The isotopy problem of braids

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N-KOOK Seminar, Osaka State University, May 16, 2015

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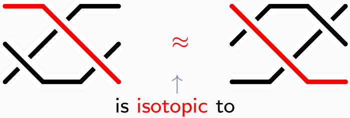

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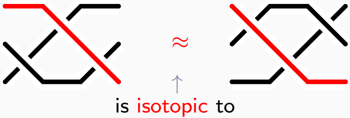

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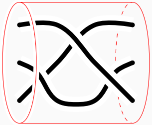
 $\leftarrow$ 

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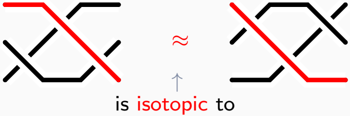

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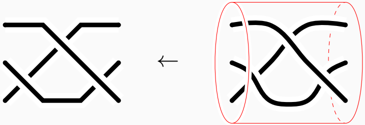

and consider ambient isotopy leaving the end-disks fixed.

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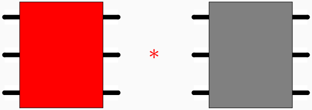

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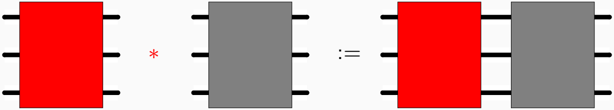

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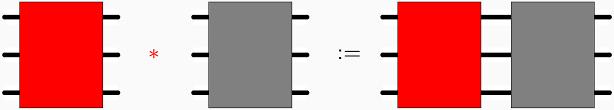

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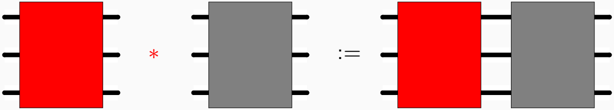

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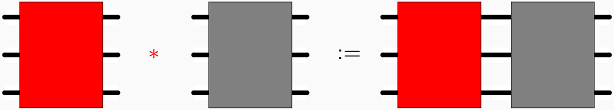

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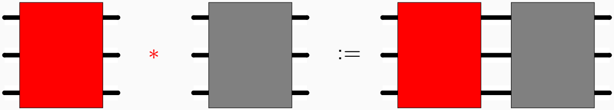

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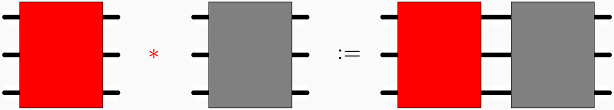

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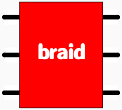

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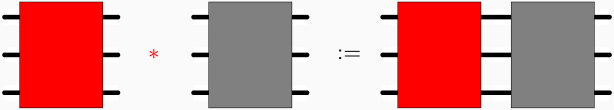

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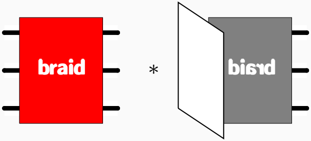

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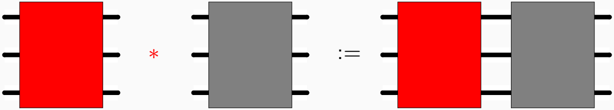

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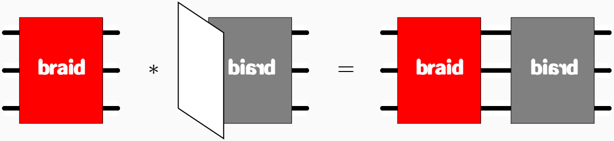

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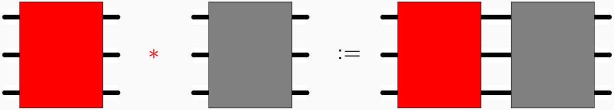

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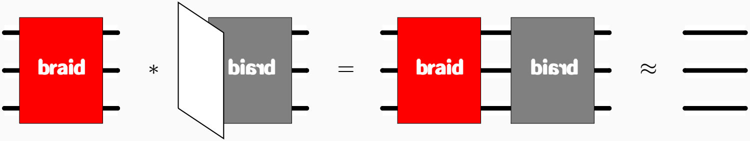

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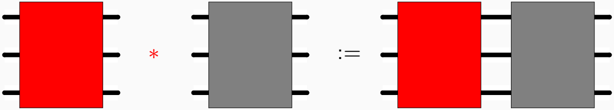

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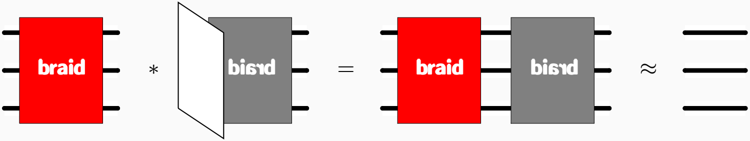

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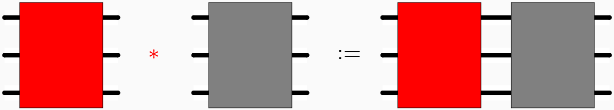

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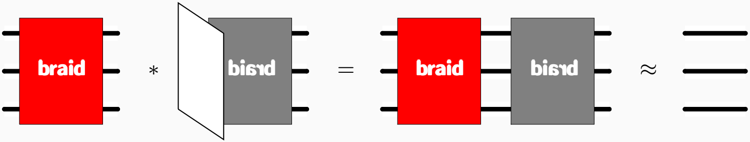

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↑ isotopy class of braid diagrams

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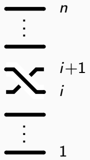

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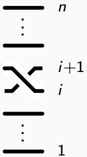
σ_i : 

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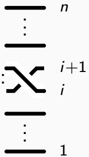
σ_i^{-1} : 

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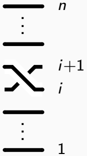
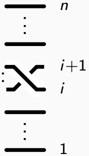
σ_i : 

or σ_i^{-1} : 

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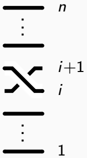
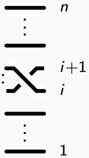
σ_i : 

or σ_i^{-1} : 

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  - Proof: Isotopy of piecewise linear diagrams is generated by Δ-moves. □

Plan:

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- Proof: Show that B_n^+ is cancellative and admits common multiples. □

- An effective way of reducing from B_n to B_n^+:

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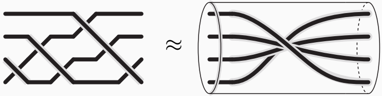

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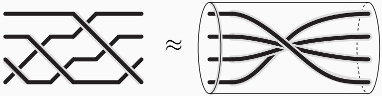

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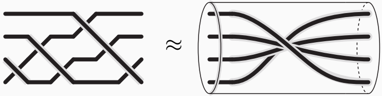

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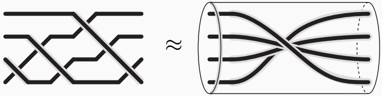

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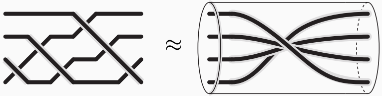

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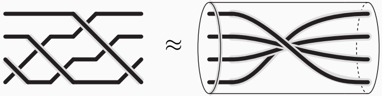

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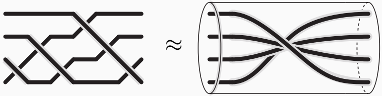

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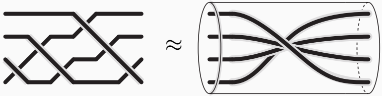

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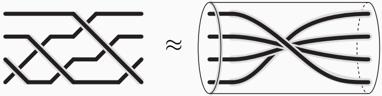

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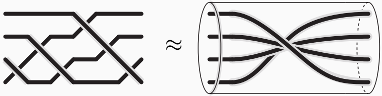

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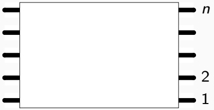

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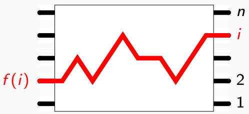

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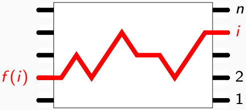

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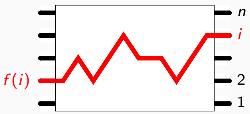

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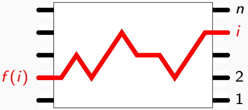

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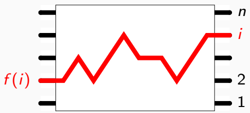

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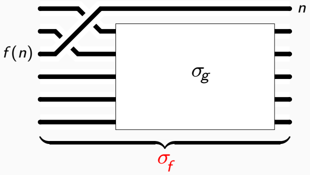

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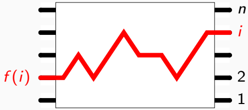

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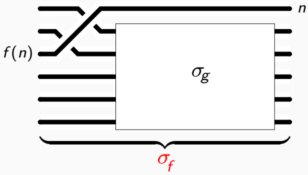

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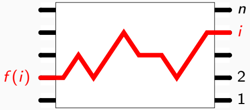

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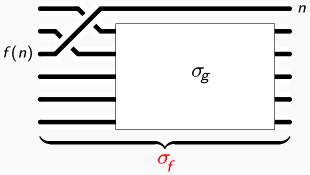

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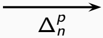

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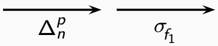

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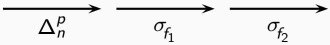

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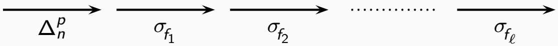

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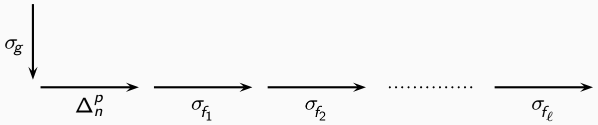

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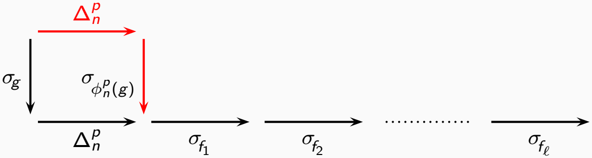

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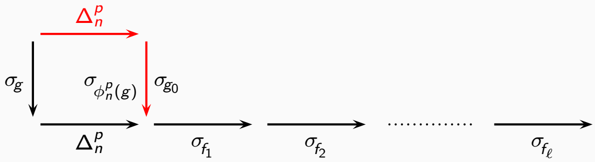

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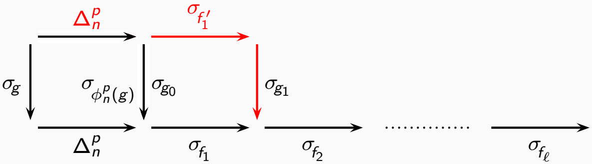

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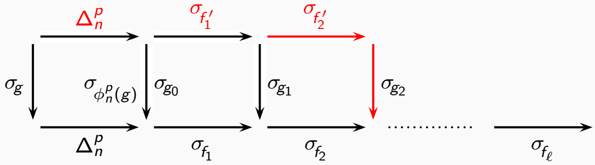

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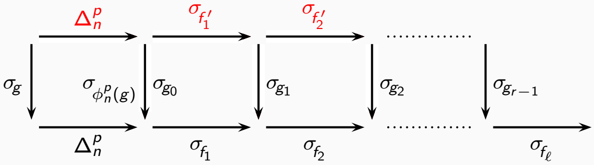

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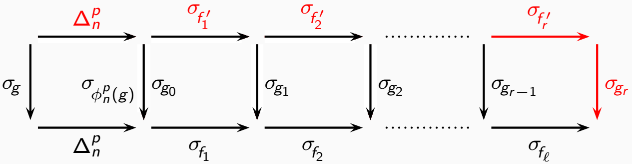

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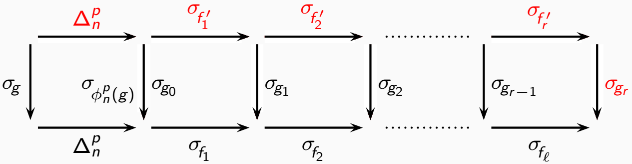

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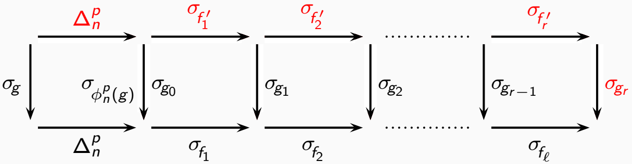

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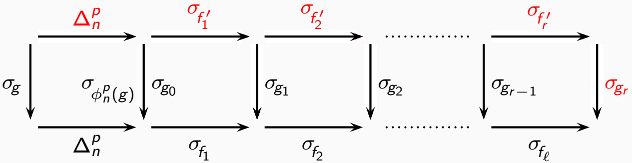

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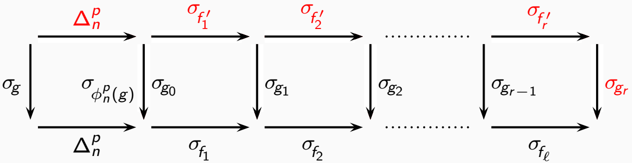

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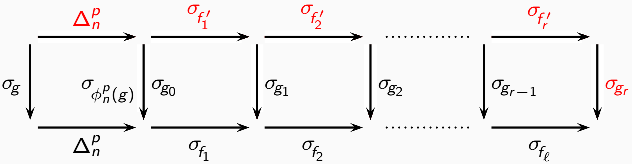

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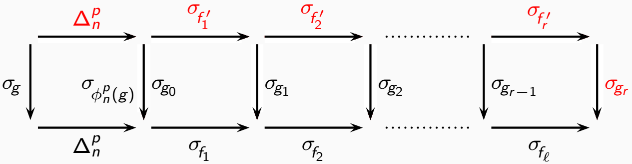

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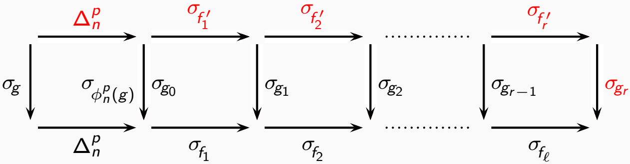

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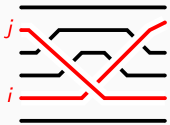

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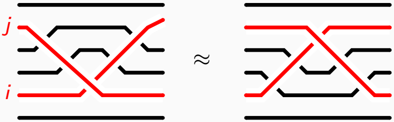

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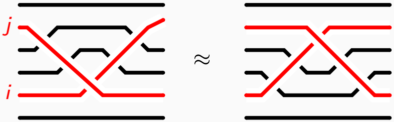

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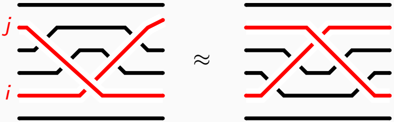

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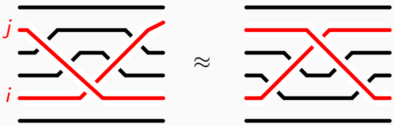

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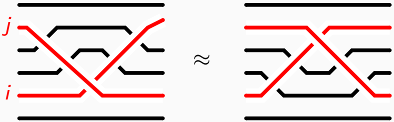

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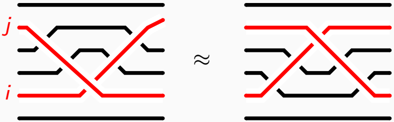

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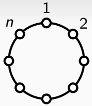
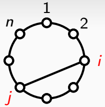
- Chord representation of the Birman–Ko–Lee generators: $a_{i,j} \mapsto$ 

- Chord representation of the Birman–Ko–Lee generators: $a_{i,j} \mapsto$ 

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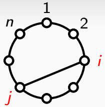
- Chord representation of the Birman–Ko–Lee generators: $a_{i,j} \mapsto$ 

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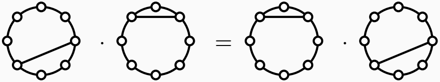

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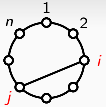
- Chord representation of the Birman–Ko–Lee generators: $a_{i,j} \mapsto$ 

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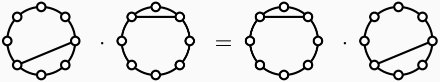

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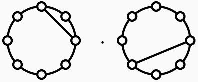

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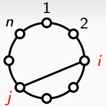

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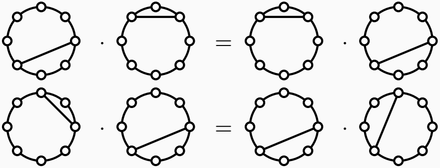

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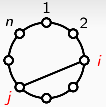
- Chord representation of the Birman–Ko–Lee generators: $a_{i,j} \mapsto$ 

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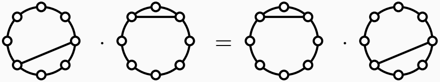

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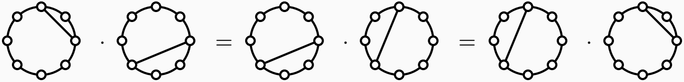

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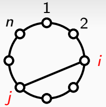
- Chord representation of the Birman–Ko–Lee generators: $a_{i,j} \mapsto$ 

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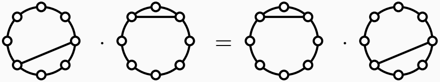

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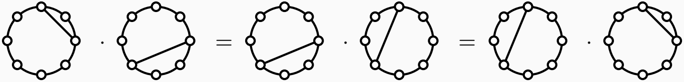

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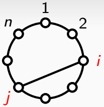
- Chord representation of the Birman–Ko–Lee generators: $a_{i,j} \mapsto$ 

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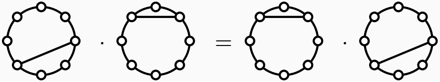

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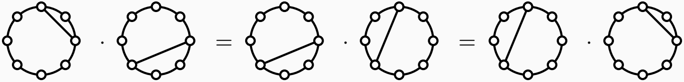

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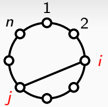
- Chord representation of the Birman–Ko–Lee generators: $a_{i,j} \mapsto$ 

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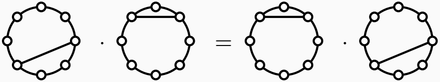

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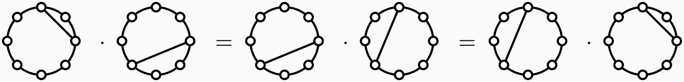

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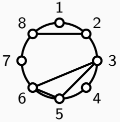
$a_{2,8}a_{3,5}a_{5,6} \leftrightarrow$ 

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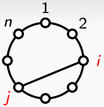

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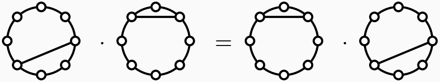

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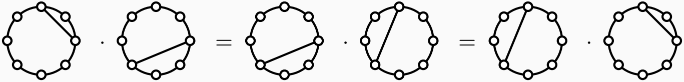

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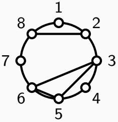

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equivalently: a non-crossing partition of $\{1, ..., n\}$

Plan:

- 1. The braid isotopy problem
- 2. Greedy normal form and the Garside structure
- 3. Dynnikov's coordinates
- 4. Bressaud's relaxation algorithm

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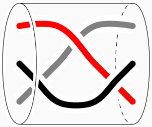

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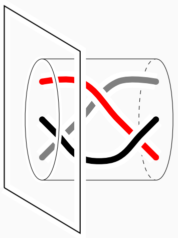

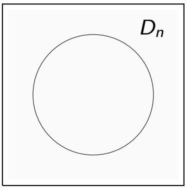

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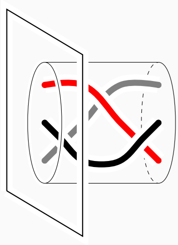

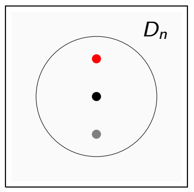

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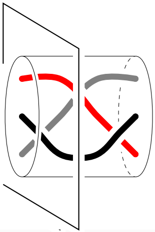

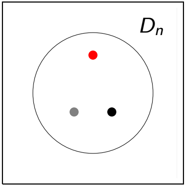

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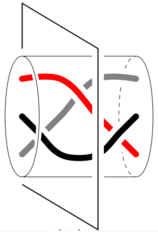

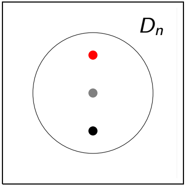

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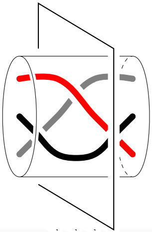

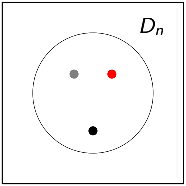

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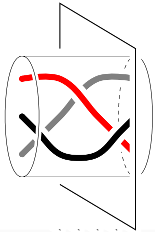

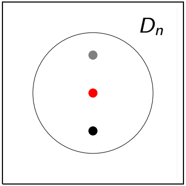

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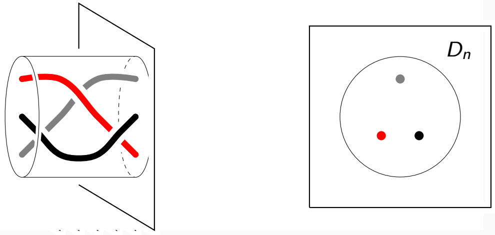

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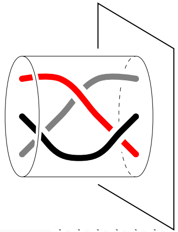

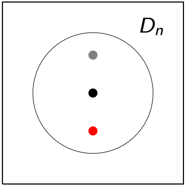

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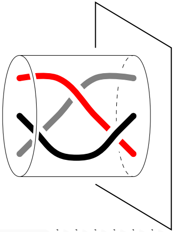

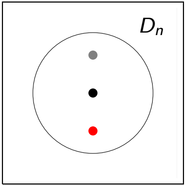

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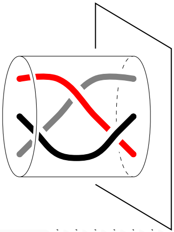

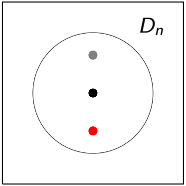

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↑ disk with n marked points

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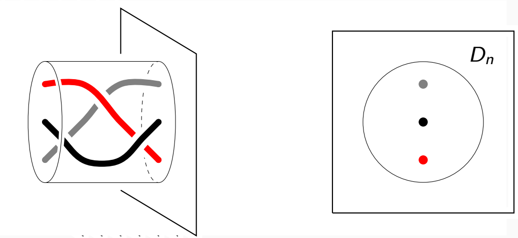

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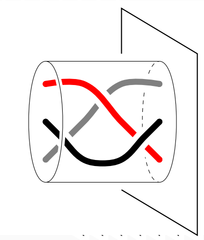

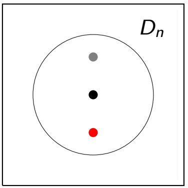

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↑ disk with n marked points ↑ boundary of D_n

- Proposition: *The group B_n is (isomorphic to) the mapping class group of D_n.*

- Viewing B_n as a group of (isotopy classes of) homeomorphisms of D_n:

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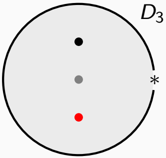

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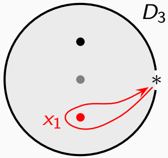

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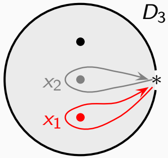

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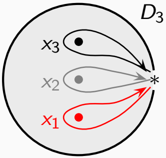

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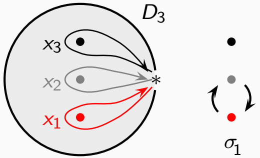

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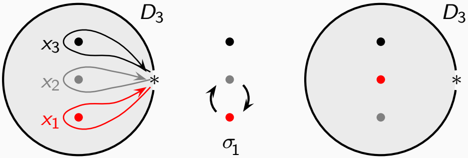

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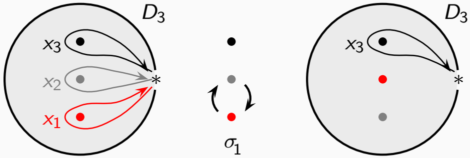

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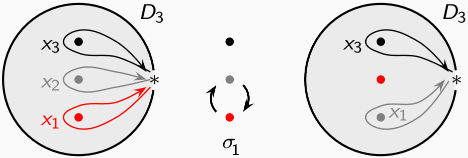

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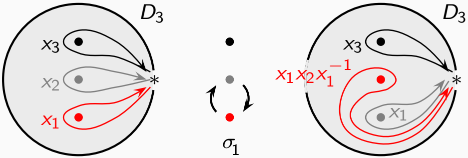

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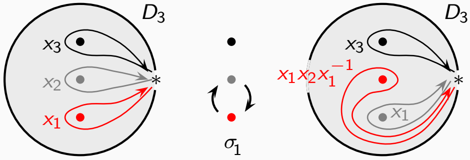

- From there: a homomorphism ρ from B_n to $\mathrm{Aut}(F_n)$:

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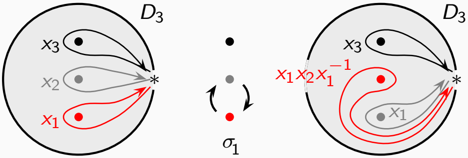

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$$\rho(\sigma_i) : \begin{cases} x_i & \mapsto \; x_i x_{i+1} x_i^{-1}, \\ & \end{cases}$$

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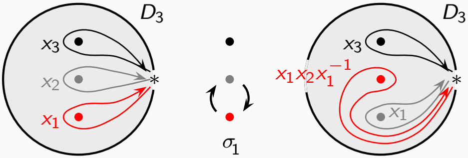

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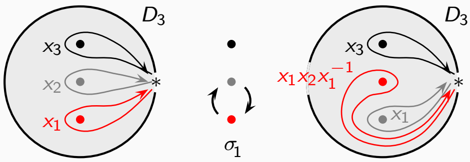

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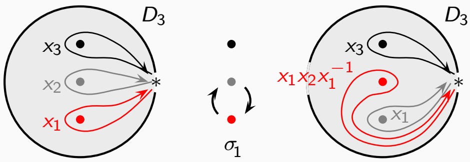

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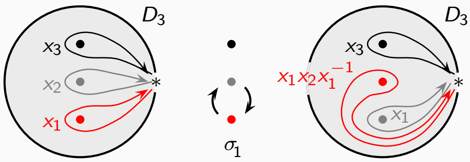

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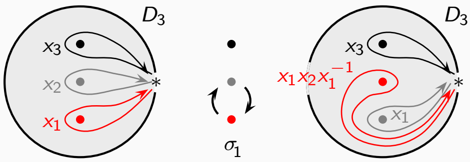

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- Theorem (Artin): *The homomorphism ρ is injective.*
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- Define an action of n-strand braid words on $\mathbb{Z}^{2n}$ by
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- Definition: *The coordinates of an n-strand braid word w are $(0, 1, 0, 1, \ldots, 0, 1) * w$.*

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- Definition: *The coordinates of an n-strand braid word w are* $(0, 1, 0, 1, ..., 0, 1) * w$.

- Theorem (Dynnikov 2000): *The coordinates of w only depend on the braid represented by w,*

- For $x \in \mathbb{Z}$, put $x^+ = \max(0, x)$, $x^- = \min(x, 0)$, and

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- Define an action of n-strand braid words on $\mathbb{Z}^{2n}$ by

$$(a_1, b_1, ..., a_n, b_n) * \sigma_i^e = (a'_1, b'_1, ..., a'_n, b'_n)$$

with $a'_k = a_k$ and $b'_k = b_k$ for $k \neq i, i{+}1$, and $(a'_i, b'_i, a'_{i+1}, b'_{i+1}) = F^e(a_i, b_i, a_{i+1}, b_{i+1})$.

- Definition: *The coordinates of an n-strand braid word w are $(0, 1, 0, 1, ..., 0, 1) * w$.*

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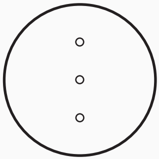
  - An extremely efficient method: "linear space, quadratic time complexity"

- Braid=homeomorphism of D_n ▶ acts on curves drawn in D_n.



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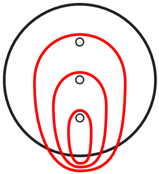

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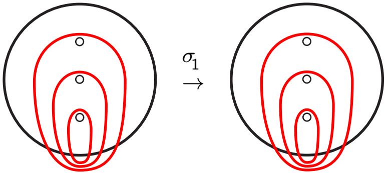

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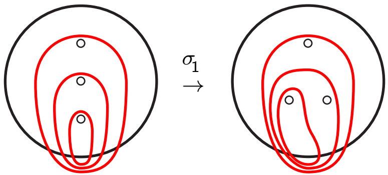

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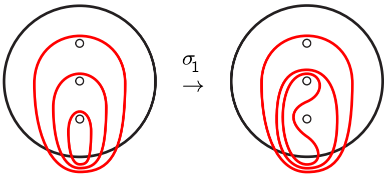

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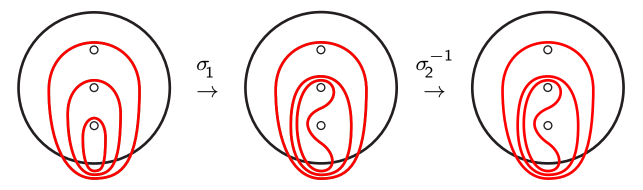

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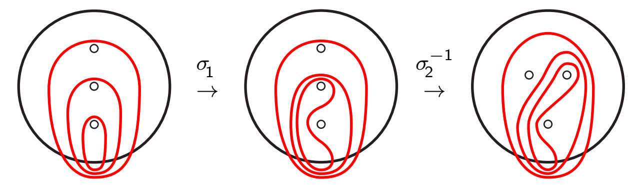

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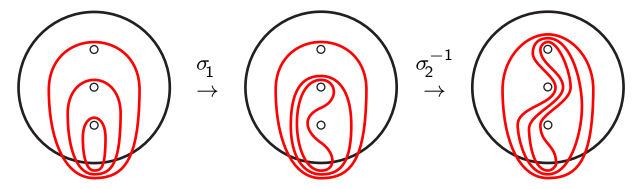

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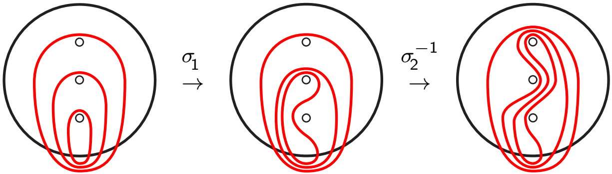

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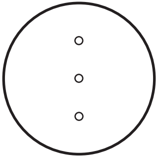

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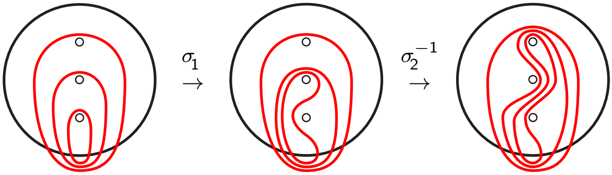

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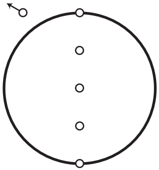

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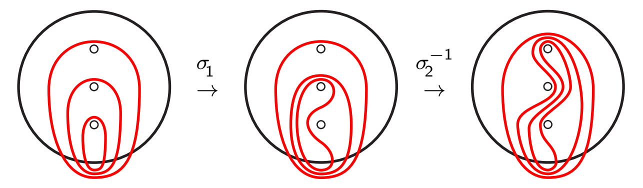

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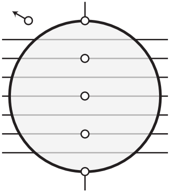

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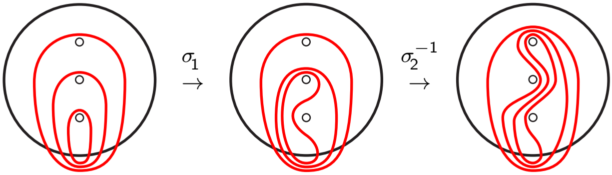

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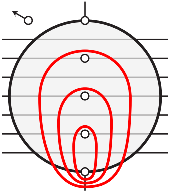

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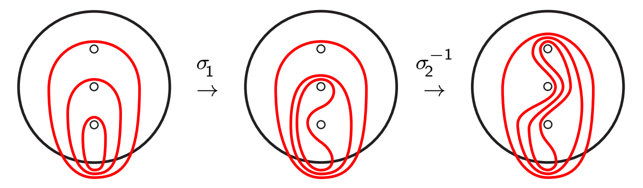

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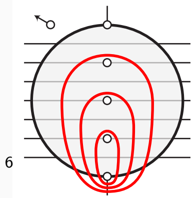

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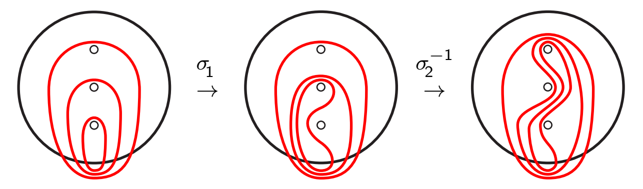

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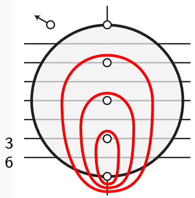

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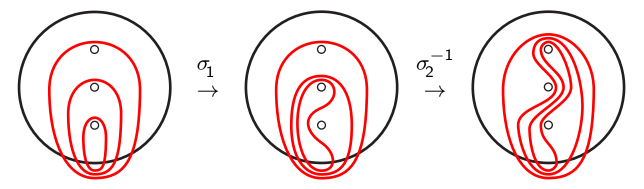

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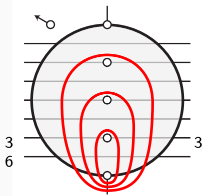

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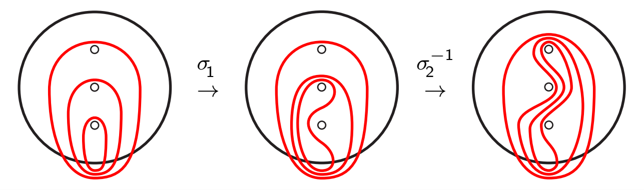

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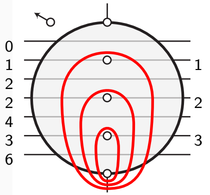

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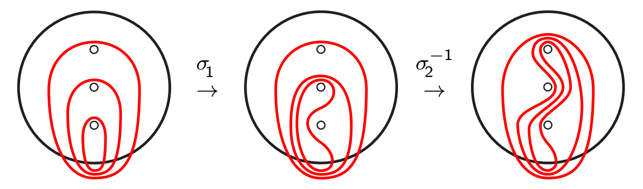

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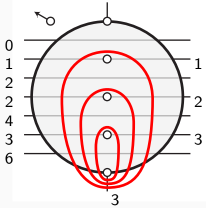

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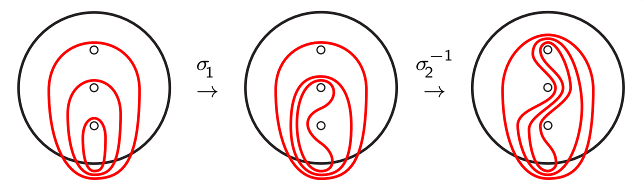

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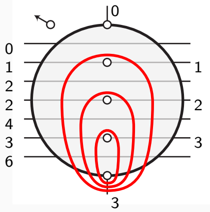

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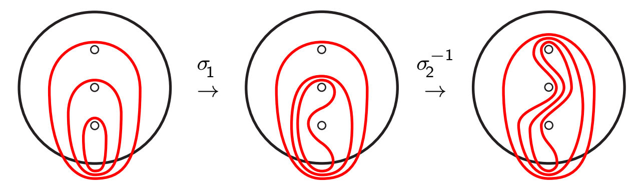

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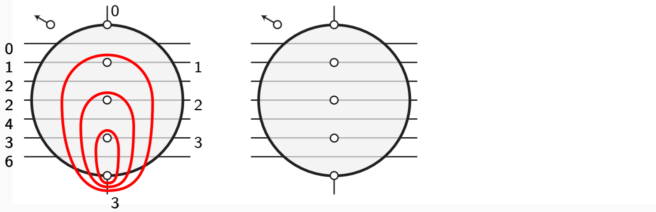

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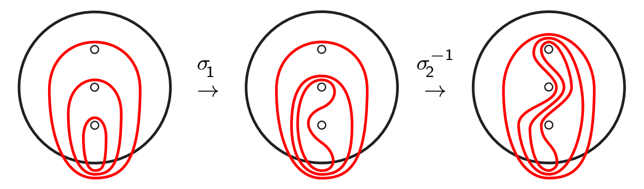

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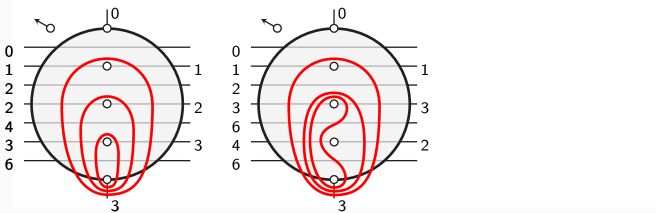

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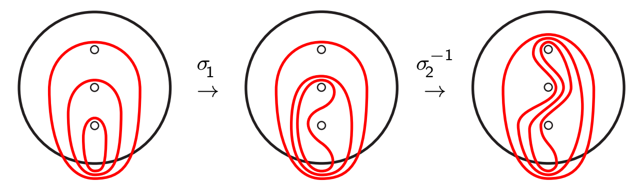

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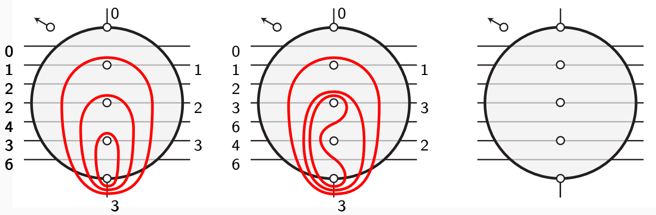

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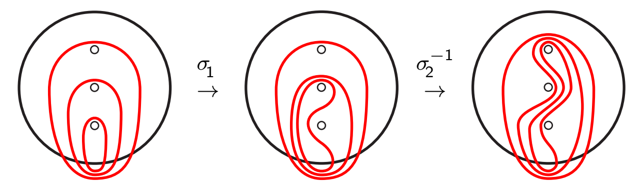

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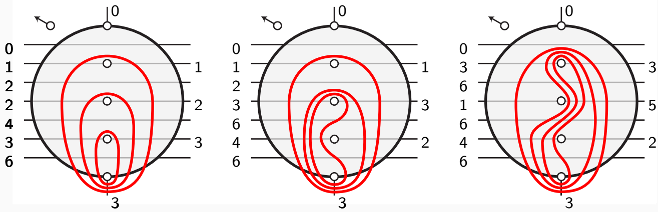

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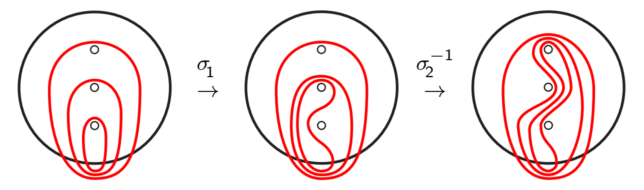

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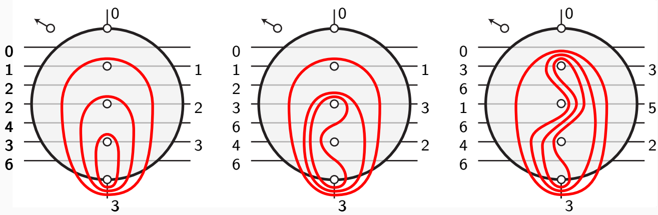

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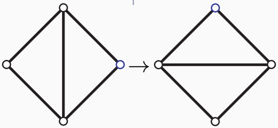

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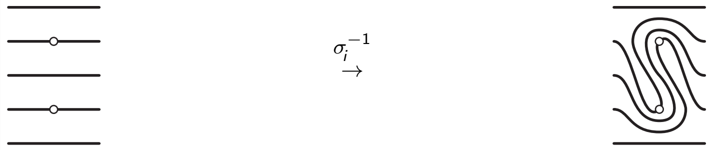

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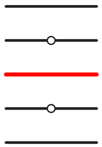

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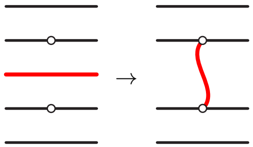

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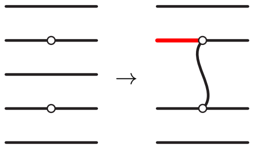

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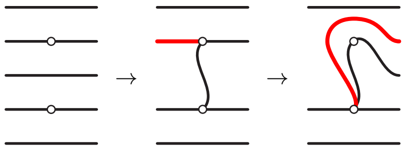

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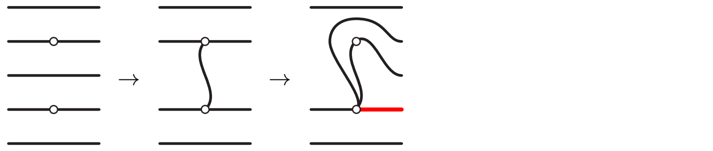

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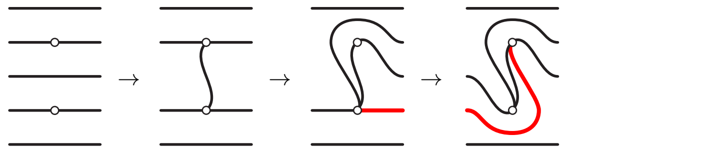

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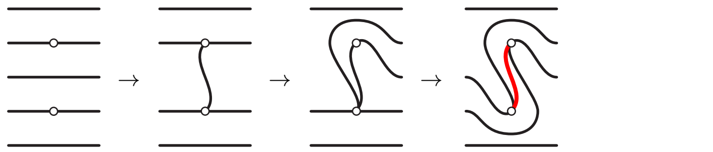

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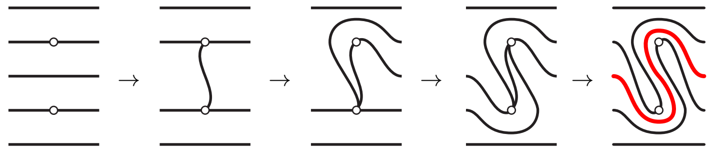

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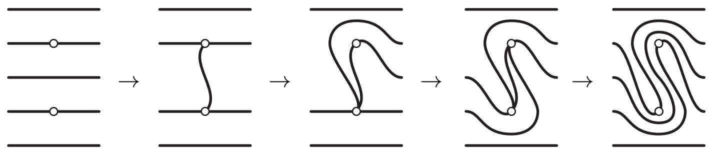

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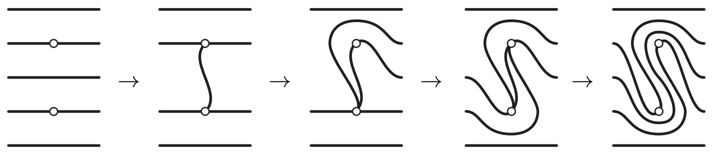

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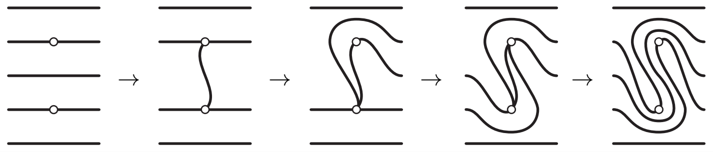

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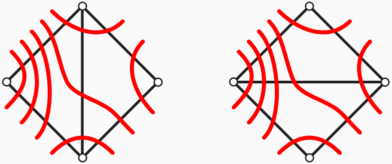

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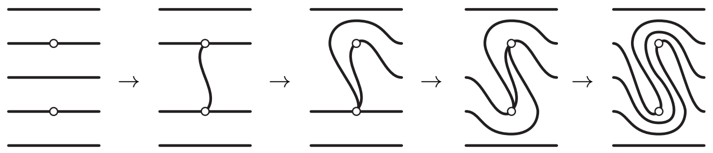

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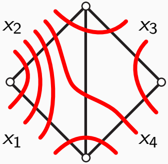

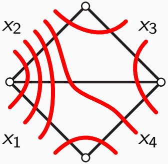

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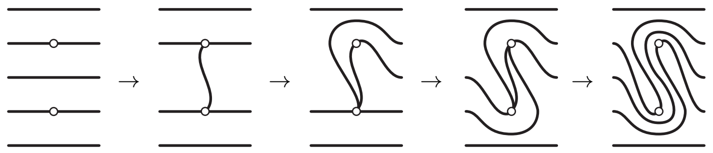

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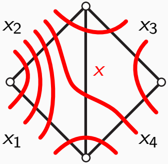

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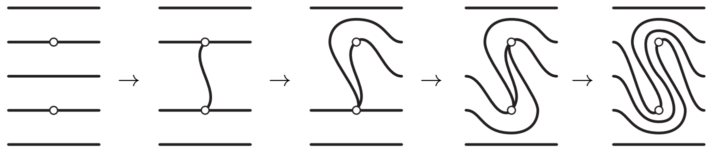

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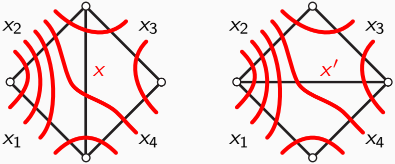

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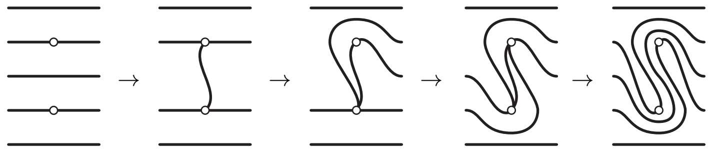

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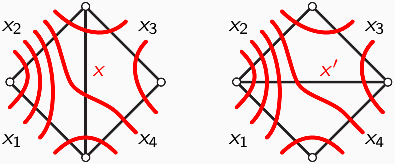

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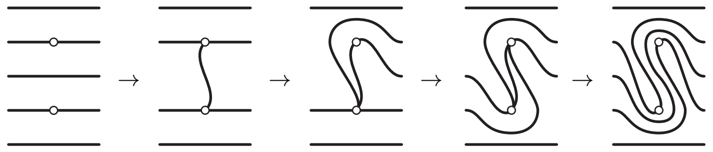

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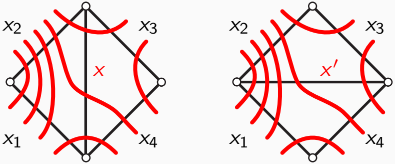

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  - ▸ Dynnikov's formulas when iterating four times (four flips).

Plan:

- 1. The braid isotopy problem
- 2. Greedy normal form and the Garside structure
- 3. Dynnikov's coordinates
- 4. Bressaud's relaxation algorithm

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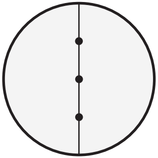

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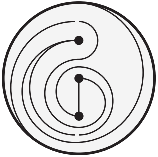

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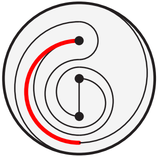

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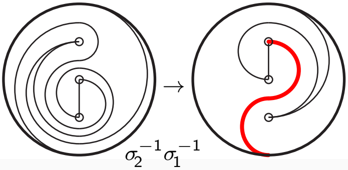

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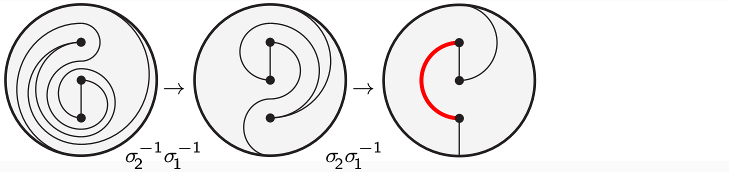

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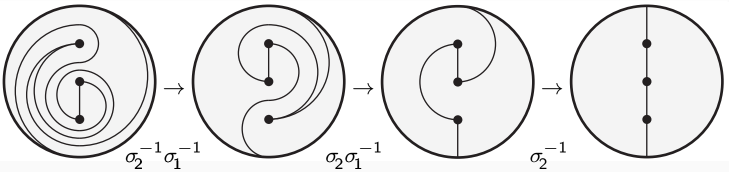

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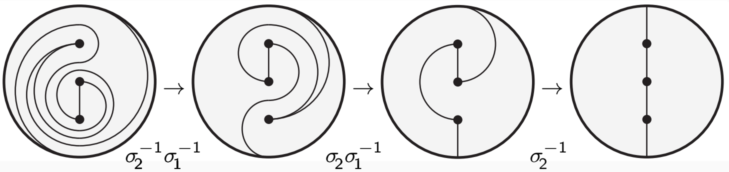

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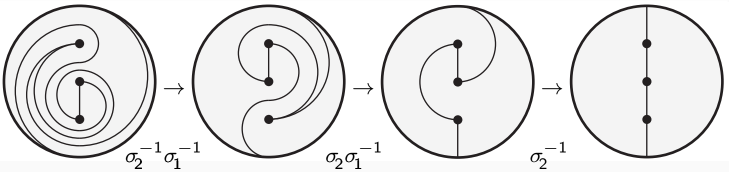

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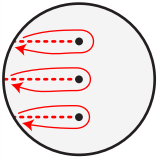

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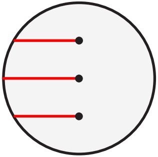

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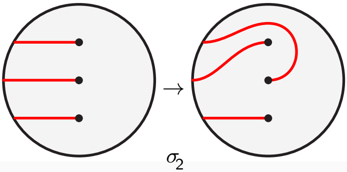

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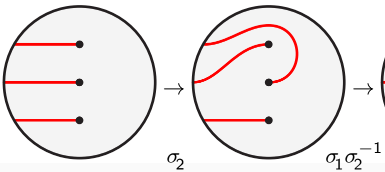

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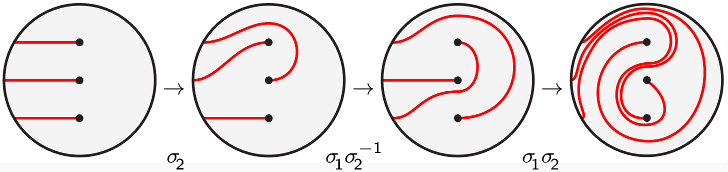

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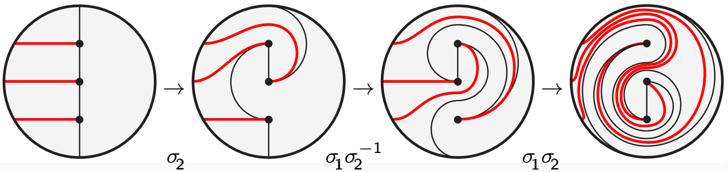

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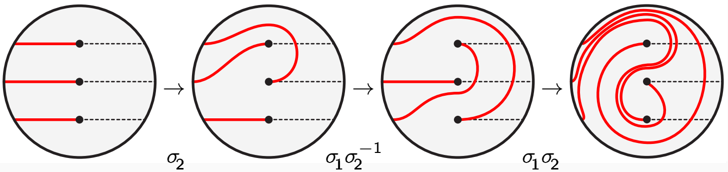

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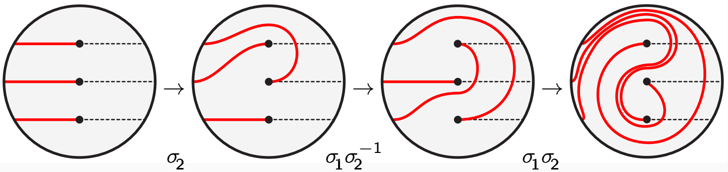

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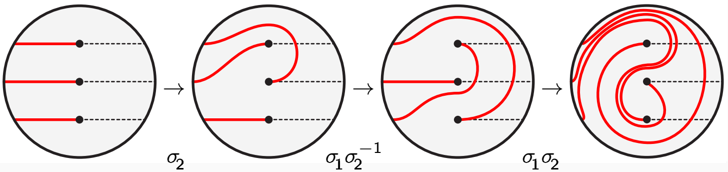

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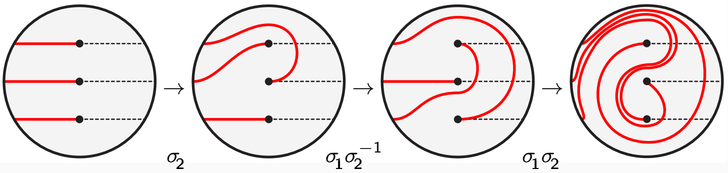

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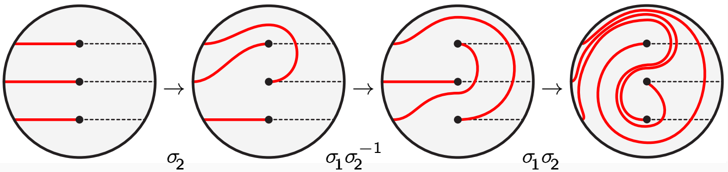

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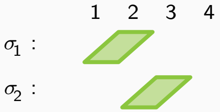
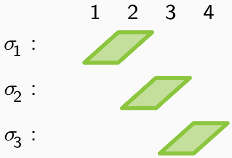
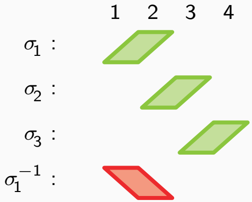
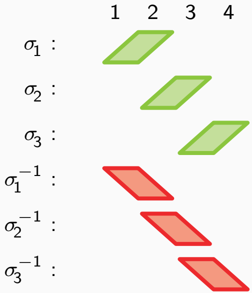
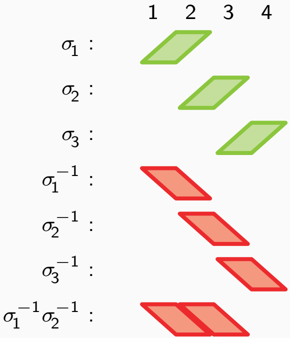
σ_1 :

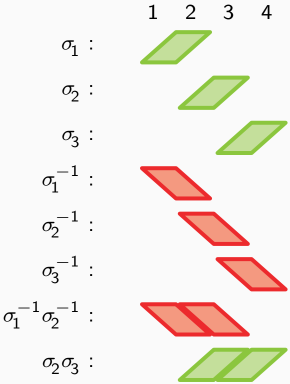

1 2 3 4
σ1 :
σ2 :


1 2 3 4
σ1 :
σ2 :
σ3 :

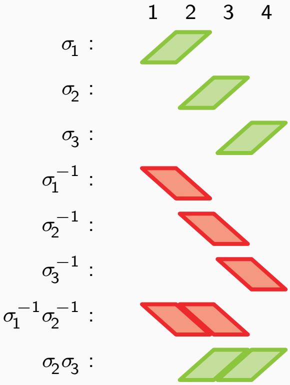

1 2 3 4
σ_1 :
σ_2 :
σ_3 :
σ_1^{-1} :


1 2 3 4
σ_1 :
σ_2 :
σ_3 :
σ_1^{-1} :
σ_2^{-1} :
σ_3^{-1} :

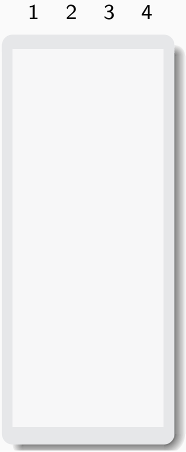

1 2 3 4
σ_1 :
σ_2 :
σ_3 :
σ_1^{-1} :
σ_2^{-1} :
σ_3^{-1} :
$\sigma_1^{-1}\sigma_2^{-1}$:



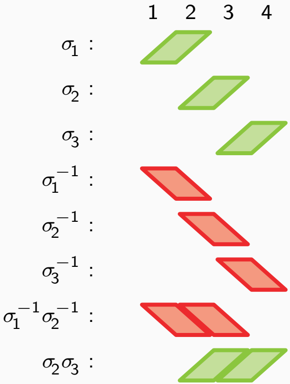
etc. (12 pieces:
6 positive, 6 negative)



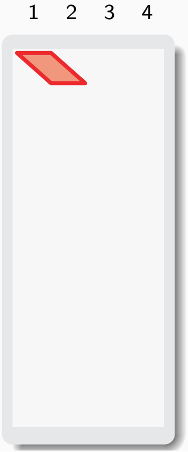
etc. (12 pieces:
6 positive, 6 negative)



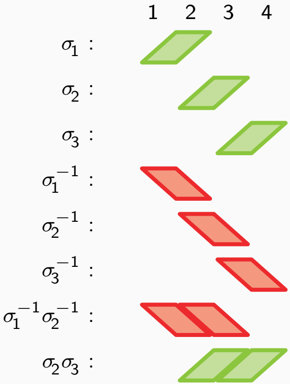
- Normal form of ε $= \varepsilon$.



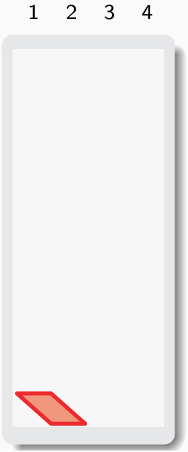
etc. (12 pieces:
6 positive, 6 negative)



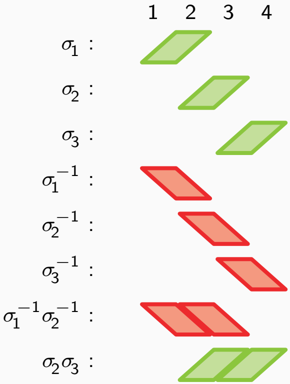
- Normal form of σ_1^{-1}



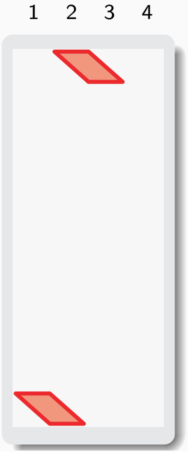
etc. (12 pieces:
6 positive, 6 negative)



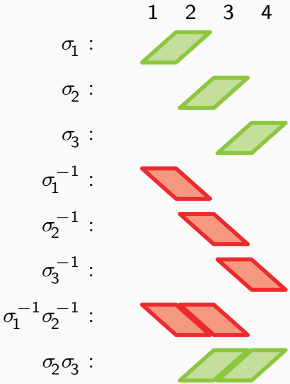
- Normal form of σ_1^{-1} $= \sigma_1^{-1}$.



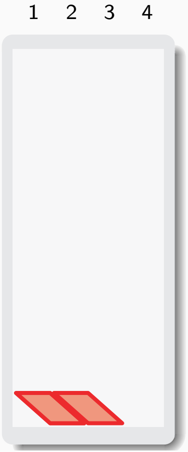
etc. (12 pieces:
6 positive, 6 negative)



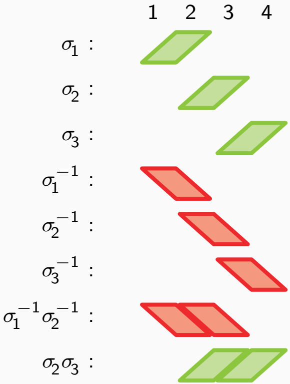
- Normal form of $\sigma_1^{-1}\sigma_2^{-1}$



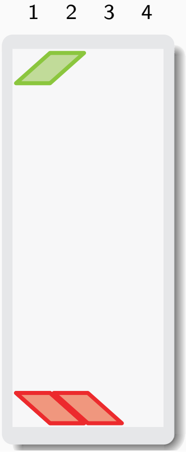
etc. (12 pieces:
6 positive, 6 negative)



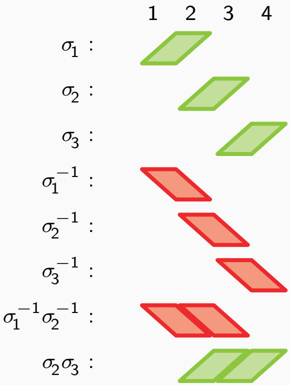
- Normal form of $\sigma_1^{-1}\sigma_2^{-1}$ $= \sigma_1^{-1}\sigma_2^{-1}$.



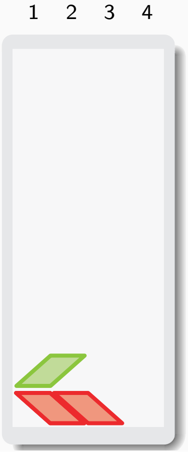
etc. (12 pieces:
6 positive, 6 negative)



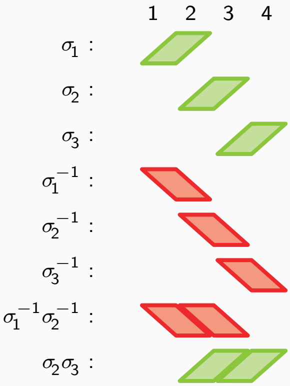
- **Normal** form of $\sigma_1^{-1}\sigma_2^{-1}\sigma_1$



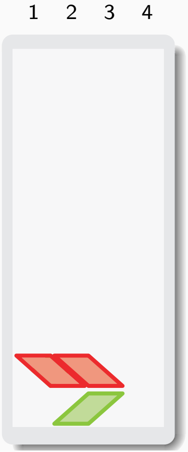
etc. (12 pieces:
6 positive, 6 negative)



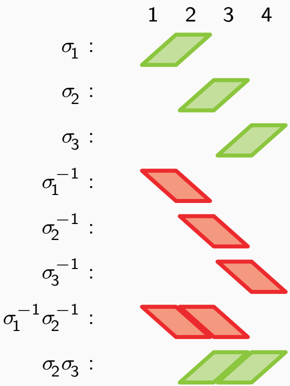
- Normal form of $\sigma_1^{-1}\sigma_2^{-1}\sigma_1$



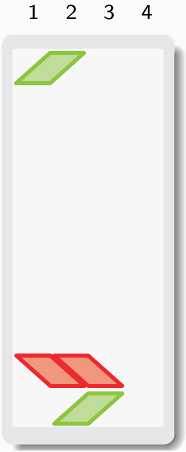
etc. (12 pieces:
6 positive, 6 negative)



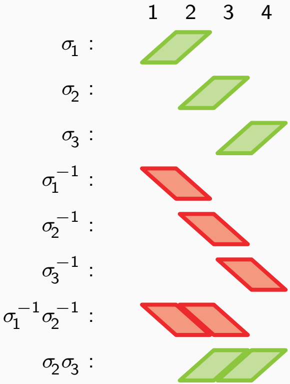
- **Normal form of** $\sigma_1^{-1}\sigma_2^{-1}\sigma_1 = \sigma_2.\sigma_1^{-1}\sigma_2^{-1}$.



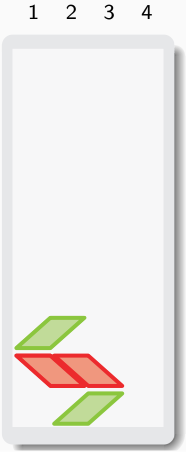
etc. (12 pieces:
6 positive, 6 negative)



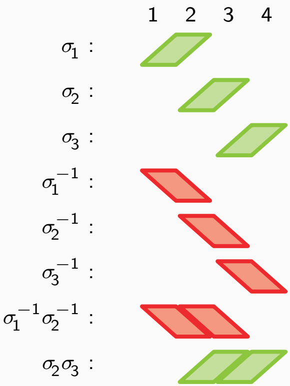
- **Normal** form of $\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_1$



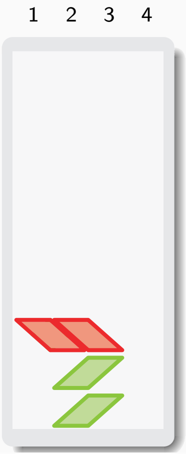
etc. (12 pieces:
6 positive, 6 negative)



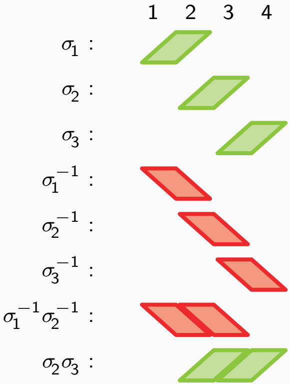
- **Normal** form of $\sigma_1^{-1}\sigma_2^{-1}\sigma_1{\color{red}\sigma_1}$



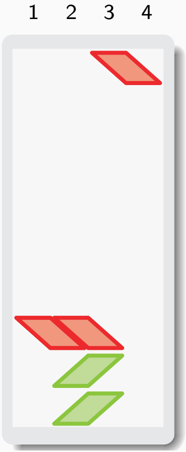
etc. (12 pieces:
6 positive, 6 negative)



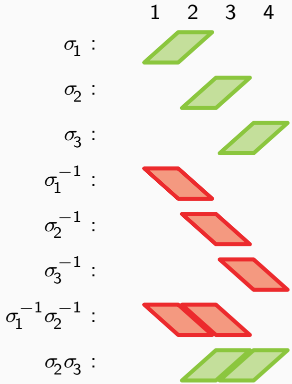
- **Normal** form of $\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_1$ $= \sigma_2.\sigma_2.\sigma_1^{-1}\sigma_2^{-1}$.



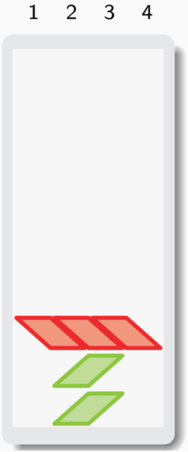
etc. (12 pieces:
6 positive, 6 negative)



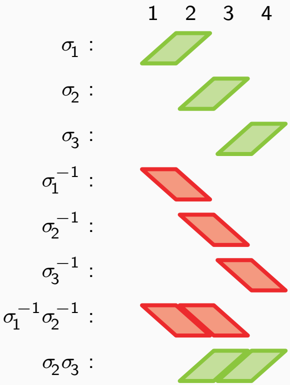
- **Normal** form of $\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_1\sigma_3^{-1}$



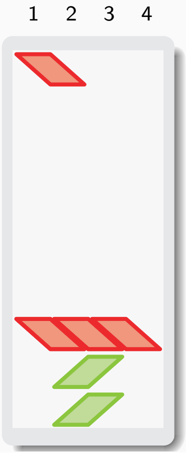
etc. (12 pieces:
6 positive, 6 negative)



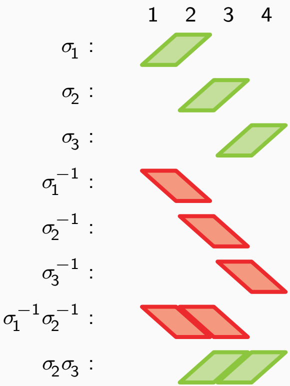
- **Normal** form of $\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_1\sigma_3^{-1}$ $= \sigma_2.\sigma_2.\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}$.



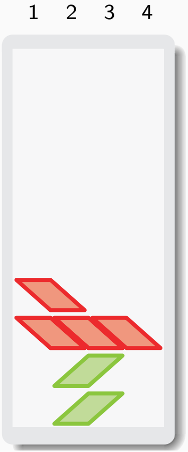
etc. (12 pieces:
6 positive, 6 negative)



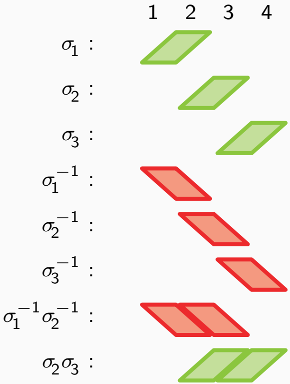
- **Normal** form of $\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_1\sigma_3^{-1}\sigma_1^{-1}$



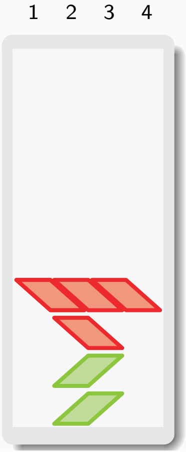
etc. (12 pieces:
6 positive, 6 negative)



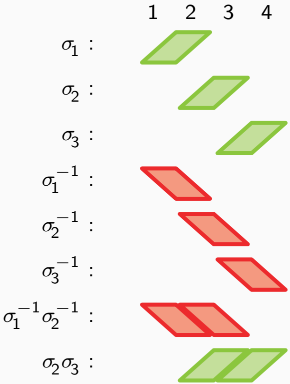
- **Normal** form of $\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_1\sigma_3^{-1}\sigma_1^{-1}$



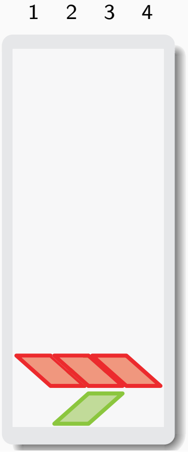
etc. (12 pieces:
6 positive, 6 negative)



- **Normal** form of $\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_1\sigma_3^{-1}\sigma_1^{-1}$

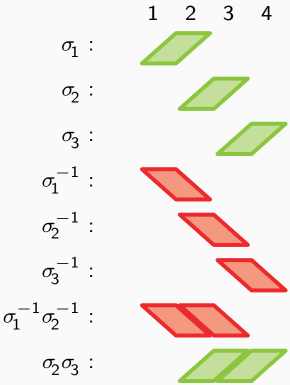


etc. (12 pieces:
6 positive, 6 negative)

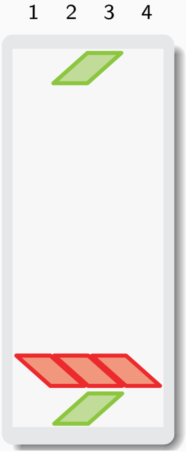


- **Normal** form of $\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_1\sigma_3^{-1}\sigma_1^{-1}$ $= \sigma_2.\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}$.

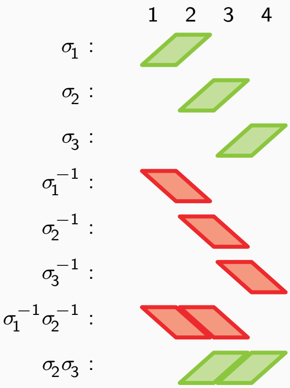
Bressaud's algorithm



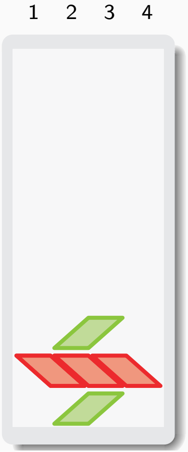
etc. (12 pieces:
6 positive, 6 negative)



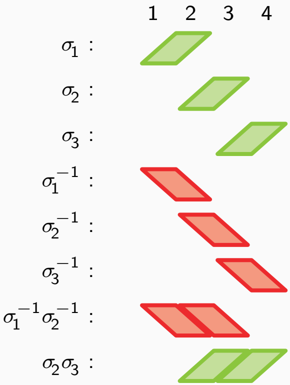
- **Normal** form of $\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_1\sigma_3^{-1}\sigma_1^{-1}\sigma_2$



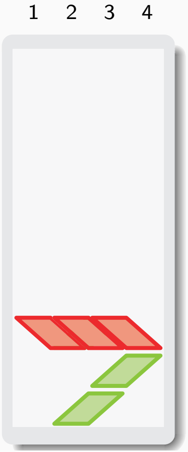
etc. (12 pieces:
6 positive, 6 negative)



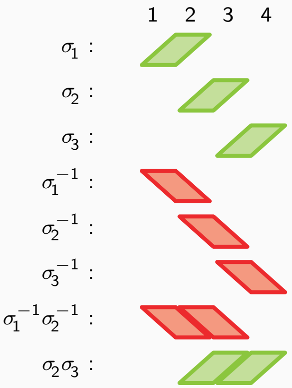
- **Normal** form of $\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_1\sigma_3^{-1}\sigma_1^{-1}\sigma_2$



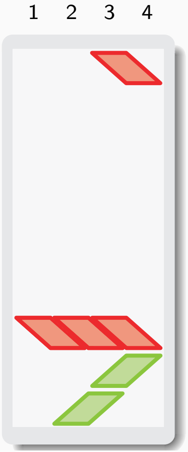
etc. (12 pieces:
6 positive, 6 negative)



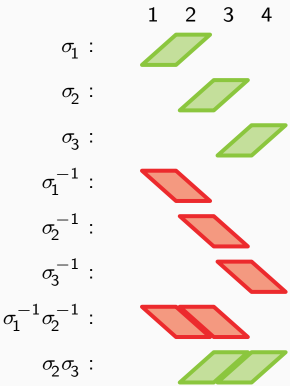
- **Normal** form of $\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_1\sigma_3^{-1}\sigma_1^{-1}\sigma_2$ $= \sigma_2.\sigma_3.\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}$.



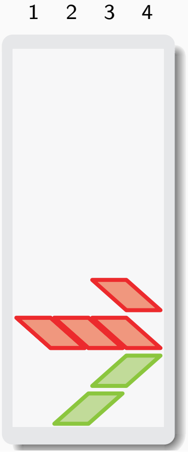
etc. (12 pieces:
6 positive, 6 negative)



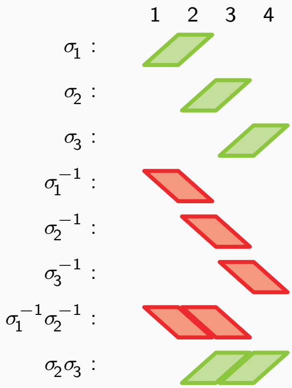
- **Normal** form of $\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_1\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_3^{-1}$



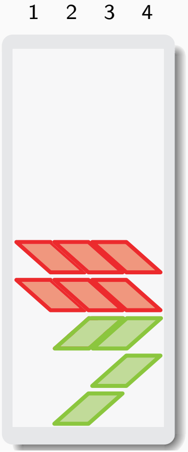
etc. (12 pieces:
6 positive, 6 negative)



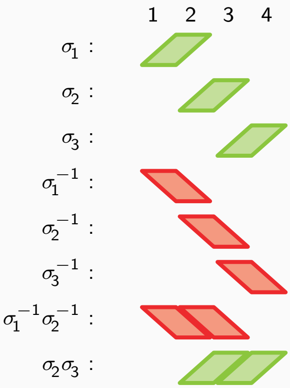
- **Normal** form of $\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_1\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_3^{-1}$



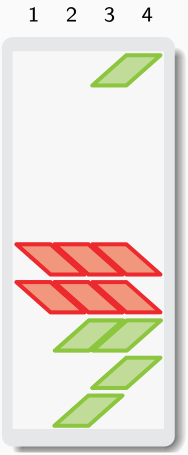
etc. (12 pieces:
6 positive, 6 negative)



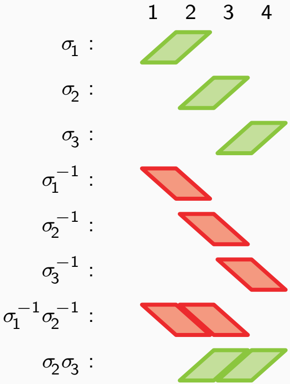
- **Normal** form of $\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_1\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_3^{-1}$ $= \sigma_2.\sigma_3.\sigma_3\sigma_2.\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}.\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}.$



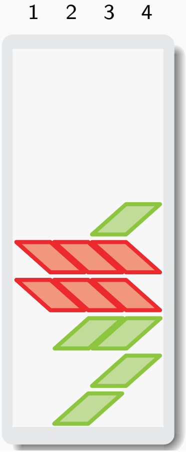
etc. (12 pieces:
6 positive, 6 negative)



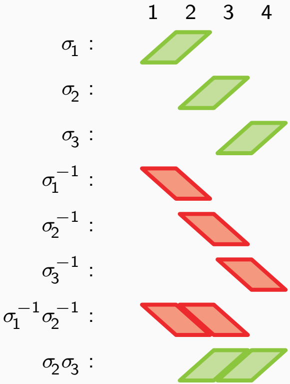
- **Normal** form of $\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_1\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_3$



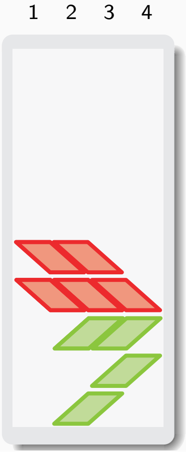
etc. (12 pieces:
6 positive, 6 negative)



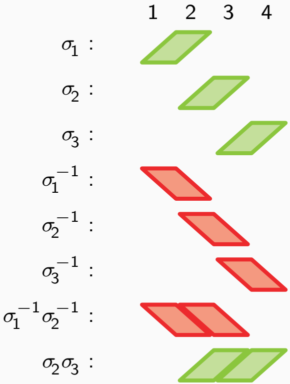
- **Normal** form of $\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_1\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_3^{-1}\textcolor{red}{\sigma_3}$



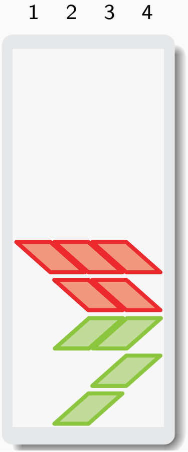
etc. (12 pieces:
6 positive, 6 negative)



- **Normal** form of $\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_1\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_3$



etc. (12 pieces:
6 positive, 6 negative)



- **Normal** form of $\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_1\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_3$

σ_1 :

σ_2 :

σ_3 :

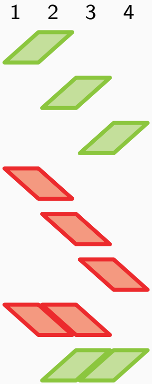
σ_1^{-1} :

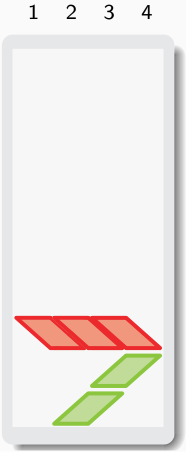
σ_2^{-1} :

σ_3^{-1} :

$\sigma_1^{-1}\sigma_2^{-1}$:

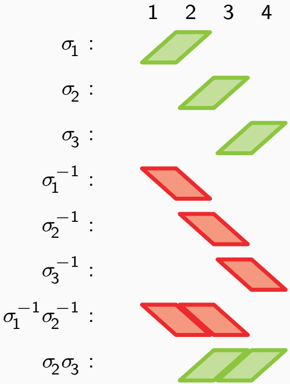
$\sigma_2\sigma_3$:



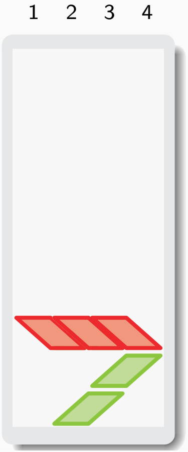


etc. (12 pieces:
6 positive, 6 negative)

- **Normal** form of $\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_1\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_3 \quad = \sigma_2.\sigma_3.\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}$.



etc. (12 pieces:
6 positive, 6 negative)



- **Normal** form of $\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_1\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_3 \quad = \sigma_2.\sigma_3.\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}$.

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