Laver tables

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• Finite objects with a simple description, discovered through set theory, with combinatorial properties that (so far) are only established using unprovable large cardinal hypotheses, and with (potential) applications in low-dimensional topology.

Plan :

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- 1. Combinatorial description of Laver tables

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- 2. Laver tables and set theory

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$\rightsquigarrow$ monogenerated substructures are trivial.

- Q : Is conjugacy of a free group characterized by selfdistributivity and idempotency? No (Drápal-Kepka-Musilek 1994, Larue 1999), it obeys

$$((x * y) * y) * (x * z) = (x * y) * ((y * x) * z), \dots$$

- A binary operation on $\{1, 2, 3, 4\}$:

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$*$	1	2	3	4
1				
2				
3				
4				

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$*$	1	2	3	4
1	2			
2	3			
3	4			
4	1			

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  $4 * 2 =$

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1	2			
2	3			
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$$4 * 2 = 4 * (1 * 1)$$

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$$\begin{aligned} 4 * 2 &= 4 * (1 * 1) = (4 * 1) * (4 * 1) = 1 * 1 = 2, \\ 4 * 3 &= 4 * (2 * 1) = (4 * 2) * (4 * 1) = 2 * 1 = 3, \\ 4 * 4 &= 4 * (3 * 1) \end{aligned}$$

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4 * 4 &= 4 * (3 * 1) = (4 * 3) * (4 * 1) = 3 * 1 = 4,\\
3 * 2 &= 3 * (1 * 1) = (3 * 1) * (3 * 1) = 4 * 4 = 4, \ldots
\end{aligned}
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2	3			
3	4	4		
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$$\begin{aligned}
4*2 &= 4*(1*1) = (4*1)*(4*1) = 1*1 = 2,\\
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4*4 &= 4*(3*1) = (4*3)*(4*1) = 3*1 = 4,\\
3*2 &= 3*(1*1) = (3*1)*(3*1) = 4*4 = 4, \ldots
\end{aligned}$$

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1	2			
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$$4 * 4 = 4 * (3 * 1) = (4 * 3) * (4 * 1) = 3 * 1 = 4,$$
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1	2			
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4*4 = 4*(3*1) = (4*3)*(4*1) = 3*1 = 4,\\
3*2 = 3*(1*1) = (3*1)*(3*1) = 4*4 = 4, \dots
\end{aligned}
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4 * 4 = 4 * (3 * 1) = (4 * 3) * (4 * 1) = 3 * 1 = 4,\\
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\end{aligned}
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- Proposition (Laver).— (i) For every N, there exists a unique binary operation $*$ on $\{1, ..., N\}$ satisfying

$$x * 1 = x + 1 \bmod N \quad \text{and}$$
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if and only if N is a power of 2.

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$\rightsquigarrow$ the Laver table with $1, 2, 4, 8, 16, 32, \ldots$ elements.

A_0	1
1	1

A_0	1
1	1

A_1	1	2
1	2	2
2	1	2

A_0	1
1	1

A_1	1	2
1	2	2
2	1	2

A_2	1	2	3	4
1	2	4	2	4
2	3	4	3	4
3	4	4	4	4
4	1	2	3	4

A_0	1
1	1

A_1	1	2
1	2	2
2	1	2

A_2	1	2	3	4
1	2	4	2	4
2	3	4	3	4
3	4	4	4	4
4	1	2	3	4

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

Laver tables: examples

A_0	1
1	1

A_1	1	2
1	2	2
2	1	2

A_2	1	2	3	4
1	2	4	2	4
2	3	4	3	4
3	4	4	4	4
4	1	2	3	4

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

A_4	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	2	12	14	16	2	12	14	16	2	12	14	16	2	12	14	16
2	3	12	15	16	3	12	15	16	3	12	15	16	3	12	15	16
3	4	8	12	16	4	8	12	16	4	8	12	16	4	8	12	16
4	5	6	7	8	13	14	15	16	5	6	7	8	13	14	15	16
5	6	8	14	16	6	8	14	16	6	8	14	16	6	8	14	16
6	7	8	15	16	7	8	15	16	7	8	15	16	7	8	15	16
7	8	16	8	16	8	16	8	16	8	16	8	16	8	16	8	16
8	9	10	11	12	13	14	15	16	9	10	11	12	13	14	15	16
9	10	12	14	16	10	12	14	16	10	12	14	16	10	12	14	16
10	11	12	15	16	11	12	15	16	11	12	15	16	11	12	15	16
11	12	16	12	16	12	16	12	16	12	16	12	16	12	16	12	16
12	13	14	15	16	13	14	15	16	13	14	15	16	13	14	15	16
13	14	16	14	16	14	16	14	16	14	16	14	16	14	16	14	16
14	15	16	15	16	15	16	15	16	15	16	15	16	15	16	15	16
15	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
16	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

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$\rightsquigarrow$ think of $\mathbb{Z}/p\mathbb{Z}$ in the associative world

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2	3	4	7	8	3	4	7	8
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4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
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4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
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$\rightsquigarrow \pi_3(8) = 8$

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4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

$\rightsquigarrow \pi_3(7) = 1$

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4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

$\rightsquigarrow \pi_3(6) = 2$

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4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

$\rightsquigarrow \pi_3(5) = 2$
$\rightsquigarrow \pi_3(6) = 2$
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3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

$\rightsquigarrow \pi_3(4) = 4$

$\rightsquigarrow \pi_3(5) = 2$

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2	3	4	7	8	3	4	7	8	
3	4	8	4	8	4	8	4	8	$\rightsquigarrow \pi_3(3) = 2$
4	5	6	7	8	5	6	7	8	$\rightsquigarrow \pi_3(4) = 4$
5	6	8	6	8	6	8	6	8	$\rightsquigarrow \pi_3(5) = 2$
6	7	8	7	8	7	8	7	8	$\rightsquigarrow \pi_3(6) = 2$
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A_3	1	2	3	4	5	6	7	8	
1	2	4	6	8	2	4	6	8	
2	3	4	7	8	3	4	7	8	$\rightsquigarrow \pi_3(2) = 4$
3	4	8	4	8	4	8	4	8	$\rightsquigarrow \pi_3(3) = 2$
4	5	6	7	8	5	6	7	8	$\rightsquigarrow \pi_3(4) = 4$
5	6	8	6	8	6	8	6	8	$\rightsquigarrow \pi_3(5) = 2$
6	7	8	7	8	7	8	7	8	$\rightsquigarrow \pi_3(6) = 2$
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A_3	1	2	3	4	5	6	7	8	
1	2	4	6	8	2	4	6	8	$\rightsquigarrow \pi_3(1) = 4$
2	3	4	7	8	3	4	7	8	$\rightsquigarrow \pi_3(2) = 4$
3	4	8	4	8	4	8	4	8	$\rightsquigarrow \pi_3(3) = 2$
4	5	6	7	8	5	6	7	8	$\rightsquigarrow \pi_3(4) = 4$
5	6	8	6	8	6	8	6	8	$\rightsquigarrow \pi_3(5) = 2$
6	7	8	7	8	7	8	7	8	$\rightsquigarrow \pi_3(6) = 2$
7	8	8	8	8	8	8	8	8	$\rightsquigarrow \pi_3(7) = 1$
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- A few values of the periods of 1 and 2:

n	
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n	0
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n	0	1
$\pi_n(1)$	1	1
$\pi_n(2)$	—	2

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n	0	1	2
$\pi_n(1)$	1	1	2
$\pi_n(2)$	—	2	2

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n	0	1	2	3
$\pi_n(1)$	1	1	2	4
$\pi_n(2)$	—	2	2	4

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n	0	1	2	3	4
$\pi_n(1)$	1	1	2	4	4
$\pi_n(2)$	—	2	2	4	4

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n	0	1	2	3	4	5
$\pi_n(1)$	1	1	2	4	4	8
$\pi_n(2)$	—	2	2	4	4	8

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- A few values of the periods of 1 and 2:

n	0	1	2	3	4	5	6
$\pi_n(1)$	1	1	2	4	4	8	8
$\pi_n(2)$	–	2	2	4	4	8	8

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n	0	1	2	3	4	5	6	7
$\pi_n(1)$	1	1	2	4	4	8	8	8
$\pi_n(2)$	—	2	2	4	4	8	8	16

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n	0	1	2	3	4	5	6	7	8
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n	0	1	2	3	4	5	6	7	8	9
$\pi_n(1)$	1	1	2	4	4	8	8	8	8	16
$\pi_n(2)$	—	2	2	4	4	8	8	16	16	16

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n	0	1	2	3	4	5	6	7	8	9	10
$\pi_n(1)$	1	1	2	4	4	8	8	8	8	16	16
$\pi_n(2)$	—	2	2	4	4	8	8	16	16	16	16

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n	0	1	2	3	4	5	6	7	8	9	10	11	⋯
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n	0	1	2	3	4	5	6	7	8	9	10	11	⋯
$\pi_n(1)$	1	1	2	4	4	8	8	8	8	16	16	16	⋯
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- Question 1 : Does $\pi_n(2) \geqslant \pi_n(1)$ always hold?

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n	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
$\pi_n(1)$	1	1	2	4	4	8	8	8	8	16	16	16	$\cdots$
$\pi_n(2)$	—	2	2	4	4	8	8	16	16	16	16	16	$\cdots$

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$\pi_n(2)$	–	2	2	4	4	8	8	16	16	16	16	16	···

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n	0	1	2	3	4	5	6	7	8	9	10	11	···
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$\pi_n(2)$	–	2	2	4	4	8	8	16	16	16	16	16	···

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- Theorem (Laver, 1995).—

  the answer to the above questions is positive.

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- A few values of the periods of 1 and 2:

n	0	1	2	3	4	5	6	7	8	9	10	11	⋯
$\pi_n(1)$	1	1	2	4	4	8	8	8	8	16	16	16	⋯
$\pi_n(2)$	–	2	2	4	4	8	8	16	16	16	16	16	⋯

- Question 1 : Does $\pi_n(2) \geqslant \pi_n(1)$ always hold?
- Question 2 : Does $\pi_n(1)$ tend to ∞ with n ? Does it reach 32 ?

- Theorem (Laver, 1995).— If there exists a selfsimilar set, then the answer to the above questions is positive.

Plan :

- 1. Combinatorial description of Laver tables
- 2. Laver tables and set theory
- 3. Laver tables and low-dimensional topology

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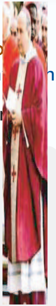

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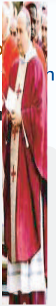

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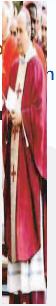

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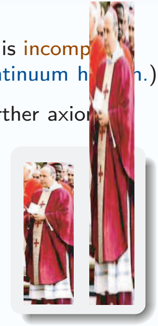

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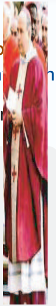

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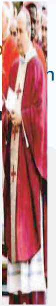

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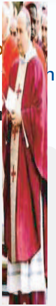

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- Hence $\mathrm{Iter}(j)/{\equiv_n}$ is an LD-structure with 2^n elements s.t. $j_{[p]} * j = j_{[p+1 \bmod 2^n]}$.

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  there exists a smallest ordinal α satisfying $j(\alpha) > \alpha$: the critical ordinal $\mathrm{crit}(j)$.
- Recall: $j_{[p]} := j(j)(j)...(j)$, p terms.

- **Proposition** (Laver).— Assume that j is an embedding of a rank R.
For k, k' in $\mathrm{Iter}(j)$, declare $k \equiv_n k'$ if

" k and k' coincide up to the level of $\mathrm{crit}(j_{[2^n]})$ "

Then $\equiv_n$ is a congruence on $\mathrm{Iter}(j)$, it has 2^n classes,
which are those of $j, j_{[2]}, ..., j_{[2^n]}$, the latter also being the class of id.

exact definition of $\equiv_n$: $\forall x \in R_\gamma (k(x) \cap R_\gamma = k'(x) \cap R_\gamma)$ with $\gamma = \mathrm{crit}(j_{[2^n]})$

- Hence $\mathrm{Iter}(j)/\equiv_n$ is an LD-structure with 2^n elements s.t. $j_{[p]} * j = j_{[p+1 \bmod 2^n]}$.

- **Corollary**.— The quotient-structure $\mathrm{Iter}(j)/\equiv_n$ is (isomorphic to) the table A_n.

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- Corollary.— If there exists a selfsimilar set, the substructure generated by $(1, 1, 1, \ldots)$ in the inverse limit of all A_n is free.

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Plan :

- 1. Combinatorial description of Laver tables
- 2. Laver tables and set theory
- 3. Laver tables and low-dimensional topology

- Planar diagrams:

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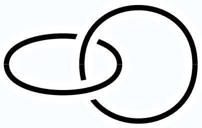

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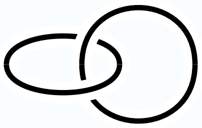 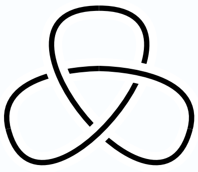

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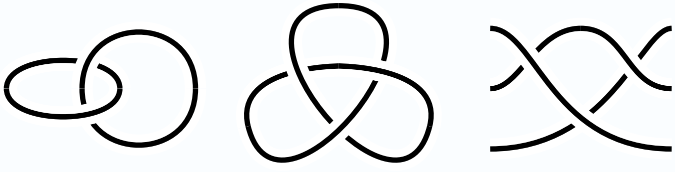

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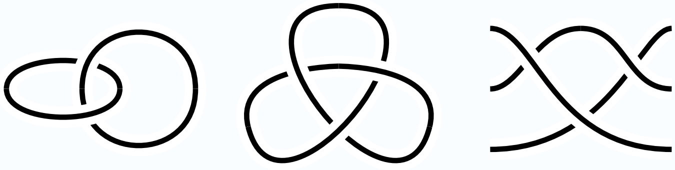

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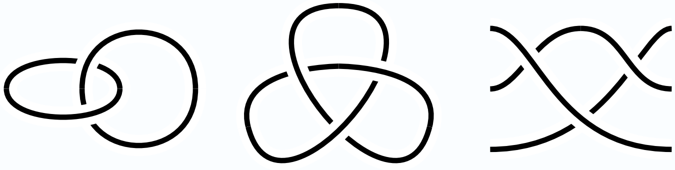

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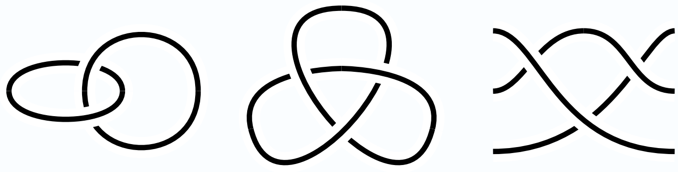

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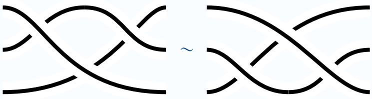

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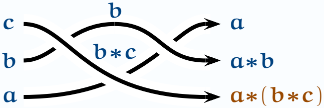

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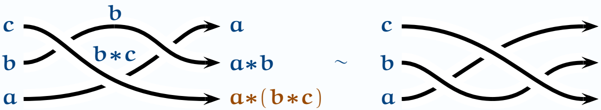

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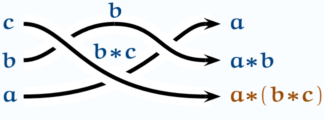

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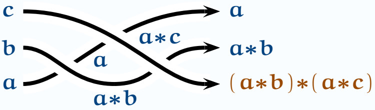

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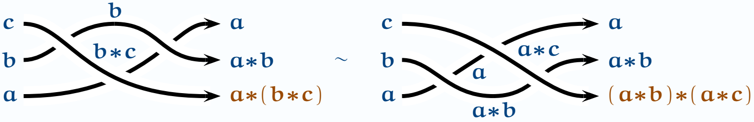

⇝ Hence: S-colorings invariant under Reidemeister move III $\Leftrightarrow$ $(S, *)$ LD-structure

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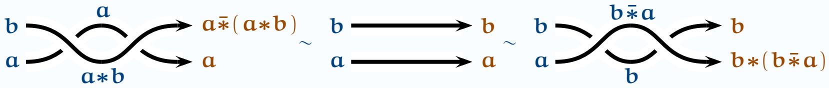

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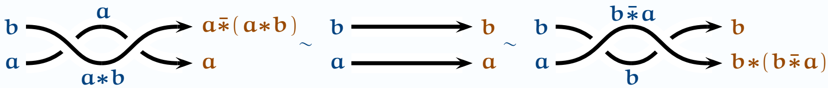

There exists $\bar{*}$ satisfying $x * (x \bar{*} y) = y$ and $x \bar{*} (x * y) = y$ iff the left-translations of $(S, *)$ are bijections.

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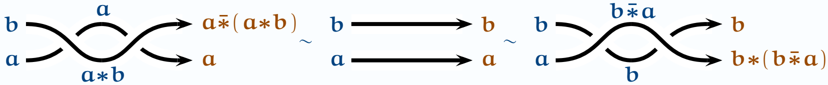

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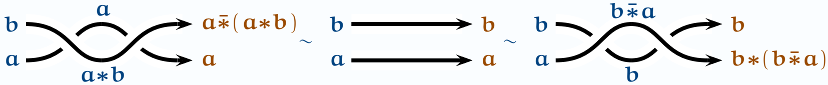

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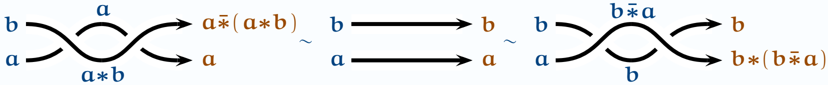

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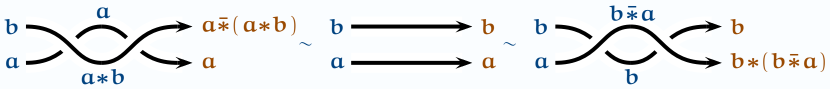

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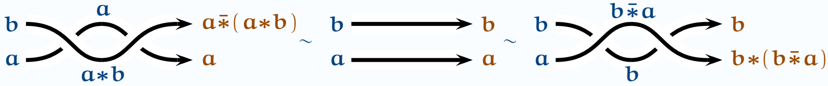

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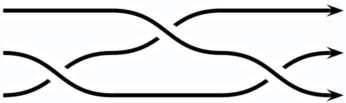

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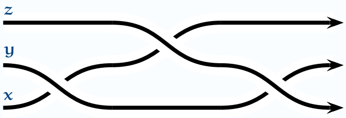

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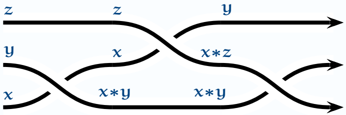

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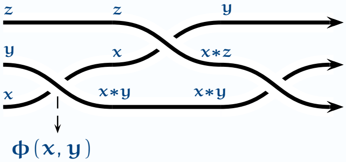

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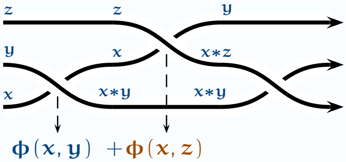

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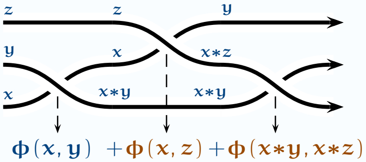

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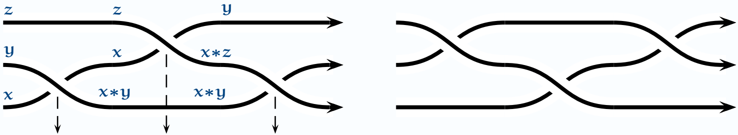

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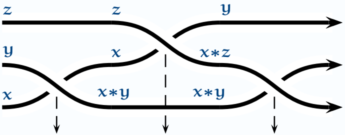

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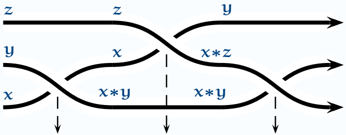

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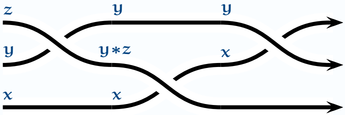

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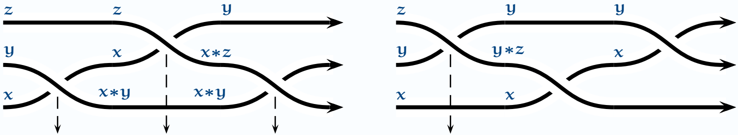

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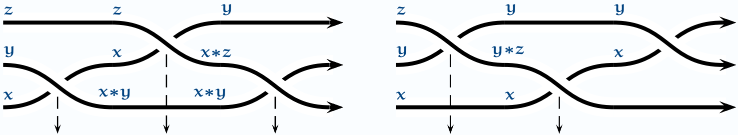

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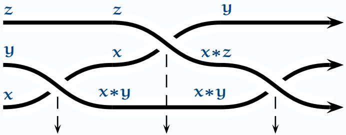

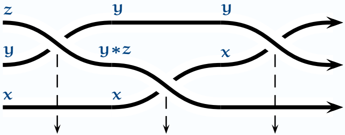

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$\psi_{1,3}$	1	2	3	4	5	6	7	8
1	1	·	·	·	·	·	·	·
2	1	·	·	·	·	·	·	·
3	1	·	·	·	·	·	·	·
4	1	·	·	·	·	·	·	·
5	1	·	·	·	·	·	·	·
6	1	·	·	·	·	·	·	·
7	1	·	·	·	·	·	·	·
8	·	·	·	·	·	·	·	·

$\psi_{2,3}$	1	2	3	4	5	6	7	8
1	·	1	·	·	·	·	·	·
2	1	1	·	·	1	·	·	·
3	1	1	·	·	1	·	·	·
4	·	1	·	·	·	·	·	·
5	1	1	·	·	1	·	·	·
6	1	1	·	·	1	·	·	·
7	1	1	·	·	1	·	·	·
8	·	·	·	·	·	·	·	·

$\psi_{3,3}$	1	2	3	4	5	6	7	8
1	1	·	1	·	1	·	·	·
2	·	·	1	·	·	·	·	·
3	1	·	1	·	1	·	·	·
4	·	·	1	·	·	·	·	·
5	1	·	1	·	1	·	·	·
6	1	·	1	·	1	·	·	·
7	1	·	1	·	1	·	·	·
8	·	·	·	·	·	·	·	·

$\psi_{4,3}$	1	2	3	4	5	6	7	8
1	·	·	·	1	·	·	·	·
2	·	·	·	1	·	·	·	·
3	·	1	·	1	·	1	·	·
4	·	·	·	1	·	·	·	·
5	·	1	·	1	·	1	·	·
6	·	1	·	1	·	1	·	·
7	1	1	1	1	1	1	1	·
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2	1	·	·	·	·	·	·	·
3	1	·	·	·	·	·	·	·
4	1	·	·	·	·	·	·	·
5	1	·	·	·	·	·	·	·
6	1	·	·	·	·	·	·	·
7	1	·	·	·	·	·	·	·
8	·	·	·	·	·	·	·	·

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1	·	1	·	·	·	·	·	·
2	1	1	·	·	1	·	·	·
3	1	1	·	·	1	·	·	·
4	·	1	·	·	·	·	·	·
5	1	1	·	·	1	·	·	·
6	1	1	·	·	1	·	·	·
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3	1	·	1	·	1	·	·	·
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7	1	·	1	·	1	·	·	·
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1	·	·	·	1	·	·	·	·
2	·	·	·	1	·	·	·	·
3	·	1	·	1	·	1	·	·
4	·	·	·	1	·	·	·	·
5	·	1	·	1	·	1	·	·
6	·	1	·	1	·	1	·	·
7	1	1	1	1	1	1	1	·
8	·	·	·	·	·	·	·	·

$\psi_{5,3}$	1	2	3	4	5	6	7	8
1	1	·	·	·	1	·	·	·
2	1	·	·	·	1	·	·	·
3	1	·	·	·	1	·	·	·
4	·	·	·	·	·	·	·	·
5	1	·	·	·	1	·	·	·
6	1	·	·	·	1	·	·	·
7	1	·	·	·	1	·	·	·
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1	·	1	·	·	·	1	·	·
2	·	1	·	·	·	1	·	·
3	1	1	1	·	1	1	1	·
4	·	·	·	·	·	·	·	·
5	·	1	·	·	·	1	·	·
6	·	1	·	·	·	1	·	·
7	1	1	1	·	1	1	1	·
8	·	·	·	·	·	·	·	·

$\psi_{7,3}$	1	2	3	4	5	6	7	8
1	1	·	1	·	1	·	1	·
2	·	·	·	·	·	·	·	·
3	1	·	1	·	1	·	1	·
4	·	·	·	·	·	·	·	·
5	1	·	1	·	1	·	1	·
6	·	·	·	·	·	·	·	·
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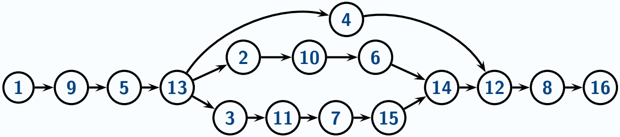

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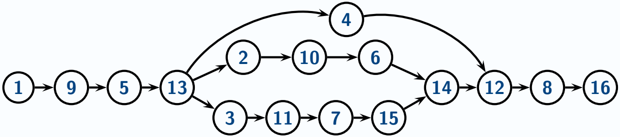

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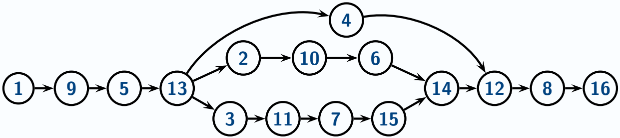

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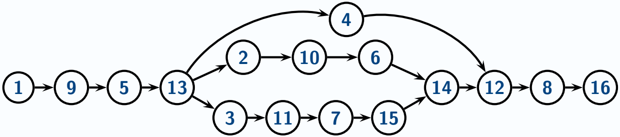

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- Conclusion : Reasonable hope of applying Laver tables in low-dimensional topology.

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