Braid combinatorics, permutations, and noncrossing partitions

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- A few combinatorial questions involving braids and their Garside structures:
  the classical Garside structure, connected with permutations,
  the dual Garside structure, connected with noncrossing partitions.

- Plan :

    1. Braid combinatorics: Artin generators
    2. Braid combinatorics: Garside generators
    3. Braid combinatorics: Birman–Ko–Lee generators

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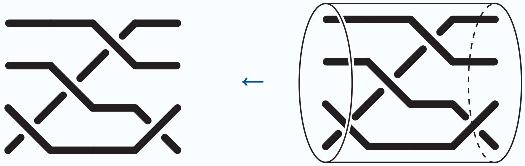
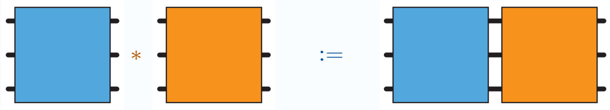
- a 4-strand braid diagram = 2D-projection of a 3D-figure:



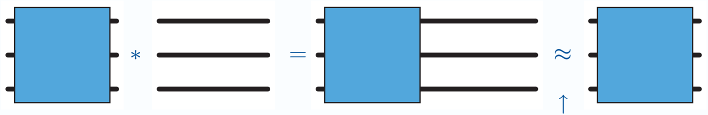
- isotopy = move the strands but keep the ends fixed:

- a braid := an isotopy class ▸ represented by 2D-diagram,
  but different 2D-diagrams may give rise to the same braid.

- Product of two braids:

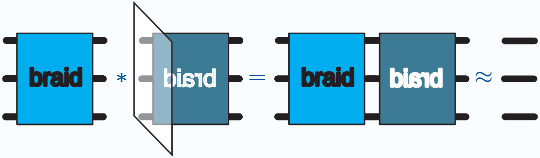


- Then well-defined (with respect to isotopy), associative, admits a unit:



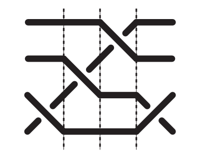
↑ isotopic to

and inverses:



▸ For each n, the group B_n of n-strand braids (E. Artin, 1925).

- Artin generators of B_n:



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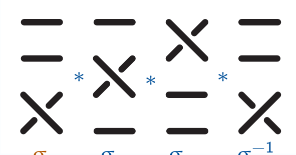
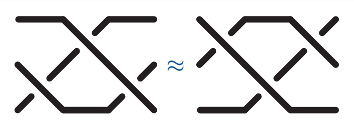


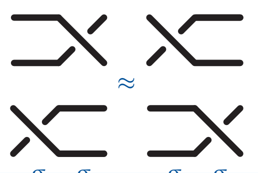
$\sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_1^{-1}$

- **Theorem (Artin).**— The group B_n is generated by $\sigma_1, \ldots, \sigma_{n-1}$,

$$\text{subject to} \begin{cases} \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j & \text{for } |i - j| = 1, \\ \sigma_i \sigma_j = \sigma_j \sigma_i & \text{for } |i - j| \geqslant 2. \end{cases}$$



$\sigma_1 \quad \sigma_2 \quad \sigma_1 \qquad \sigma_2 \quad \sigma_1 \quad \sigma_2$



$\sigma_1 \quad \sigma_3 \qquad \sigma_3 \quad \sigma_1$

- For $n \geqslant 2$, the group B_n is infinite ▶ consider finite subsets.

- $B_n^+ :=$ monoid of classes of n-strand positive diagrams

  ↑ all crossings have a positive orientation

- Theorem (Garside, 1967).— As a monoid, B_n^+ admits the presentation... (as B_n); it is cancellative, and admits lcms and gcds.

- Hence: Equivalent positive braid words have the same length,
  ▶ every positive braid β has a well-defined length $\|\beta\|^{\mathrm{Art}}$ w.r.t. Artin generators σ_i.

- Question: Determine $N_{n,\ell}^{\mathrm{Art}+} := \#\{\beta \in B_n^+ \mid \|\beta\|^{\mathrm{Art}} = \ell\}$ and/or the associated generating series.

- **Theorem** (Deligne, 1972).— For every n, the g.f. of $N_{n,\ell}^{\mathrm{Art}+}$ is rational.

- Proof: For β in B_n^+, define $M(\beta) := \{\beta\gamma \mid \gamma \in B_n^+\}$ = right-multiples of β.
  - Then $B_n^+ \setminus \{1\} = \bigcup_i M(\sigma_i)$, and $M(\sigma_i) \cap M(\sigma_j) = M(\mathrm{lcm}(\sigma_i, \sigma_j))$.
  - By inclusion–exclusion, get induction $N_{n,\ell}^{\mathrm{Art}+} = c_1 N_{n,\ell-1}^{\mathrm{Art}+} + \cdots + c_K N_{n,\ell-K}^{\mathrm{Art}+}$. $\square$

- More precisely: for every n, the generating series of $N_{n,\ell}^{\mathrm{Art}+}$ is the inverse of a polynomial $P_n(t)$.

- **Proposition** (Bronfman, 2001).— Starting from $P_0(t) = P_1(t) = 1$, one has

$$P_n(t) = \sum_{i=1}^{n} (-1)^{i+1} t^{\frac{i(i-1)}{2}} P_{n-i}(t).$$

- Same question for B_n instead of B_n^+; all representatives dont have the same length
  - ▶ define $\|\beta\|^{\text{Art}}$:= the minimal length of a word representing β.

- Question: Determine $N_{n,\ell}^{\text{Art}} := \#\{\beta \in B_n \mid \|\beta\|^{\text{Art}} = \ell\}$
  and/or determine the associated generating series.

- Proposition (Mairesse–Matheus, 2005).— The generating series of $N_{3,\ell}^{\text{Art}}$ is

$$1+\frac{2t(2-2t-t^2)}{(1-t)(1-2t)(1-t-t^2)}.$$

- Then open, even $N_{4,\ell}^{\text{Art}}$: (Mairesse) no rational fraction with degree $\leqslant 13$ denominator.

- “Explanation”: Artin generators are not the right generators...
  ▶ change generators

- Plan :

  1. Braid combinatorics: Artin generators
  2. Braid combinatorics: Garside generators
  3. Braid combinatorics: Birman–Ko–Lee generators

• Definition: A Garside structure in a group G is a subset S of G s.t. every element g of G admits an S-normal decomposition, meaning $g = s_p^{-1} \cdots s_1^{-1} t_1 \cdots t_q$ with $s_1, ..., s_p, t_1, ..., t_q$ in S and, using "f left-divides g" for "$f^{-1}g$ lies in the submonoid $\widehat{S}$ of G generated by S",

- every element of S left-dividing $s_i s_{i+1}$ left-divides s_i,
- every element of S left-dividing $t_i t_{i+1}$ left-divides t_i,
- 1 is the only element of S left-dividing s_1 and t_1.

• When it exists, an S-normal decomposition is (essentially) unique, and geodesic.

• Every group is a Garside structure in itself: interesting only when S is small.

• Normality is local: if S is finite, S-normal sequences make a rational language

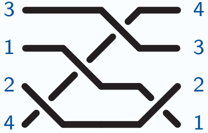
- automatic structure, solution of the word and conjugacy problems, ...
- counting problems: $\#$ elements with S-normal decompositions of length ℓ.

• Definition: A Garside structure S in a group G is bounded if there exists an element Δ ("Garside element") such that S consists of the left-divisors of Δ in $\widehat{S}$.

• In this case:

- the S-normal decomposition of g in $\widehat{S}$ is recursively given by $s_1 = \gcd(g, \Delta)$;
- (s, t) is S-normal iff 1 is the only element of S left-dividing $s^{-1}\Delta$ and t.

- Permutation associated with a braid:

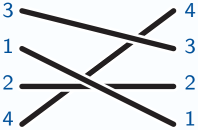


$\mapsto$ $(4, 2, 1, 3)$

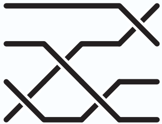
  - ▶ A surjective homomorphism $\pi_n : B_n \to \mathfrak{S}_n$.

- Lemma: Call a braid simple if it can be represented by a positive diagram in which any two strands cross at most once. Then, for every permutation f in $\mathfrak{S}_n$, there exists exactly one simple braid σ_f satisfying $\pi_n(\sigma_f) = f$.

$(4, 2, 1, 3) \mapsto$



$\mapsto$



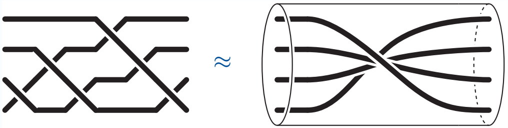
: $\sigma_1 \sigma_2 \sigma_1 \sigma_3$

  - ▶ The family S_n of all simple n-strand braids is a copy of $\mathfrak{S}_n$.

• Theorem (Garside, Adjan, Morton–ElRifai, Thurston).— For each n, the family S_n is a Garside structure in B_n, bounded by $\sigma_{(n,\dots,1)}$; the associated monoid is B_n^+.

• "Garside's fundamental braid" $\Delta_n := \sigma_{(n,\dots,1)}$, whence $\Delta_n = \Delta_{n-1} \cdot \sigma_{n-1} \dots \sigma_2 \sigma_1$:

$\Delta_1 = 1, \quad \Delta_2 = \sigma_1, \quad \Delta_2 = \sigma_1 \sigma_2 \sigma_1,$ etc.



• A new family of generators: the Garside generators σ_f

- a very redundant family: $n!$ elements, whereas only $n-1$ Artin generators;
- many expressions for a braid, but a distinguished one: the S_n-normal one;
- in terms of Garside generators, the group B_n – and the monoid B_n^+ – are presented by the relations $\sigma_f \sigma_g = \sigma_{fg}$ with $\ell(f) + \ell(g) = \ell(fg)$;

  ↑ length of $f :=$ # of inversions in f

- the poset $(S_n, \preccurlyeq)$ is isomorphic to $(\mathfrak{S}_n, \leqslant)$.

  ↑ left-divisibility in B_n^+ ↑ weak order in $\mathfrak{S}_n$

- Question: Determine $N^{\text{Gar}+}_{n,\ell} := \#\{\beta \in B_n^+ \mid \|\beta\|^{\text{Gar}} = \ell\}$ and/or its generating series, where $\|\beta\|^{\text{Gar}}$:= length of the S_n-normal decomposition.

  (and idem with $N^{\text{Gar}}_{n,\ell} := \#\{\beta \in B_n \mid \|\beta\|^{\text{Gar}} = \ell\}$.)

- An easy question (contrary to the case of Artin generators):
  - by construction, $N^{\text{Gar}+}_{n,\ell} = \#$ length ℓ normal sequences in B_n^+,
  - and normality is a local property:
    a sequence is S_n-normal iff every length 2 subsequence is S_n-normal.

- Proposition.— Let M_n be the $n! \times n!$ matrix indexed by simple braids (i.e., by permutations) s.t. $(M_n)_{s,t} = \begin{cases} 1 & \text{if } (s,t) \text{ is normal,} \\ 0 & \text{otherwise.} \end{cases}$
  Then $N^{\text{Gar}+}_{n,\ell}$ is the idth entry in $(1, ..., 1) \cdot M_n^{\ell}$.

  - For each n, the generating series of $N^{\text{Gar}+}_{n,\ell}$ is rational.

- Lemma 1: For f, g in $\mathfrak{S}_n$, the pair (σ_f, σ_g) is normal iff $\mathrm{Desc}(f) \supseteq \mathrm{Desc}(g^{-1})$.

  ↑ descents of $f := \{k \mid f(k) > f(k+1)\}$

- Hence, if $\mathrm{Desc}(g^{-1}) = \mathrm{Desc}(g'^{-1})$, the columns of g and g' in M_n are equal;
  - columns can be gathered: replace M_n (size $n!$) with M'_n (size 2^{n-1}).

- Lemma 2: The $\#$ of permutations f satisfying $\mathrm{Desc}(f) \supseteq I$ and $\mathrm{Desc}(f^{-1}) \supseteq J$ is the $\#$ of $k \times \ell$ matrices with entries in $\mathbb{N}$ s.t. the sum of the ith row is p_i and the sum of the jth column is q_j, with $(p_1, \ldots, p_k)$ the composition of I and $(q_1, \ldots, q_\ell)$ that of J.

  ↑ composition: sequence of sizes of the blocks of adjacent elements

  ↓ partition: set of sizes of the blocks of adjacent elements

- Hence $(M'_n)_{I,J}$ only depends on the partition of J;
  - can gather columns again: replace M'_n (size 2^{n-1}) with M''_n (size $p(n)$).

- Remarks:
  - Going from M_n to $M''_n \approx$ reducing the size of the automatic structure of B_n from $n!$ to $p(n)$ $\left(\sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \right)$
  - (Hohlweg) That $(M'_n)_{I,J}$ only depends on the partition of J is (another) form of Solomon's result about the descent algebra.

- The growth rate of $N_{n,\ell}^{\mathrm{Gar}+}$ is connected with the eigenvalues of M_n, hence of M''_n:

$$\begin{aligned}
\mathrm{CharPol}(M''_1) &= x-1\\
\mathrm{CharPol}(M''_2) &= \mathrm{CharPol}(M''_1)\cdot(x-1)\\
\mathrm{CharPol}(M''_3) &= \mathrm{CharPol}(M''_2)\cdot(x-2)\\
\mathrm{CharPol}(M''_4) &= \mathrm{CharPol}(M''_3)\cdot(x^2-6x+3)\\
\mathrm{CharPol}(M''_5) &= \mathrm{CharPol}(M''_4)\cdot(x^2-20x+24),\ldots
\end{aligned}$$

- **Theorem** (Hivert–Novelli–Thibon).— The characteristic polynomial of M''_n divides that of M''_{n+1}.

▸ Proof: Interpret M''_n in terms of quasi-symmetric functions in the sense of Malvenuto–Reutenauer, and determine the LU-decomposition. □

- Spectral radius:

n	2	3	4	5	6	7	8
$\rho(M_n)$	1	2	5.5	18.7	77.4	373.9	2066.6
$\rho(M_n)/(n\rho(M_{n-1}))$	0.5	0.667	0.681	0.687	0.689	0.690	0.691

▸ What is the asymptotic behaviour?

- So far: $N_{n,\ell}^{\text{Gar}+}$ with n fixed and ℓ varying;
  for ℓ fixed and n varying, different induction schemes (starting with $N_{n,1}^{\text{Gar}+} = n!$).

- Proposition.— $N_{n,2}^{\text{Gar}+} = \sum_{0}^{n-1} (-1)^{n+i+1} \binom{n}{i}^2 N_{i,2}^{\text{Gar}+}$,
  whence (Carlitz–Scoville–Vaughan) $1 + \sum_{n} N_{n,2}^{\text{Gar}+} \frac{z^n}{(n!)^2} = \frac{1}{J_0(\sqrt{z})}$.

Bessel function J_0

- Put $N_{n,\ell}^{\text{Gar}+}(s) :=$ # normal sequences in B_n^+ finishing with s:
  $N_{n,3}^{\text{Gar}+}(\Delta_{n-1}) = 2^{n-1}$, $N_{n,3}^{\text{Gar}+}(\Delta_{n-2}) \sim 2 \cdot 3^n$, $N_{n,4}^{\text{Gar}+}(\Delta_{n-1}) = \lfloor n!e \rfloor - 1$...

- Conclusion: Braid combinatorics w.r.t. Garside generators
  leads to new, interesting (?) questions about permutation combinatorics.

- Braid groups are countable, braids can be encoded in integers, and most of their (algebraic) properties can be proved in the logical framework of Peano arithmetic, and even of weaker subsystems, like $I\Sigma_1$ where induction is limited to formulas involving at most one unbounded quantifier.

- Braids admit an ordering, s.t. $(B_n^+, \leqslant)$ is a well-ordering of type $\omega^{\omega^{n-2}}$;
  - one can construct long (finite) descending sequences of positive braids;
  - but this cannot be done in $I\Sigma_1$ (reminiscent of Goodstein's sequences);
  - where is the transition from $I\Sigma_1$-provability to $I\Sigma_1$-unprovability?

- Definition: For $F : \mathbb{N} \to \mathbb{N}$, let WO_F be the statement:
  "For every ℓ, there exists m s.t. every strictly decreasing sequence $(\beta_t)_{t \geqslant 0}$ in B_3^+ satisfying $\|\beta_t\|^{\mathrm{Gar}} \leqslant \ell + F(t)$ for each t has length at most m".

- WO_0 trivially true (finite $\#$), and WO_F provable for every F using König's Lemma.

- Theorem (Carlucci, D., Weiermann).— For $r \leqslant \omega$, let $F_r(x) := \lfloor \sqrt[\mathrm{Ack}_r^{-1}(x)]{x} \rfloor$.
  Then WO_{F_r} is $I\Sigma_1$-provable for finite r, and $I\Sigma_1$-unprovable for $r = \omega$.

  - Proof: Evaluate $\#\{\beta \in B_3^+ \mid \|\beta\|^{\mathrm{Gar}} \leqslant \ell \ \&\ \beta < \Delta_3^k\}$. □

- Plan :

    1. Braid combinatorics: Artin generators
    2. Braid combinatorics: Garside generators
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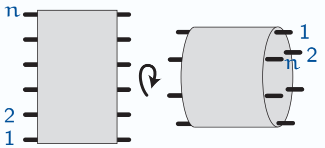
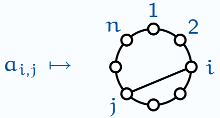
- Another family of generators for B_n: the Birman–Ko–Lee generators
$$a_{i,j} := \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1} \text{ for } 1 \leqslant i < j \leqslant n.$$

- The dual braid monoid: the submonoid B_n^{+*} of B_n generated by the elements $a_{i,j}$.

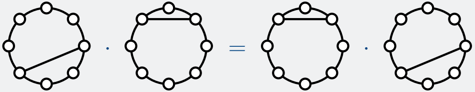
- Proposition (Birman–Ko–Lee, 1997).— Let $\delta_n = \sigma_{n-1} \cdots \sigma_2 \sigma_1$. Then the family of all divisors of δ_n in B_n^{+*} is a Garside structure in B_n; it is bounded by δ_n.

- Chord representation of the Birman–Ko–Lee generators:

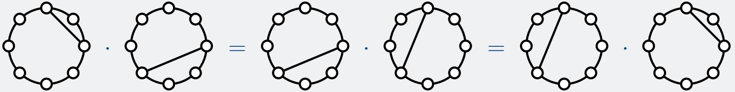




- Lemma: In terms of the BKL generators, B_n is presented by the relations

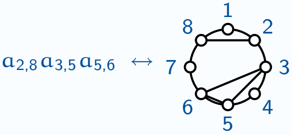


for disjoint chords,



for adjacent chords enumerated in clockwise order.

- Hence: For P a p-gon, can define a_P to be the product of the $a_{i,j}$ corresponding to $p-1$ adjacent edges of P in clockwise order;
  idem for an union of disjoint polygons.

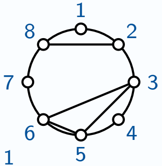


• **Proposition** (Bessis–Digne–Michel).— The elements of the Garside structure S_n^* (divisors of δ_n in B_n^{+*}) are the elements a_P with P a union of disjoint polygons with n vertices, hence in 1-1 correspondence with the Cat_n noncrossing partitions of $\{1, \ldots, n\}$.

- notation a_λ for λ a noncrossing partition

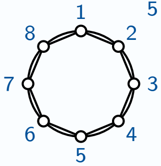
• Examples:

- $\{\{1\}, \{2, 8\}, \{3, 5, 6\}, \{4\}, \{7\}\}$ $\leftrightarrow$



$\leftrightarrow$ $a_{2,8}\, a_{3,5}\, a_{5,6}$

- $\{\{1, 2, 3, 4, 5, 6, 7, 8\}\}$ $\leftrightarrow$



$\leftrightarrow$ $\delta_8 = a_{12}\, a_{23} \cdots a_{78}$

• Remark: The permutation of the braid a_λ is the permutation associated with λ (product of cycles of the parts)

- Question: Determine $N^{BKL+}_{n,\ell} := \#\{\beta \in B^{+*}_n \mid \|\beta\|^{BKL} = \ell\}$ and its generating series, where $\|\beta\|^{BKL}$:= length of the S^*_n-normal decomposition of β.

- For instance: $N^{BKL+}_{n,1} = \#S^*_n = \mathrm{Cat}_n$.

- Exactly similar to the classical case: local property, etc.

- Proposition.— Let M^*_n be the $\mathrm{Cat}_n \times \mathrm{Cat}_n$ matrix indexed by noncrossing partitions s.t. $(M^*_n)_{\lambda,\mu} = \begin{cases} 1 & \text{if } (a_\lambda, a_\mu) \text{ is } S^*_n\text{-normal,} \\ 0 & \text{otherwise.} \end{cases}$ Then $N^{BKL+}_{n,\ell}$ is the 1_nth entry in $(1, \dots, 1) \cdot {M^*_n}^{\ell}$.

↑ the partition with n parts

  - For every n, the generating series of $N^{BKL+}_{n,\ell}$ is rational.

- When is (a_λ, a_μ) S_n^*-normal?

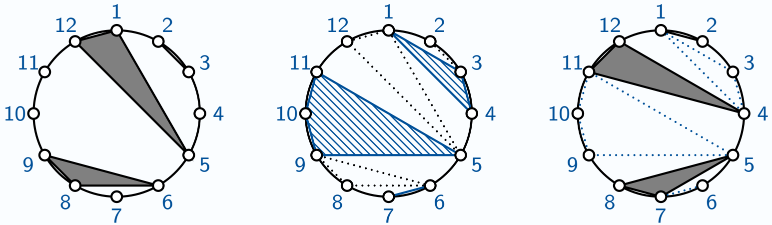
- Recall: If a Garside structure S is bounded by Δ, then (s, t) is S-normal iff ∂s and t have no nontrivial common left-divisor.

  ↑ the element s' s.t. $ss' = \Delta$

  - When does $a_{i,j}$ left-divide a_λ?
  - What is the partition of ∂a_λ in terms of that of a_λ?

- Lemma (Bessis–Digne–Michel): The element $a_{i,j}$ left- (or right-) divides a_λ iff the chord (i, j) is included in the polygon of λ.

- Lemma (Bessis–Digne–Michel): The partition of ∂a_λ is the Kreweras complement $\overline{\lambda}$ of λ.



a_λ $\quad$ $\partial a_\lambda = a_{\overline{\lambda}}$ $\quad$ $\partial^2 a_\lambda = \phi_n^*(a_\lambda)$

- **Proposition** (Biane).— The generating series $G(z)$ of $N_{n,2}^{BKL+}$ is derived from the generating series $F(z)$ of Cat_n^2 by

$$G(z) = F(zG(z)). \qquad (\#)$$

- Proof:
  - Let $G(z) = \sum_n N_{n,2}^{BKL+} z^n$,
    with $N_{n,2}^{BKL+} = \#$ length 2 normal sequences $= \#$ positive entries in M_n^*.
  - From what we saw: $(M_n^*)_{\lambda,\mu} = 1$ iff $\overline{\lambda} \wedge \mu = 0_n$. As $\lambda \to \overline{\lambda}$ is a bijection, one has also $N_{n,2}^{BKL+} = \#\{(\lambda,\mu) \in (NC_n)^2 \mid \lambda \vee \mu = 1_n\}$.
  - The number $N_{n,2}^{BKL+}$ is the nth free cumulant of $X_1^2 X_2^2$ where X_1, X_2 are independent free random variables of variance 1.
  - Hence connected to the g.f. F of pairs of noncrossing partitions under $(\#)$. □

- First values:

d	1	2	3	4	5	6	7
$N^{BKL+}_{2,d}$	2	3	4	5	6	7	8
$N^{BKL+}_{3,d}$	5	15	83	177	367	749	1 515
$N^{BKL+}_{4,d}$	14	99	556	2 856	14 122	68 927	334 632
$N^{BKL+}_{5,d}$	42	773	11 124	147 855	1 917 046	24 672 817	
$N^{BKL+}_{6,d}$	132	6 743	266 944	9 845 829	356 470 124		

- Questions about columns (OK for $d \leqslant 2$):
  - What is the behaviour of $N^{BKL+}_{n,3}$, etc.?

- Questions about rows (OK for $n \leqslant 3$):
  - Can one reduce the size of M^*_n?
  - Is M^*_n always invertible?
  - What is the asymptotic behaviour of the spectral radius of M^*_n?

n	1	2	3	4	5	6	7
$tr(M^*_n)$	1	2	5	14	42	132	429
$det(M^*_n)$	1	1	2	$2^4 \cdot 5$	$2^{16} \cdot 5^5 \cdot 7$	$2^{63} \cdot 3 \cdot 5^{21} \cdot 7^7$	$2^{247} \cdot 3^8 \cdot 5^{84} \cdot 7^{35} \cdot 11$
$\rho(M^*_n)$	1	1	2	4.83...	12.83...	35.98...	104.87...

• Whenever a group admits a finite Garside structure,
there is a finite state automaton, whence an incidence matrix.

• The associated combinatorics is likely to be interesting if the Garside structure is connected with combinatorially meaningful objects:
permutations (Garside case), noncrossing partitions (Birman–Ko–Lee case), etc.

• The family of group(oid)s that admit an interesting Garside structure is large and so far not well understood:

- for instance (Bessis, 2006) free groups do;
- also: exotic Garside structures on braid groups;
- and exotic non-Garside normal forms with local characterizations;
- most results involving braids extend to Artin–Tits groups of spherical type (i.e., associated with a finite Coxeter group);
  - many potential combinatorial problems

• Specific case of dual braid monoids and noncrossing partitions:

- (almost) nothing known so far,
- but the analogy B_n^{+*}/B_n^+ suggests that combinatorics could be interesting (?).

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