Isolated orderings on an orderable group

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- A simple scheme for constructing monoids in which left-divisibility is a linear ordering, connected with non-Noetherian Garside theory.
- Application: ordered groups whose space of orderings has an isolated point.

- Plan :

  1. The space of orderings of an orderable group
  2. Right-triangular presentations
  3. The case of braid groups

1. The space of orderings of an orderable group
2. Right-triangular presentations
3. The case of braid groups

• Definition.— A group G is orderable if there exists a linear ordering $\leqslant$ on G that is left-invariant, that is, $g \leqslant h$ implies $fg \leqslant fh$ for all f, g, h.

• Lemma.— (i) If $\leqslant$ is a left-invariant ordering on G, then $P := \{g \in G \mid g > 1\}$ is a subsemigroup of G s.t. P, P^{-1}, $\{1\}$ partition G. $\leftarrow$ P : the positive cone of $\leqslant$.
(ii) Conversely, if P is a subsemigroup of G s.t. P, P^{-1}, $\{1\}$ partition G, then $g^{-1}h \in P$ defines a left-invariant linear ordering on G.

• Definition.— A monoid M is of right-O-type if M is left-cancellative, has no nontrivial invertible element, and the left-divisibility relation $\preccurlyeq$ is a linear ordering on M.

$\uparrow$ $g \preccurlyeq h$ iff $\exists h' \, (gh' = h)$

• Lemma.— (i) If $\leqslant$ is a left-invariant ordering on G, then $\{g \in G \mid g \geqslant 1\}$ is a monoid of O-type.
(ii) Conversely, if M is a monoid of O-type, then $g^{-1}h \in M \setminus \{1\}$ defines a left-invariant linear ordering on the enveloping group of M.

$\leftarrow$ right- and left-O-type

$\rightsquigarrow$ constructing orderable groups $\Leftrightarrow$ constructing monoids of O-type

- **Definition.**— For G orderable group,
  $LO(G)$:= the family of all positive cones of left-invariant orderings on G.

$\uparrow$ a subset of $\mathfrak{P}(G)$ $\qquad$ $\uparrow$ subsets of G

$\uparrow$ $\approx \{0,1\}^G$, a compact totally disconnected space

- **Proposition** (Sikora).— The set $LO(G)$ is a closed subspace of $\{0,1\}^{G\times G}$.

- Proof: - P belongs to $LO(G)$ iff

$$P^2 \subseteq P, \text{ and } P \cup P^{-1} \cup \{1\} = G \text{ and } P \cap P^{-1} = \emptyset \text{ and } 1 \notin P.$$

- P does not belong to $LO(G)$ iff $\exists g, h \; (g \in P \;\&\; h \in P \;\&\; gh \notin P)$ or...
- base of open sets

$$U_{g_1,\dots,g_p h_1,\dots,h_q} = \{X \subseteq G \mid g_1, \dots, g_p \in X \;\&\; h_1, \dots, h_q \notin X\}. \qquad \square$$

- If G is (finite or) countable, then $\mathfrak{P}(G)$ is metrizable.

- Proposition (Linnel).— A space $\boldsymbol{LO}(G)$ cannot be countably infinite.

- Corollary.— If G is countable and orderable, the space $\boldsymbol{LO}(G)$ is
  - either finite,
  - or isomorphic to the Cantor space,
  - or isomorphic to a subspace of the Cantor space with isolated points.

- Examples:
  - $\boldsymbol{LO}(\pi_1(\text{Klein bottle}))$ $(= \boldsymbol{LO}(\mathbb{Z} \rtimes \mathbb{Z}))$ has 4 elements;
  - (Sikora) $\boldsymbol{LO}(\mathbb{Z}^n)$ is a Cantor space;
  - (McCleary, Navas) $\boldsymbol{LO}(F_n)$ is a Cantor space.

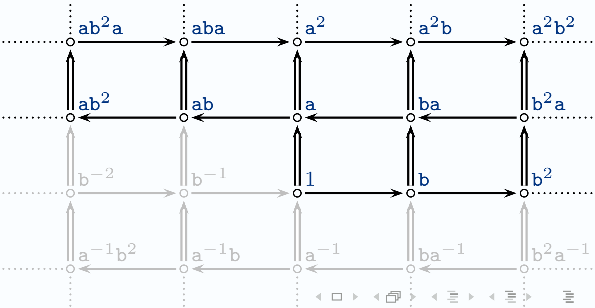
$\rightsquigarrow$ Can $\boldsymbol{LO}(G)$ be infinite with isolated points?

• Lemma.— (i) A left-invariant ordering $\leqslant$ of G is isolated iff exists a finite subset $\{g_1, ..., g_p\}$ of G s.t. $\leqslant$ is the only left-invariant ordering with $1 < g_1$, ..., $1 < g_p$.
(ii) This is true in particular if the positive cone is finitely generated as a semigroup.

• Proof: (i) $\{P_\leqslant\} = U_{g_1,...,g_p,\emptyset}$; (ii) if $P_\leqslant$ is generated by $g_1, ..., g_p$,
then every cone that contains $g_1, ..., g_p$ includes $P_\leqslant$, hence is equal to $P_\leqslant$. □

• Proposition.— Assume that the group G admits a positive presentation $\langle S \mid R \rangle$ with S finite and $\langle S \mid R \rangle^+$ of O-type. Then the subsemigroup of G generated by S is the positive cone of an isolated left-invariant ordering of G.

• Example: $\mathbb{Z} \rtimes \mathbb{Z}$
$= \langle \mathtt{a}, \mathtt{b} \mid \mathtt{ab} = \mathtt{b}^{-1}\mathtt{a} \rangle$
$= \langle \mathtt{a}, \mathtt{b} \mid \mathtt{a} = \mathtt{bab} \rangle$.



- Goal: Constructing finitely generated monoids of O-type.

  $\rightsquigarrow$ Here: consider presentations of a certain simple syntactic type.

- **Definition.**— A (positive) presentation is right-triangular if there exists an enumeration $(s_1, s_2, ...)$ of S such that R consists of relations $s_1 = s_2 w_2$, $s_2 = s_3 w_3$, ...
  (w_2, w_3, ... words in S).

- Example: $\langle \mathtt{a}, \mathtt{b}, \mathtt{c} \mid \mathtt{a} = \mathtt{bac}, \mathtt{b} = \mathtt{cba} \rangle$ is right-triangular and left-triangular.

- **Key Lemma.**— If (S, R) is right-triangular, then TFAE
  (i) $\langle S \mid R \rangle^+$ is of right-O-type;
  (ii) any two elements of $\langle S \mid R \rangle^+$ admit a common right-multiple.

- Proof: Right-reversing is necessarily complete; it necessarily provides a $\prec$-relation. $\square$

$\rightsquigarrow$ How to prove the existence of common right-multiples?

- To prove that common right-multiples exist: find a (right-pre)-Garside element.

- Lemma.— Assume that M is a left-cancellative monoid and exists Δ in M s.t.
  (i) Every right-divisor of Δ is a left-divisor of Δ,
  (ii) The left-divisors of Δ generate M.

Then any two elements of M admit a common right-multiple.

- Proof: Every element of M left-divides Δ^p for p large enough. □

- Proposition.— Assume that M is a monoid that admits a right-triangular presentation $\langle S \mid R \rangle^+$ and there exists Δ in M satisfying $s \preccurlyeq \Delta \preccurlyeq s\Delta$ for every s in S.

Then M is of right-O-type (and Δ is a right-Garside element in M).

- Proof: Construct an endomorphism ϕ of M s.t. $g\Delta = \Delta\phi(g)$ for every g. □

$\rightsquigarrow$ An easy criterion, in particular well-fitted for computer experiments

• **Proposition.**— Let $M_{p,q,r} := \langle \mathtt{a}, \mathtt{b} \mid \mathtt{a} = \mathtt{b}(\mathtt{a}^p\mathtt{b}^r)^q\rangle^+$ with $\Delta = \mathtt{a}^{p+1}$. Then $M_{p,q,r}$ is of right-O-type; for $r = 1$, it is of O-type, (and Δ is a Garside element).

• Proof: Relations $\mathtt{b} \preccurlyeq \mathtt{a} \preccurlyeq \Delta \preccurlyeq a\Delta$ straightforward; remains to check $\Delta \preccurlyeq \mathtt{b}\Delta$.
Previous proposition $\Rightarrow$ right-O-type; for $r = 1$, everything is symmetric. □

• Particular cases:

- $M_{1,1,1}$ $\mathtt{a} = \mathtt{bab}$: Klein bottle group;
- $M_{1,2,1}$ $\mathtt{a} = \mathtt{ba}^2\mathtt{b}$: braid group B_3 with $\mathtt{a} = \sigma_1\sigma_2$, $\mathtt{b} = \sigma_2^{-1}$,
  $\rightsquigarrow$ hence $\boldsymbol{LO}(B_3)$ has an isolated point;
- $M_{1,3,1}$ $\mathtt{a} = \mathtt{ba}^3\mathtt{b}$: braid group B_3 with $\mathtt{a} = \sigma_1\sigma_2\sigma_1$, $\mathtt{b} = \sigma_2^{-1}$;
- $M_{p,q,1}$ $\mathtt{x}^{p+1} = \mathtt{y}^{q+1}$ torus knot group.

- The D-ordering on B_n: a braid is larger than 1 if it admits an expression in the generators σ_i s.t. the generator with least index occurs positively only.

- **Proposition.**— (Navas) The D-ordering is the limit of its conjugates.

$\rightsquigarrow$ hence not isolated in the space $\boldsymbol{LO}(B_n)$

- **Proposition.**— (Dubrovina-Dubrovin) The submonoid $B_n^{\oplus}$ of $\boldsymbol{B_n}$ generated by $\sigma_1\sigma_2...\sigma_{n-1}$, $(\sigma_2...\sigma_{n-1})^{-1}$, $\sigma_3...\sigma_{n-1}$, $(\sigma_4...\sigma_{n-1})^{-1}$, ... is of O-type.

$\rightsquigarrow$ hence isolated in the space $\boldsymbol{LO}(B_n)$

- The monoid $B_3^{\oplus}$ admits the presentations $\langle \mathtt{a}, \mathtt{b} \mid \mathtt{a} = \mathtt{ba}^2\mathtt{b}\rangle^+$ and $\langle \mathtt{a}, \mathtt{b} \mid \mathtt{ba}^3\mathtt{b}\rangle^+$.

$\rightsquigarrow$ $=$ the monoids of O-type obtained above

- **Proposition.**— The monoid $B_4^{\oplus}$ admits no right-triangular presentation with respect to the generators $\sigma_1\sigma_2\sigma_3$, $(\sigma_1\sigma_2)^{-1}$, σ_3.

$\rightsquigarrow$ many orderings escape to the current approach

- Definition.— An element Δ of a cancellative monoid $\boldsymbol{M}$ is a Garside element in $\boldsymbol{M}$ if
  - the left- and right-divisors of Δ coincide,
  - the divisors of Δ generate $\boldsymbol{M}$,
  - for every $\boldsymbol{g}$ in $\boldsymbol{M}$, the elements $\boldsymbol{g}$ and Δ admit a left-gcd.

- **Proposition.**— Every submonoid of O-type of B_n admits $\Delta_n^{\pm 2}$ as a Garside element.

- Proof: The generators σ_i are pairwise conjugated under roots of Δ_n^{2p}. □

$\rightsquigarrow$ many exotic (non-Noetherian) Garside structures on B_n.

For isolated orderings:

- A. Navas, *A remarkable family of left-ordered groups: central extensions of Hecke groups*, J. Algebra, 328 (2011) 31-42
- T. Ito, *Dehornoy-like left-orderings and isolated left-orderings*, J. Algebra, 374 (2013) 42-58
- T. Ito, *Construction of isolated left-orderings via partially central cyclic amalgamation*, arXiv:1107.0545

For monoids of O-type and right-triangular presentations:

- P. Dehornoy; *Monoids of O-type, subword reversing, and ordered groups*; arXiv:1204.3211

For orderings on the braid groups:

- P. Dehornoy, with I. Dynnikov, D. Rolfsen, B. Wiest, Braid ordering, Math. Surveys and Monographs vol. 148, Amer. Math. Soc. 2008

For non-Noetherian Garside structures:

- P. Dehornoy, with F. Digne, E. Godelle, D. Krammer, J. Michel, Foundations of Garside Theory, submitted, www.math.unicaen.fr/∼dehornoy/