The alternating normal form of braids

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- An introduction for T. Ito's talk in IDLT...

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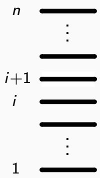

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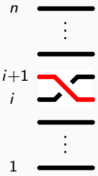

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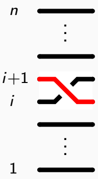

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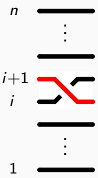

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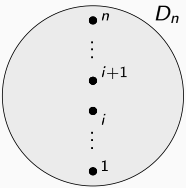

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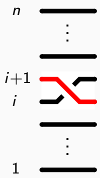

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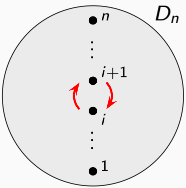

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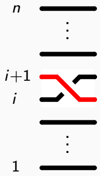

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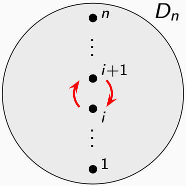

- Definition: B_n^+ := submonoid of B_n generated by $\sigma_1, \ldots, \sigma_{n-1}$ (positive braids).

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- Definition: In the above framework, call β_1 the S-tail of β.

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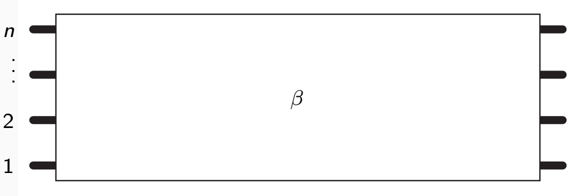

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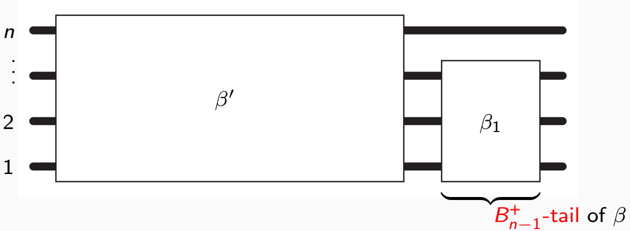

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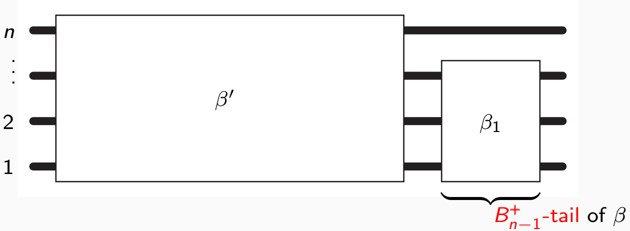

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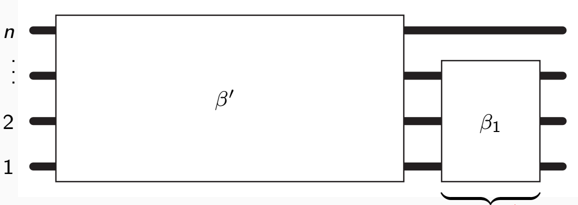

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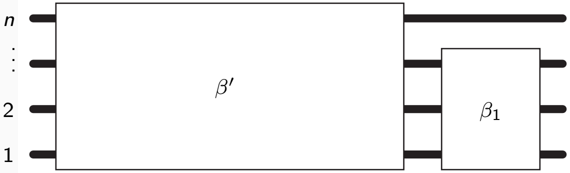

B_{n-1}^{+}-tail of β

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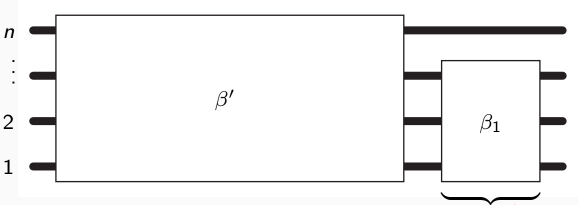

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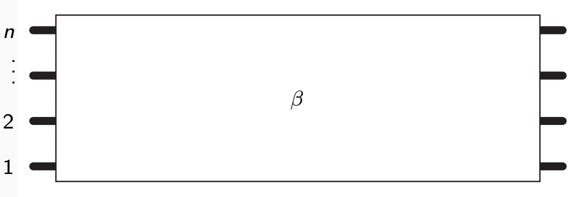

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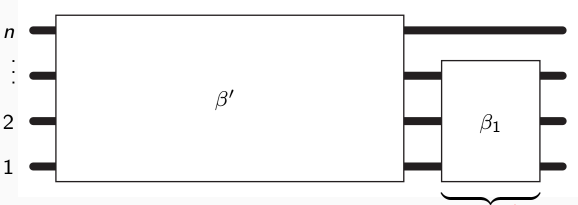

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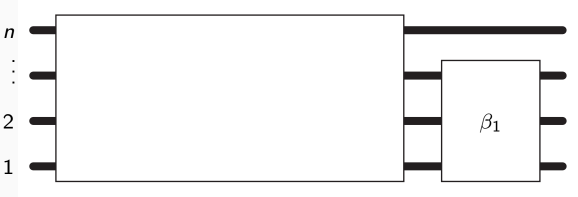

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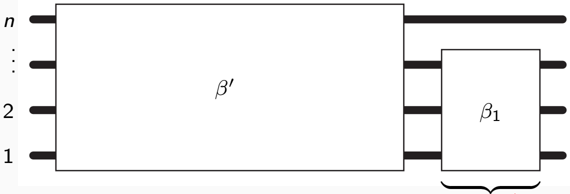

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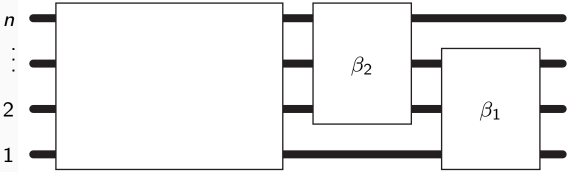

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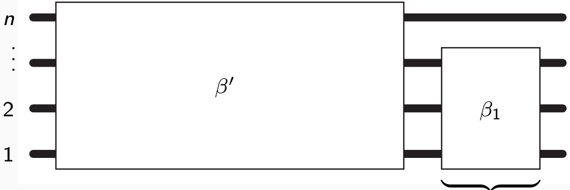

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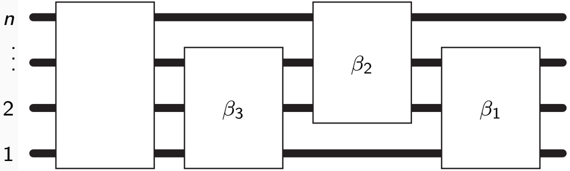

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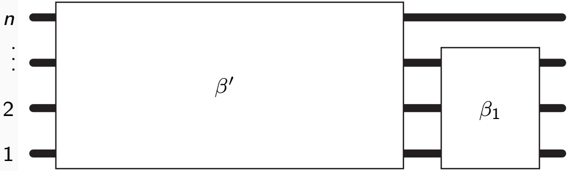

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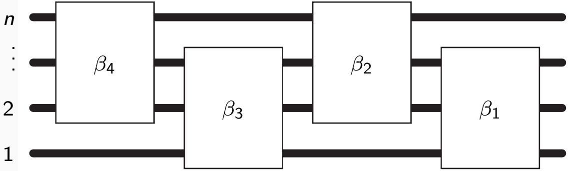

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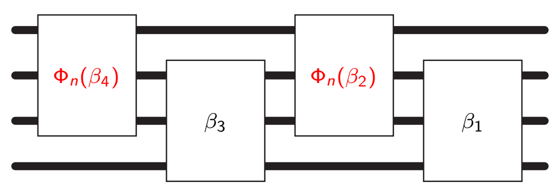

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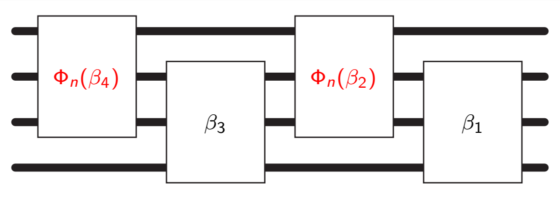

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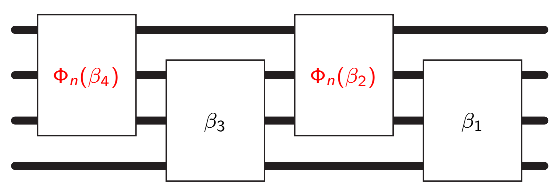

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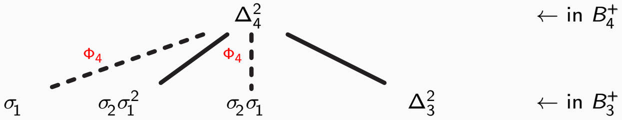

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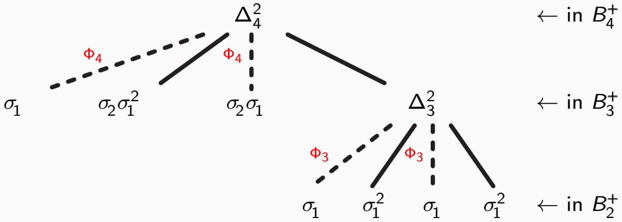

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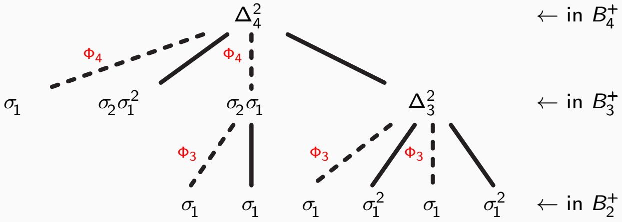

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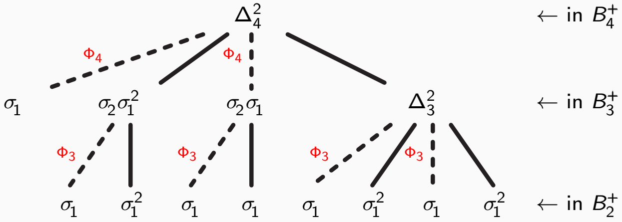

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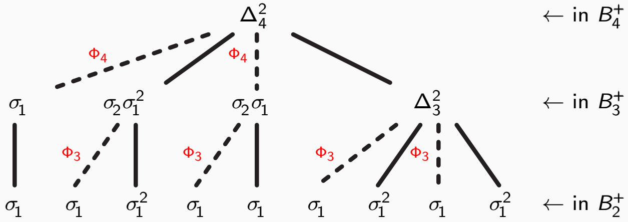

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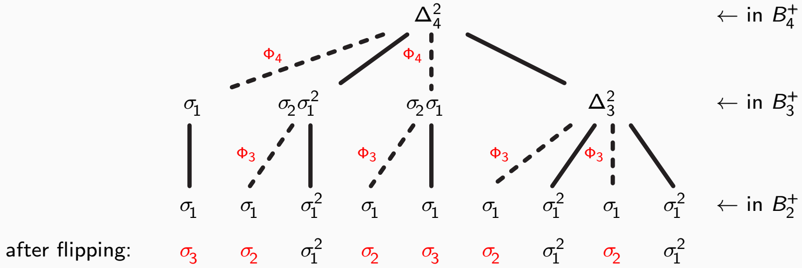

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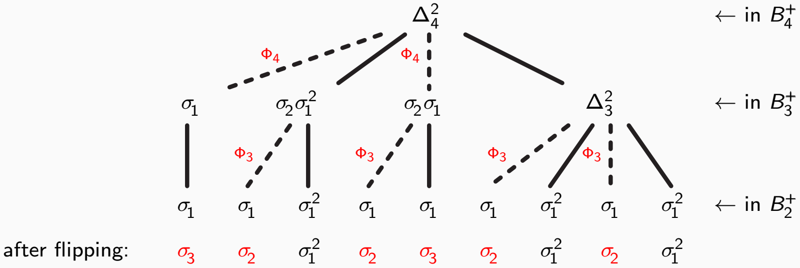

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Plan:

- 1. The alternating normal form
- 2. Connection with the standard braid order
- 3. Application to unprovability statements
- 4. The rotating normal form

• <u>Definition</u>: For x, x' in B_∞, declare $x <_{\mathrm{D}} x'$ if, among all braid words that represent $x^{-1}x'$, at least one is such that the generator σ_i with highest index appears positively only.

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• Remark: replacing "maximal index" with "minimal index" in the definition
amounts to flipping the order: for β, γ in B_n, $\beta <'_D \gamma$ iff $\Phi_n(\beta) <_D \Phi_n(\gamma)$.

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- The braid order is effective (there is an algorithm deciding $<_{\mathrm{D}}$), but complicated.

- In particular: The well-order property gives a distinguished element ($<_{\mathrm{D}}$-smallest elt) in every nonempty subset of B_n^+ (e.g., in each conjugacy class) but cannot be computed in practice (?).

- Typically: $<_{\mathrm{D}}$ is not well connected with the greedy normal form:
  - If β, γ are divisors of Δ_n, then $\beta <_{\mathrm{D}} \gamma$ iff $\mathrm{perm}(\beta) <^{\mathrm{Lex}} \mathrm{perm}(\gamma)$, (good!)
  - ... but does not extend to arbitrary positive braids, viewed as sequences of divisors of Δ_n. (bad!)

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• Corollary: *The braid order can be read from the alternating normal form.*

Plan:

- 1. The alternating normal form
- 2. Connection with the standard braid order
- 3. Application to unprovability statements
- 4. The rotating normal form

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  (reminiscent of Goodstein's sequences and Hydra battles)

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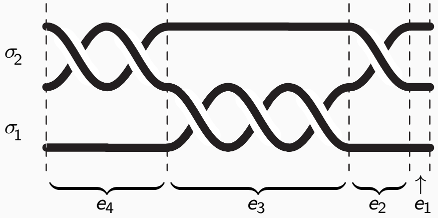

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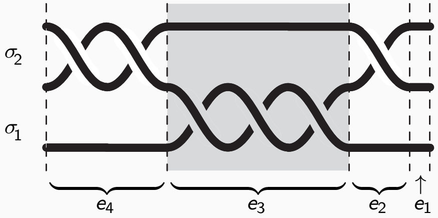

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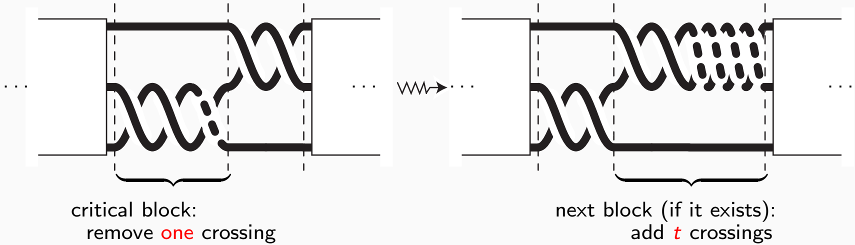

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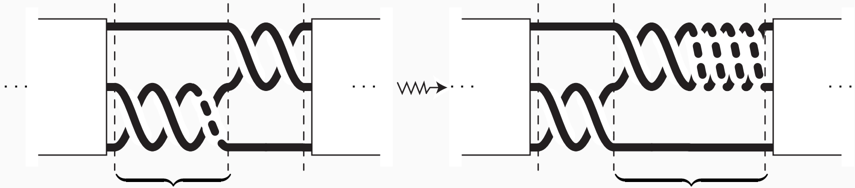

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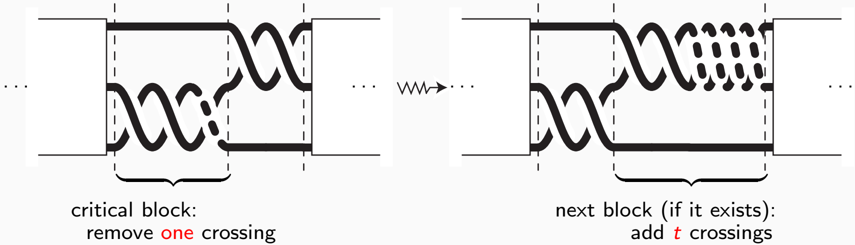

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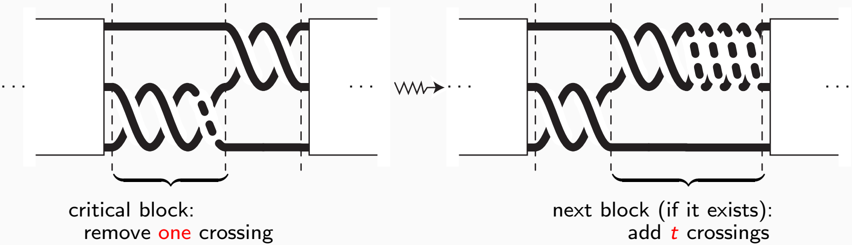

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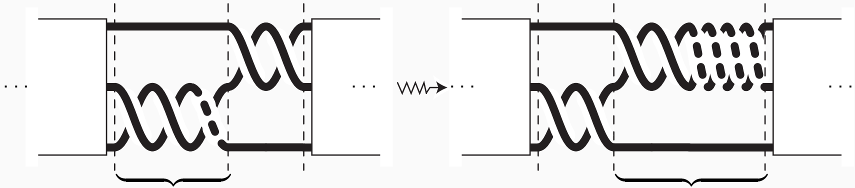

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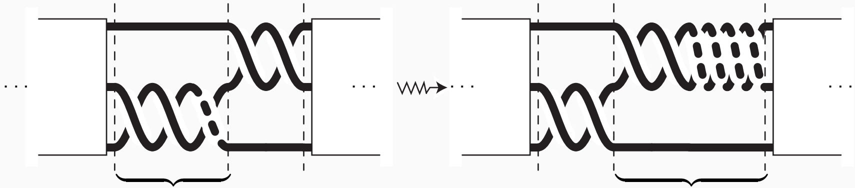

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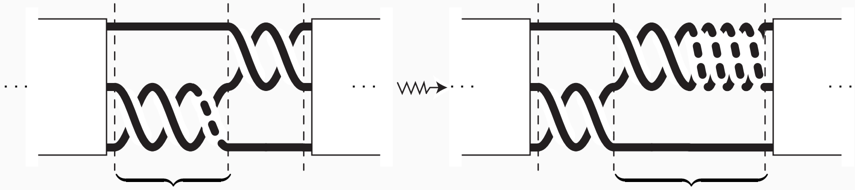

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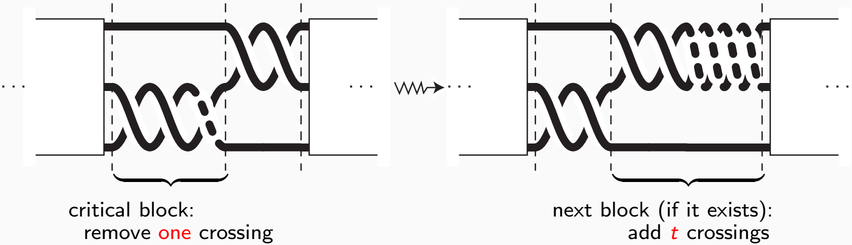

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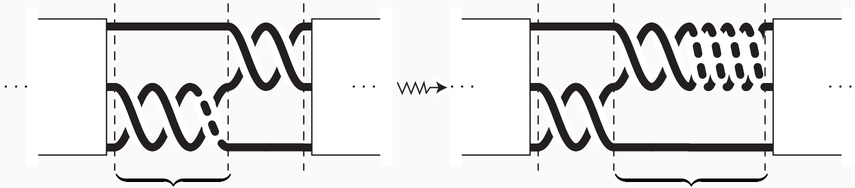

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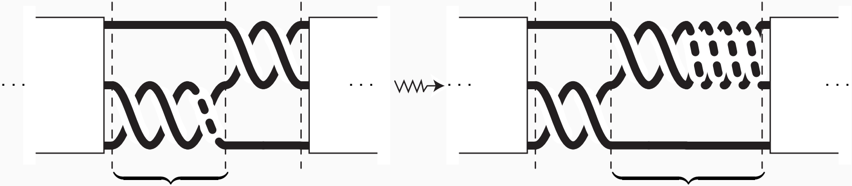

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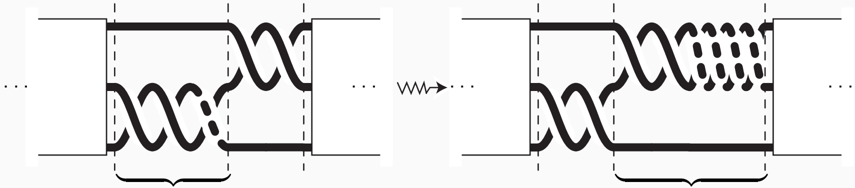

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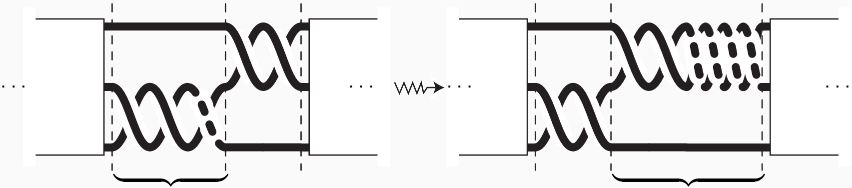

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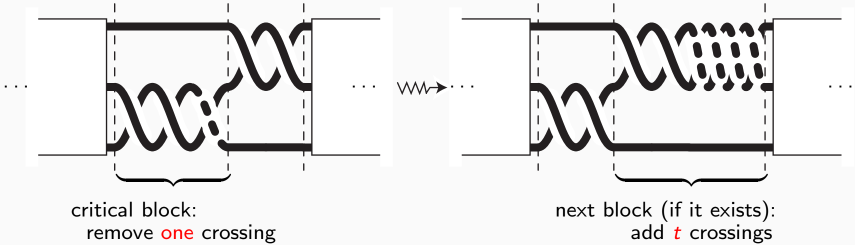

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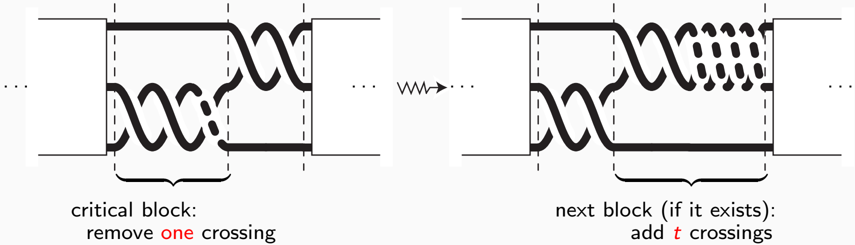

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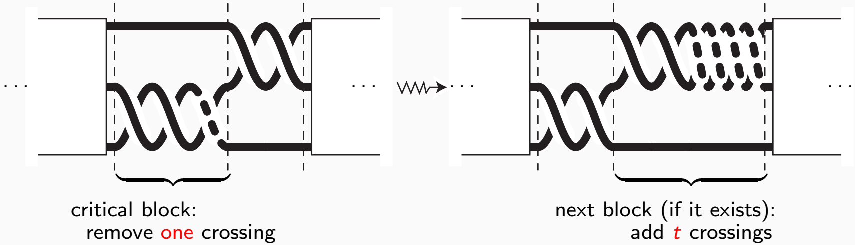

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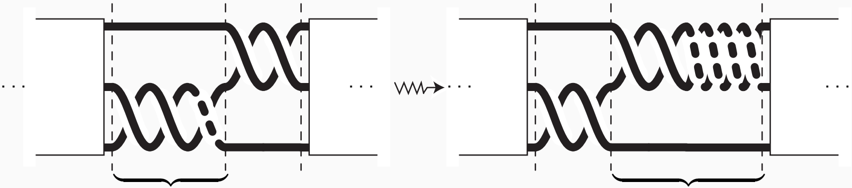

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Contrasting with the folklore result:

- Proposition: All usual (algebraic) properties of braids can be proved in $I\Sigma_1$.

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- Key point for the proof: Fine counting arguments in B_3^+, namely evaluating

$$\#\{\beta \in B_3^+ \mid \|\beta\| \leqslant \ell \quad \text{and} \quad \beta <_{\mathrm{D}} \Delta_3^k\}.$$

Plan:

- 1. The alternating normal form
- 2. Connection with the standard braid order
- 3. Application to unprovability statements
- 4. The rotating normal form

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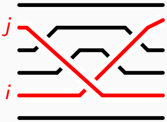

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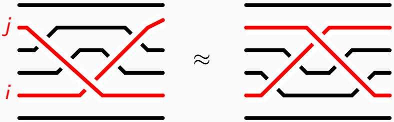

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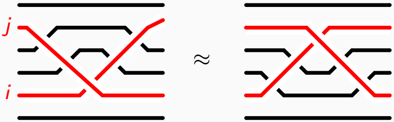

- Definition: (dual braid monoid) $B_n^{+*} :=$ the submonoid of B_n generated by the $a_{i,j}$s.

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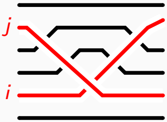
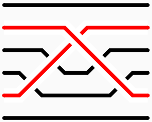


$\approx$



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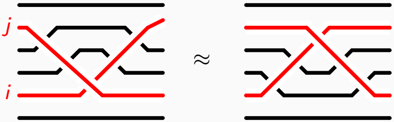

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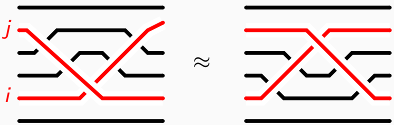

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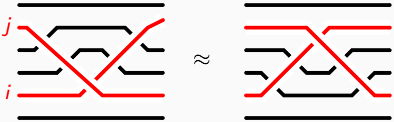

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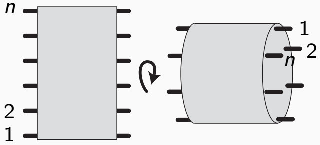

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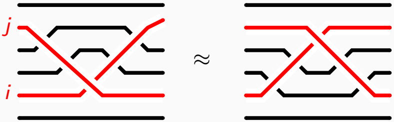

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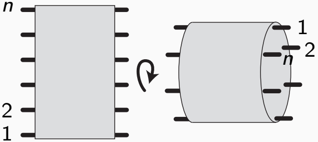

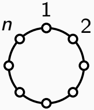

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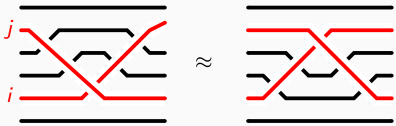

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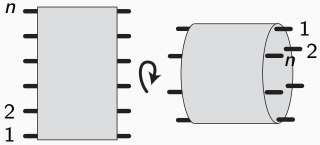

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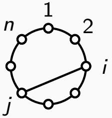

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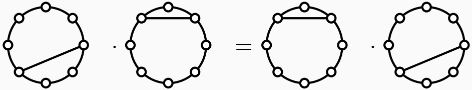

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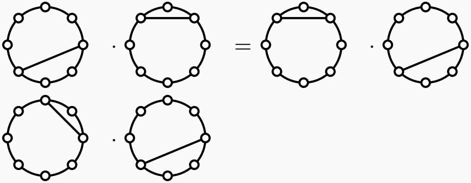

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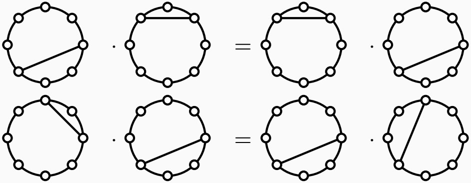

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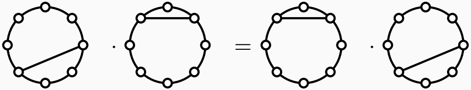

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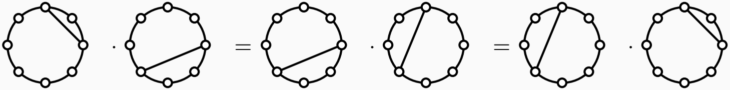

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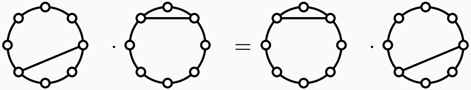

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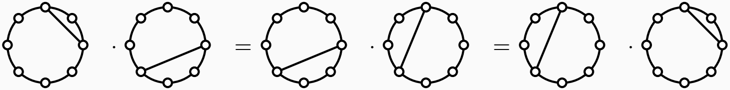

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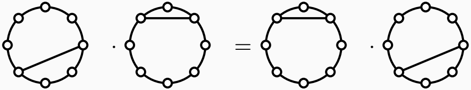

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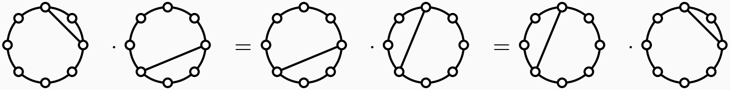

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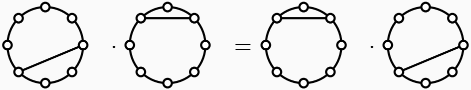

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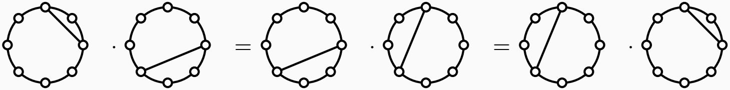

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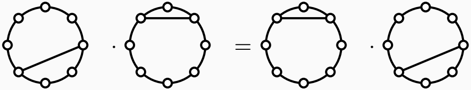

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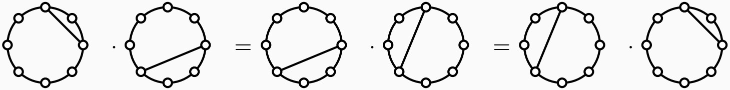

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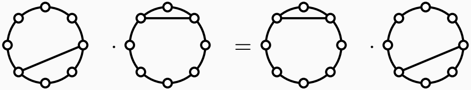

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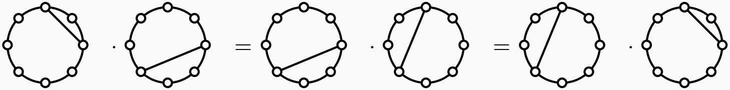

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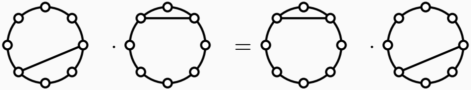

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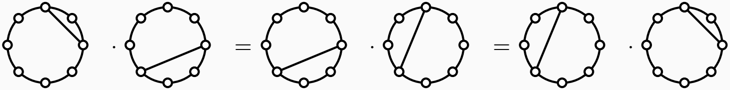

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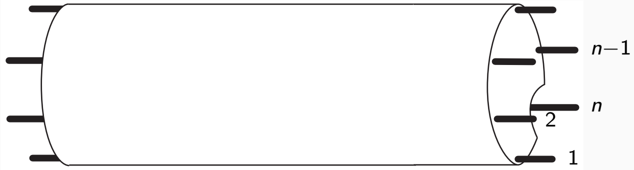

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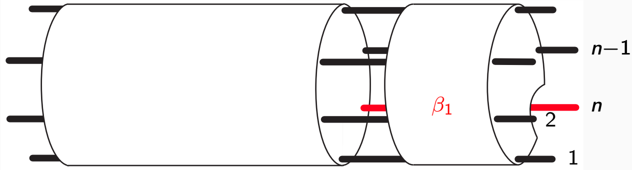

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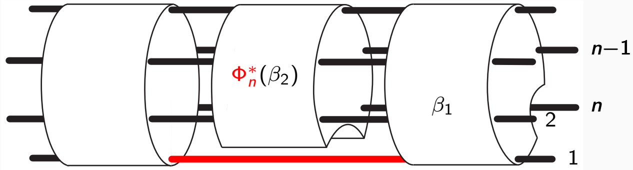

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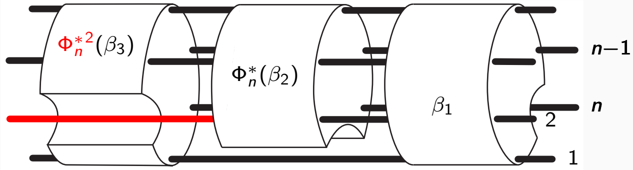

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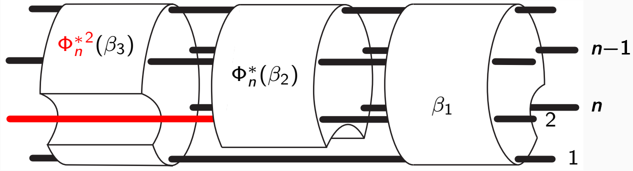

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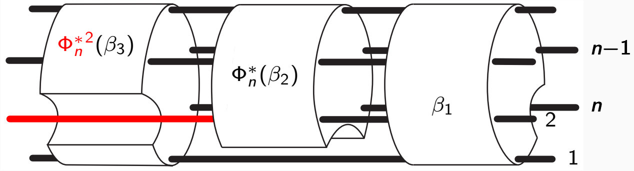

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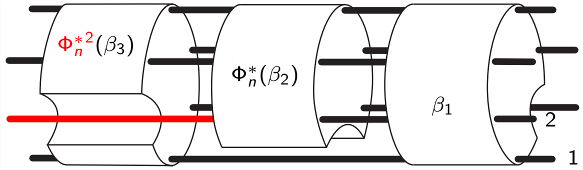

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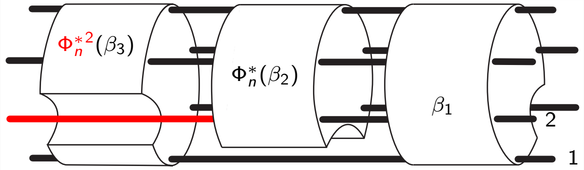


• Theorem (Fromentin 2008): For β, γ in B_n^{+*} with Φ^*-splittings

$$\beta = \Phi_n^{*\,p-1}(\beta_p) \cdot \ldots . \cdot \Phi_n^{*}(\beta_2) \cdot \beta_1, \quad \gamma = \Phi_n^{*\,q-1}(\gamma_q) \cdot \ldots . \cdot \Phi_n^{*}(\gamma_2) \cdot \gamma_1,$$

$\beta <_{\mathrm{D}} \gamma$ holds iff either $p < q$,

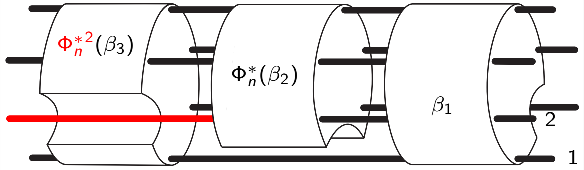
or $p = q$ and there exists r s.t. $\beta_i = \gamma_i$ for $i > r$ and $\beta_r <_{\mathrm{D}} \gamma_r$.

• Iterating: the rotating normal form...

- Proposition (Fromentin): Every braid β in B_n^{+*} admits a unique decomposition

$$\beta = \Phi_n^{*\,p-1}(\beta_p) \cdot \ldots. \cdot \Phi_n^{*\,2}(\beta_3) \cdot \Phi_n^{*}(\beta_2) \cdot \beta_1, \qquad \leftarrow \text{the } \Phi^*\text{-splitting of } \beta$$

with $\beta_i \in B_{n-1}^{+*}$ s.t. $\Phi_n^{*\,p-k}(\beta_p) \cdot \ldots \cdot \beta_k$ is right-divisible by no $a_{i,j}$ with $i, j \neq n-1$.



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- Iterating: the rotating normal form... and applications.

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