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Conference ILDT, Kyoto, May 21, 2015

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• An introduction to some of the many aspects of the standard braid order, with an emphasis on the known connections with knot theory.

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Plan :

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- The Braid Order in Antiquity

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- The Braid Order in Modern Times (Knot Applications)

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- The set-theoretical roots

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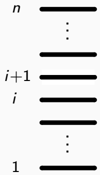

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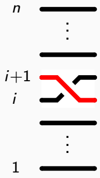

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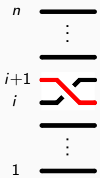

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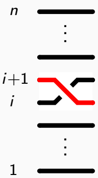

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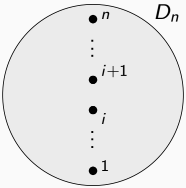

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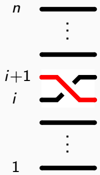

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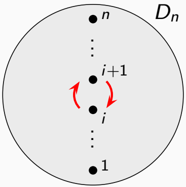

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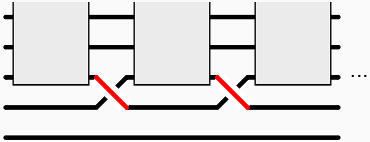

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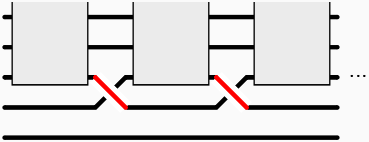

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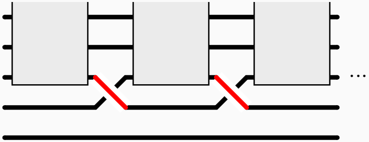

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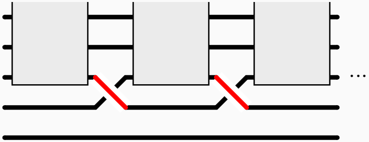

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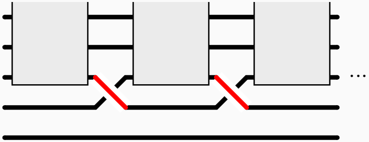

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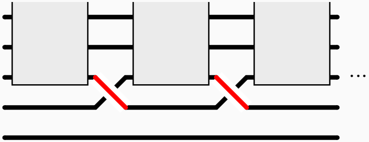

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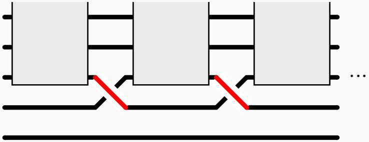

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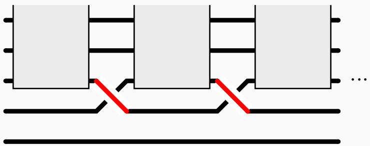

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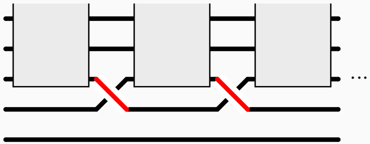

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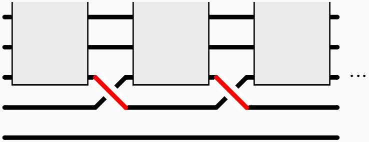

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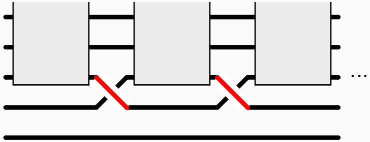

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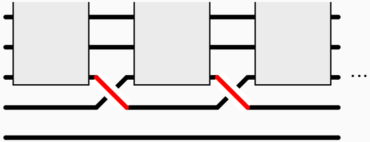

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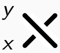

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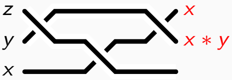

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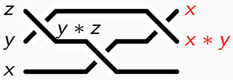

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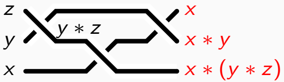

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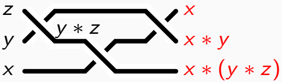

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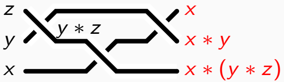

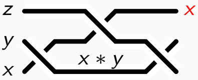

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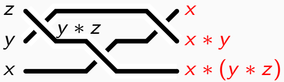

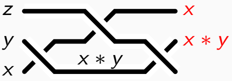

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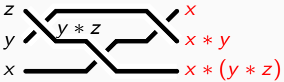

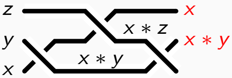

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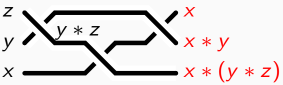

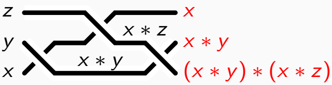

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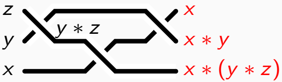

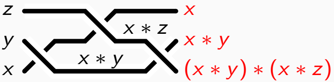

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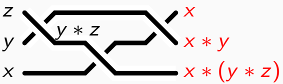

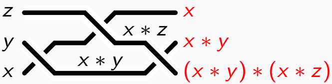

- Fact: *One obtains an action of B_n^+ iff $*$ satisfies the left self-distributivity law (LD):*

$$x * (y * z) = (x * y) * (x * z).$$

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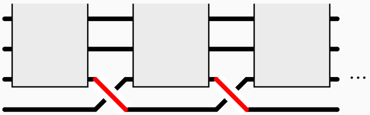

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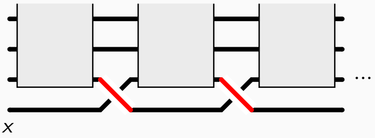

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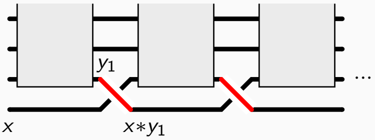

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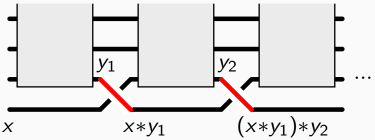

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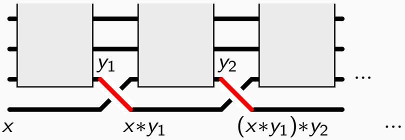

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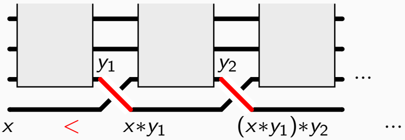

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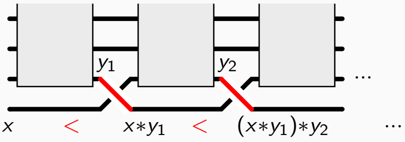

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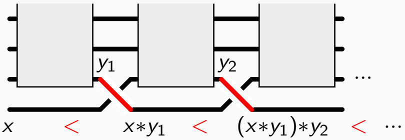

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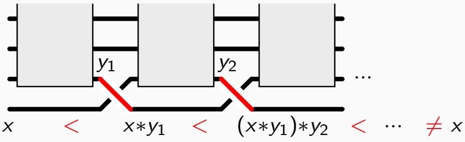

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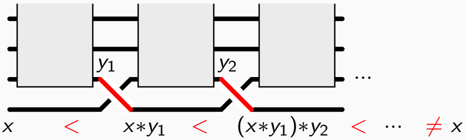

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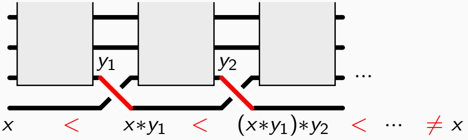

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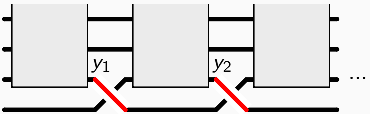

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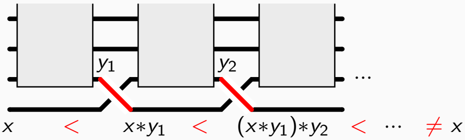

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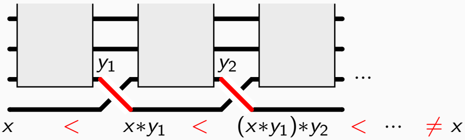

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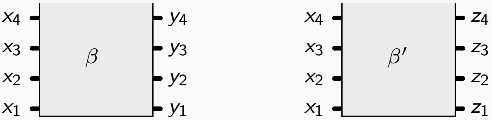

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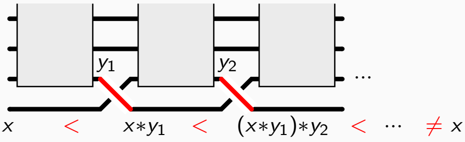

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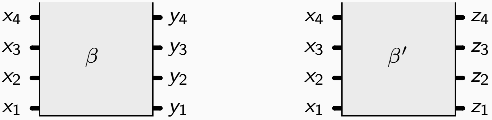

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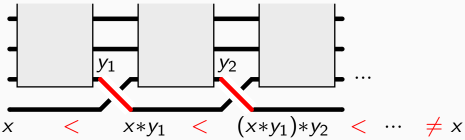

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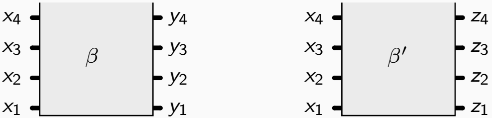
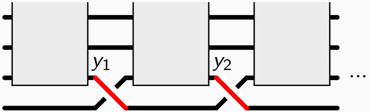


Then $\beta <_{\mathrm{D}} \beta'$ iff $(y_1, y_2, \ldots) <^{\mathrm{Lex}} (z_1, z_2, \ldots)$

- Claim: *Theorem 1 (braid order) directly comes from Theorem $\frac{1}{2}$ (orderable shelf).*

- Ingredient 1 : A σ-positive braid word never represents 1.



$x < x*y_1 < (x*y_1)*y_2 < \cdots \neq x$

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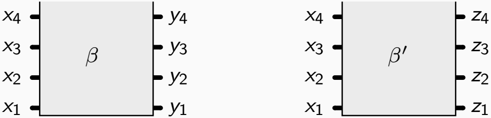

Then $\beta <_{\mathrm{D}} \beta'$ iff $(y_1, y_2, \ldots) <^{\mathrm{Lex}} (z_1, z_2, \ldots)$

- Question: OK, but then, why to look for orderable shelves?

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↑ closure of $\{j\}$ under application: $j(j)$, $j(j)(j)$...

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  - Because the latter extends Artin's braid group: Theorem 1 (braid order).

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II. The Braid Order in the Middle Ages:

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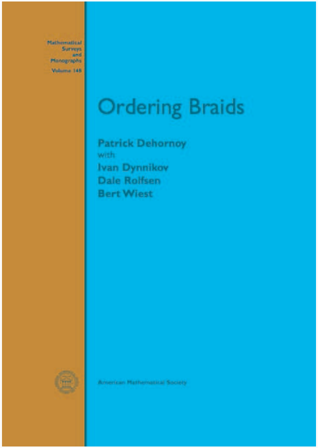

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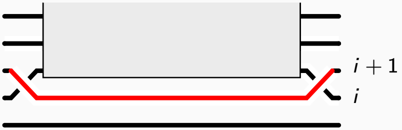

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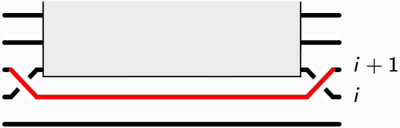

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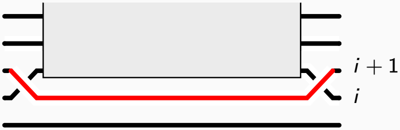

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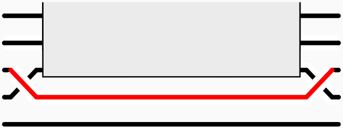

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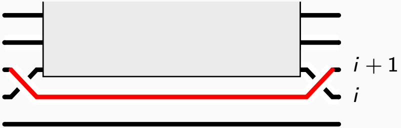

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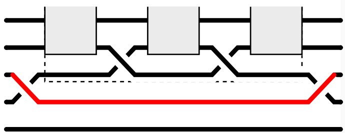

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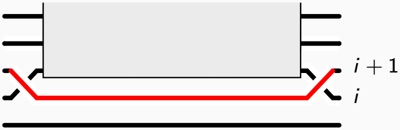

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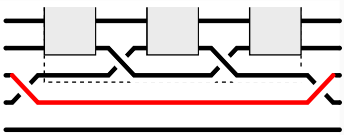

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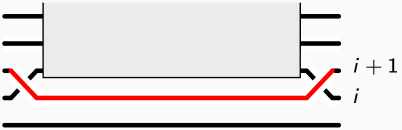

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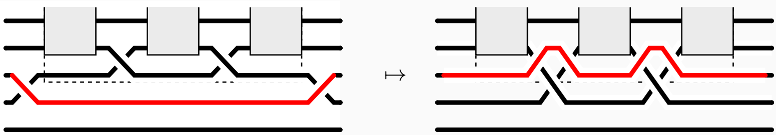

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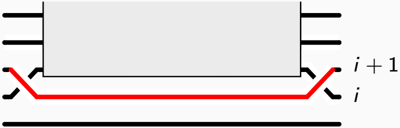

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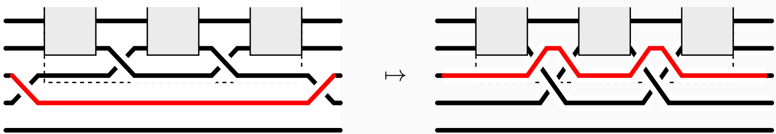

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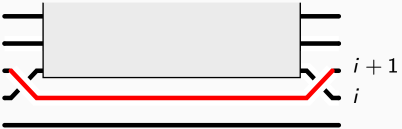

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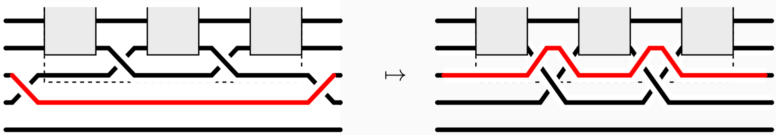

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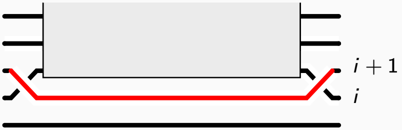

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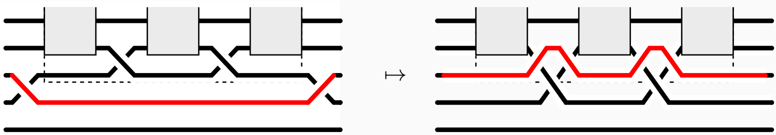

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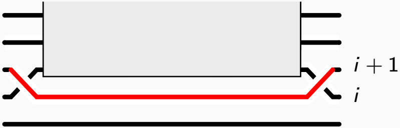

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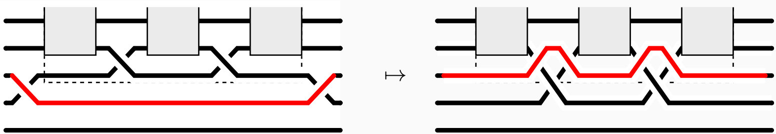

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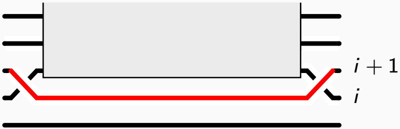

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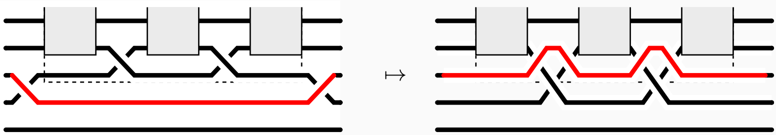

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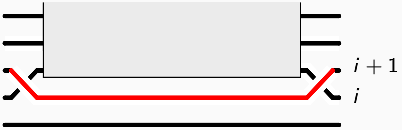

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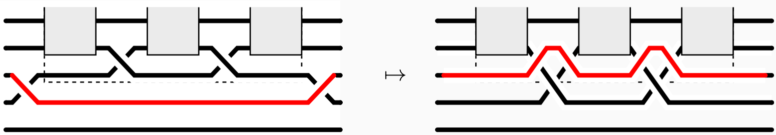

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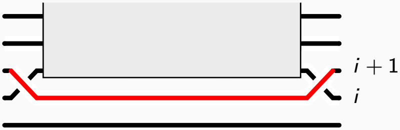

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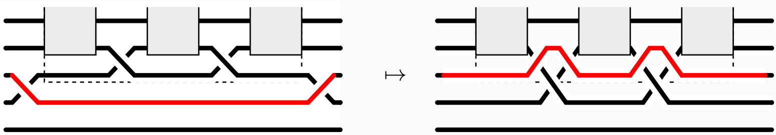

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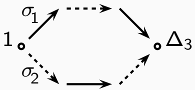

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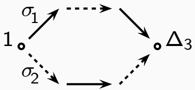
  - Example: Cayley(Δ_3) = 

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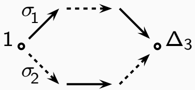

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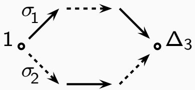

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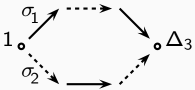

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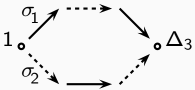

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- Lemma: (i) *Every n-strand braid word is drawn in Cayley(Δ_n^d) for $d \gg 0$.*
(ii) *For every β, the words drawn from β in Cayley(Δ_n^d) are closed under handle reduction.*

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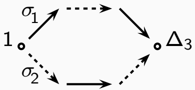

  - Braid word drawn in Cayley(Δ_n^d) from some prescribed vertex: $\sigma_1\sigma_2\sigma_2^{-1}$ is drawn from 1 in Cayley(Δ_3), but $\sigma_1\sigma_1$ is not.

- Lemma: (i) *Every n-strand braid word is drawn in Cayley(Δ_n^d) for $d \gg 0$.*
(ii) *For every β, the words drawn from β in Cayley(Δ_n^d) are closed under handle reduction.*

- Hence: In a sequence of handle reductions, all words remain drawn in some finite fragment of the Cayley graph of B_n.

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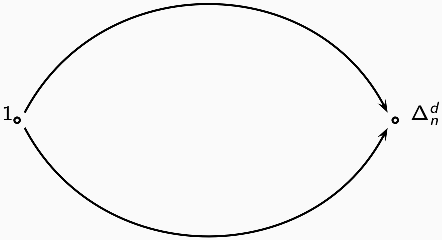

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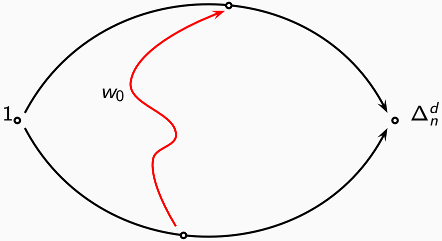

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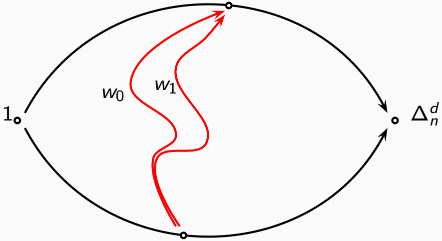

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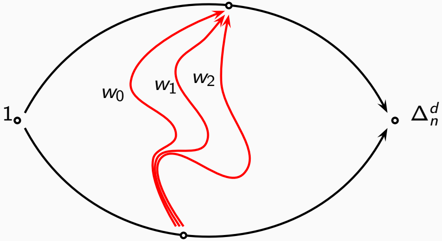

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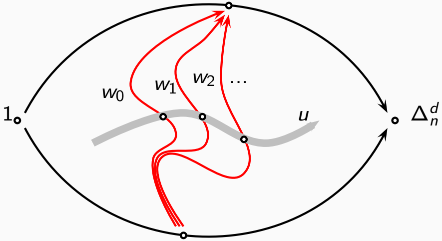

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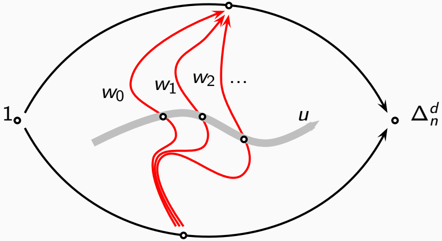

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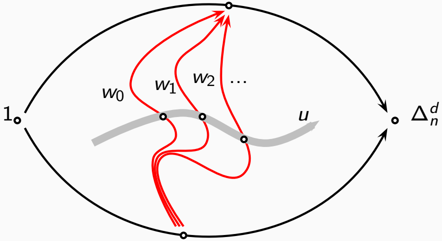

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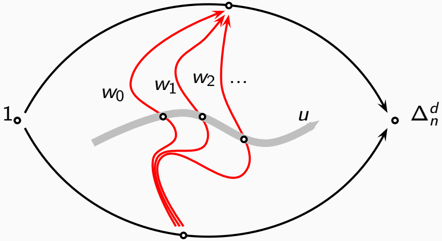

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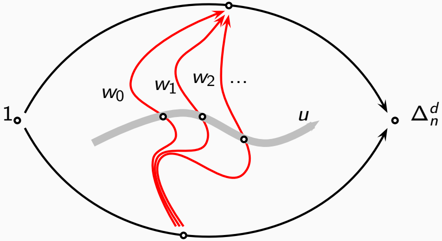

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III. The Braid Order in Modern Times:

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- The floor (after Malyutin–Netstvetaev and Ito)
- Conjugacy via the μ function

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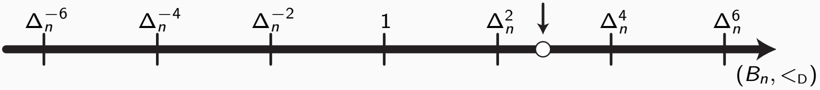

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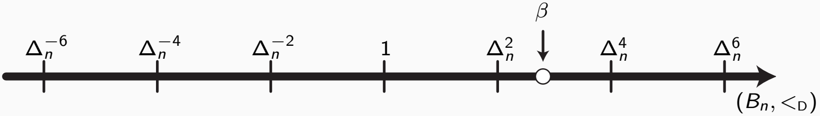

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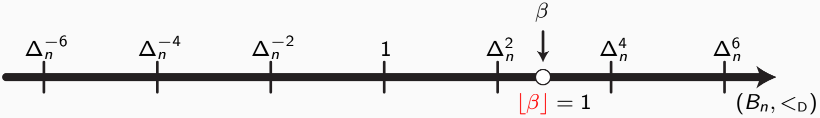

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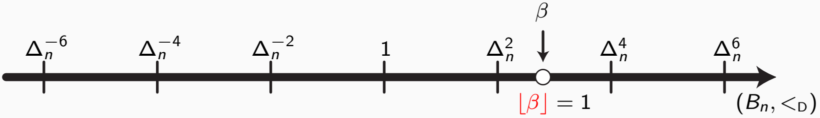

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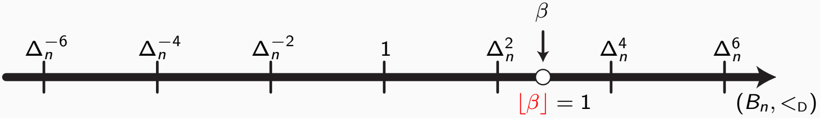

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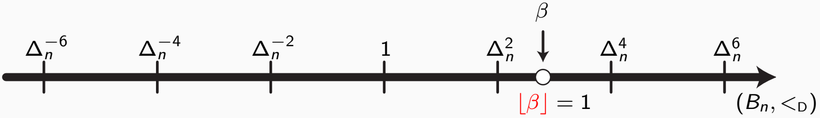

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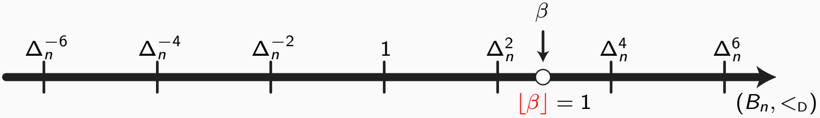

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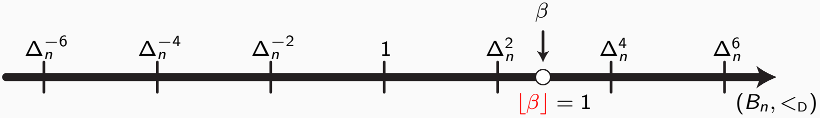

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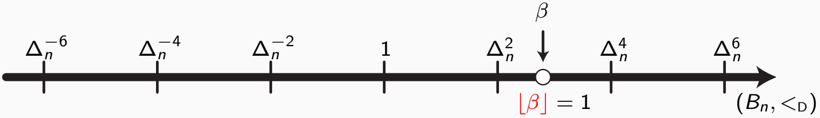

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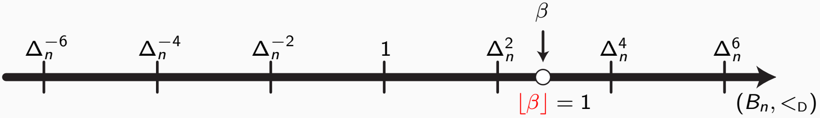

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  - Then $\beta \sim \Delta_n \gamma \sigma_{n-1} \Delta_n^{-1} = \mathrm{sh}(\gamma')\sigma_1$, where $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$ for each i.
  - Now $1 <_D \mathrm{sh}(\gamma')\sigma_1$, since $\mathrm{sh}(\gamma')\sigma_1$ is σ-positive.
  - And $\mathrm{sh}(\gamma')\sigma_1 <_D \Delta_n^2$, since $\sigma_1^{-1}\mathrm{sh}(\gamma'^{-1})\Delta_n^2 = \sigma_1^{-1}\Delta_n^2\mathrm{sh}(\gamma'^{-1})$ is σ-positive.
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- Proposition (Malyutin–Netsvetaev, Ito):

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conjectured (Ito) $r(n) \leqslant n-1$ for each n.

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Useful only if it can be computed...

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Useful only if it can be computed...

- Conjecture (D., Fromentin, Gebhardt, 2009): For β in B_3^+,

$$\mu(\beta\Delta_3^2) = \sigma_1\sigma_2^2\sigma_1 \cdot \mu(\beta) \cdot \sigma_1^2.$$

- Theorem (Laver, 1995): *For every braid β and every i, one has $\beta^{-1}\sigma_i\beta >_D 1$.*

- Corollary: *The restriction of the braid order to B_n^+ is a well-ordering.*

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- If successful for conjugacy, try the same approach for Markov equivalence...

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