Braid ordering: history and connections with knots

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• An introduction to some of the many aspects of the standard braid order, with an emphasis on the known connections with knot theory.

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Plan :

- The Braid Order in Antiquity
- The Braid Order in the Middle Ages
- The Braid Order in Modern Times (Knot Applications)

I. The Braid Order in Antiquity: 1985-92

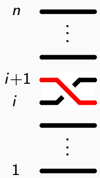
- The set-theoretical roots

- Definition (Artin 1925/1948): The braid group B_n is the group with presentation

$$\left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle.$$

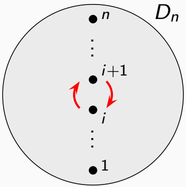
$\simeq$ { braid diagrams } / isotopy:

$\sigma_i \quad \leftrightarrow$

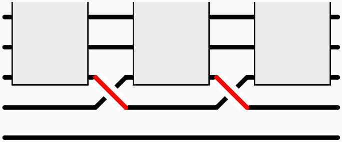


$\simeq$ mapping class group of D_n (disk with n punctures):

$\sigma_i \quad \leftrightarrow$



- Definition: A σ-positive braid diagram:



... ←all bottom crossings $(= \sigma_i$ with minimal $i)$ are positive (no σ_i^{-1})

- Theorem 1 (D. 1992): *For β, β' in B_n, declare $\beta <_{\mathrm{D}} \beta'$ if $\beta^{-1}\beta'$ can be represented by a σ-positive diagram. Then $<$ is a left-invariant linear ordering on B_n.*

↑ $\beta <_{\mathrm{D}} \beta'$ implies $\alpha\beta <_{\mathrm{D}} \alpha\beta'$

- Example: Let $\beta = \sigma_1$, $\beta' = \sigma_2\sigma_1$. Then $\beta^{-1}\beta' = \sigma_1^{-1}\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2^{-1}$, so $\beta <_{\mathrm{D}} \beta'$.

- Question: Where does this order come from?

- Theorem 0 (D. 1986): *If j is a self-embedding of a self-similar rank, then the LD-structure of Iter(j) implies Π^1_1-determinacy.*

??????

- Braid diagram colorings:
  - start with a set S ("colors"),
  - apply colors at the left ends of the strands in a braid diagram,
  - propagate the colors to the right,
  - compare the initial and final colors.

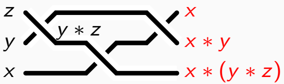
- Option 1: Colors are preserved in crossings:

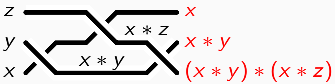
  - permutation of colors: $B_n \twoheadrightarrow \mathfrak{S}_n$

- Option 2: (Joyce, Matveev, Brieskorn 1980s) Colors change under the rule

  where $*$ is a (fixed) binary operation on S.

- For an action of B_n on S^n, one needs compatibility with the braid relations:





- Fact: *One obtains an action of B_n^+ iff $*$ satisfies the left self-distributivity law (LD):*

$$x * (y * z) = (x * y) * (x * z).$$

- Classical shelves (or LD-systems) (= sets with an operation obeying the LD-law):
  - $x * y = y$, leads to $B_n \twoheadrightarrow \mathfrak{S}_n$.
  - $x * y = xyx^{-1}$, leads to $B_n \to \mathrm{Aut}(F_n)$ (Artin representation)
  - $x * y = (1-t)x + ty$, leads to $B_n \to \mathrm{GL}_n(\mathbb{Z}[t, t^{-1}])$ (Burau representation)

Note: in these examples, $x * x = x$ always holds.

- Definition: A shelf $(S, *)$ is orderable if there exists a (left-invariant) linear ordering $<$ on S satisfying $x < x * y$ for all x, y.

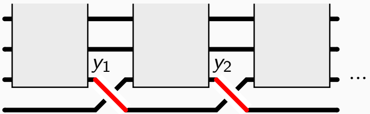
Note: (if they exist), orderable shelves are very different: $x < x * x \neq x$.

- Theorem $\frac{1}{2}$: *Orderable shelves exist: free shelves are orderable.*

- Claim: *Theorem 1 (braid order) directly comes from Theorem $\frac{1}{2}$ (orderable LD).*

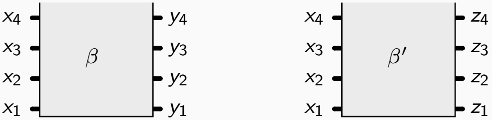
- <u>Claim</u>: *Theorem 1 (braid order) directly comes from Theorem $\frac{1}{2}$ (orderable shelf).*

- Ingredient 1 : A σ-positive braid word never represents 1.



$x \;<\; x{*}y_1 \;<\; (x{*}y_1){*}y_2 \;<\; \cdots \;\neq x$

- Ingredient 2 : Any two braids are comparable.



Then $\beta <_{\mathrm{D}} \beta'$ iff $(y_1, y_2, \ldots) <^{\mathrm{Lex}} (z_1, z_2, \ldots)$

- <u>Question</u>: OK, but then, why to look for orderable shelves?

… because Set Theory told us!

- Set Theory is a theory of infinity;
  - Axiomatized in the Zermelo-Fraenkel system ZF (1922), which is incor
  - Some statements are neither provable, nor refutable from ZF (e.g., CH
  - Hence: discover more properties of infinity and add further axioms to Z

- Typically, large cardinal axioms = (various) solutions to

$$\frac{\text{ultra-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}.$$

(inaccessible cardinals, measurable cardinals, etc.)

- General principle: "being self-similar implies being large".
  - A is infinite iff $\exists j : A \to A$ injective not bijective;

    a (self-)embedding of A ↓
  - A is ultra-infinite ("self-similar") iff $\exists j : A \to A$ injective not bijective and preserving every notion that is definable from $\in$.

- Example: $\mathbb{N}$ is infinite, but not ultra-infinite: if $j : \mathbb{N} \to \mathbb{N}$ preserves every notion that is definable from $\in$, then j preserves $0, 1, 2$, etc. hence j is the identity map.

- Definition: A rank is a set R such that $f : R \to R$ implies $f \in R$. ??????

- Assume that there exists a self-similar set
  (= a set with a nontrivial self-embedding). Then:
  - There exists a self-similar rank, say R;
  - If i, j are self-embeddings of R, then $i : R \to R$ and $j \in R$, hence
    we can apply i to j, obtaining $i(j)$;
  - "Being a self-embedding" is definable from $\in$, so $i(j)$ is a self-embedding:
    ("application" is a binary operation on self-embeddings of R).
  - "Being the image of" is definable from $\in$, so $\ell = j(k)$ implies $i(\ell) = i(j)(i(k))$,
    that is, $i(j(k)) = i(j)(i(k))$: the "application" operation satisfies the LD law.

- Proposition: *If j is a self-embedding of a self-similar rank, then Iter(j) is a shelf.*

↑ closure of $\{j\}$ under application: $j(j)$, $j(j)(j)$...

- Remember the question: why to look for orderable shelves (Theorem $\frac{1}{2}$)?

- Proposition (D. 1989): *If there exists at least one orderable shelf, then the Word Problem for LD is solvable.*

  ↑ deciding whether two terms are equal modulo LD

- Proposition (Laver 1989): *If j is a self-embedding of a self-similar rank, then Iter(j) is an orderable shelf.*

- Corollary: *If there exists a self-similar set, the Word Problem for LD is solvable.*

- But the existence of a self-similar set is an unprovable axiom (Gödel), so the corollary does not solve the Word Problem for LD.
  - Construct another orderable shelf (a real one!): Theorem $\frac{1}{2}$ (orderable shelf).
  - Done by investigating a certain "geometry group of LD".
  - Because the latter extends Artin's braid group: Theorem 1 (braid order).

- Question: Why care about Iter(j) and prove the previous propositions?

- Theorem 0 (D. 1986): *If j is a self-embedding of a self-similar rank, then the LD-structure of Iter(j) implies Π^1_1-determinacy.*

meaning: " the shelf Iter(j) is not trivial "

- Thus: a continuous path from Theorem 0 (about sets) to Theorem 1 (about braids).

- Question: Is the braid order an application of Set Theory?
  - Formally, no: braids appear when sets disappear.
  - In essence, yes: orderable shelves have been investigated because Set Theory showed they might exist and be involved in deep phenomena.

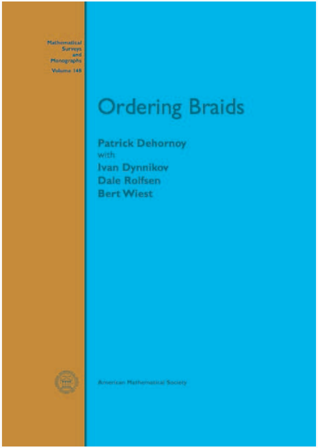
- Analogy:
  - In physics: using physical intuition and/or evidence, guess some statement, then pass it to the mathematician for a formal proof.
  - Here: using logical intuition and/or evidence ($\exists$ self-similar set), guess some statement ($\exists$ orderable shelf), then pass it to the mathematician for a formal proof.

II. The Braid Order in the Middle Ages: 1992–2000

- Handle reduction

- The braid order is a complicated object: non-Archimedian, non-Conradian, ...

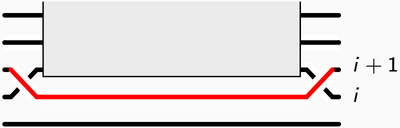
$\uparrow$ $\exists \beta, \beta' > 1\ \forall p\ (\beta^p <_{\mathrm{D}} \beta')$ $\uparrow$ $\exists \beta, \beta' > 1\ \forall p\ (\beta < \beta' \beta^p)$



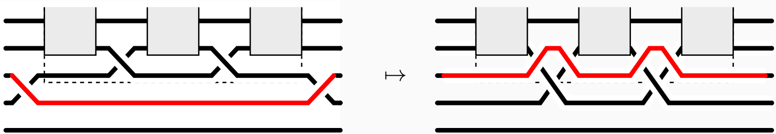
- Theorems (Burckel, D., Dynnikov, Fenn, Fromentin, Funk, Greene, Larue, Rolfsen, Rourke, Short, Wiest, ...):
  "Many different approaches
  lead to the same braid order".

- Theorems (Clay, Dubrovina–Dubrovin, Ito, Navas, Rolfsen, Short, Wiest, ...):
  "There exist many different braid orders
  making an interesting space".

- A σ_i-handle:



- Reducing a handle:



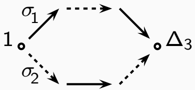
- Handle reduction is an isotopy;
- It extends free group reduction;
- Words with no handle are: the empty word, σ-positive words, σ-negative words.

• Theorem (D. 1995): *A braid β satisfies $\beta = 1$ (resp. $\beta > 1$) iff some/any sequence of handle reductions from some/any braid word representing β finishes with the empty word (resp. with a σ-positive word).*

- Aim: Show that there is no infinite sequence of handle reductions.

- Cayley graph of B_n: vertices = braids; edge $\overset{\beta}{\circ} \xrightarrow{\sigma_i} \overset{\beta'}{\circ}$ for $\beta\sigma_i = \beta'$.

- Cayley(Δ_n^d): restriction of the Cayley graph of B_n to the divisors of Δ_n^d (in the sense of the monoid B_n^+)
  - Example: Cayley(Δ_3) =



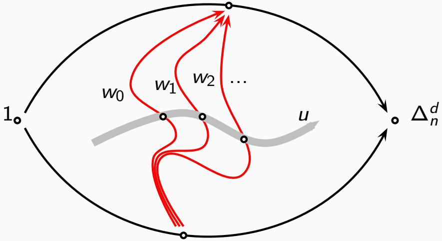
  - Braid word drawn in Cayley(Δ_n^d) from some prescribed vertex: $\sigma_1\sigma_2\sigma_2^{-1}$ is drawn from 1 in Cayley(Δ_3), but $\sigma_1\sigma_1$ is not.

- Lemma: (i) *Every n-strand braid word is drawn in Cayley*(Δ_n^d) *for* $d \gg 0$.
(ii) *For every* β*, the words drawn from* β *in Cayley*(Δ_n^d) *are closed under handle reduction.*

- Hence: In a sequence of handle reductions, all words remain drawn in some finite fragment of the Cayley graph of B_n.

- Aim: Show that there is no infinite sequence of handle reductions.

- Let $\overrightarrow{w} = (w_0, w_1, \ldots)$ be a sequence of handle reductions; all w_i drawn in Cayley(Δ_n^d).
  - Point: Show that $N :=$ # reductions of the first σ_1-handle in $\overrightarrow{w}$ is finite.
  - Reason: There exists a (transverse) witness-word u, drawn in Cayley(Δ_n^d),
    s.t. u contains no letter σ_1^{-1}, and exactly N letters σ_1:



  - Now: a path with no σ_1^{-1} cannot cross the same σ_1-edge twice,
  - As $\#\{\sigma_1\text{-edges}\}$ in Cayley(Δ_n^d) is finite, N must be finite. □

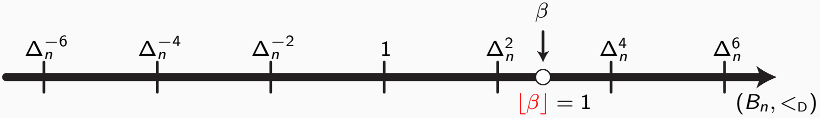
- Question: What is the complexity? Find the "real" convergence proof.

III. The Braid Order in Modern Times: 2000-...

- The floor (after Malyutin–Netstvetaev and Ito)
- Conjugacy via the μ function

- <u>Definition</u>: For β in B_n, the floor $\lfloor\beta\rfloor$ is the unique m satisfying

$$\Delta_n^{2m} \leqslant_{\mathrm{D}} \beta <_{\mathrm{D}} \Delta_n^{2m+2}.$$



- <u>Proposition</u> (Malyutin–Netsvetaev, 2000):
  (i) *The floor is a quasi-character with defect* 1 *on* B_n: $\big|\lfloor\beta\gamma\rfloor - \lfloor\beta\rfloor - \lfloor\gamma\rfloor\big| \leqslant 1$.
  (ii) *If* β *and* β' *are conjugate, then* $|\lfloor\beta\rfloor - \lfloor\beta'\rfloor| \leqslant 1$.

- <u>Corollary</u>: *The stable floor* $\lfloor\beta\rfloor_s = \lim_p \lfloor\beta^p\rfloor/p$ *is the only pseudo-character on* B_n *that is positive on braids* $>_{D} 1$ *and is* 1 *on* Δ_n^2.

- <u>Principle</u> for using the floor in knot theory:

> If $|\lfloor\beta\rfloor|$ is large, then the properties of $\widehat{\beta}$ can be read from those of β.

- Lemma: *If* $||\lfloor\beta\rfloor|| > 1$, *then* $\widehat{\beta}$ *admits no destabilisation.*

  $\uparrow$

  (assuming $\beta \in B_n$) β is conjugate to no braid $\gamma\sigma_{n-1}^{\pm 1}$ with $\gamma \in B_{n-1}$

- Proof: Assume $\beta \sim \gamma\sigma_{n-1}$ with $\gamma \in B_{n-1}$.
  - Then $\beta \sim \Delta_n \gamma \sigma_{n-1} \Delta_n^{-1} = \mathrm{sh}(\gamma')\sigma_1$, where $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$ for each i.
  - Now $1 <_{\mathrm{D}} \mathrm{sh}(\gamma')\sigma_1$, since $\mathrm{sh}(\gamma')\sigma_1$ is σ-positive.
  - And $\mathrm{sh}(\gamma')\sigma_1 <_{\mathrm{D}} \Delta_n^2$, since $\sigma_1^{-1}\mathrm{sh}(\gamma'^{-1})\Delta_n^2 = \sigma_1^{-1}\Delta_n^2\mathrm{sh}(\gamma'^{-1})$ is σ-positive.
  - Hence, $1 <_{\mathrm{D}} \mathrm{sh}(\gamma')\sigma_1 \leqslant \Delta_n^2$, that is, $\lfloor \mathrm{sh}(\gamma')\sigma_1 \rfloor = 0$.
  - Hence, $||\lfloor\beta\rfloor|| \leqslant 1$. Idem for $\beta \sim \gamma\sigma_{n-1}^{-1}$... □

- Proposition (Malyutin–Netsvetaev, Ito):

  (i) *If* $||\lfloor\beta\rfloor|| > 1$, *then* $\widehat{\beta}$ *admits no exchange move.*

  (ii) *If* $||\lfloor\beta\rfloor|| > 2$, *then* $\widehat{\beta}$ *admits no flype.*

- Theorem (Malyutin–Netsvetaev, 2004).—
  If β satisfies $||\lfloor\beta\rfloor|| > 1$, then $\widehat{\beta}$ is prime, non-split, and nontrivial.

- Proof: For χ a pseudo-character on B_n satisfying $\chi|_{B_{n-1}} = 0$, then
  $|\chi(\beta)| > \mathrm{defect}(\chi)$ implies that $\widehat{\beta}$ is prime. Apply to $\lfloor\ \rfloor_s$. □

- Theorem (Malyutin–Netsvetaev, 2004).— *For every n, there exists $r(n)$ such that for every β in B_n with $||\lfloor\beta\rfloor|| \geqslant r(n)$, $\widehat{\beta}$ is represented by a unique conjugacy class in B_n.*

$$\forall\beta,\beta'\in B_n\ (\widehat{\beta'} \approx \widehat{\beta} \Rightarrow \beta' \sim \beta)$$

- Proof: For each template move M, there exists r s.t.
  $||\lfloor\beta\rfloor|| > r(n)$ implies that $\widehat{\beta}$ is not eligible for M.
  By the Birman-Menasco MTWS theory, $\exists$ finitely template moves for each n. □

- (M.-N., 2000) $r(3) \leqslant 3$; (Matsuda, 2008) $r(4) \leqslant 4$; (Ito, 2009) $r(3) = 2$.
  conjectured (Ito) $r(n) \leqslant n-1$ for each n.

- Theorem (Ito, 2012): *For every β in B_n:*

$$||\lfloor\beta\rfloor|| \leqslant \frac{4\cdot\mathrm{genus}(\widehat{\beta})}{n+2} - \frac{2}{n+2} + \frac{3}{2} \leqslant \mathrm{genus}(\widehat{\beta}) + 1.$$

"The closure of a large braid is a complicated knot"

- Theorem (Ito, 2012): *If β satisfies $||\lfloor\beta\rfloor|| \geqslant 2$ and $\widehat{\beta}$ is a knot, then*
  - *β is periodic iff $\widehat{\beta}$ is a torus knot,*
  - *β is reducible iff $\widehat{\beta}$ is a satellite knot,*
  - *β is pseudo-Anosov iff $\widehat{\beta}$ is hyperbolic.*

False in general: the trefoil knot is the closure of σ_1^3 (periodic), of $\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2$ (reducible), and of $\sigma_1^3\sigma_2^{-1}$ (pseudo-Anosov).

• Theorem (Ito, 2014): *If H is a nontrivial, non-central normal subgroup of B_n, then, for every γ in B_n, the set $\{\widehat{\beta\gamma} \mid \beta \in H\}$ contains infinitely many (hyperbolic) knots.*

• Proof (sketch):

- The subgroup H is unbounded with respect to $<_{\mathrm{D}}$: $\forall\gamma{\in}B_n\ \exists\beta{\in}H\ (\gamma <_{\mathrm{D}} \beta)$.
  nontrivial: uses the alternating normal form of braids...
- Then $\{\beta\gamma \mid \beta \in H\}$ is also unbounded.
- Hence, $\{\widehat{\beta\gamma} \mid \beta \in H\}$ contains knots of arbitrarily high genus,
  hence certainly infinitely many knots.
- Moreover, one may assume β pseudo-Anosov, hence $\widehat{\beta}$ hyperbolic. □

• Corollary (Ito, 2014): *Let $\rho_1, ..., \rho_k$ be non-faithful quantum representations of B_n. Then, for every isotopy type τ, there exist infinitely many hyperbolic knots of type τ on which the invariants derived from $\rho_1, ..., \rho_k$ agree.*

• Corollary (Ito, 2014): *If the Burau representation of B_4 is not faithful, then there exists a nontrivial knot with trivial Jones polynomial.*

- Theorem (Laver, 1995): *For every braid β and every i, one has $\beta^{-1}\sigma_i\beta >_D 1$.*

- Corollary: *The restriction of the braid order to B_n^+ is a well-ordering.*

  ↑ the submonoid of B_n generated by $\sigma_1, \ldots, \sigma_{n-1}$

  ↑ every nonempty subset has a minimal element

- Definition: For β in B_n^+, put

$$\mu(\beta) = \min\{\beta' \in B_n^+ \mid \beta' \text{ conjugate to } \beta\}.$$

Useful only if it can be computed...

- Conjecture (D., Fromentin, Gebhardt, 2009): For β in B_3^+,

$$\mu(\beta\Delta_3^2) = \sigma_1\sigma_2^2\sigma_1 \cdot \mu(\beta) \cdot \sigma_1^2.$$

...more generally, a reasonable hope of computing μ using the alternating normal form, and its analog for the dual braid monoid (Fromentin's rotating normal form).

- If successful for conjugacy, try the same approach for Markov equivalence...

- P. Dehornoy, with I. Dynnikov, D. Rolfsen, B. Wiest, *Ordering braids*
Math. Surveys and Monographs vol. 148, Amer. Math. Soc. (2008)

- A. Malyutin and N. Netsvetaev, Dehornoy's ordering on the braid group and braid moves,
St. Peterburg Math. J. 15 (2004) 437-448.

- T. Ito, Braid ordering and knot genus, *J. Knot Th. Ramif.* 20 (2011) 1311-1323.

- T. Ito, Braid ordering and the geometry of closed braids, *Geom. Topol.* 15 (2011) 473-498.

- T. Ito, Kernel of braid representation and knot polynomial, *Math. Zeitschr.* (2015) to appear
DOI 10 1007/s 00209-015-1426-7

- J. Fromentin, Every braid admits a short sigma-definite expression,
J. Europ. Math. Soc. 13 (2011) 1591-1631.

www.math.unicaen.fr/∼dehornoy