Braid combinatorics, permutations, and noncrossing partitions

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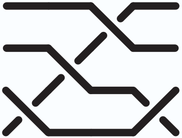

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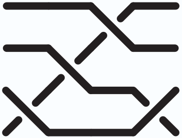

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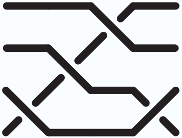

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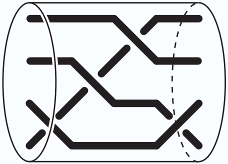

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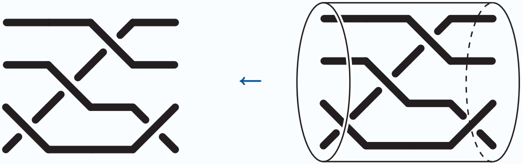

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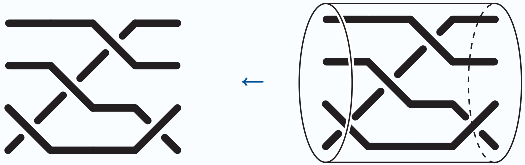

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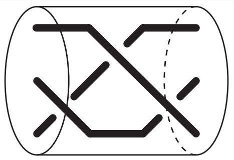

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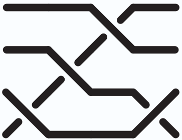

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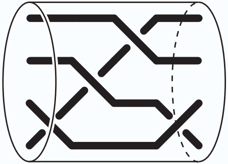

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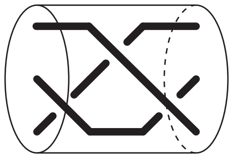

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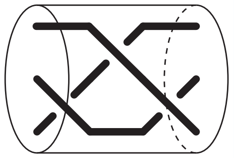

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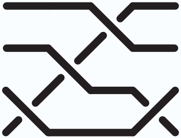

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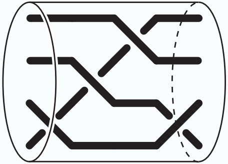

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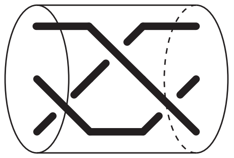

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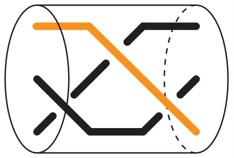

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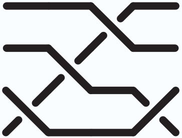

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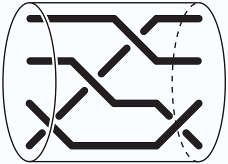

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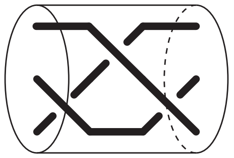

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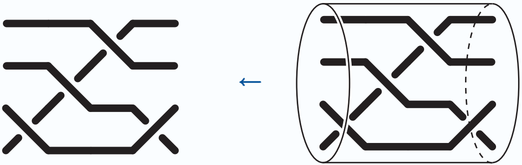

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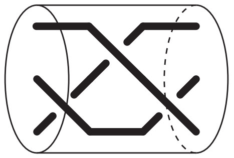

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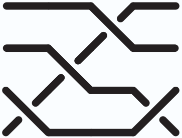

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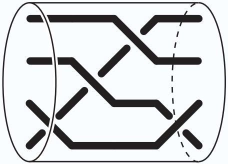

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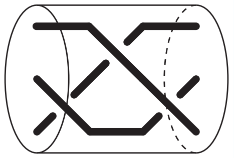

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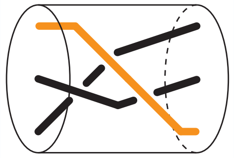

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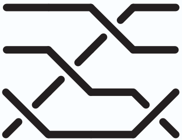

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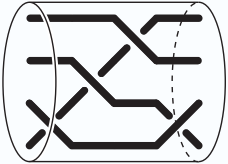

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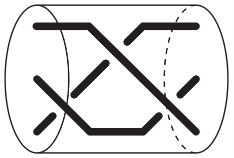

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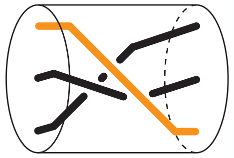

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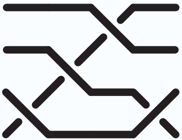

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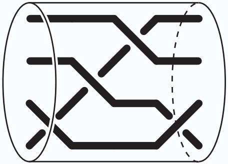

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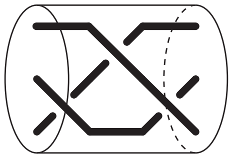

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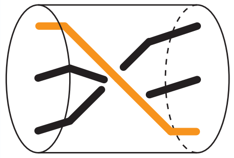

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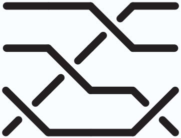

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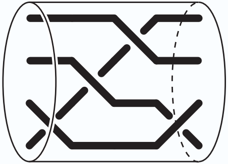

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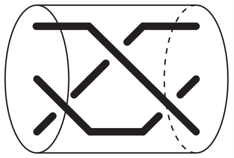

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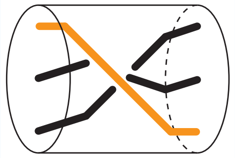

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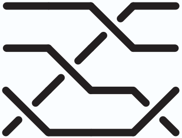

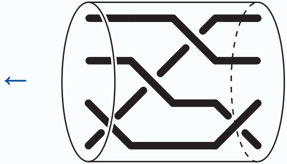

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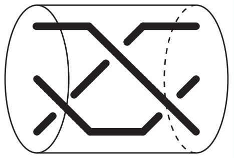

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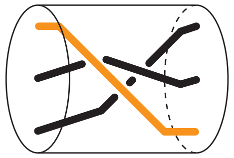

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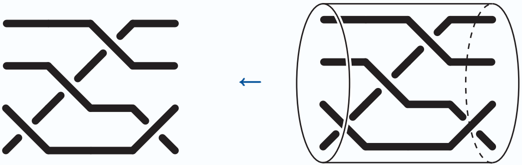

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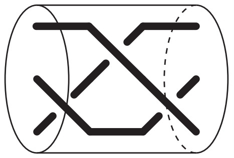

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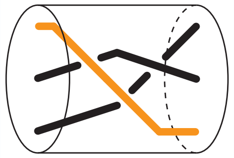

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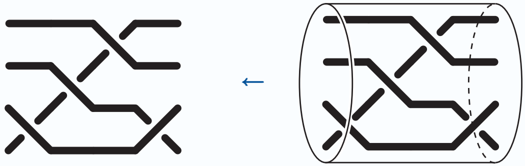

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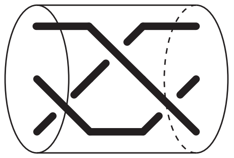

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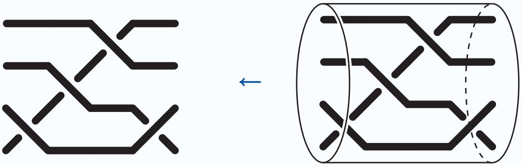

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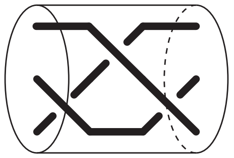

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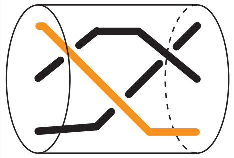

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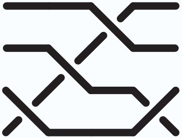

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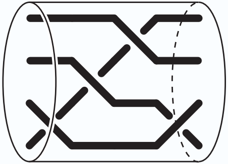

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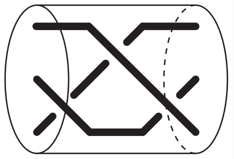

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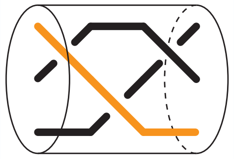

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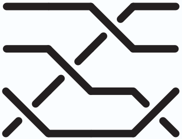

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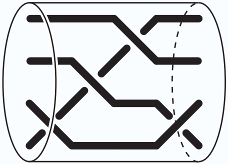

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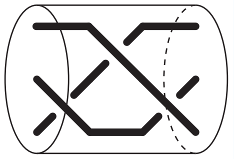

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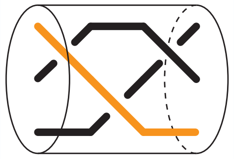

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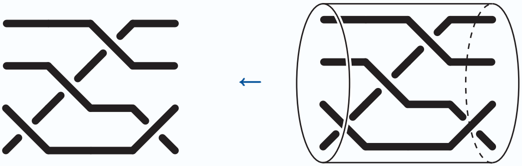

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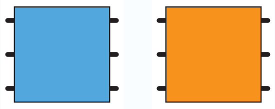
  but different 2D-diagrams may give rise to the same braid.

- Product of two braids:



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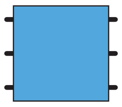
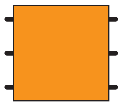
 $*$  $:=$

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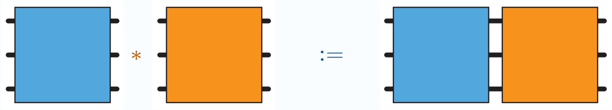

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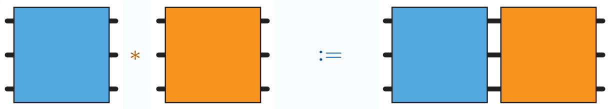

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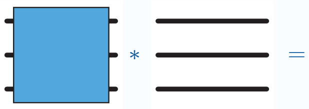

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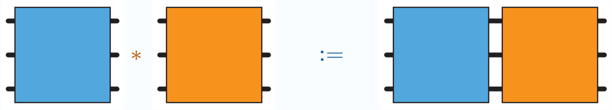

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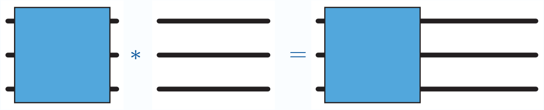

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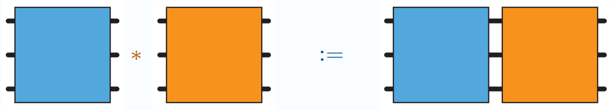

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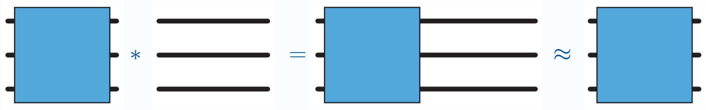

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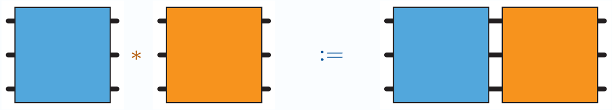

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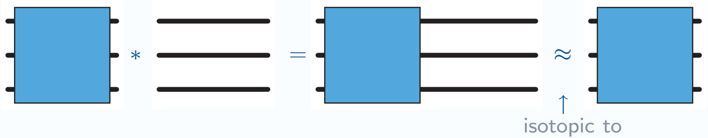

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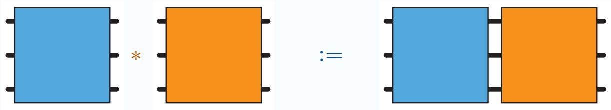

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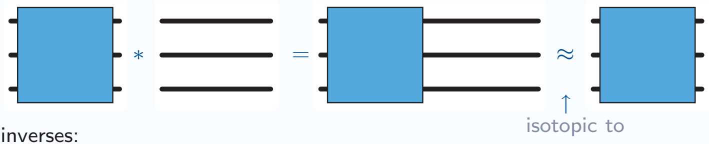

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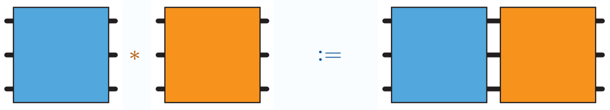

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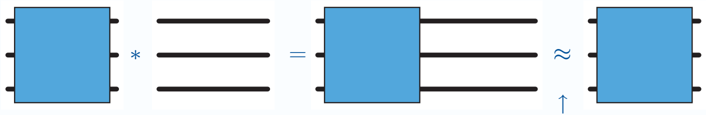

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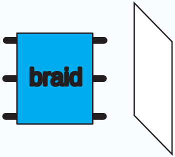

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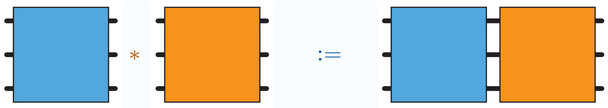

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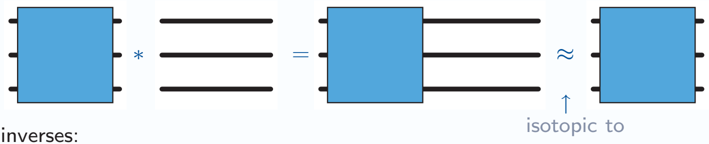

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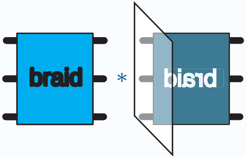

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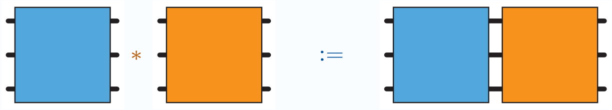

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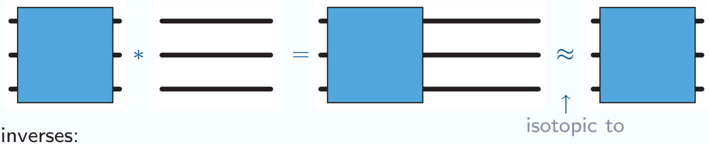

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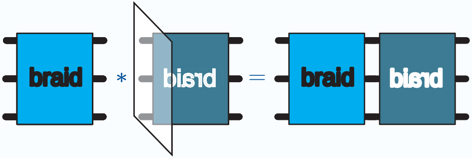

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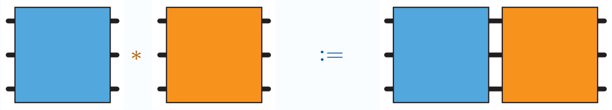

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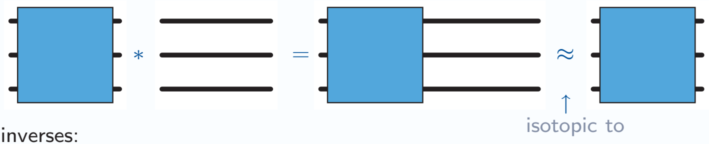

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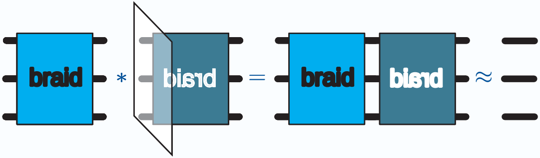

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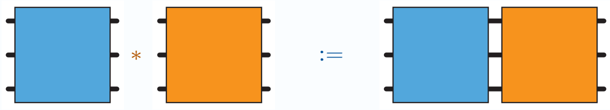

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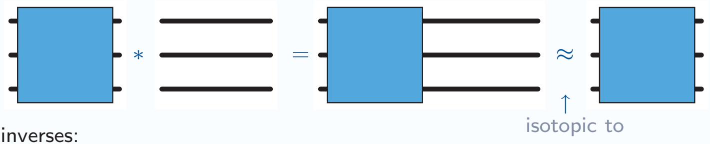

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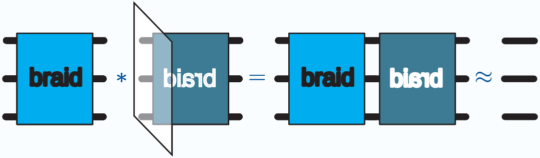

▸ For each n, the group B_n of n-strand braids (E. Artin, 1925).

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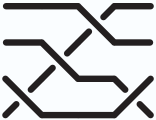

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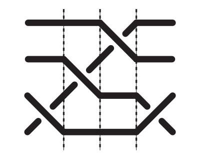

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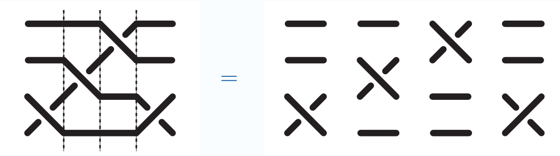

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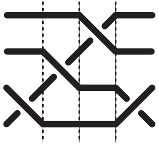

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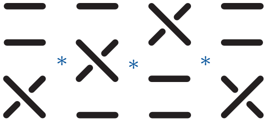

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σ_1

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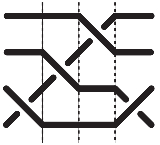

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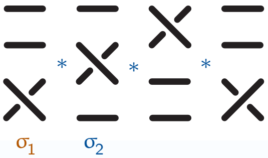

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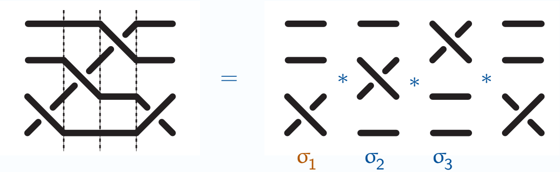

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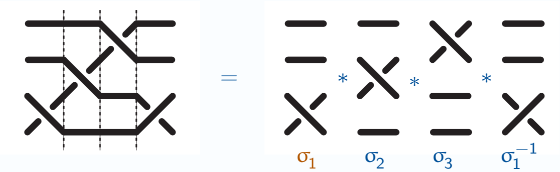

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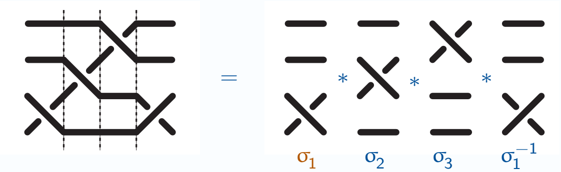

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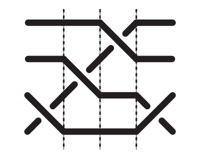

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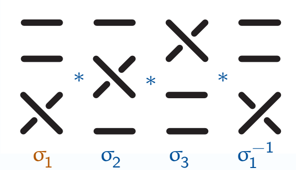

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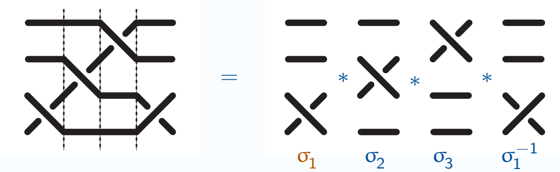

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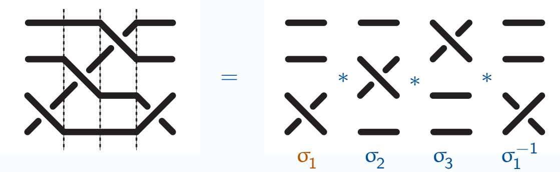

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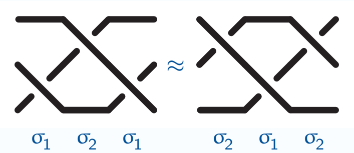

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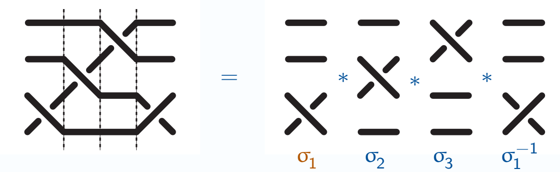

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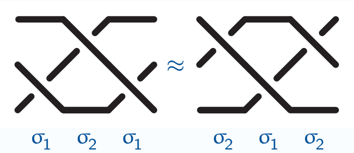

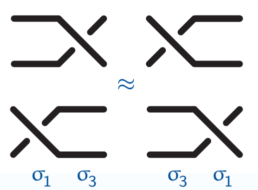

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- Plan :
  1. Braid combinatorics: Artin generators
  2. Braid combinatorics: Garside generators
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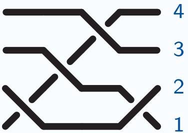

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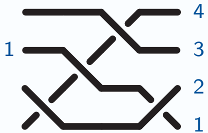

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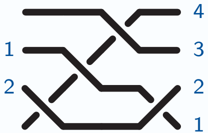

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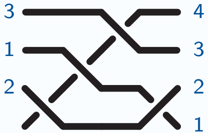

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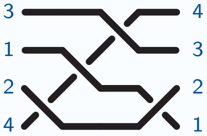

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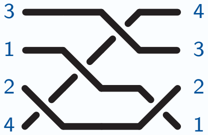

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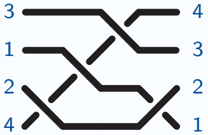

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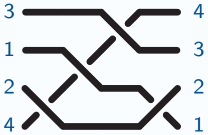

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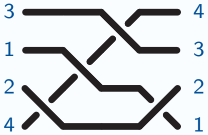

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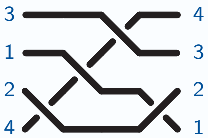

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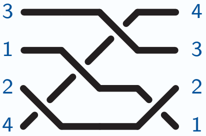

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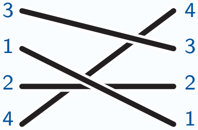

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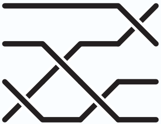

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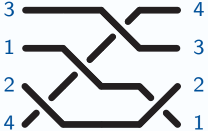

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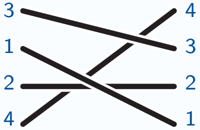

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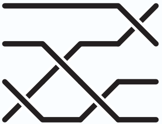

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- ▶ The family S_n of all simple n-strand braids is a copy of $\mathfrak{S}_n$.

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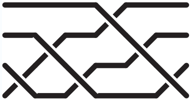

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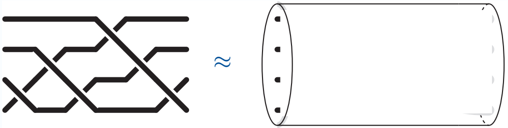

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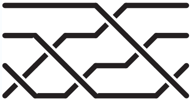 $\approx$ 

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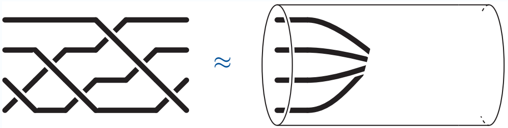

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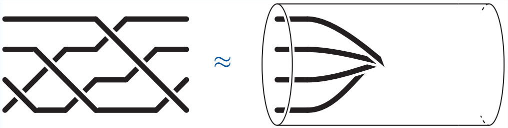

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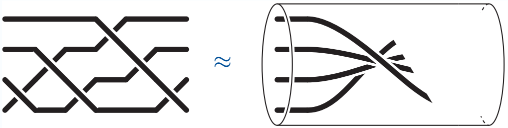

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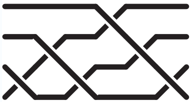

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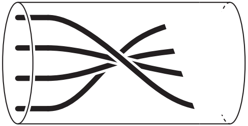

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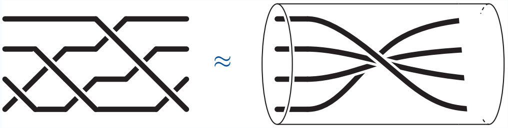

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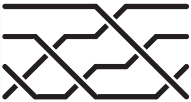
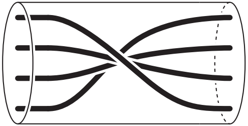
 $\approx$ 

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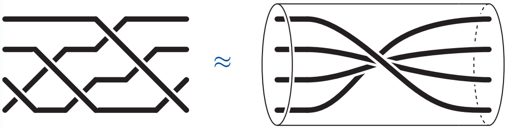

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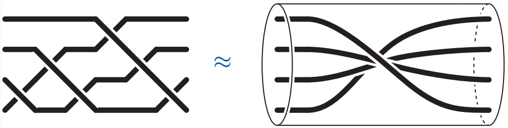

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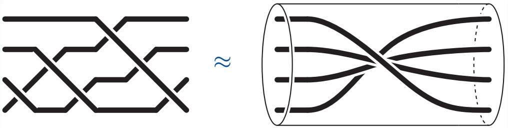

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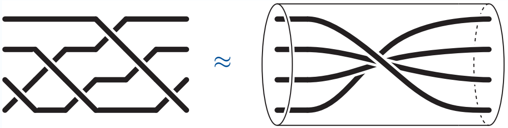

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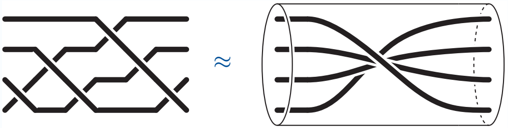

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  ↑ left-divisibility in B_n^+ ↑ weak order in $\mathfrak{S}_n$

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▸ What is the asymptotic behaviour?

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- Conclusion: Braid combinatorics w.r.t. Garside generators
  leads to new, interesting (?) questions about permutation combinatorics.

• Braid groups are countable, braids can be encoded in integers, and most of their (algebraic) properties can be proved in the logical framework of Peano arithmetic, and even of weaker subsystems, like $\mathrm{I}\Sigma_1$ where induction is limited to formulas involving at most one unbounded quantifier.

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  - Proof: Evaluate $\#\{\beta \in B_3^+ \mid \|\beta\|^{\text{Gar}} \leqslant \ell \ \& \ \beta < \Delta_3^k\}$. □

- Plan :

    1. Braid combinatorics: Artin generators
    2. Braid combinatorics: Garside generators
    3. Braid combinatorics: Birman–Ko–Lee generators

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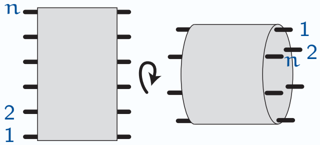

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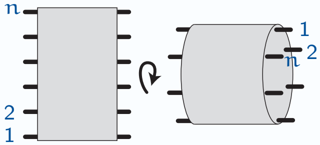

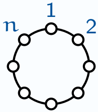

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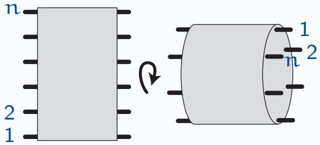

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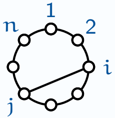

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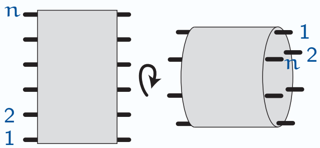
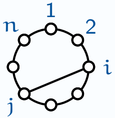


$a_{i,j} \mapsto$ 

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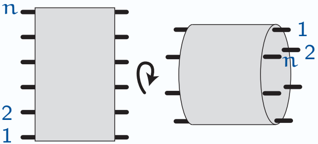
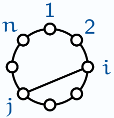


$a_{i,j} \mapsto$ 

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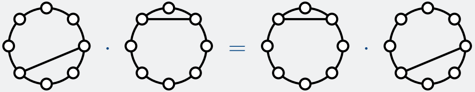 for disjoint chords,

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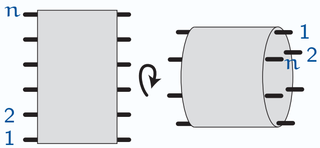

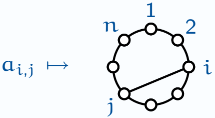

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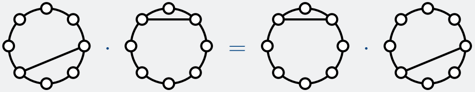

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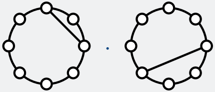

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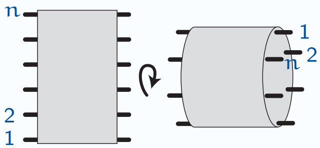

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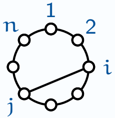

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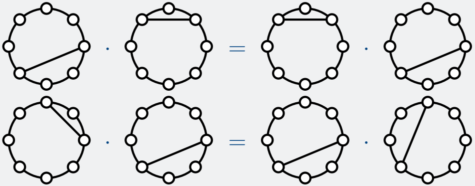

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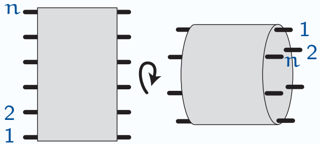

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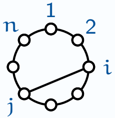

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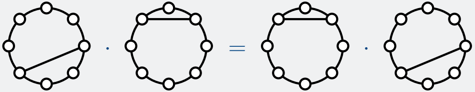

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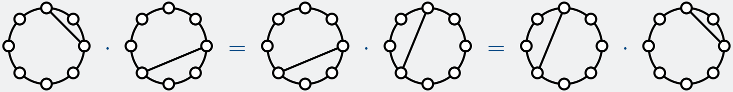

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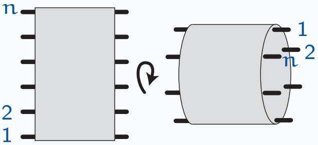

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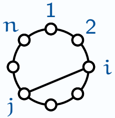

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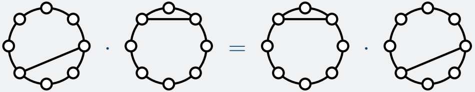

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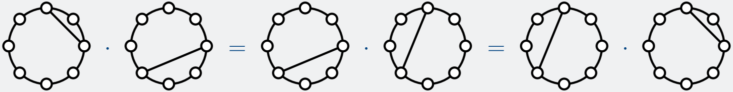

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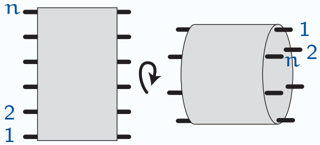
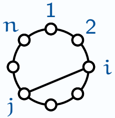


$a_{i,j} \mapsto$ 

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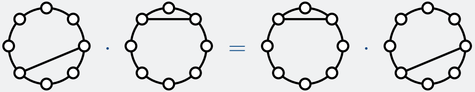

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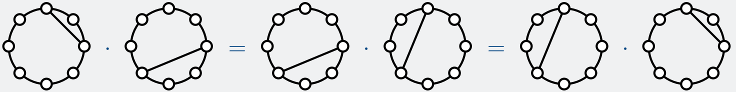

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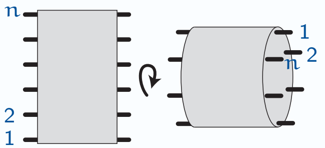

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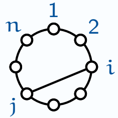

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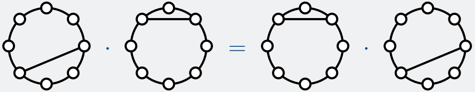

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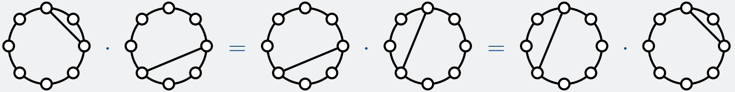

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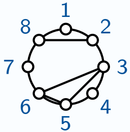

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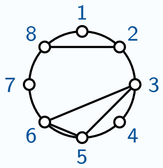

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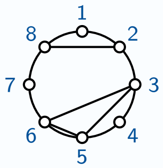
- $\{\{1\}, \{2, 8\}, \{3, 5, 6\}, \{4\}, \{7\}\}$ $\leftrightarrow$ 

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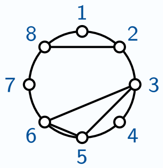

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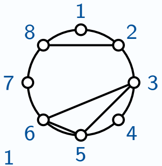

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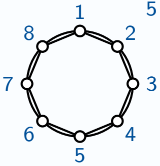

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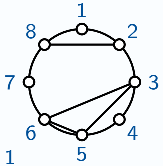
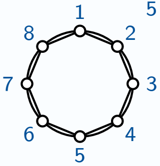
- $\{\{1\}, \{2, 8\}, \{3, 5, 6\}, \{4\}, \{7\}\}$ $\leftrightarrow$  $\leftrightarrow$ $a_{2,8}\, a_{3,5}\, a_{5,6}$

- $\{\{1, 2, 3, 4, 5, 6, 7, 8\}\}$ $\leftrightarrow$  $\leftrightarrow$ $\delta_8 = a_{12}\, a_{23} \cdots a_{78}$

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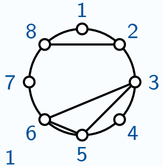
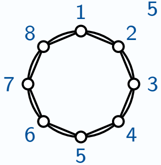
- $\{\{1\}, \{2, 8\}, \{3, 5, 6\}, \{4\}, \{7\}\} \leftrightarrow$  $\leftrightarrow a_{2,8}\, a_{3,5}\, a_{5,6}$

- $\{\{1, 2, 3, 4, 5, 6, 7, 8\}\} \leftrightarrow$  $\leftrightarrow \delta_8 = a_{12}\, a_{23} \cdots a_{78}$

• Remark: The permutation of the braid a_λ is the permutation associated with λ

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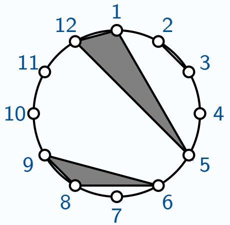

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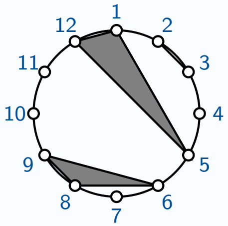

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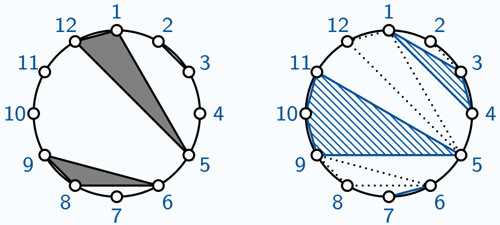

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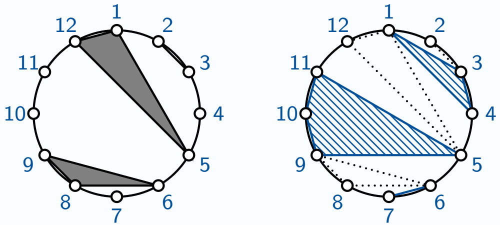

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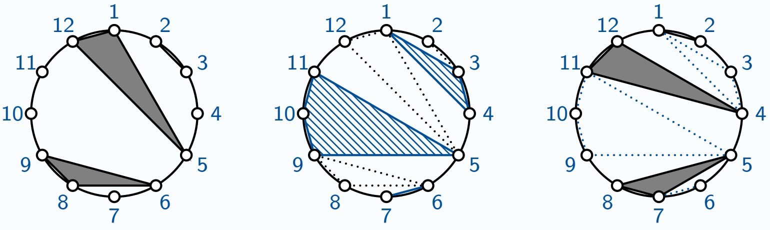

a_λ $\quad$ $\partial a_\lambda = a_{\overline{\lambda}}$ $\quad$ $\partial^2 a_\lambda = \phi_n^*(a_\lambda)$

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- Hence connected to the g.f. F of pairs of noncrossing partitions under $(\#)$. $\square$

- First values:

d	1	2	3	4	5	6	7
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- J. Mairesse & F. Matheus, Growth series for Artin groups of dihedral type
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- P. Dehornoy, Combinatorics of normal sequences of braids,
  J. Combinatorial Th. Series A, 114 (2007) 389–409.

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