Isolated orderings on an orderable group

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- A simple scheme for constructing monoids in which left-divisibility is a linear ordering, connected with non-Noetherian Garside theory.
- Application: ordered groups whose space of orderings has an isolated point.

- Plan :

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    1. The space of orderings of an orderable group

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  2. Right-triangular presentations

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⇝ constructing orderable groups ⇔ constructing monoids of O-type

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$\rightsquigarrow$ Can $\boldsymbol{LO}(G)$ be infinite with isolated points?

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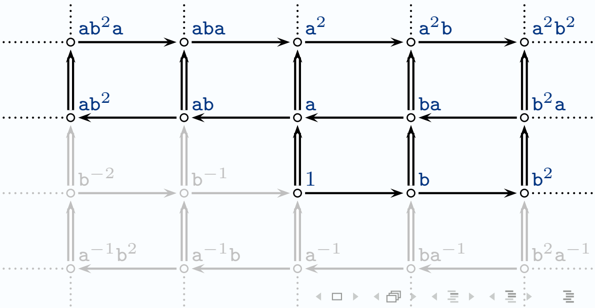

1. The space of orderings of an orderable group
2. Right-triangular presentations
3. The case of braid groups

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$\rightsquigarrow$ How to prove the existence of common right-multiples?

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1. The space of orderings of an orderable group
2. Right-triangular presentations
3. The case of braid groups

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$\rightsquigarrow$ many exotic (non-Noetherian) Garside structures on B_n.

For isolated orderings:

• A. Navas, *A remarkable family of left-ordered groups: central extensions of Hecke groups*, J. Algebra, 328 (2011) 31-42

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For non-Noetherian Garside structures:

- P. Dehornoy, with F. Digne, E. Godelle, D. Krammer, J. Michel, Foundations of Garside Theory, submitted, www.math.unicaen.fr/∼dehornoy/