DEHORNOY. St Andrews minicourse -I.1-
May 2014 -

Thompson's groups and its cousins as geometry groups of algebraic laws.

Aim: Revisit R. Thompson's construction of F (and V): associate with every algebraic law I (or family of algebraic laws) a monoid/group G_I that captures its specific geometry.

- Case of a simple law: ⇝ interesting group
  typically (A) $x(yz) = (xy)z$ ⇝ $G_A = F$
- Case of a complicated law: ⇝ more about I
  typically (LD) $x(yz) = (xy)(xz)$ ⇝ solution of the Word Problem

Plan: I. The Thompson monoid of an algebraic law
II. Presentations
III. A few cousins of F

I. The Thompson monoid of an algebraic law

1. Algebraic laws

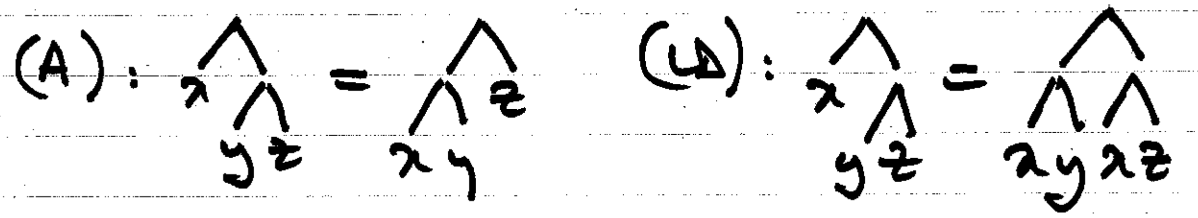
A law I is a pair of terms (T_L, T_R) // or $T_L = T_R$

terms: parenthesized expressions involving variables and operators

Here: only <u>one</u> <u>binary</u> operator

<u>Ex</u>: (A) $x(yz) = (xy)z$, (LD) $x(yz) = (xy)(xz)$ ...

View terms as binary <u>trees</u>:

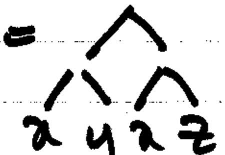
(A):  (LD):

<u>Notation</u>:

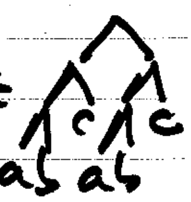
- $\mathcal{T}_X$:= {all terms with variables in X}
  (possibly $X = \mathbb{N}$ "$x_0, x_1, \ldots$")

- $\mathbb{A}$:= {all binary <u>addresses</u>} $= \{0,1\}^*$
  ↑ finite seq. of 0s and 1s

- For T in $\mathcal{T}_X$ and α in $\mathbb{A}$ small enough
  $T/_\alpha$:= the α-subterm of T

<u>Ex</u>: $T =$  $T/_0 =$ , $T/_{00} = x$, $T/_{000}\not\downarrow$
($\downarrow$ = "is defined")

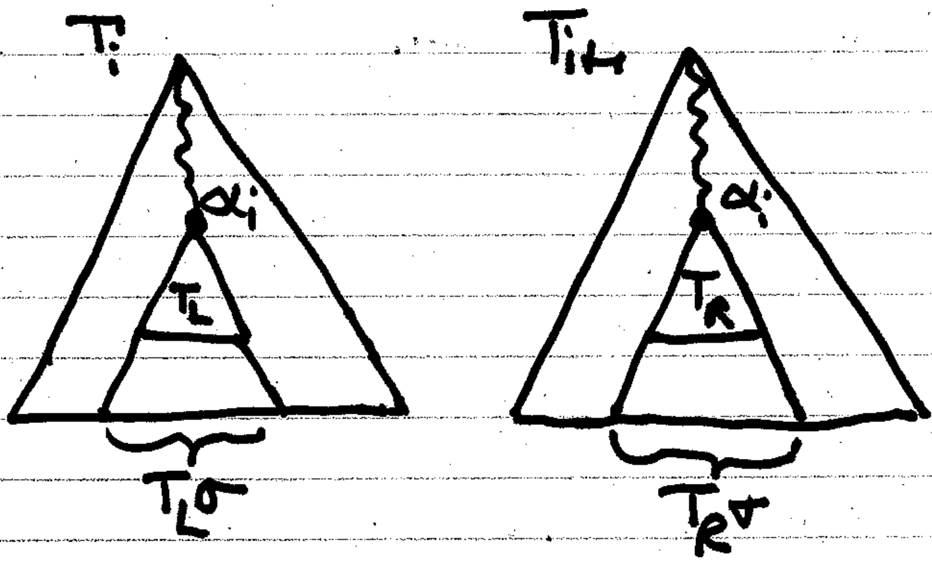
- For T in $\mathcal{T}_{\mathbb{N}}$ and $\sigma : \mathbb{N} \to \mathcal{T}_X$:
  T^σ := result of replacing every variable i of T with $\sigma(i)$

<u>Ex</u>: $T =$  $\sigma(1) =$ , $\sigma(2) = \sigma(3) = c$: $T^\sigma =$ 

Lemma.– Assume that I is an algebraic law, say $I = (T_L, T_R)$. Define an X-valued *instance* of I to be any pair $(T_L\sigma, T_R\sigma)$ with $\sigma : \mathbb{N} \to \mathcal{T}_X$, and let $=_{I,X}$ be the congruence on $\mathcal{T}_X$ generated by all X-valued instances of I.

(i) $\mathcal{T}_X / =_{I,X}$ is a free I-structure based on X.

(ii) Two terms T, T' are $=_{I,X}$-equivalent iff there exists a finite sequence $T_0, \dots, T_n$ with $T_0 = T$, $T_n = T'$ and for every i, there exists an address α_i and a sign e_i $(=\pm)$ s.t. $T_i/\gamma = T_{i+1}/\gamma$ for $\gamma \perp \alpha_i$ and $(T_i/\alpha_i, T_{i+1}/\alpha_i)$ is an instance of I^{e_i}.

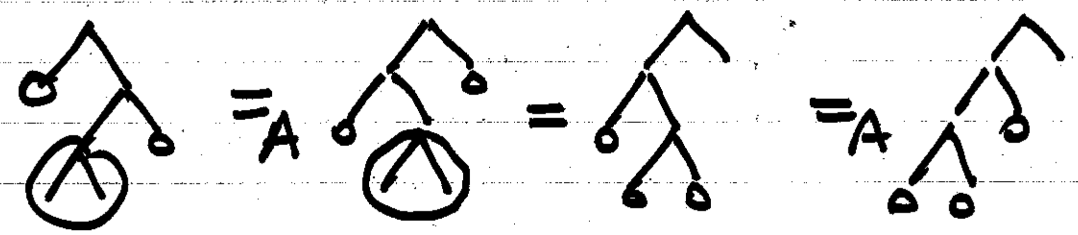


Proof. (i) Let S obey I and $F : X \to S$.
As $\mathcal{T}_X$ is the absolutely free algebra based on X

$$\begin{tikzcd}
\mathcal{T}_X \arrow[r, "\hat{F}"] \arrow[d] & S \\
\mathcal{T}_X / =_{I,X} \arrow[ur, dotted, "\exists"'] &
\end{tikzcd}$$

(ii) Write $\equiv$ for $\exists$ sequence ...
Then $T \equiv T'$ implies $T =_{I,X} T'$ since $T_i =_{I,X} T_{i+1}$.
Conversely $\equiv$ is a congruence on $\mathcal{T}_X$ that contains all X-valued instances of X, hence $\equiv$ includes $=_{I,X}$. ∎

Ex. (A) $x((yz)t) =_A ((xy)z)t$

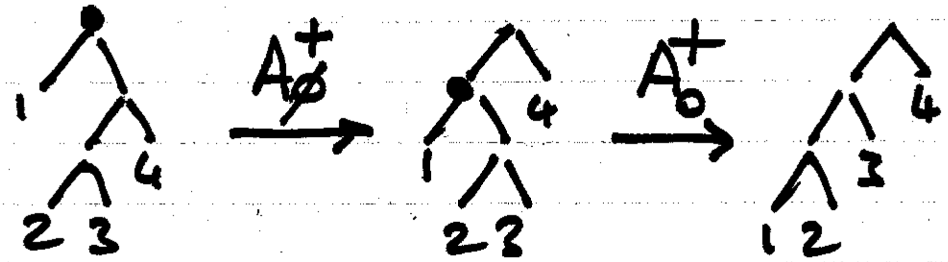


2- The monoid $\mathcal{G}_I$ ("Thompson monoid")

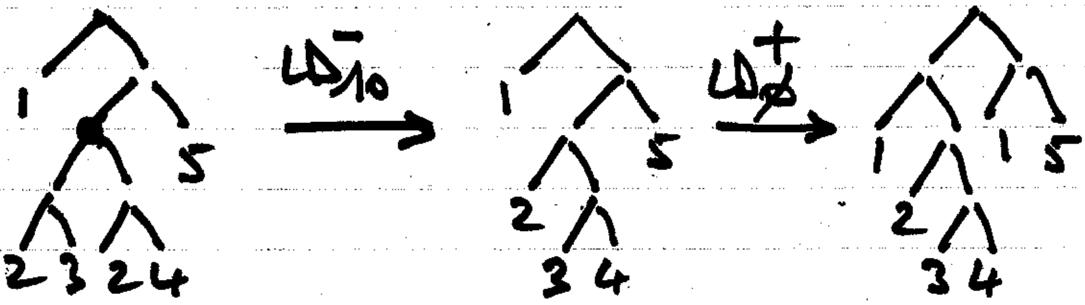
View the application of I as a group action?
ans: YES, rather a monoid partial action

Defi.- Assume that I is an algebraic law. For α in A and e in $\{\pm\}$, define I_α^e to be the partial map on $\mathcal{T}$ ($= \mathcal{T}_{\mathbb{N}}$) "apply I^e at position α". The geometry (or Thompson) monoid of I, written $\mathcal{G}_I$, is the monoid gene'd by all I_α^e.

Ex (A)



(LD)



Proposition.— For every algebraic law I, TFAE
(i) $T =_{I,X} T'$ holds;
(ii) some element of $\mathcal{G}_I$ maps T to T'.

(obvious)

Notation. For $w = \alpha_1^{e_1}|\dots|\alpha_n^{e_n}$ (a word in $(A \cup \overline{A})^*$), write I_w for $I_{\alpha_1}^{e_1} \dots I_{\alpha_n}^{e_n}$ (action on the right = reversed composition)
e.g., $A_{\emptyset|10}$, $LD_{\overline{10}|\emptyset}$, ...

Remarks. (i) Partial maps: cannot apply I_α when term skeleton is too small
($\leadsto$ can be solved by considering infinite trees)

... and in case of variable clashes
(⇝ cannot be solved in general).

- Could use a category: objects = terms, morphisms = triples (T, w, T') with $T \bullet I_w = T'$
(⇝ but then identify instance, and same problems: no advantage)

<u>Lemma</u>.– Assume I is an algebraic law. Then, for every w in $(A \cup \bar{A})^*$, there exist two terms $T_L(w)$, $T_R(w)$ s.t. I_w (as a set of pairs) is the collection of all instances of $(T_L(w), T_R(w))$.

<u>Proof</u>. Induction on the length of w. For $|w| = 1$ and $w = \emptyset$, by def. $T_L(w) = T_L$, $T_R(w) = T_R$ (where $I = (T_L, T_R)$). For $w = \alpha$....
Assume $w = uv$:

$$T_L(u)\sigma \xmapsto{I_u} T_R(u)\sigma \qquad T_L(v)\tau \xmapsto{I_v} T_R(v)\tau$$

<u>Claim</u>. If $\exists \sigma, \tau \; (T_R(u)\sigma = T_L(v)\tau)$
(a "unifier" of $T_R(u)$ and $T_L(v)$),
then $\exists$ <u>least</u> one σ_0, τ_0. Then

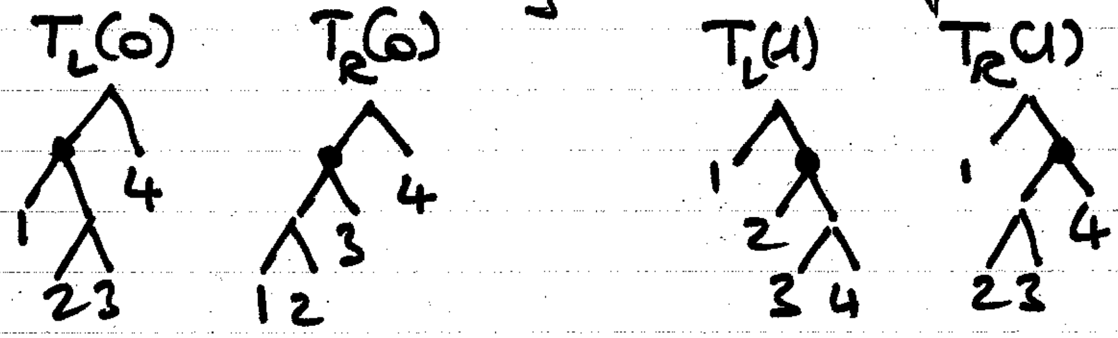
$$T_L(uv) = T_L(u)\sigma_0, \quad T_R(uv) = T_R(v)\tau_0. \blacksquare$$

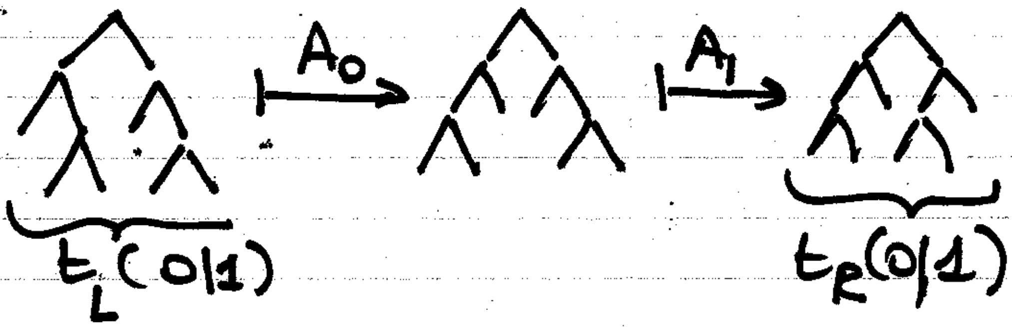
__Ex__ (A) T_L $(=T_L(\emptyset))$ and T_R $(=T_R(\emptyset))$ are __injective__ terms

↖ no variable repeated more than once

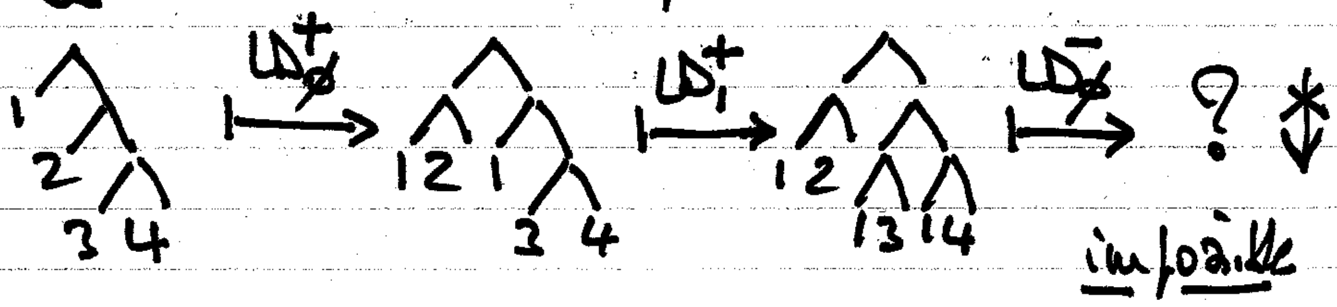
Inductively: $T_L(w)$ and $T_R(w)$ injective for every w

⇝ in __this__ case, unification simply means taking the union of skeletons





__Ex__ (LD) More complicated:



3. <u>Making</u> $\mathcal{G}_I$ <u>into a group</u>

<u>Fact</u>. $\mathcal{G}_I$ is an inverse monoid

$$\forall g \exists g' (g g' g = g \ \& \ g' g g' = g')$$

but <u>not</u> a group: $gg' = \underline{id}_{Dom(g)} \neq \underline{id}_{\mathcal{G}}$

<u>Lemma</u>. For g, g' in $\mathcal{G}_I$, declare $g \approx g'$ if $\exists T (T.g\downarrow$ and $T.g'\downarrow)$ and $T.g = T.g'$ whenever defined. Assume

(i) $\forall g_1, \dots, g_n \exists T (T.g_1\downarrow \ \& \dots \& \ T.g_n\downarrow)$

(ii) $\exists T (T.g = T.g') \Rightarrow g \approx g'$.

Then $\approx$ is a congruence on $\mathcal{G}_I$, and $\mathcal{G}_I/\approx$ is a group.

<u>Proof</u>. Assume $g \approx g' \approx g''$.

By (i) $\exists T (T.g\downarrow$ and $T.g'\downarrow$ and $T.g''\downarrow)$

Then $T.g = T.g' = T.g''$

By (ii) $g \approx g''$: $\approx$ is equival. rel.

Assume $g \approx g'$.

By (i) $\exists_T (T.fg\downarrow \ \& \ T.fg'\downarrow)$.

Then $T.fg = T.fg'$.

By (ii) $fg \approx fg'$: $\approx$ compatible with mult.

For g in $\mathcal{G}_I$:
By (i) $\exists T\, (T.\underline{gg}^{-1}\downarrow$ & $T.\underline{id}\downarrow)$
Then $T.\,gs^{-1} = T$
By (ii) $gs^{-1} \approx \underline{id}$ ■

<u>Ex</u> (A) satisfies (i) + (ii)
(i) $T.A_W\downarrow$ iff $\underline{Skel}(T) \geq \underline{Skel}(\Pi(w))$
(ii) $T_L(w), T_R(w)$ injective.... . ■

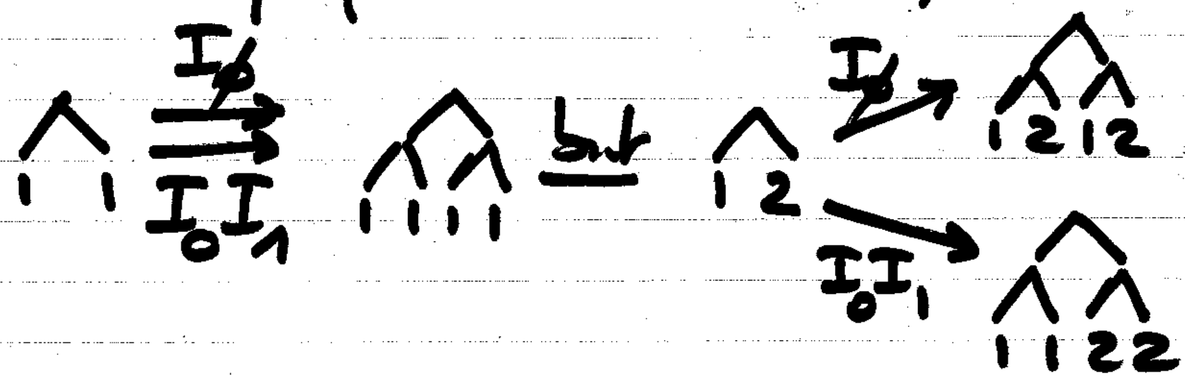
Hence: $\mathcal{G}_A/\approx$ is a group

<u>Prop</u> (R. Thompson) $\mathcal{G}_A/\approx\; = F$.

<u>Proof</u>. $\approx$-class of $w \longmapsto$
$$\left(\bigcap_{w'\approx w} \underline{Skel}(T_L(w')), \bigcap_{w'\approx w} \underline{Skel}(T_R(w')) \right)$$
gives bijection $\mathcal{G}_A/\approx \;\longleftrightarrow$ pairs of trees ■

<u>Remarks</u>. (i) may fail
$T.\,LD_\emptyset^-\downarrow$ and $T.\,LD_1^- LD_\emptyset^-\downarrow$ impossible.

(ii) may fail: $I = (x, x \wedge x)$



Exercise. $\mathcal{G}_{A,C} = \mathcal{V}$

commutativity $xy = yx$

II. Presentations.

Principle: As there is no clear way of making a group out of $\mathcal{G}_I$ in general, find a (presentation) of $\mathcal{G}_I$ by relations R_I and then consider the abstract group G_I presented by R_I. If R_I correctly guessed it should (?) be possible to use G_I instead of $\mathcal{G}_I$.

Case 1: $\mathcal{G}_I/\approx$ is a group
⇝ Expect $G_I = \mathcal{G}_I/\approx$ and results on G_I

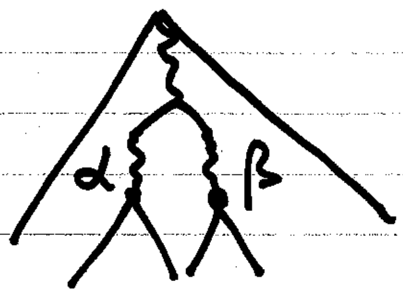
Case 2: $\mathcal{G}_I/\approx$ is not a group
⇝ Expect results about I

1. The relations R_I

Principle: Fix a law I, for α, β in $\mathbb{A}$, look for relations

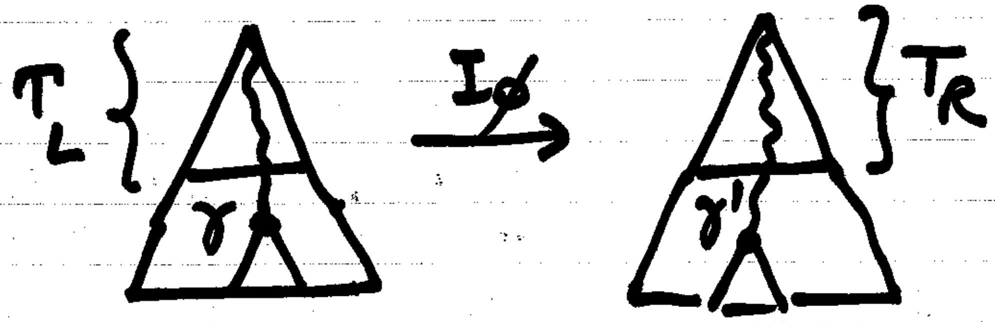
$$I_\alpha \dots = I_\beta \dots \quad (\text{"confluence"})$$

(ii) "Parallel" case: $\alpha \perp \beta$



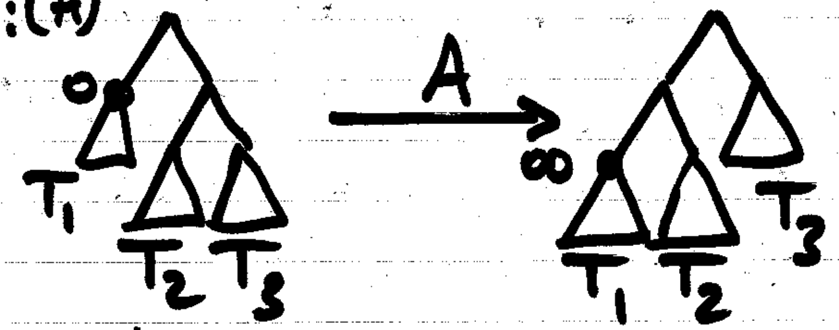
$$I_\alpha I_\beta = I_\beta I_\alpha$$

(ii) "Nested" case: $I = (T_L, T_R)$ and $\gamma \notin \underline{\text{Skel}}(T_L)$



$I_\gamma I_\emptyset = I_\emptyset I_{\gamma'} \dots$

Ex: (A)

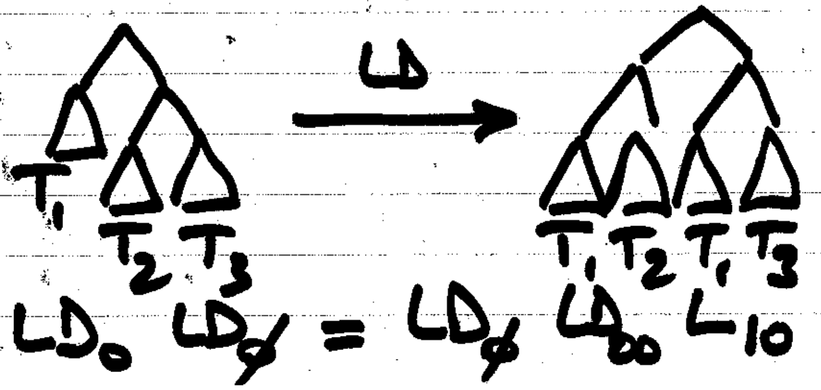


$A_0 A = A A_{00}$

more generally: $A_{0\gamma} A_\emptyset = A_\emptyset A_{00\gamma}$

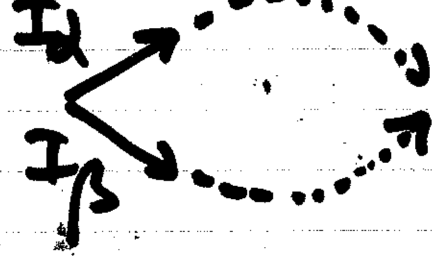
" " : $A_{\alpha 0 \gamma} A_\alpha = A_\alpha A_{\alpha 00 \gamma}$

Attention! Maybe variables repetitions

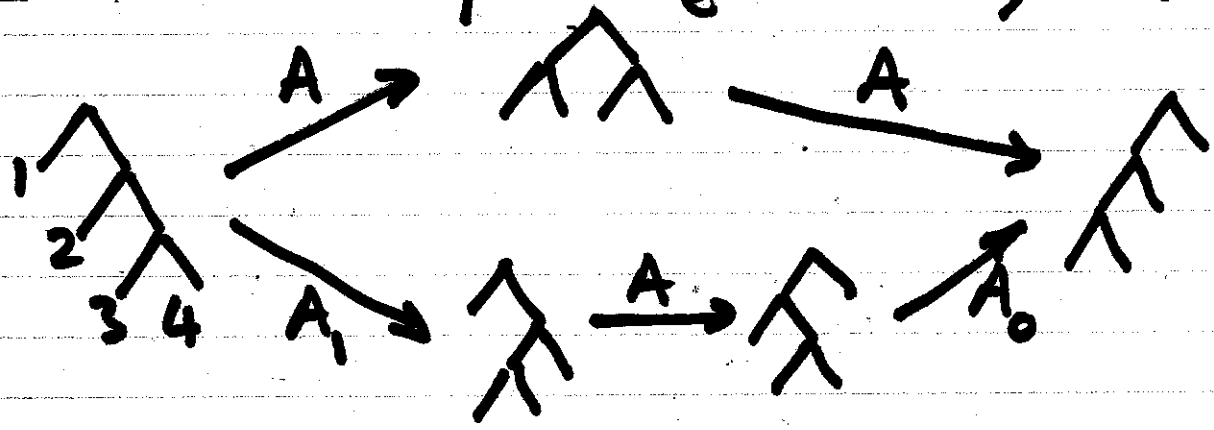


$LD_0 LD_\emptyset = LD_\emptyset LD_{00} L_{10} \dots$

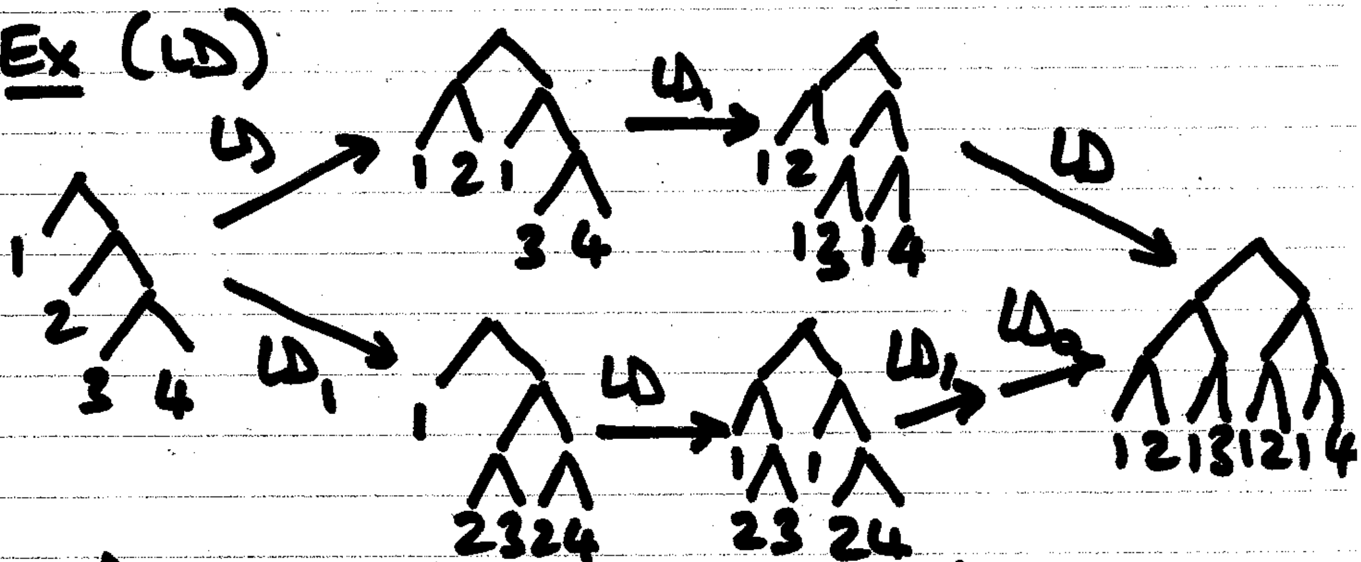
(iii') "<u>Overlapping</u>" case : try to ensure "local confluence"



<u>Ex</u> (A) Only missing case $A_\emptyset / A_1$



<u>Ex</u> (LD)



<u>Def</u>. $G_I := \langle \mathcal{A} \mid R_I \rangle$ (group)

$G_I^+ := \langle \mathcal{A} \mid R_I \rangle^+$ (monoid)

2. The blueprint method

What is the connection $\mathcal{G}_I / G_I$? $\mathcal{G}_I^+ / G_I^+$?
(Is $G_A = F$?)

$\rightsquigarrow$ All relations of R_I valid in $\mathcal{G}_I$
but maybe further relations (?).

Qu: Does $I_w \approx I_{w'}$ imply $w \equiv_{R_I} w'$?

Assume $\chi : \underset{\text{terms}}{\mathcal{T}} \hookrightarrow \underset{\text{words}}{(A \cup \bar{A})^*}$

s.t.

$$\chi(T \bullet I_w) \equiv_{R_I} \chi(T) \cdot w$$

(external) action on terms $\quad$ (internal) multiplication.

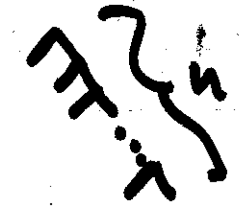
Then $I_w \approx I_{w'}$ implies $\exists T (T \cdot w = T \cdot w')$

hence $\chi(T) w \equiv_{R_I} \chi(T \bullet I_w) = \chi(T \bullet I_{w'}) \equiv_{R_I} \chi(T) w'$

hence $w \equiv_{R_I} w'$.

i.e., R_I gives presentation of $\mathcal{G}_I$

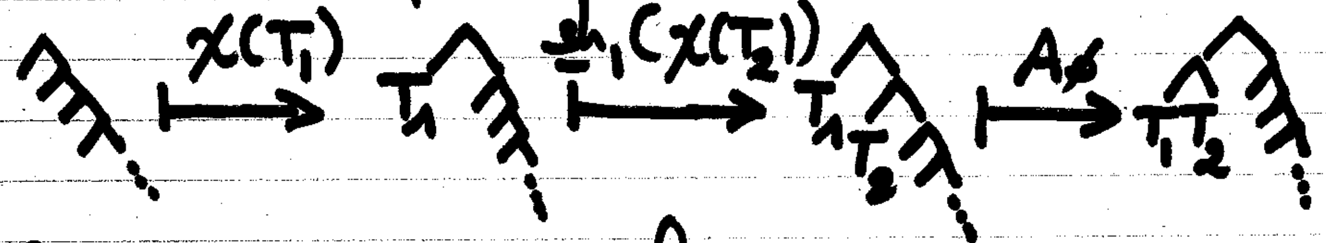
<u>Ex</u> (A) Every size n term is A-equivalent to $x^{[n]}$



⇝ define $\chi(T)$ as a distinguished word u s.t. $x^{[n]} \xmapsto[A_u]{} T$

Rather than $\xmapsto{\chi(T)} T$, ensure $\xmapsto{\chi(T)} T$

⇝ easier for an induction:



So: inductive def

$$\chi(T_1{}^\wedge T_2) = \chi(T_1).\,\underline{sh}_1(\chi(T_2)).\,A_\emptyset$$

left-shift all addresses by 1

<u>Exercise</u>: $\chi(\ \ ) = A_{\emptyset|1|1}$

<u>lemma</u>. $\chi(T \cdot A_\alpha) \equiv_{R_A} \chi(T) \cdot A_{0\alpha}$

$$\begin{tikzcd}
 & T \arrow[d, "A_{0\alpha}"] \\
\bullet \arrow[ur, "\chi(T)"] \arrow[r, "\chi(T\cdot A_\alpha)"'] & T\cdot A_\alpha
\end{tikzcd}$$

Hence $w \approx w' \Rightarrow \underline{sh}_0(w) \equiv_{R_A} \underline{sh}_0(w')$

$\Rightarrow w \equiv_{R_A} w'$

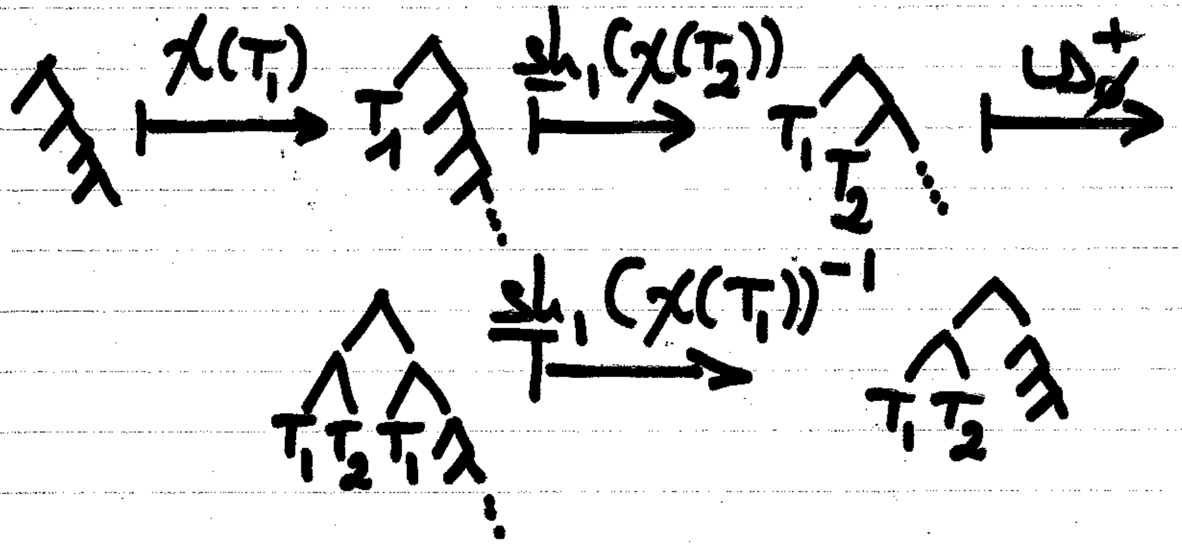
(because w, w' positive)

$\uparrow$

no A_α^{-1}

<u>Prop</u>. $F = G_A \ (:= \langle A \mid R_A \rangle)$.

Ex (LD) Blueprint? YES



Lemma $\chi(T \cdot LD_\alpha) \equiv_{R_{LD}} \chi(T) \cdot 0\alpha$

So $w \approx w' \Rightarrow \underline{sh}_0(w) \equiv_{R_{LD}} \underline{sh}_0(w')$

$\overset{?}{\Rightarrow} w \equiv_{R_{LD}} w'$

DIFFICULT because not positive

$\rightsquigarrow$ requires controlling G_{LD}.

and connection G^+_{LD} / G_{LD}

Typical question: Is G^+_{LD} cancellative?

3. Right-complemented presentations

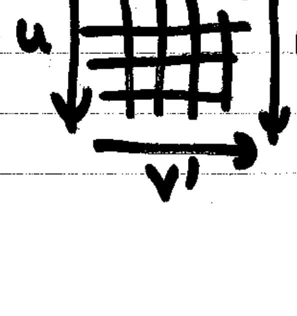
$$G^+_I := \langle A \mid R_I \rangle^+$$

Assume (true for (A), for (LD), ...)
$\forall \alpha, \beta\ \exists!$ relation ($I_\alpha \ldots = I_\beta \ldots$).
Can we exploit this? YES

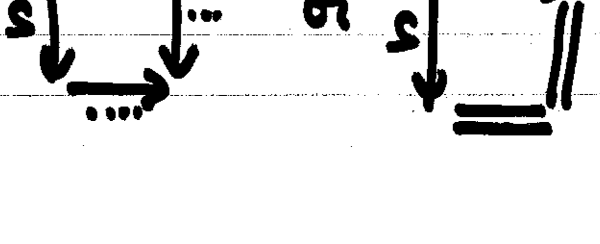
all relations $u = v$, $u, v \in S^* \neq \varepsilon$

Def: A positive presentation (S, R) is *right-complemented* if, for all s, t in S, there exists (at most) one relation $s \ldots = t \ldots$ in R.

Def: Assume (S, R) right-complem'd. For u, v, u', v' in S^*, say that (u, v) $\curvearrowright_R$ *R-reverses* to (u', v') if $\exists$ grid diagram



with all squares of the form

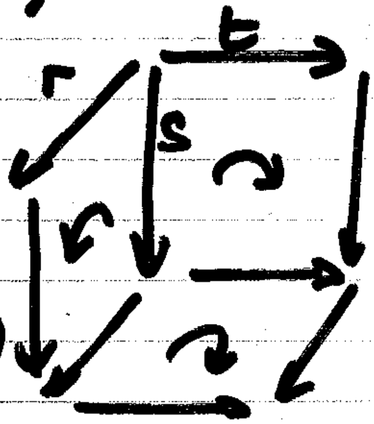
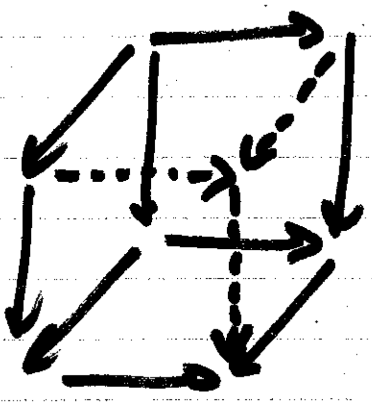


Remark. If $(u,v) \curvearrowright_R (\varepsilon,\varepsilon)$, then $u \equiv_R v$

Thm. Assume (S,R) right-compl'd, and

(i) $\exists \lambda : S^* \longrightarrow \underline{\mathrm{Ord}}$ (λ $\equiv_R$-invariant and $\forall s \in S\ \forall u \in S^*$ ($\lambda(su) > \lambda(u)$);

(ii) For all r, s, t in S

If  , then 

"cube condition"

Then $u \equiv_R v \iff (u,v) \curvearrowright_R (\varepsilon,\varepsilon)$.

Coro. The monoid $\langle S | R \rangle^+$ is left-cancellative

Proof. $su \equiv_R sv \Rightarrow (su, sv) \curvearrowright_R (\varepsilon,\varepsilon)$
$\Rightarrow (u,v) \curvearrowright_R (\varepsilon,\varepsilon) \Rightarrow u \equiv_R v.$ ∎

Application. G^+_{LD} is left-cancellative ....
$\underline{sh}_0(w) \equiv_R \underline{sh}_0(w') \Rightarrow w \equiv_R w'$.

4. Using the geometry group G_I

Ex (A) Here the law is simple (Word Problem obvious, etc.)
Benefit: Understand G_A ($= F$) better.

Proposition. – G_A ($= F$) is a group of fractions for G_A^+, which is a lattice w.r.t. divisibility

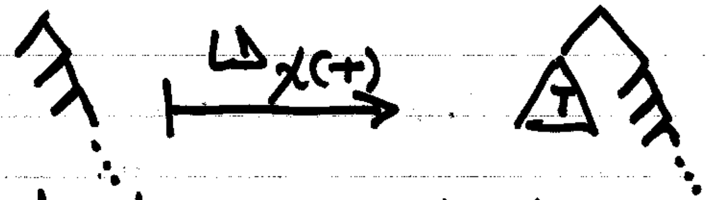
Coro. – For every n, the $=_A$-class of $x^{[n]}$ is a lattice.

↑ the Tamari lattice (cf. associahedron)

Ex (LD) Here the law is complicated
Benefit: Solve the Word Problem of I
+ construct concrete I-structures

⇝ Use blueprint again.

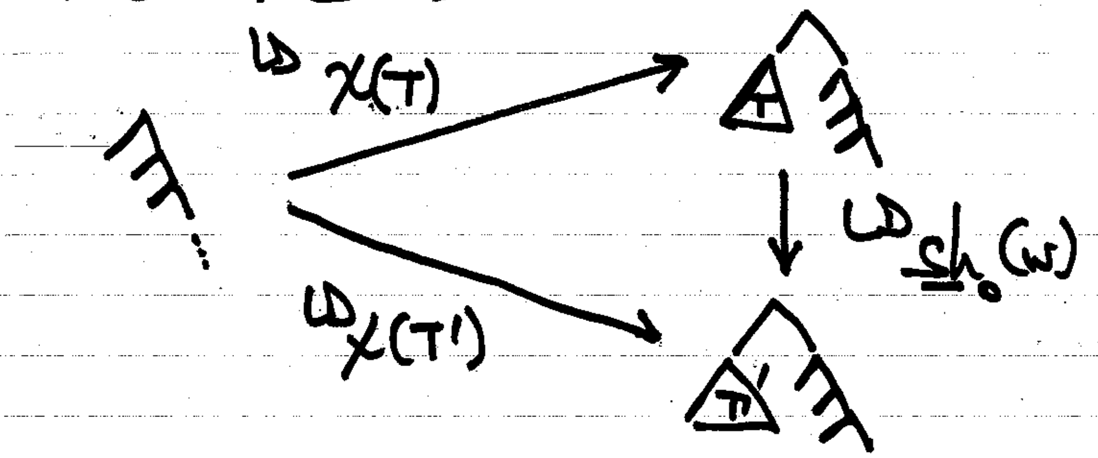
Recall : $\chi : \mathcal{T}_x \hookrightarrow (\mathbb{A} \cup \overline{\mathbb{A}})^*$ s.t.



with inductive construction:

$$\chi(x) = \varepsilon, \quad \chi(T_1{}^\wedge T_2) = \chi(T_1).\underline{sh}_1(\chi(T_2)).\mathrm{LD}_\emptyset.\underline{sh}_1(\chi(T_1))^{-1}.$$

Assume $T' = T \bullet w$. Then



So expect — and check —

$$\chi(T') \equiv_{\mathcal{R}_{LD}} \chi(T).\underline{sh}_0(w).$$

Hence, <u>in</u> G_{LD} : $T =_{LD} T'$ implies

$$\chi(T)^{-1}\,\chi(T') \in \underline{sh}_0(G_{LD})$$

$\rightsquigarrow$ If we collapse $\underline{sh}_0(G_{LD})$ in G_{LD}, we get an LD-<u>invariant</u>.

<u>Proposition</u>. For f, g in $\underline{sh}_0(G_{LD}) \setminus G_{LD}$ (coset set) put:
$$f * g = f \cdot \underline{sh}_1(g) \cdot [LD_\infty] \cdot \underline{sh}_1(f)^{-1}.$$
Then $*$ obeys LD.

In fact, in <u>this</u> case: (*) $\underline{sh}_0(G_{LD})$ is a normal subgroup, so the coset set is a group, namely Artin's braid group B_∞
(**) $T =_{LD} T' \Rightarrow \chi(T)^{-1}\chi(T') \in \underline{sh}_0(G_{LD})$
is an equivalence, so one solves the WP: $T =_{LD} T'$ iff the B_∞-evaluations of T and T' coincide.

In other words: $(B_\infty, *)$ includes a <u>free</u> LD-structure.

III. A few cousins of Thompson's groups

Consider further laws — or, more generally, rewrite systems
(P.D., V. van Oostrom, MSCS 18 (2008) 1133-1167)

1. (A) $\rightsquigarrow$ F

2. (A, C) $\rightsquigarrow$ V

3. (A, S) $\rightsquigarrow$ $\hat{V}$ (subgroup fixing I^∞)
S: semi-commutativity $x(yz) = y(xz)$

4. (CD) "central duplication" $x(yz) = (xy)(yz)$
Here solution of the WP, realization of the free CD. structure in a $\mathfrak{sh}_0(G_{CD}) \backslash G_{CD}$ (not normal) (Alg. Univ. 48 (2002) 223-248)

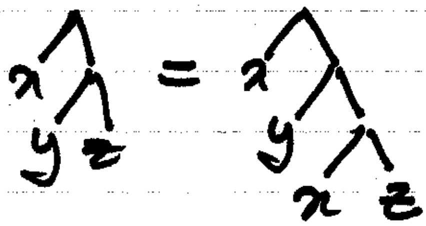
5. (A, C$^\#$) $\longrightarrow$ BV
C$^\#$: "twisted commutativity"
$xy = x[y]\,x$
[y]: second operation obeying LD

6. (A, S$^\#$) $\rightsquigarrow$ $\widehat{BV}$
S$^\#$: $x(yz) = x[y](xz)$

see JPAA 203 (2005) 1-44
AdvMath. 205 (2006) 354-409

7. A new chain — and another "strand splitting" group

(I) : $x(yz) = x(y(xz))$

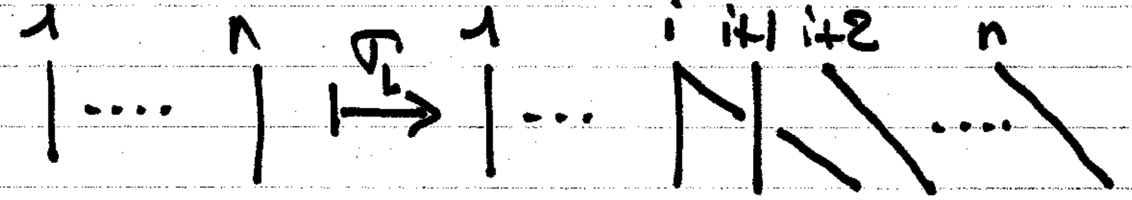


Consider the action on right vines = action on sequences by "permutation + duplication"

$(x, y, z, \dots) \mapsto (x, y, x, z, \dots)$

Consider the subgroup H (and the submonoid H^+) of G_I gener'd by

$$\sigma_i : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_i, x_{i+1}, x_i, x_{i+2}, \dots, x_n)$$



<u>Presentation of $H^{(+)}$</u> :

$$(\#)\begin{cases} \sigma_j\sigma_i = \sigma_i\sigma_{j+1} & \text{for } j > i+1, \\ \sigma_j\sigma_i\sigma_j = \sigma_i\sigma_j\sigma_{j+2} & \text{for } j = i+1. \end{cases}$$

⇝ contain of F : $\sigma_j\sigma_i = \sigma_i\sigma_{j+1}$ for $j \geq i+1$

⇝ contain of B_∞ : $\sigma_j\sigma_i\sigma_j = \sigma_i\sigma_j\sigma_i$ for $j = i+1$

Here : " $\dfrac{H}{B_\infty} = \dfrac{F}{\mathbb{Z}^n}$ " (shift)

<u>Proposition</u>. The group H is a group of fractions for H^+, which is a lattice w.r.t. divisibility, etc.

↑

the monoid H^+ has good "Garside properties" as B_∞^+ does

<u>Exercise</u>. Check the (left- and right-) cube conditions for the relations (#)

⇝ This new contain <u>seems</u> interesting (?)

Remarks. (i) There is no flexibility: all good results that hold with $\sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_4$ fail with $\sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_3$ (even cancellativity)

(ii) When one considers "duplication without permutation", i.e., $(x, y, \dots) \mapsto (x, x, y, \dots)$, then the F-relations $a_j a_i = a_i a_{j+1}$ for $j > i$ hold, but F is not the associated group as one has (trivial) extra relations like $a_1^2 = a_1 a_2$.

Suggestion: Investigate the new domain further ...

Reference

– (book) Birkhäuser PM 192 (1999)

– Methods for investigating presented monoids: IJAC 21 (2011) 71 – 118