Laver tables

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• Finite objects with a simple description, discovered through set theory, with combinatorial properties that (so far) are only established using unprovable large cardinal hypotheses, and with (potential) applications in low-dimensional topology.

Plan :

- 1. Combinatorial description of Laver tables
- 2. Laver tables and set theory
- 3. Laver tables and low-dimensional topology

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- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \qquad \textbf{(LD)}$$

cf. associativity: $x * (y * z) = (x * y) * z$.

- Classical examples:
  - S arbitrary and $x * y := y$, or more generally $x * y = f(y)$;
  - E module and $x * y := (1 - \lambda)x + \lambda y$;
  - G group and $x * y := xyx^{-1}$.

- Remark : These operations obey $x * x = x$ ("idempotency")

⇝ monogenerated substructures are trivial.

- Q : Is conjugacy of a free group characterized by selfdistributivity and idempotency? No (Drápal-Kepka-Musilek 1994, Larue 1999), it obeys

$$((x * y) * y) * (x * z) = (x * y) * ((y * x) * z), \ldots$$

- A binary operation on $\{1,2,3,4\}$: the four element Laver table

$*$	1	2	3	4
1	2	4	2	4
2	3	4	3	4
3	4	4	4	4
4	1	2	3	4

- Start with $+1 \bmod 4$ in the first column,
  and complete so as to obey the rule $x * (y * 1) = (x * y) * (x * 1)$:

$$\begin{aligned}
4 * 2 = 4 * (1 * 1) = (4 * 1) * (4 * 1) = 1 * 1 = 2,\\
4 * 3 = 4 * (2 * 1) = (4 * 2) * (4 * 1) = 2 * 1 = 3,\\
4 * 4 = 4 * (3 * 1) = (4 * 3) * (4 * 1) = 3 * 1 = 4,\\
3 * 2 = 3 * (1 * 1) = (3 * 1) * (3 * 1) = 4 * 4 = 4, \ldots
\end{aligned}$$

- The same construction works for every size and it provides a selfdistributive structure for powers of 2:

- Proposition (Laver).— (i) For every N, there exists a unique binary operation $*$ on $\{1, \ldots, N\}$ satisfying

$$x * 1 = x + 1 \bmod N \quad \text{and}$$
$$x * (y * 1) = (x * y) * (x * 1).$$

(ii) The operation thus obtained obeys the law

$$x * (y * z) = (x * y) * (x * z) \qquad \text{(LD)}$$

if and only if N is a power of 2.

$\rightsquigarrow$ the Laver table with 1, 2, 4, 8, 16, 32,... elements.

Laver tables: examples

A_0	1
1	1

A_1	1	2
1	2	2
2	1	2

A_2	1	2	3	4
1	2	4	2	4
2	3	4	3	4
3	4	4	4	4
4	1	2	3	4

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

A_4	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	2	12	14	16	2	12	14	16	2	12	14	16	2	12	14	16
2	3	12	15	16	3	12	15	16	3	12	15	16	3	12	15	16
3	4	8	12	16	4	8	12	16	4	8	12	16	4	8	12	16
4	5	6	7	8	13	14	15	16	5	6	7	8	13	14	15	16
5	6	8	14	16	6	8	14	16	6	8	14	16	6	8	14	16
6	7	8	15	16	7	8	15	16	7	8	15	16	7	8	15	16
7	8	16	8	16	8	16	8	16	8	16	8	16	8	16	8	16
8	9	10	11	12	13	14	15	16	9	10	11	12	13	14	15	16
9	10	12	14	16	10	12	14	16	10	12	14	16	10	12	14	16
10	11	12	15	16	11	12	15	16	11	12	15	16	11	12	15	16
11	12	16	12	16	12	16	12	16	12	16	12	16	12	16	12	16
12	13	14	15	16	13	14	15	16	13	14	15	16	13	14	15	16
13	14	16	14	16	14	16	14	16	14	16	14	16	14	16	14	16
14	15	16	15	16	15	16	15	16	15	16	15	16	15	16	15	16
15	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
16	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

- For $n \geqslant 1$, one has $1 * 1 = 2 \neq 1$ in A_n: not idempotent.

  ⇝ quite différent from group conjugacy and other classical LD-structures

- Proposition (Laver).— The LD-structure A_n is generated by 1 and admits the presentation $\langle 1 \mid 1_{[2^n]} = 1 \rangle$, with $x_{[k]} = (\ldots((x{*}x){*}x)\ldots){*}x$, k terms.

- Proposition (Drápal).— There exists an (explicit) list of constructions $\mathcal{L}$ (direct product, ...) such that every finite monogenerated LD-structure can be obtained from Laver tables using constructions from $\mathcal{L}$.

⇝ think of $\mathbb{Z}/p\mathbb{Z}$ in the associative world

- **Proposition** (Laver).— For every $p \leqslant 2^n$, there exists a number $\pi_n(p)$, a power of 2, such that the pth row in (the table of) A_n is the repetition of $\pi_n(p)$ values increasing from $p+1 \bmod 2^n$ to 2^n.

- Example :

A_3	1	2	3	4	5	6	7	8	
1	2	4	6	8	2	4	6	8	$\rightsquigarrow \pi_3(1) = 4$
2	3	4	7	8	3	4	7	8	$\rightsquigarrow \pi_3(2) = 4$
3	4	8	4	8	4	8	4	8	$\rightsquigarrow \pi_3(3) = 2$
4	5	6	7	8	5	6	7	8	$\rightsquigarrow \pi_3(4) = 4$
5	6	8	6	8	6	8	6	8	$\rightsquigarrow \pi_3(5) = 2$
6	7	8	7	8	7	8	7	8	$\rightsquigarrow \pi_3(6) = 2$
7	8	8	8	8	8	8	8	8	$\rightsquigarrow \pi_3(7) = 1$
8	1	2	3	4	5	6	7	8	$\rightsquigarrow \pi_3(8) = 8$

- The map $x \mapsto x \bmod 2^{n-1}$ is a surjective homomorphism from A_n to A_{n-1}.
  - ⇝ the inverse limit of the A_n is an LD operation on 2-adic numbers;
  - ⇝ one always has $\pi_n(p) \geqslant \pi_{n-1}(p)$.

- A few values of the periods of 1 and 2:

n	0	1	2	3	4	5	6	7	8	9	10	11	⋯
$\pi_n(1)$	1	1	2	4	4	8	8	8	8	16	16	16	⋯
$\pi_n(2)$	–	2	2	4	4	8	8	16	16	16	16	16	⋯

- Question 1 : Does $\pi_n(2) \geqslant \pi_n(1)$ always hold?
- Question 2 : Does $\pi_n(1)$ tend to ∞ with n ? Does it reach 32 ?

- Theorem (Laver, 1995).— If there exists a selfsimilar set, then the answer to the above questions is positive.

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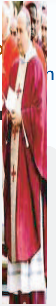

- Set theory is a theory of infinity;
it was axiomatized in the Zermelo-Fraenkel system **ZF** (1922), which is incomp
some statements are neither provable nor refutable from **ZF** (e.g., continuum h h.)

$\rightsquigarrow$ Discover more properties of infinity and complete **ZF** with further axio

- Typically, large cardinals axioms = various solutions to the equation

$$\frac{\text{ultra-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}.$$

Examples: inaccessible cardinals, measurable cardinals, etc.

- General principle: "being selfsimilar implies being large".
  - $\boldsymbol{A}$ is infinite iff $\exists \mathbf{j} : \boldsymbol{A} \to \boldsymbol{A}$ injective not bijective;

    a (self)embedding of $\boldsymbol{A}$
    ↓
  - $\boldsymbol{A}$ is ultra-infinite ("selfsimilar") iff $\exists \mathbf{j} : \boldsymbol{A} \to \boldsymbol{A}$ injective not bijective
and preserving every notion that is definable from $\in$.

- Example: $\mathbb{N}$ infinite, but not ultra-infinite: if $\mathbf{j} : \mathbb{N} \to \mathbb{N}$ preserves every notion that is definable from $\in$, then $\mathbf{j}$ preserves $0, 1, 2$, etc. hence $\mathbf{j}$ is the identity map.

- Definition.— A rank is a set $\mathbf{R}$ such that $\mathbf{f} : \mathbf{R} \to \mathbf{R}$ implies $\mathbf{f} \in \mathbf{R}$. (this exists...)
- Assume that there exists a selfsimilar set:
  - then there exists a selfsimilar rank, say $\mathbf{R}$;
  - if $\mathbf{i}$, $\mathbf{j}$ are embeddings of $\mathbf{R}$, then $\mathbf{i} : \mathbf{R} \to \mathbf{R}$ and $\mathbf{j} \in \mathbf{R}$,
    hence we can apply $\mathbf{i}$ to $\mathbf{j}$;
  - "being an embedding" is definable from $\in$,
    hence $\mathbf{i}(\mathbf{j})$ is an embedding;
  - "being the image of" is definable from $\in$,
    hence $\boldsymbol{\ell} = \mathbf{j}(\mathbf{k})$ implies $\mathbf{i}(\boldsymbol{\ell}) = \mathbf{i}(\mathbf{j})(\mathbf{i}(\mathbf{k}))$, *i.e.*, $\mathbf{i}(\mathbf{j}(\mathbf{k})) = \mathbf{i}(\mathbf{j})(\mathbf{i}(\mathbf{k}))$: **LD**-law.

- Proposition.— If $\mathbf{j}$ is an embedding of a rank $\mathbf{R}$,
  then the iterates of $\mathbf{j}$ make an LD-structure $\mathrm{Iter}(\mathbf{j})$.

↑ closure of $\{\mathbf{j}\}$ under the "apply" operation: $\mathbf{j}(\mathbf{j})$, $\mathbf{j}(\mathbf{j})(\mathbf{j})$...

- An embedding j maps every ordinal α to an ordinal $j(\alpha) \geqslant \alpha$;
  there exists a smallest ordinal α satisfying $j(\alpha) > \alpha$: the critical ordinal $crit(j)$.
- Recall: $j_{[p]} := j(j)(j)...(j)$, p terms.

- Proposition (Laver).— Assume that j is an embedding of a rank R.

For k, k' in $Iter(j)$, declare $k \equiv_n k'$ if

" k and k' coincide up to the level of $crit(j_{[2^n]})$ "

Then $\equiv_n$ is a congruence on $Iter(j)$, it has 2^n classes,
which are those of $j, j_{[2]}, ..., j_{[2^n]}$, the latter also being the class of id.

exact definition of $\equiv_n$: $\forall x{\in}R_\gamma(k(x){\cap}R_\gamma{=}k'(x){\cap}R_\gamma)$ with $\gamma = crit(j_{[2^n]})$

- Hence $Iter(j)/{\equiv_n}$ is an LD-structure with 2^n elements s.t. $j_{[p]} * j = j_{[p+1 \bmod 2^n]}$.

- Corollary.— The quotient-structure $Iter(j)/{\equiv_n}$ is (isomorphic to) the table A_n.

• Lemma 1.— If j is an embedding, then, for $m \leqslant n$ and $p \leqslant 2^n$, TFAE
- the embedding $j_{[p]}$ maps $\mathrm{crit}(j_{[2^m]})$ to $\mathrm{crit}(j_{[2^n]})$
- the period of p jumps from 2^m to 2^{m+1} between A_n and A_{n+1}.

• Lemma 2.— If j is an embedding, then $j(j)(\alpha) \leqslant j(\alpha)$ holds for every ordinal α.

• Proof: There exists β satisfying $j(\beta) > \alpha$, hence there exists a smallest such β, which therefore satisfies $j(\beta) > \alpha$ and

$$\forall \gamma < \beta \ (j(\gamma) \leqslant \alpha). \qquad (*)$$

Applying j to $(*)$ gives

$$\forall \gamma < j(\beta) \ (j(j)(\gamma) \leqslant j(\alpha)). \qquad (**)$$

Taking $\gamma = \alpha$ in $(**)$ yields $j(j)(\alpha) \leqslant j(\alpha)$. □

• Proposition (Laver).— If there exists a selfsimilar set,
then $\pi_n(2) \geqslant \pi_n(1)$ holds for every n.

- Theorem (Steel, Laver).— If j is an embedding of a rank R,
then the sequence $\mathrm{crit}(j_{[2^n]})$ is unbounded in R.

- Proposition (Laver).— If there exists a selfsimilar set,
the sequence of periods $\pi_n(1)$ tends to ∞ with n.

- Corollary.— If there exists a selfsimilar set,
the substructure generated by $(1, 1, 1, ...)$ in the inverse limit of all A_n is free.

- Did we answer the questions about Laver tables?
  — No, because the existence of a selfsimilar set is a large cardinal axiom,
  hence unprovable, and whose non-contradiction cannot be proved from ZF.

- Is the large cardinal assumption necessary?
  — Probably not... So far, we cannot avoid it, but nothing indicates that it should be necessary; and there is no systematic method for avoiding it.

- An attempt: Drápal's program, three steps completed so far...

- A similar example: the orderability of free LD-structures, first established using a selfsimilar set, then using a direct argument (based on braid groups).

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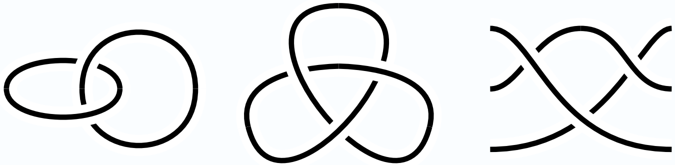
- Planar diagrams:



⇝ projections of curves embedded in $\mathbb{R}^3$

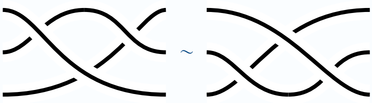
- Generic question: recognizing whether two diagrams are (projections of) isotopic figures
  ⇝ find isotopy invariants.

- Two diagrams represent isotopic figures iff one can go from the former to the latter using finitely many Reidemeister moves:

- type I :

- type II :

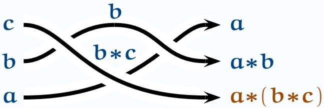
- type III :



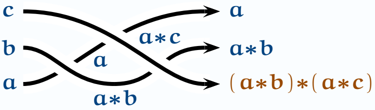
- Fix a set (of colors) S equipped with two operations $*, \bar{*}$,
  and color the strands in diagrams obeying the rules:

et

- Action of Reidemeister moves on colors:

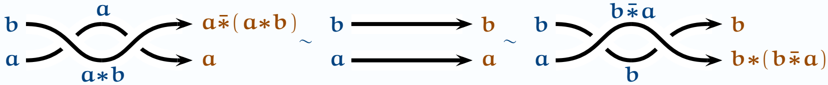


$\sim$



⇝ Hence: S-colorings invariant under Reidemeister move III $\Leftrightarrow$ $(S, *)$ LD-structure

- Idem for Reidemeister move II:



There exists $\bar{*}$ satisfying $x * (x \,\bar{*}\, y) = y$ and $x \,\bar{*}\, (x * y) = y$
iff the left-translations of $(S, *)$ are bijections.

⇝ Hence: S-colorings invariant under Reidemeister moves II+III ⇔
$(S, *)$ is an LD-structure with bijective left-translations
↑
a rack (Fenn–Rourke)

- Idem for Reidemeister move I:

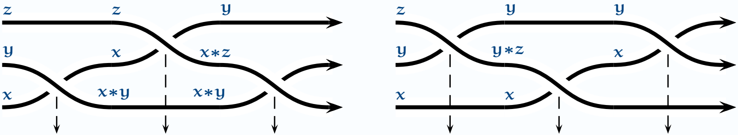
⇝ Hence: S-colorings invariant under Reidemeister moves I+II+III ⇔
$(S, *)$ is an idempotent rack
↑
a quandle (Joyce)

- Theoretical (Joyce, Matveev): The "fundamental quandle" is a complete invariant w.r.t. isotopy up to mirror symmetry.

- Practical (Carter, Kamada): use (co)-homology of LD-structures.

- Definition.— A 2-cocycle on an LD-systructure $(S, *)$ is a map $\phi : S^2 \to \mathbb{Z}$ satisfying $\phi(x, z) + \phi(x*y, x*z) = \phi(y, z) + \phi(x, y*z)$.

- Every 2-cocycle provides an invariant w.r.t. Reidemeister move III (and more...):



$\phi(x, y) + \phi(x, z) + \phi(x*y, x*z)$ $\quad$ $\phi(y, z) + \phi(x, y*z) + \phi(x, y)$

- Laver tables are LD-structures, but neither racks (nor quandles):
  - ⇝ not obvious to use them in topology, but possible (Przytycki, ...),
  - ⇝ step 1 : determine the associated cocycles.

- **Proposition** (D., Lebed).— The 2-cocycles for A_n make a free $\mathbb{Z}$-module of rank 2^n, with an explicit basis made of $\{0, 1\}$-valued functions.

$\psi_{1,3}$	1	2	3	4	5	6	7	8
1	1	·	·	·	·	·	·	·
2	1	·	·	·	·	·	·	·
3	1	·	·	·	·	·	·	·
4	1	·	·	·	·	·	·	·
5	1	·	·	·	·	·	·	·
6	1	·	·	·	·	·	·	·
7	1	·	·	·	·	·	·	·
8	·	·	·	·	·	·	·	·

$\psi_{2,3}$	1	2	3	4	5	6	7	8
1	·	1	·	·	·	·	·	·
2	1	1	·	·	1	·	·	·
3	1	1	·	·	1	·	·	·
4	·	1	·	·	·	·	·	·
5	1	1	·	·	1	·	·	·
6	1	1	·	·	1	·	·	·
7	1	1	·	·	1	·	·	·
8	·	·	·	·	·	·	·	·

$\psi_{3,3}$	1	2	3	4	5	6	7	8
1	1	·	1	·	1	·	·	·
2	·	·	1	·	·	·	·	·
3	1	·	1	·	1	·	·	·
4	·	·	1	·	·	·	·	·
5	1	·	1	·	1	·	·	·
6	1	·	1	·	1	·	·	·
7	1	·	1	·	1	·	·	·
8	·	·	·	·	·	·	·	·

$\psi_{4,3}$	1	2	3	4	5	6	7	8
1	·	·	·	1	·	·	·	·
2	·	·	·	1	·	·	·	·
3	·	1	·	1	·	1	·	·
4	·	·	·	1	·	·	·	·
5	·	1	·	1	·	1	·	·
6	·	1	·	1	·	1	·	·
7	1	1	1	1	1	1	1	·
8	·	·	·	·	·	·	·	·

$\psi_{5,3}$	1	2	3	4	5	6	7	8
1	1	·	·	·	1	·	·	·
2	1	·	·	·	1	·	·	·
3	1	·	·	·	1	·	·	·
4	·	·	·	·	·	·	·	·
5	1	·	·	·	1	·	·	·
6	1	·	·	·	1	·	·	·
7	1	·	·	·	1	·	·	·
8	·	·	·	·	·	·	·	·

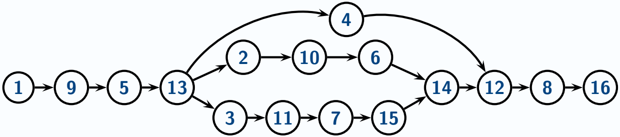
$\psi_{6,3}$	1	2	3	4	5	6	7	8
1	·	1	·	·	·	1	·	·
2	·	1	·	·	·	1	·	·
3	1	1	1	·	1	1	1	·
4	·	·	·	·	·	·	·	·
5	·	1	·	·	·	1	·	·
6	·	1	·	·	·	1	·	·
7	1	1	1	·	1	1	1	·
8	·	·	·	·	·	·	·	·

$\psi_{7,3}$	1	2	3	4	5	6	7	8
1	1	·	1	·	1	·	1	·
2	·	·	·	·	·	·	·	·
3	1	·	1	·	1	·	1	·
4	·	·	·	·	·	·	·	·
5	1	·	1	·	1	·	1	·
6	·	·	·	·	·	·	·	·
7	1	·	1	·	1	·	1	·
8	·	·	·	·	·	·	·	·

- These cocycles are not trivial: for instance, the "period" cocycle ψ_n s.t. $\psi_n(x, y) = 1$ iff y is a multiple of the period of x in A_n.

$\exists z\,(y = z * x)$ ↓

- Proofs: Relie on the right-divisiblity relation of A_n, which is a partial order:



- Analogous results for 3-cocycles.

- Question : What do these new positive braid invariants count?

- Conclusion : Reasonable hope of applying Laver tables in low-dimensional topology.

- Are the properties of periods in Laver tables an application of set theory?
  - So far, yes;
  - In the future, formally no if one finds alternative proofs that do not use large cardinals.
  - But, in any case, it is set theory that made the properties first accessible: even if one does not **believe** that large cardinals exist, they can provide valuable intuitions and simple arguments.

- An analogy:
  - In physics: using a physical intuition, guess statements, then pass them to the mathematician for a formal proof.
  - Here: using a logical intuition (existence of a selfsimiliar set), guess statements (periods tend to ∞ in Laver tables), then pass them to the mathematician for a formal proof.

Richard Laver
(1942-2012)

- R. Laver, *On the algebra of elementary embeddings of a rank into itself*, Advances in Math. 110 (1995) 334–346

- P. Dehornoy, Braids and self-distributivity, Progress in math. vol 192, Birkhaüser (1999), chapters X and XIII

- P. Dehornoy & V. Lebed, *Two- and three-cocycles for Laver tables*, J. Knot Theory and Ramifications, *to appear*, arXiv:1401.2335

www.math.unicaen.fr/~dehornoy