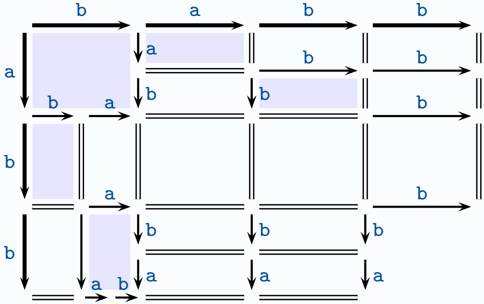

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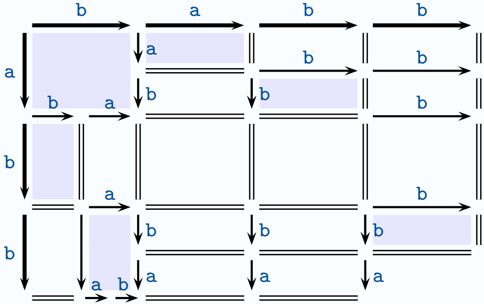

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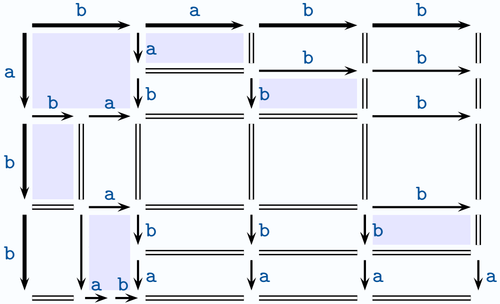

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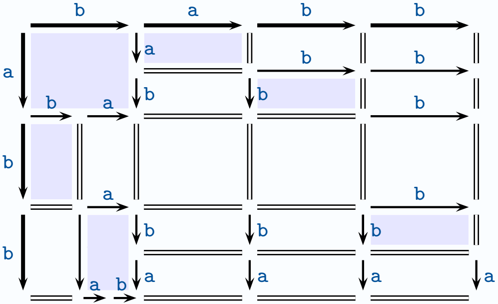

- Theorem.— Reversing terminates in quadratic time (in this specific case).

- Proof:

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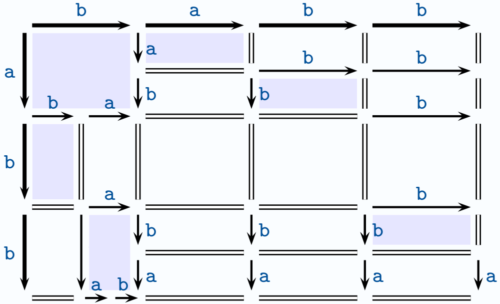

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- Proof: Return to the baby case

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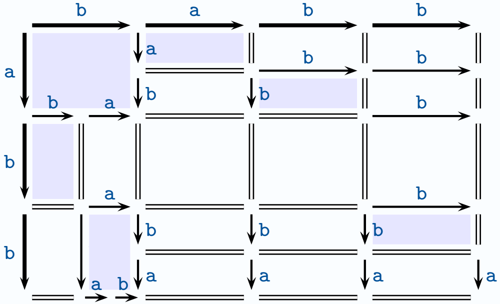

- Theorem.— Reversing terminates in quadratic time (in this specific case).

- Proof: Return to the baby case = find a (finite) set of words S that includes the alphabet and closed under reversing.

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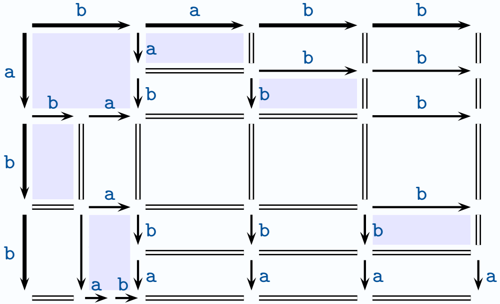

- Theorem.— Reversing terminates in quadratic time (in this specific case).

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↑ for all u, v in S, exist u', v' in S s.t. $\exists$ reversing grid

$$\begin{tikzcd} \cdot \arrow[r, "v"] \arrow[d, "u"'] & \cdot \arrow[d, "u'"] \\ \cdot \arrow[r, "v'"'] & \cdot \end{tikzcd}$$

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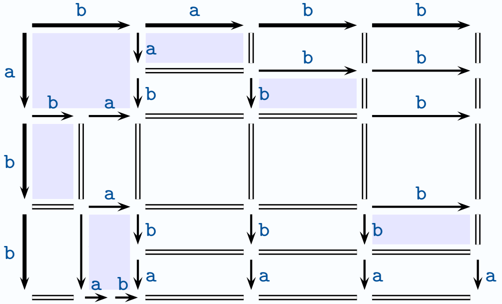

• Theorem.— Reversing terminates in quadratic time (in this specific case).

• Proof: Return to the baby case = find a (finite) set of words $\boldsymbol{S}$ that includes the alphabet and closed under reversing.

for all $\boldsymbol{u}, \boldsymbol{v}$ in $\boldsymbol{S}$, exist $\boldsymbol{u}', \boldsymbol{v}'$ in $\boldsymbol{S}$ s.t. $\exists$ reversing grid
$$\begin{tikzcd} \cdot \arrow[r, "\boldsymbol{v}"] \arrow[d, "\boldsymbol{u}"'] & \cdot \arrow[d, "\boldsymbol{u}'"] \\ \cdot \arrow[r, "\boldsymbol{v}'"'] & \cdot \end{tikzcd}$$

Here: works with $\boldsymbol{S} = \{\mathtt{a}, \mathtt{b}, \mathtt{ab}, \mathtt{ba}\}$. $\square$

- Always like that?

- Always like that? Not really...

- Example 3:

Alphabet `a`, `b`, `A`, `B`,

- Always like that? Not really...

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Alphabet $\mathtt{a}, \mathtt{b}, \mathtt{A}, \mathtt{B}$, rules $\mathtt{Aa} \longrightarrow \varepsilon$, $\mathtt{Bb} \longrightarrow \varepsilon$,

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- Definition.— Assume (A, R) semigroup presentation and, for all $s \neq t$ in A, there is exactly one relation $s... = t...$ in R, say $sC(s,t) = tC(t,s)$. Then reversing is the rewrite system on $A \cup \overline{A}$ (a copy of A, here : capitalized letters) with rules $\overline{s}s \to \varepsilon$ and $\overline{s}t \to C(s,t)\overline{C(t,s)}$ for $s \neq t$ in A.

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- Question.— What can YOU say about reversing?

For the Polish Algorithm:

• P. Dehornoy, Braids and selfdistributivity, Progress in math. vol 192, Birkhaüser 2000 (Chapter VIII)

• O. Deiser, Notes on the Polish Algorithm, deiser@tum.de (Technishe Universität München)

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...venez au groupe de travail du vendredi !