The group of parenthesized braids

The group of parenthesized braids

Patrick Dehornoy

Laboratoire de Mathématiques Nicolas Oresme
Université de Caen, France

The group of parenthesized braids

Patrick Dehornoy

Laboratoire de Mathématiques Nicolas Oresme
Université de Caen, France

Topology Seminar, Tokyo University, May 7, 2015

The group of parenthesized braids

Patrick Dehornoy

Laboratoire de Mathématiques Nicolas Oresme
Université de Caen, France

Topology Seminar, Tokyo University, May 7, 2015

- A group $B_\bullet$ that extends both Artin's braid group B_∞ and Thompson's group F,

The group of parenthesized braids

Patrick Dehornoy

Laboratoire de Mathématiques Nicolas Oresme
Université de Caen, France

Topology Seminar, Tokyo University, May 7, 2015

• A group $B_\bullet$ that extends both Artin's braid group B_∞ and Thompson's group F, occurring in various contexts:

- "geometry group of an algebraic law",

The group of parenthesized braids

Patrick Dehornoy

Laboratoire de Mathématiques Nicolas Oresme
Université de Caen, France

Topology Seminar, Tokyo University, May 7, 2015

• A group $B_\bullet$ that extends both Artin's braid group B_∞ and Thompson's group F, occurring in various contexts:

- "geometry group of an algebraic law",
- subgroup of M. Brin's braided Thompson group BV,

The group of parenthesized braids

Patrick Dehornoy

Laboratoire de Mathématiques Nicolas Oresme
Université de Caen, France

Topology Seminar, Tokyo University, May 7, 2015

• A group $B_\bullet$ that extends both Artin's braid group B_∞ and Thompson's group F, occurring in various contexts:

- "geometry group of an algebraic law",
- subgroup of M. Brin's braided Thompson group BV,
- quotient of a group of Greenberg–Sergiescu and Funar–Kapoudjian, ...

The group of parenthesized braids

Patrick Dehornoy

Laboratoire de Mathématiques Nicolas Oresme
Université de Caen, France

Topology Seminar, Tokyo University, May 7, 2015

- A group $B_\bullet$ that extends both Artin's braid group B_∞ and Thompson's group F, occurring in various contexts:
  - "geometry group of an algebraic law",
  - subgroup of M. Brin's braided Thompson group BV,
  - quotient of a group of Greenberg–Sergiescu and Funar–Kapoudjian, ...
- Here: insist on similarity with B_∞ (use of self-distributivity)

The group of parenthesized braids

Patrick Dehornoy

Laboratoire de Mathématiques Nicolas Oresme
Université de Caen, France

Topology Seminar, Tokyo University, May 7, 2015

• A group $B_\bullet$ that extends both Artin's braid group B_∞ and Thompson's group F, occurring in various contexts:

- "geometry group of an algebraic law",
- subgroup of M. Brin's braided Thompson group BV,
- quotient of a group of Greenberg–Sergiescu and Funar–Kapoudjian, ...

• Here: insist on similarity with B_∞ (use of self-distributivity) and connection with homeomorphisms of $S^2\setminus$Cantor.

Plan:

Plan:

- 1. Artin's braid group B_∞

Plan:

- 1. Artin's braid group B_∞
- 2. Thompson's group F

Plan:

- 1. Artin's braid group B_∞
- 2. Thompson's group F
- 3. The parenthesized braid group $B_\bullet$

Plan:

- 1. Artin's braid group B_∞
- 2. Thompson's group F
- 3. The parenthesized braid group $B_\bullet$
- 4. The Artin representation of $B_\bullet$

Plan:

- 1. Artin's braid group B_∞
- 2. Thompson's group F
- 3. The parenthesized braid group $B_\bullet$
- 4. The Artin representation of $B_\bullet$

- <u>Definition</u> (Artin): For $n \geqslant 1$, the braid group B_n

- Definition (Artin): For $n \geqslant 1$, the braid group B_n
  $= \{$ n-strand braid diagrams $\}$ / isotopy

- Definition (Artin): For $n \geqslant 1$, the braid group B_n

  $= \{$ n-strand braid diagrams $\}$ / isotopy

  $= \pi_1($configuration space of n points of $\mathbb{C}$ mod. action of $S_n)$

- Definition (Artin): For $n \geqslant 1$, the braid group B_n

  $= \{$ n-strand braid diagrams $\}$ / isotopy

  $= \pi_1$(configuration space of n points of $\mathbb{C}$ mod. action of S_n)

  $= \mathcal{MCG}(D_n)$

  $\uparrow$

  $\{$homeomorphisms of an n-punctured disk that fix $\partial D_n\}$/isotopy

- Definition (Artin): For $n \geqslant 1$, the braid group B_n

$= \{$ n-strand braid diagrams $\}$ / isotopy

$= \pi_1$(configuration space of n points of $\mathbb{C}$ mod. action of S_n)

$= \mathcal{MCG}(D_n)$

$\uparrow$ {homeomorphisms of an n-punctured disk that fix ∂D_n}/isotopy

$$= \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle.$$

- Definition (Artin): For $n \geqslant 1$, the braid group B_n

$$= \{\ n\text{-strand braid diagrams}\ \}\ /\ \text{isotopy}$$

$$= \pi_1(\text{configuration space of } n \text{ points of } \mathbb{C} \text{ mod. action of } S_n)$$

$$= \mathcal{MCG}(D_n)$$

$\uparrow$ {homeomorphisms of an n-punctured disk that fix ∂D_n}/isotopy

$$= \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle.$$

σ_i :

1 $\quad i \quad i{+}1 \quad n$

- Definition (Artin): For $n \geqslant 1$, the braid group B_n

$$= \{\ n\text{-strand braid diagrams}\ \}\ /\ \text{isotopy}$$

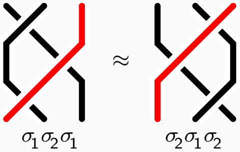
$$= \pi_1(\text{configuration space of } n \text{ points of } \mathbb{C} \text{ mod. action of } S_n)$$

$$= \mathcal{MCG}(D_n)$$

$\uparrow$ {homeomorphisms of an n-punctured disk that fix ∂D_n}/isotopy

$$= \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle.$$

σ_i :



- Definition (Artin): For $n \geqslant 1$, the braid group B_n

$$= \{\ n\text{-strand braid diagrams}\ \}\ /\ \text{isotopy}$$

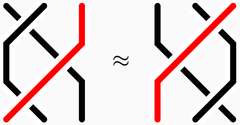
$$= \pi_1(\text{configuration space of } n \text{ points of } \mathbb{C} \text{ mod. action of } S_n)$$

$$= \mathcal{MCG}(D_n)$$

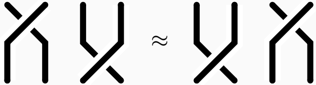
$\uparrow$ {homeomorphisms of an n-punctured disk that fix ∂D_n}/isotopy

$$= \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle.$$

σ_i :

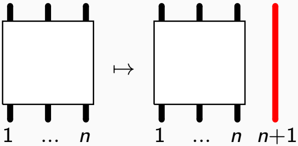


$\sigma_1\sigma_2\sigma_1$ $\approx$ $\sigma_2\sigma_1\sigma_2$

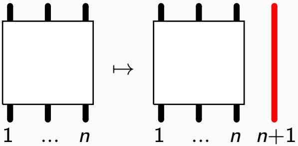


$\sigma_1\sigma_3$ $\approx$ $\sigma_3\sigma_1$

- Embedding B_n into B_{n+1}: add a trivial $(n+1)$st strand

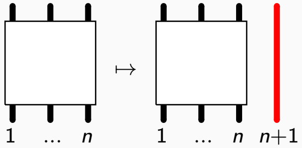


- Embedding B_n into B_{n+1}: add a trivial $(n+1)$st strand



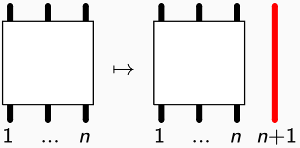
- Then $B_\infty := \varinjlim B_n$

- Embedding B_n into B_{n+1}: add a trivial $(n+1)$st strand



- Then $B_\infty := \varinjlim B_n = \left\langle \sigma_1, \sigma_2, \ldots \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle.$

- Embedding B_n into B_{n+1}: add a trivial $(n+1)$st strand



- Then $B_\infty := \varinjlim B_n = \left\langle \sigma_1, \sigma_2, \dots \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle.$

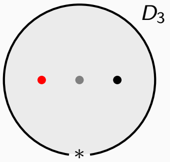
- Equivalently: Identify B_n with a subgroup of B_{n+1}, and put

$$B_\infty = \bigcup_n B_n.$$

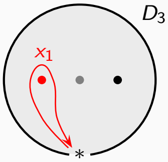
- Viewing B_n as a group of (isotopy classes of) homeomorphisms of D_n:

- Viewing B_n as a group of (isotopy classes of) homeomorphisms of D_n:
  - action of B_n on the fundamental group of D_n,

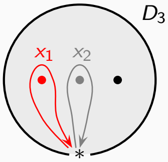
- Viewing B_n as a group of (isotopy classes of) homeomorphisms of D_n:
  - action of B_n on the fundamental group of D_n, a free group of rank n.



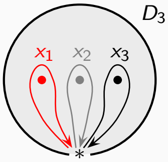
- Viewing B_n as a group of (isotopy classes of) homeomorphisms of D_n:
  - action of B_n on the fundamental group of D_n, a free group of rank n.



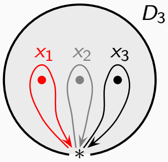
- Viewing B_n as a group of (isotopy classes of) homeomorphisms of D_n:
  - action of B_n on the fundamental group of D_n, a free group of rank n.

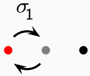


- Viewing B_n as a group of (isotopy classes of) homeomorphisms of D_n:
  - action of B_n on the fundamental group of D_n, a free group of rank n.

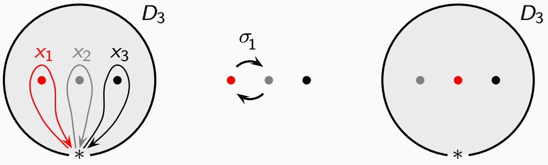


- Viewing B_n as a group of (isotopy classes of) homeomorphisms of D_n:
  - action of B_n on the fundamental group of D_n, a free group of rank n.

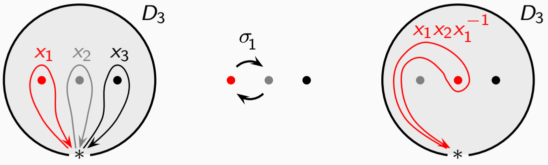




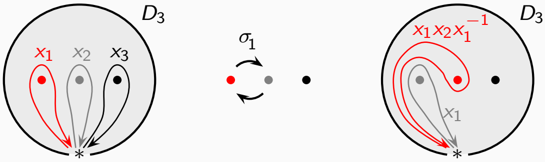
- Viewing B_n as a group of (isotopy classes of) homeomorphisms of D_n:
  - action of B_n on the fundamental group of D_n, a free group of rank n.



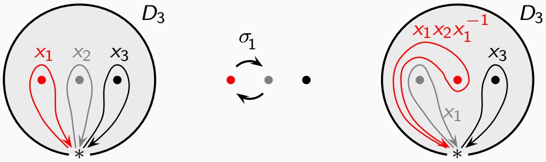
- Viewing B_n as a group of (isotopy classes of) homeomorphisms of D_n:
  - action of B_n on the fundamental group of D_n, a free group of rank n.



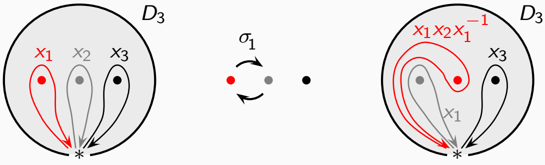
- Viewing B_n as a group of (isotopy classes of) homeomorphisms of D_n:
  - action of B_n on the fundamental group of D_n, a free group of rank n.



- Viewing B_n as a group of (isotopy classes of) homeomorphisms of D_n:
  - action of B_n on the fundamental group of D_n, a free group of rank n.

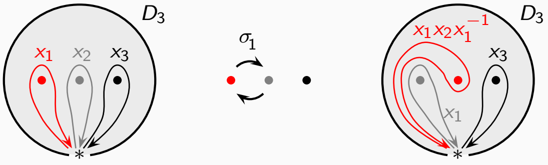


- Viewing B_n as a group of (isotopy classes of) homeomorphisms of D_n:
  - action of B_n on the fundamental group of D_n, a free group of rank n.



- From there: homomorphism ρ from B_n to $\mathrm{Aut}(F_n)$:

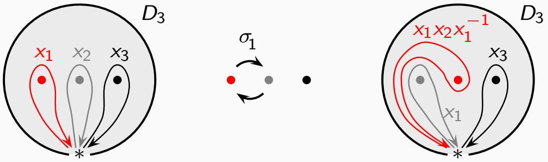
- Viewing B_n as a group of (isotopy classes of) homeomorphisms of D_n:
  - action of B_n on the fundamental group of D_n, a free group of rank n.



- From there: homomorphism ρ from B_n to $\mathrm{Aut}(F_n)$:

$$\rho(\sigma_i) : \begin{cases} x_i & \mapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} & \mapsto x_i, \\ x_k & \mapsto x_k \quad \text{for } k \neq i, i+1. \end{cases}$$

- Viewing B_n as a group of (isotopy classes of) homeomorphisms of D_n:
  - action of B_n on the fundamental group of D_n, a free group of rank n.



- From there: homomorphism ρ from B_n to $\mathrm{Aut}(F_n)$:

$$\rho(\sigma_i) : \begin{cases} x_i & \mapsto \; x_i x_{i+1} x_i^{-1}, \\ x_{i+1} & \mapsto \; x_i, \\ x_k & \mapsto \; x_k \quad \text{for } k \neq i, i+1. \end{cases}$$

- Theorem (Artin): *The homomorphism ρ is injective.*

Plan:

- 1. Artin's braid group B_∞
- 2. Thompson's group F
- 3. The parenthesized braid group $B_\bullet$
- 4. The Artin representation of $B_\bullet$

- Definition (Richard Thompson, 1965):

$$F := \langle a_0, a_1, \dots \mid a_j a_i = a_i a_{j+1} \text{ for } j > i \rangle. \qquad (*)$$

- Definition (Richard Thompson, 1965):

$$F := \langle a_0, a_1, \dots \mid a_j a_i = a_i a_{j+1} \text{ for } j > i \rangle. \qquad (*)$$

  - occurs in the construction of a f.p. group with unsolvable word problem

- Definition (Richard Thompson, 1965):

$$F := \langle a_0, a_1, \ldots \mid a_j a_i = a_i a_{j+1} \text{ for } j > i \rangle. \qquad (*)$$

  - occurs in the construction of a f.p. group with unsolvable word problem

- Fact: *The group F is a group of right fractions for the monoid F^+.*

  $\uparrow$ the monoid presented by $(*)$

- Definition (Richard Thompson, 1965):

$$F := \langle a_0, a_1, \ldots \mid a_j a_i = a_i a_{j+1} \text{ for } j > i \rangle. \qquad (*)$$

  - occurs in the construction of a f.p. group with unsolvable word problem

- Fact: *The group F is a group of right fractions for the monoid F^+.*

  ↑ the monoid presented by $(*)$

- Fact: *Every element of F has a unique expression of the form*

$$a_0^{p_0} a_1^{p_1} \cdots a_n^{p_n} a_n^{-q_n} \cdots a_1^{-q_1} a_0^{-q_0}$$

such that $((p_k \neq 0$ and $q_k \neq 0)$ implies $(p_{k+1} \neq 0)$ or $(q_{k+1} \neq 0))$.

- Definition (Richard Thompson, 1965):

$$F := \langle a_0, a_1, \dots \mid a_j a_i = a_i a_{j+1} \text{ for } j > i \rangle. \qquad (*)$$

  - occurs in the construction of a f.p. group with unsolvable word problem

- Fact: *The group F is a group of right fractions for the monoid F^+.*

  ↑ the monoid presented by $(*)$

- Fact: *Every element of F has a unique expression of the form*

$$a_0^{p_0} a_1^{p_1} \cdots a_n^{p_n} a_n^{-q_n} \cdots a_1^{-q_1} a_0^{-q_0}$$

such that $((p_k \neq 0$ and $q_k \neq 0)$ implies $(p_{k+1} \neq 0)$ or $(q_{k+1} \neq 0))$.

- Fact: *The group F is finitely presented:*

- Definition (Richard Thompson, 1965):

$$F := \langle a_0, a_1, \ldots \mid a_j a_i = a_i a_{j+1} \text{ for } j > i \rangle. \qquad (*)$$

  - occurs in the construction of a f.p. group with unsolvable word problem

- Fact: *The group F is a group of right fractions for the monoid F^+.*

  ↑ the monoid presented by $(*)$

- Fact: *Every element of F has a unique expression of the form*

$$a_0^{p_0} a_1^{p_1} \cdots a_n^{p_n} a_n^{-q_n} \cdots a_1^{-q_1} a_0^{-q_0}$$

such that $((p_k \neq 0$ and $q_k \neq 0)$ implies $(p_{k+1} \neq 0)$ or $(q_{k+1} \neq 0))$.

- Fact: *The group F is finitely presented:*
  - *generated by a_0 and a_1, since $a_n = a_1^{a_0^n}$ for $n \geqslant 2$;*

- Definition (Richard Thompson, 1965):

$$F := \langle a_0, a_1, \dots \mid a_j a_i = a_i a_{j+1} \text{ for } j > i \rangle. \qquad (*)$$

  - occurs in the construction of a f.p. group with unsolvable word problem

- Fact: *The group F is a group of right fractions for the monoid F^+.*
  ↑ the monoid presented by $(*)$

- Fact: *Every element of F has a unique expression of the form*

$$a_0^{p_0} a_1^{p_1} \cdots a_n^{p_n} a_n^{-q_n} \cdots a_1^{-q_1} a_0^{-q_0}$$

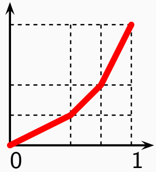
such that $((p_k \neq 0$ and $q_k \neq 0)$ implies $(p_{k+1} \neq 0)$ or $(q_{k+1} \neq 0))$.

- Fact: *The group F is finitely presented:*
  - *generated by a_0 and a_1, since $a_n = a_1^{a_0^n}$ for $n \geqslant 2$;*
  - *relations: "$a_2^{a_1} = a_3$" and "$a_3^{a_1} = a_4$", that is, $a_1^{a_0 a_1} = a_1^{a_0^2}$ and $a_1^{a_0^2 a_1} = a_1^{a_0^3}$.*

- $F \simeq \{$ piecewise linear orientation preserving homeomorphisms of $[0, 1]$ with discontinuities of the derivative and slopes of the form 2^k $\}$.

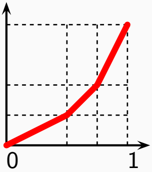
- $F \simeq \{$ piecewise linear orientation preserving homeomorphisms of $[0,1]$ with discontinuities of the derivative and slopes of the form 2^k $\}$.

$a_0 \mapsto$

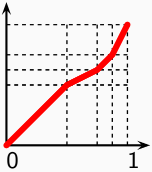


- $F \simeq \{$ piecewise linear orientation preserving homeomorphisms of $[0, 1]$ with discontinuities of the derivative and slopes of the form 2^k $\}$.

$a_0 \mapsto$

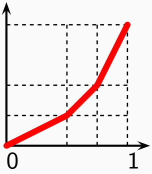


$a_1 \mapsto$

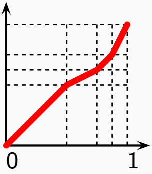


- $F \simeq \{$ piecewise linear orientation preserving homeomorphisms of $[0,1]$ with discontinuities of the derivative and slopes of the form 2^k $\}$.

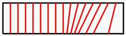
$a_0 \mapsto$



$a_1 \mapsto$

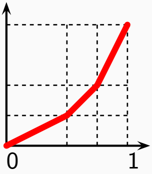


also represented as

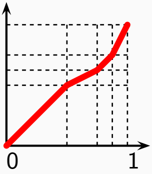


- $F \simeq \{$ piecewise linear orientation preserving homeomorphisms of $[0,1]$ with discontinuities of the derivative and slopes of the form 2^k $\}$.

$a_0 \mapsto$



$a_1 \mapsto$

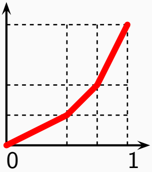


also represented as

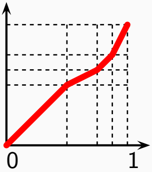
- An element of F = a pair of dyadic decompositions of $[0,1]$:

- $F \simeq \{$ piecewise linear orientation preserving homeomorphisms of $[0,1]$ with discontinuities of the derivative and slopes of the form 2^k $\}$.

$a_0 \mapsto$



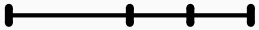
$a_1 \mapsto$



also represented as

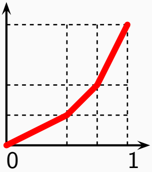
- An element of F = a pair of dyadic decompositions of $[0,1]$:

$a_0 \mapsto$

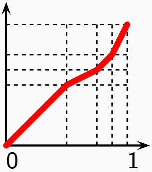


- $F \simeq \{$ piecewise linear orientation preserving homeomorphisms of $[0,1]$ with discontinuities of the derivative and slopes of the form 2^k $\}$.

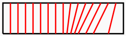
$a_0 \mapsto$



$a_1 \mapsto$

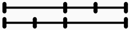


also represented as



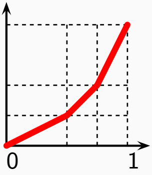
- An element of F = a pair of dyadic decompositions of $[0,1]$:

$a_0 \mapsto$

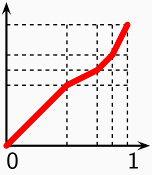


- $F \simeq \{$ piecewise linear orientation preserving homeomorphisms of $[0,1]$ with discontinuities of the derivative and slopes of the form 2^k $\}$.

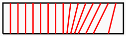
$a_0 \mapsto$



$a_1 \mapsto$

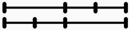


also represented as



- An element of F = a pair of dyadic decompositions of $[0,1]$:

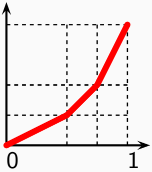
$a_0 \mapsto$



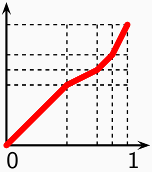
$a_1 \mapsto$

- $F \simeq \{$ piecewise linear orientation preserving homeomorphisms of $[0,1]$ with discontinuities of the derivative and slopes of the form 2^k $\}$.

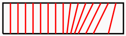
$a_0 \mapsto$



$a_1 \mapsto$

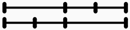


also represented as

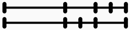


- An element of F = a pair of dyadic decompositions of $[0,1]$:

$a_0 \mapsto$

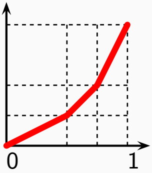


$a_1 \mapsto$

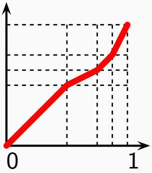


- $F \simeq \{$ piecewise linear orientation preserving homeomorphisms of $[0,1]$ with discontinuities of the derivative and slopes of the form 2^k $\}$.

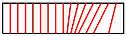
$a_0 \mapsto$



$a_1 \mapsto$



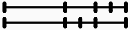
also represented as



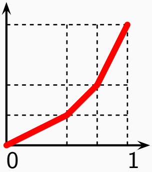
- An element of F = a pair of dyadic decompositions of $[0,1]$:
  = a pair of finite binary rooted trees:

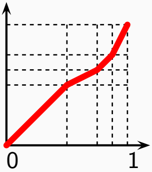
$a_0 \mapsto$

$a_1 \mapsto$

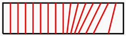


- $F \simeq \{$ piecewise linear orientation preserving homeomorphisms of $[0,1]$ with discontinuities of the derivative and slopes of the form 2^k $\}$.

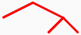
$a_0 \mapsto$ 

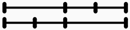
$a_1 \mapsto$ 

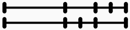
also represented as



- An element of F = a pair of dyadic decompositions of $[0,1]$:
  = a pair of finite binary rooted trees:

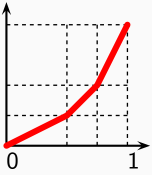


$a_0 \mapsto$ 

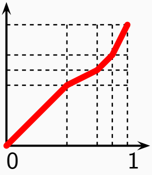
$a_1 \mapsto$ 

- $F \simeq \{$ piecewise linear orientation preserving homeomorphisms of $[0,1]$ with discontinuities of the derivative and slopes of the form 2^k $\}$.

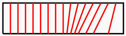
$a_0 \mapsto$



$a_1 \mapsto$

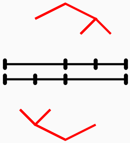


also represented as

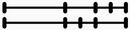


- An element of F = a pair of dyadic decompositions of $[0,1]$:
  = a pair of finite binary rooted trees:

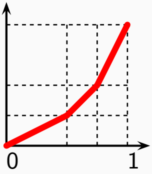
$a_0 \mapsto$

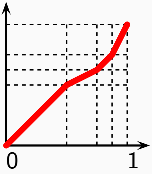


$a_1 \mapsto$

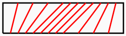


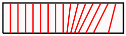
- $F \simeq \{$ piecewise linear orientation preserving homeomorphisms of $[0,1]$ with discontinuities of the derivative and slopes of the form 2^k $\}$.

$a_0 \mapsto$ 

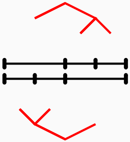
$a_1 \mapsto$ 

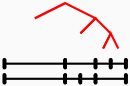
also represented as





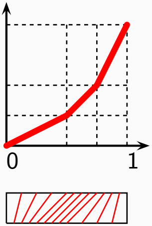
- An element of F = a pair of dyadic decompositions of $[0,1]$:
  = a pair of finite binary rooted trees:

$a_0 \mapsto$ 

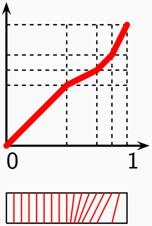
$a_1 \mapsto$ 

- $F \simeq \{$ piecewise linear orientation preserving homeomorphisms of $[0,1]$ with discontinuities of the derivative and slopes of the form 2^k $\}$.

$a_0 \mapsto$



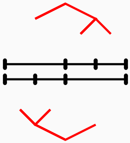
$a_1 \mapsto$



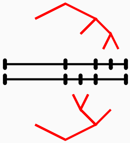
also represented as

- An element of F = a pair of dyadic decompositions of $[0,1]$:
  = a pair of finite binary rooted trees:

$a_0 \mapsto$



$a_1 \mapsto$



- Fact: *The center of F is trivial.*
  - Point: every homeomorphism commuting with x_1 fixes $1/2$.

- Fact: *The center of F is trivial.*
  - Point: every homeomorphism commuting with x_1 fixes $1/2$.

- Fact: *Commutators in F correspond to homeomorphisms with slope* 1 *near* 0 *and* 1.

- Fact: *The center of F is trivial.*
  - Point: every homeomorphism commuting with x_1 fixes $1/2$.

- Fact: *Commutators in F correspond to homeomorphisms with slope 1 near 0 and 1.*
  - Hence $F/[F,F] \simeq \mathbb{Z} \oplus \mathbb{Z}$.

- Fact: *The center of F is trivial.*
  - Point: every homeomorphism commuting with x_1 fixes $1/2$.

- Fact: *Commutators in F correspond to homeomorphisms with slope 1 near 0 and 1.*
  - Hence $F/[F,F] \simeq \mathbb{Z} \oplus \mathbb{Z}$.

- Proposition: *Every proper quotient of F is abelian.*
  - Point: Every normal subgroup contains all commutators.

- Fact: *The center of F is trivial.*
  - Point: every homeomorphism commuting with x_1 fixes $1/2$.

- Fact: *Commutators in F correspond to homeomorphisms with slope 1 near 0 and 1.*
  - Hence $F/[F,F] \simeq \mathbb{Z} \oplus \mathbb{Z}$.

- Proposition: *Every proper quotient of F is abelian.*
  - Point: Every normal subgroup contains all commutators.

- Theorem: *The subgroup $[F,F]$ is simple.*
  - Point: A normal subgroup of $[F,F]$ contains all commutators.

- Fact: *The center of F is trivial.*
  - Point: every homeomorphism commuting with x_1 fixes $1/2$.

- Fact: *Commutators in F correspond to homeomorphisms with slope 1 near 0 and 1.*
  - Hence $F/[F, F] \simeq \mathbb{Z} \oplus \mathbb{Z}$.

- Proposition: *Every proper quotient of F is abelian.*
  - Point: Every normal subgroup contains all commutators.

- Theorem: *The subgroup $[F, F]$ is simple.*
  - Point: A normal subgroup of $[F, F]$ contains all commutators.

- Theorem (Brin–Squier, 1985): *The group F includes no free subgroup of rank $\geqslant 2$.*

- Fact: *The center of F is trivial.*
  - Point: every homeomorphism commuting with x_1 fixes $1/2$.

- Fact: *Commutators in F correspond to homeomorphisms with slope 1 near 0 and 1.*
  - Hence $F/[F, F] \simeq \mathbb{Z} \oplus \mathbb{Z}$.

- Proposition: *Every proper quotient of F is abelian.*
  - Point: Every normal subgroup contains all commutators.

- Theorem: *The subgroup $[F, F]$ is simple.*
  - Point: A normal subgroup of $[F, F]$ contains all commutators.

- Theorem (Brin–Squier, 1985): *The group F includes no free subgroup of rank $\geqslant 2$.*
  - In fact: Every non-abelian subgroup of F includes a copy of $\mathbb{Z}^{\infty}$.

- Fact: *The center of F is trivial.*
  - Point: every homeomorphism commuting with x_1 fixes $1/2$.

- Fact: *Commutators in F correspond to homeomorphisms with slope 1 near 0 and 1.*
  - Hence $F/[F, F] \simeq \mathbb{Z} \oplus \mathbb{Z}$.

- Proposition: *Every proper quotient of F is abelian.*
  - Point: Every normal subgroup contains all commutators.

- Theorem: *The subgroup $[F, F]$ is simple.*
  - Point: A normal subgroup of $[F, F]$ contains all commutators.

- Theorem (Brin–Squier, 1985): *The group F includes no free subgroup of rank $\geqslant 2$.*
  - In fact: Every non-abelian subgroup of F includes a copy of $\mathbb{Z}^\infty$.
  - Compare with: F^+ includes a free monoid of rank 2.

- <u>Question 1</u> (Gersten): Is F automatic? (F is not word hyperbolic)

  ↑ ∃ finite state automaton computing a normal form for the elements

- Question 1 (Gersten): Is F automatic? (F is not word hyperbolic)

  ↑ $\exists$ finite state automaton computing a normal form for the elements

- Theorem (Guba 2005): *The Dehn function of F is quadratic.*

  ↑ $\Phi(n) := \sup\{\text{area}(w) \mid \text{length}(w) = n \text{ and } w \text{ represents } 1 \text{ in } F\}$

- <u>Question 1</u> (Gersten): Is F automatic? (F is not word hyperbolic)
  $\uparrow$
  $\exists$ finite state automaton computing a normal form for the elements

- <u>Theorem</u> (Guba 2005): *The Dehn function of F is quadratic.*
  $\uparrow$
  $\Phi(n) := \sup\{\text{area}(w) \mid \text{length}(w) = n \text{ and } w \text{ represents } 1 \text{ in } F\}$

- <u>Question 2</u> (Geoghegan): Is F amenable?
  $\uparrow$
  $\exists$ left-invariant $[0,1]$-measure on $\mathfrak{P}(F)$

- Question 1 (Gersten): Is F automatic? (F is not word hyperbolic)
  ↑
  ∃ finite state automaton computing a normal form for the elements

- Theorem (Guba 2005): *The Dehn function of F is quadratic.*
  ↑
  $\Phi(n) := \sup\{\text{area}(w) \mid \text{length}(w) = n \text{ and } w \text{ represents } 1 \text{ in } F\}$

- Question 2 (Geoghegan): Is F amenable?
  ↑
  ∃ left-invariant $[0,1]$-measure on $\mathfrak{P}(F)$

- Question 3: What is the growth rate of F^+ and F w.r.t. $\{a_0^{\pm 1}, a_1^{\pm 1}\}$?
  ↑
  $\sqrt[n]{a_n}$, with $a_n :=$ #{elements with length n expression}

- Question 1 (Gersten): Is F automatic? (F is not word hyperbolic)
  $\uparrow$
  $\exists$ finite state automaton computing a normal form for the elements

- Theorem (Guba 2005): *The Dehn function of F is quadratic.*
  $\uparrow$
  $\Phi(n) := \sup\{\text{area}(w) \mid \text{length}(w) = n \text{ and } w \text{ represents } 1 \text{ in } F\}$

- Question 2 (Geoghegan): Is F amenable?
  $\uparrow$
  $\exists$ left-invariant $[0,1]$-measure on $\mathfrak{P}(F)$

- Question 3: What is the growth rate of F^+ and F w.r.t. $\{a_0^{\pm 1}, a_1^{\pm 1}\}$?
  $\uparrow$
  $\sqrt[n]{a_n}$, with $a_n := \#\{\text{elements with length } n \text{ expression}\}$

- Theorem: - (i) (Burillo) *The growth rate of F^+ is* $\frac{1}{2\sin(\pi/14)} \approx 2.24\ldots$

- Question 1 (Gersten): Is F automatic? (F is not word hyperbolic)
  $\uparrow$
  $\exists$ finite state automaton computing a normal form for the elements

- Theorem (Guba 2005): *The Dehn function of F is quadratic.*
  $\uparrow$
  $\Phi(n) := \sup\{\text{area}(w) \mid \text{length}(w) = n \text{ and } w \text{ represents } 1 \text{ in } F\}$

- Question 2 (Geoghegan): Is F amenable?
  $\uparrow$
  $\exists$ left-invariant $[0,1]$-measure on $\mathfrak{P}(F)$

- Question 3: What is the growth rate of F^+ and F w.r.t. $\{a_0^{\pm 1}, a_1^{\pm 1}\}$?
  $\uparrow$
  $\sqrt[n]{a_n}$, with $a_n :=$ #{elements with length n expression}

- Theorem: - (i) (Burillo) *The growth rate of F^+ is* $\frac{1}{2\sin(\pi/14)} \approx 2.24\ldots$.
  - (ii) (Guba) *The growth rate of F lies between* $\frac{3+\sqrt{5}}{2} \approx 2.618\ldots$ *and* 3.

Plan:

- 1. Artin's braid group B_∞
- 2. Thompson's group F
- 3. The parenthesized braid group $B_\bullet$
- 4. The Artin representation of $B_\bullet$

- Ordinary braid diagrams:

- Ordinary braid diagrams:

← initial positions: • • • •

- Ordinary braid diagrams:

← initial positions: • • • •

← final positions: • • • •

- Ordinary braid diagrams:

← initial positions: • • • •

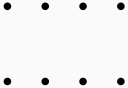
← final positions: • • • •

  - one elementary pattern: crossing σ_i:

- Ordinary braid diagrams:

← initial positions:

← final positions:



  - ▸ one elementary pattern: crossing σ_i:

- Parenthesized braid diagrams: (possibly) non-equidistant positions:

- Ordinary braid diagrams:

← initial positions: • • • •

← final positions: • • • •

  - ▶ one elementary pattern: crossing σ_i:

- Parenthesized braid diagrams: (possibly) non-equidistant positions:

- Ordinary braid diagrams:

← initial positions: • • • •

← final positions: • • • •

  - one elementary pattern: crossing σ_i:

- Parenthesized braid diagrams: (possibly) non-equidistant positions:

← initial positions: (••) •

- Ordinary braid diagrams:

← initial positions: • • • •

← final positions: • • • •

  - ▸ one elementary pattern: crossing σ_i:

- Parenthesized braid diagrams: (possibly) non-equidistant positions:

← initial positions: (••) •

← final positions: • (••)

- Ordinary braid diagrams:

← initial positions: • • • •

← final positions: • • • •

▸ one elementary pattern: crossing σ_i:

- Parenthesized braid diagrams: (possibly) non-equidistant positions:

← initial positions: (••) •

← final positions: • (••)

▸ two elementary patterns: crossing σ_i :

- Ordinary braid diagrams:

← initial positions: • • • •

← final positions: • • • •

  - ▶ one elementary pattern: crossing σ_i:

- Parenthesized braid diagrams: (possibly) non-equidistant positions:

← initial positions: (••) •

← final positions: • (••)

  - ▶ two elementary patterns: crossing σ_i :

grouping a_i :

- Ordinary braid diagrams:

← initial positions: • • • •

← final positions: • • • •

▶ one elementary pattern: crossing σ_i:

- Parenthesized braid diagrams: (possibly) non-equidistant positions:

← initial positions: (••) •

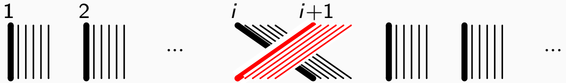
← final positions: • (••)

▶ two elementary patterns: crossing σ_i :

grouping a_i :

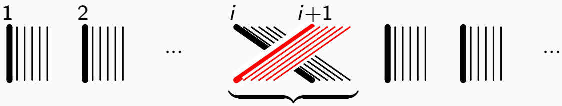
(and their inverses)

- More precisely:

$\sigma_i \;\rightarrow$ 

- More precisely:

$\sigma_i \rightarrow$

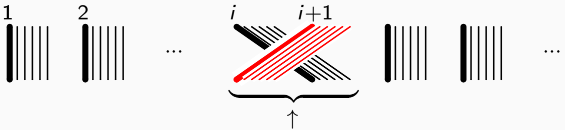


$\uparrow$

everything in $[i+1, i+2)$ crosses over everything in $[i, i+1)$

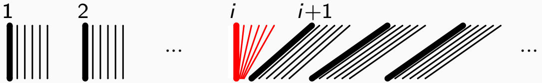
- More precisely:

$\sigma_i \rightarrow$



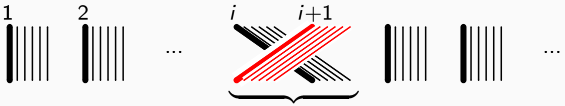
everything in $[i+1, i+2)$ crosses
over everything in $[i, i+1)$

$a_i \rightarrow$



- More precisely:

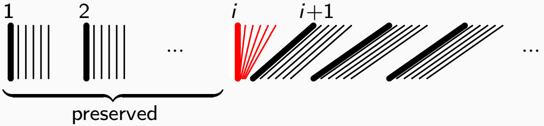
$\sigma_i \quad \rightarrow$



$\uparrow$

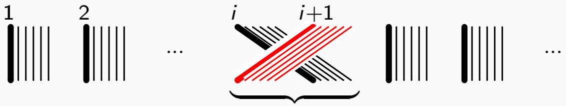
everything in $[i+1, i+2)$ crosses
over everything in $[i, i+1)$

$a_i \quad \rightarrow$



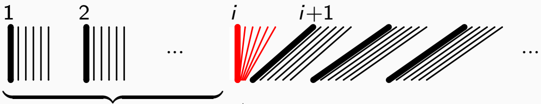
- More precisely:

$\sigma_i \rightarrow$



↑
everything in $[i+1, i+2)$ crosses
over everything in $[i, i+1)$

$a_i \rightarrow$

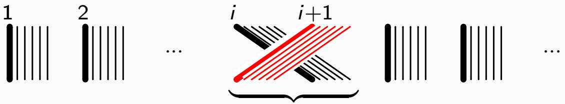


preserved

↑
everything in $[i, i+1)$
is shrinked by an ε-factor

- More precisely:

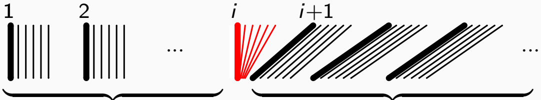
$\sigma_i \;\rightarrow$



$\uparrow$

everything in $[i+1, i+2)$ crosses
over everything in $[i, i+1)$

$a_i \;\rightarrow$

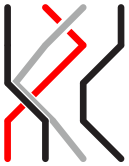


preserved $\uparrow$ translated

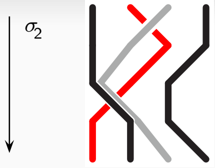
everything in $[i, i+1)$
is shrinked by an ε-factor

- A typical parenthesized braid diagram:

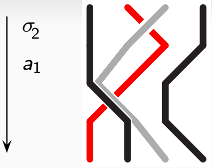
- A typical parenthesized braid diagram:



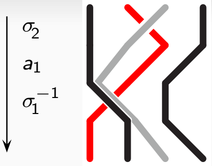
- A typical parenthesized braid diagram:



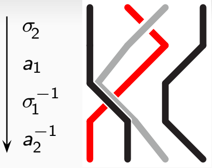
- A typical parenthesized braid diagram:



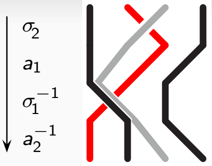
- A typical parenthesized braid diagram:



- A typical parenthesized braid diagram:

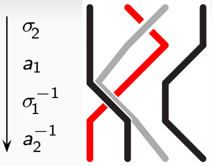


- A typical parenthesized braid diagram:



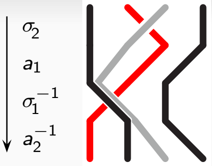
- Connection with binary trees: positions correspond to nodes in a binary tree;

- A typical parenthesized braid diagram:



- Connection with binary trees: positions correspond to nodes in a binary tree;
  - Enumerated starting from the root and descending the right branch.

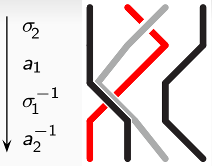
- A typical parenthesized braid diagram:



- Connection with binary trees: positions correspond to nodes in a binary tree;
  - Enumerated starting from the root and descending the right branch.

σ_i : ...  ...

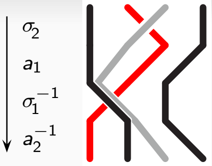
- A typical parenthesized braid diagram:



- Connection with binary trees: positions correspond to nodes in a binary tree;
  - Enumerated starting from the root and descending the right branch.

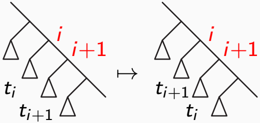
σ_i : ...  ...

- A typical parenthesized braid diagram:

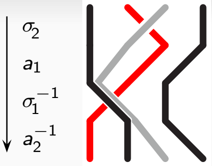


- Connection with binary trees: positions correspond to nodes in a binary tree;
  - Enumerated starting from the root and descending the right branch.

σ_i : ... 

... corresponds to 

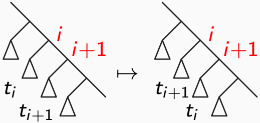
- A typical parenthesized braid diagram:



- Connection with binary trees: positions correspond to nodes in a binary tree;
  - Enumerated starting from the root and descending the right branch.

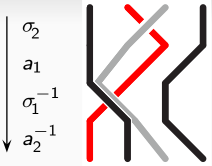
σ_i : ... 

... corresponds to



σ_i = switching subtrees

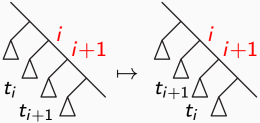
- A typical parenthesized braid diagram:



- Connection with binary trees: positions correspond to nodes in a binary tree;
  - Enumerated starting from the root and descending the right branch.

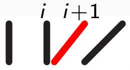
σ_i : ...

... corresponds to



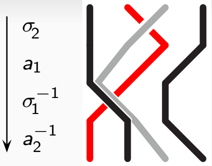
σ_i = switching subtrees

a_i : ...



...

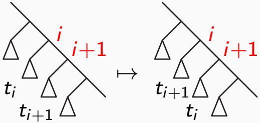
- A typical parenthesized braid diagram:



- Connection with binary trees: positions correspond to nodes in a binary tree;
  - Enumerated starting from the root and descending the right branch.

σ_i : ...

... corresponds to

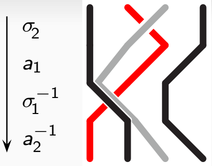


σ_i = switching subtrees

a_i : ...

...

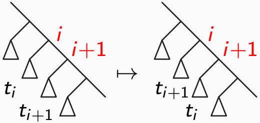
- A typical parenthesized braid diagram:



- Connection with binary trees: positions correspond to nodes in a binary tree;
  - ▶ Enumerated starting from the root and descending the right branch.

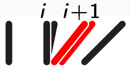
σ_i : ...

... corresponds to

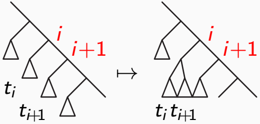


σ_i = switching subtrees

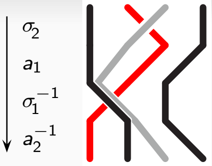
a_i : ...



... corresponds to



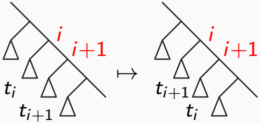
- A typical parenthesized braid diagram:



- Connection with binary trees: positions correspond to nodes in a binary tree;
  - Enumerated starting from the root and descending the right branch.

σ_i : ...

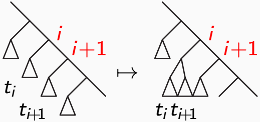
... corresponds to



σ_i = switching subtrees

a_i : ...

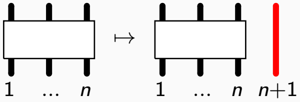
... corresponds to



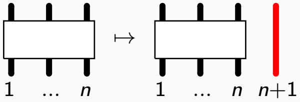
a_i = applying associativity

- For ordinary braid diagrams, only one completion:

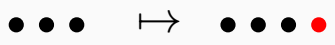
- For ordinary braid diagrams, only one completion:



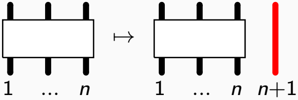
- For ordinary braid diagrams, only one completion:



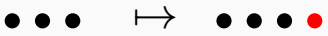
corresponding to



- For ordinary braid diagrams, only one completion:

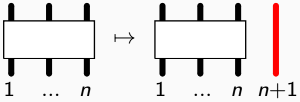


corresponding to

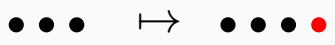


- For parenthesized braid diagrams, several completions:

- For ordinary braid diagrams, only one completion:



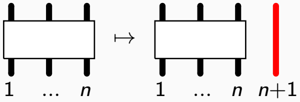
corresponding to



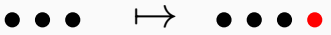
- For parenthesized braid diagrams, several completions:
  - index positions by sequences of integers

    (or, equivalently, infinitesimals)

- For ordinary braid diagrams, only one completion:

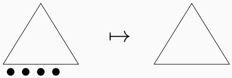


corresponding to

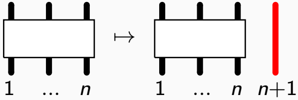


- For parenthesized braid diagrams, several completions:
  - index positions by sequences of integers

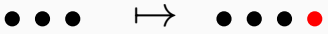
(or, equivalently, infinitesimals)



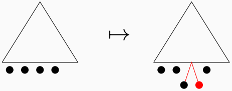
- For ordinary braid diagrams, only one completion:



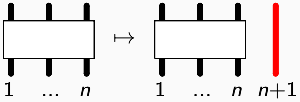
corresponding to



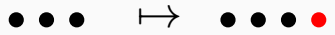
- For parenthesized braid diagrams, several completions:
  - ▸ index positions by sequences of integers (or, equivalently, infinitesimals)



- For ordinary braid diagrams, only one completion:

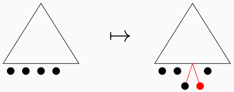


corresponding to

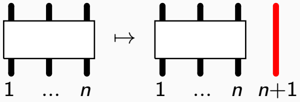


- For parenthesized braid diagrams, several completions:
  - index positions by sequences of integers

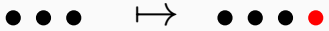
    (or, equivalently, infinitesimals)



- For ordinary braid diagrams, only one completion:

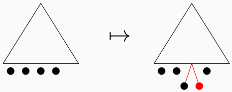


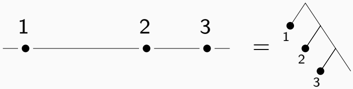
corresponding to



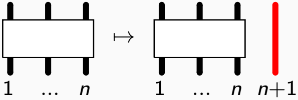
- For parenthesized braid diagrams, several completions:
  - index positions by sequences of integers

    (or, equivalently, infinitesimals)

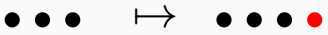




- For ordinary braid diagrams, only one completion:

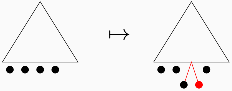


corresponding to

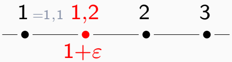


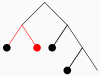
- For parenthesized braid diagrams, several completions:
  - index positions by sequences of integers

    (or, equivalently, infinitesimals)

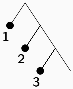


complete at 1:

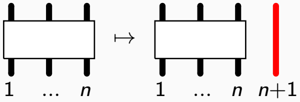


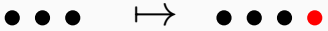


=

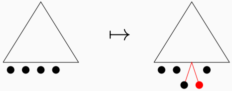


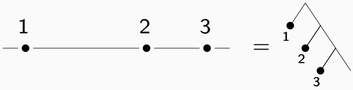
- For ordinary braid diagrams, only one completion:

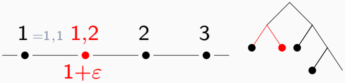


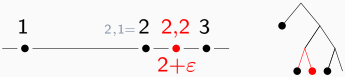
corresponding to 

- For parenthesized braid diagrams, several completions:
  - index positions by sequences of integers
    (or, equivalently, infinitesimals)

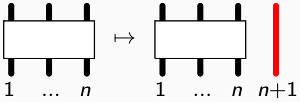


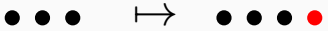


complete at 1: 

complete at 2: 

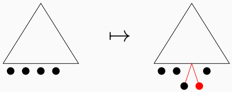
- For ordinary braid diagrams, only one completion:

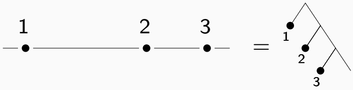


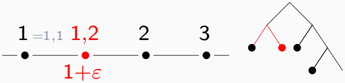
corresponding to 

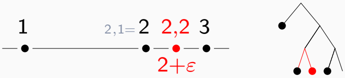
- For parenthesized braid diagrams, several completions:
  - index positions by sequences of integers

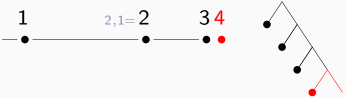
    (or, equivalently, infinitesimals)



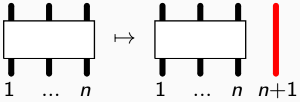


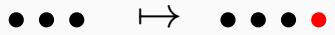
complete at 1: 

complete at 2: 

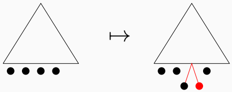
complete at 3: 

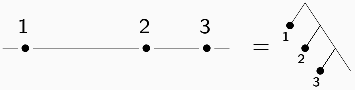
- For ordinary braid diagrams, only one completion:

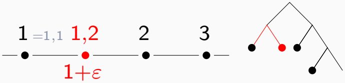


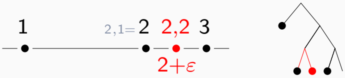
corresponding to 

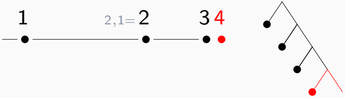
- For parenthesized braid diagrams, several completions:
  - index positions by sequences of integers
    (or, equivalently, infinitesimals)





complete at 1: 

complete at 2: 

complete at 3: 

- Parenthesized braid diagrams form a groupoid (small category with inverses):

$$\mathcal{B} := \{\text{ parenthesized braid diagrams }\}/\text{isotopy.}$$

- Parenthesized braid diagrams form a groupoid (small category with inverses):

$$\mathcal{B} := \{\, \text{parenthesized braid diagrams} \,\}/\text{isotopy}.$$

  - Two isotopy classes D, D' can be composed when the final positions of D coincide with the initial positions of D'.

- Parenthesized braid diagrams form a groupoid (small category with inverses):

$$\mathcal{B} := \{ \text{parenthesized braid diagrams} \}/\text{isotopy}.$$

  - Two isotopy classes D, D' can be composed when the final positions of D coincide with the initial positions of D'.

- To make a group:

- Parenthesized braid diagrams form a groupoid (small category with inverses):

$$\mathcal{B} := \{\text{ parenthesized braid diagrams }\}/\text{isotopy}.$$

  - Two isotopy classes D, D' can be composed when the final positions of D coincide with the initial positions of D'.

- To make a group:
  - Going from $\coprod B_n$ (groupoid) to B_∞: embed B_n into $B_{n'}$ for $n \leqslant n'$.

- Parenthesized braid diagrams form a groupoid (small category with inverses):

$$\mathcal{B} := \{\, \text{parenthesized braid diagrams} \,\}/\text{isotopy}.$$

  - Two isotopy classes D, D' can be composed when
    the final positions of D coincide with the initial positions of D'.

- To make a group:
  - Going from $\coprod B_n$ (groupoid) to B_∞: embed B_n into $B_{n'}$ for $n \leqslant n'$.
  - Going from $\coprod B_t$ (groupoid) to a group: embed B_t into $B_{t'}$ for $t \subseteq t'$.
    $\uparrow$
    the family of (isotopy classes) of diagrams with initial positions t (a binary tree)

- Parenthesized braid diagrams form a groupoid (small category with inverses):

$$\mathcal{B} := \{\,\text{parenthesized braid diagrams}\,\}/\text{isotopy}.$$

  - Two isotopy classes D, D' can be composed when the final positions of D coincide with the initial positions of D'.

- To make a group:
  - Going from $\coprod B_n$ (groupoid) to B_∞: embed B_n into $B_{n'}$ for $n \leqslant n'$.
  - Going from $\coprod B_t$ (groupoid) to a group: embed B_t into $B_{t'}$ for $t \subseteq t'$.

  ↑ the family of (isotopy classes) of diagrams with initial positions t (a binary tree)

- Definition: The group $B_\bullet$ of parenthesized braids is

$$\varinjlim \{\,\text{parenthesized braid diagrams}\,\}/\text{isotopy}.$$

- Proposition: *A presentation of $B_\bullet$ in terms of the generators a_i and σ_i is*

- Proposition: *A presentation of $B_\bullet$ in terms of the generators a_i and σ_i is*

  for $x = \sigma$ or a: $\quad \sigma_i x_j = x_j \sigma_i$

- Proposition: *A presentation of $B_\bullet$ in terms of the generators a_i and σ_i is*

  for $x = \sigma$ or a: $\sigma_i x_j = x_j \sigma_i$ *and* $a_i x_{j-1} = x_j a_i$

- Proposition: *A presentation of $B_\bullet$ in terms of the generators a_i and σ_i is*

$$\text{for } x = \sigma \text{ or } a: \quad \sigma_i x_j = x_j \sigma_i \quad \text{and} \quad a_i x_{j-1} = x_j a_i \quad \text{for } j \geqslant i+2,$$

- Proposition: *A presentation of $B_\bullet$ in terms of the generators a_i and σ_i is*

$$\text{for } x = \sigma \text{ or } a: \quad \sigma_i x_j = x_j \sigma_i \quad \text{and} \quad a_i x_{j-1} = x_j a_i \quad \text{for } j \geqslant i+2,$$

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j,$$

- Proposition: *A presentation of $B_\bullet$ in terms of the generators a_i and σ_i is*

$$\text{for } x = \sigma \text{ or } a: \quad \sigma_i x_j = x_j \sigma_i \quad \text{and} \quad a_i x_{j-1} = x_j a_i \quad \text{for } j \geqslant i+2,$$

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, \quad \sigma_i \sigma_j a_i = a_j \sigma_i,$$

- Proposition: *A presentation of $B_\bullet$ in terms of the generators a_i and σ_i is*

$$\text{for } x = \sigma \text{ or } a: \quad \sigma_i x_j = x_j \sigma_i \quad \text{and} \quad a_i x_{j-1} = x_j a_i \quad \text{for } j \geqslant i+2,$$

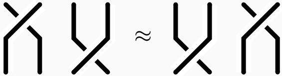
$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, \quad \sigma_i \sigma_j a_i = a_j \sigma_i, \quad \sigma_j \sigma_i a_j = a_i \sigma_i \quad \text{for } j = i+1.$$

• Proposition: *A presentation of $B_\bullet$ in terms of the generators a_i and σ_i is*

$$\text{for } x = \sigma \text{ or } a: \quad \sigma_i x_j = x_j \sigma_i \quad \text{and} \quad a_i x_{j-1} = x_j a_i \quad \text{for } j \geqslant i+2,$$

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, \quad \sigma_i \sigma_j a_i = a_j \sigma_i, \quad \sigma_j \sigma_i a_j = a_i \sigma_i \quad \text{for } j = i+1.$$

▶ commutation relations:



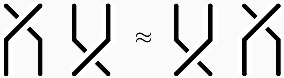
$\sigma_i \sigma_j = \sigma_j \sigma_i$

- Proposition: *A presentation of $B_\bullet$ in terms of the generators a_i and σ_i is*

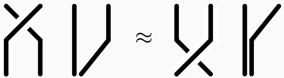
$$\text{for } x = \sigma \text{ or } a: \quad \sigma_i x_j = x_j \sigma_i \quad \text{and} \quad a_i x_{j-1} = x_j a_i \quad \text{for } j \geqslant i+2,$$

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, \quad \sigma_i \sigma_j a_i = a_j \sigma_i, \quad \sigma_j \sigma_i a_j = a_i \sigma_i \quad \text{for } j = i+1.$$

- commutation relations:



$\sigma_i \sigma_j = \sigma_j \sigma_i$



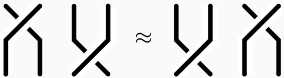
$\sigma_i a_j = a_j \sigma_i$

- <u>Proposition</u>: *A presentation of $B_\bullet$ in terms of the generators a_i and σ_i is*

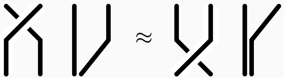
$$\text{for } x = \sigma \text{ or } a: \quad \sigma_i x_j = x_j \sigma_i \quad \text{and} \quad a_i x_{j-1} = x_j a_i \quad \text{for } j \geqslant i+2,$$

$$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, \quad \sigma_i\sigma_j a_i = a_j\sigma_i, \quad \sigma_j\sigma_i a_j = a_i\sigma_i \quad \text{for } j = i+1.$$

▶ commutation relations:



$\sigma_i\sigma_j = \sigma_j\sigma_i$



$\sigma_i a_j = a_j\sigma_i$

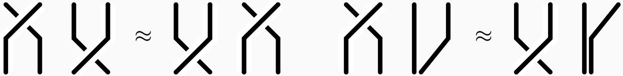
for $j \geqslant i+2$

- <u>Proposition</u>: *A presentation of $B_\bullet$ in terms of the generators a_i and σ_i is*

$$\text{for } x = \sigma \text{ or } a: \quad \sigma_i x_j = x_j \sigma_i \quad \text{and} \quad a_i x_{j-1} = x_j a_i \quad \text{for } j \geqslant i+2,$$

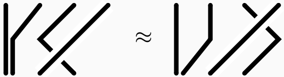
$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, \quad \sigma_i \sigma_j a_i = a_j \sigma_i, \quad \sigma_j \sigma_i a_j = a_i \sigma_i \quad \text{for } j = i+1.$$

▶ commutation relations:



$\sigma_i \sigma_j = \sigma_j \sigma_i$ $\qquad$ $\sigma_i a_j = a_j \sigma_i$ $\qquad$ for $j \geqslant i+2$

▶ semi-commutation relations ("Thompson" relations):



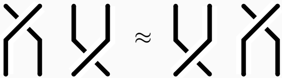
$a_i \sigma_{j-1} = \sigma_j a_i$

- Proposition: *A presentation of $B_\bullet$ in terms of the generators a_i and σ_i is*

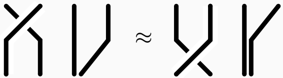
$$\text{for } x = \sigma \text{ or } a: \quad \sigma_i x_j = x_j \sigma_i \quad \text{and} \quad a_i x_{j-1} = x_j a_i \quad \text{for } j \geqslant i+2,$$

$$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, \quad \sigma_i\sigma_j a_i = a_j\sigma_i, \quad \sigma_j\sigma_i a_j = a_i\sigma_j \quad \text{for } j = i+1.$$

- commutation relations:



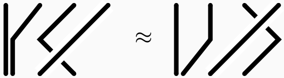
$\sigma_i\sigma_j = \sigma_j\sigma_i$



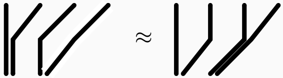
$\sigma_i a_j = a_j\sigma_i$

for $j \geqslant i+2$

- semi-commutation relations ("Thompson" relations):



$a_i\sigma_{j-1} = \sigma_j a_i$



$a_i a_{j-1} = a_j a_i$

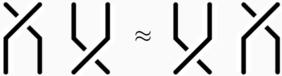
for $j \geqslant i+2$

- Proposition: *A presentation of $B_\bullet$ in terms of the generators a_i and σ_i is*

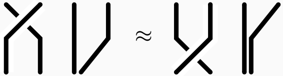
$$\text{for } x = \sigma \text{ or } a: \quad \sigma_i x_j = x_j \sigma_i \quad \text{and} \quad a_i x_{j-1} = x_j a_i \quad \text{for } j \geqslant i+2,$$

$$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, \quad \sigma_i\sigma_j a_i = a_j\sigma_i, \quad \sigma_j\sigma_i a_j = a_i\sigma_j \quad \text{for } j = i+1.$$

- commutation relations:



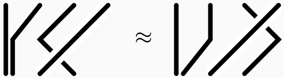
$\sigma_i\sigma_j = \sigma_j\sigma_i$



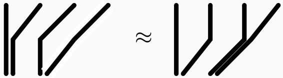
$\sigma_i a_j = a_j\sigma_i$

for $j \geqslant i+2$

- semi-commutation relations ("Thompson" relations):



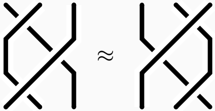
$a_i\sigma_{j-1} = \sigma_j a_i$



$a_i a_{j-1} = a_j a_i$

for $j \geqslant i+2$

- braid relations:



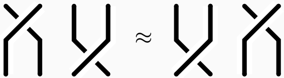
$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

- Proposition: *A presentation of $B_\bullet$ in terms of the generators a_i and σ_i is*

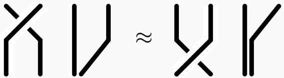
$$\text{for } x = \sigma \text{ or } a: \quad \sigma_i x_j = x_j \sigma_i \quad \text{and} \quad a_i x_{j-1} = x_j a_i \quad \text{for } j \geqslant i+2,$$

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, \quad \sigma_i \sigma_j a_i = a_j \sigma_i, \quad \sigma_j \sigma_i a_j = a_i \sigma_j \quad \text{for } j = i+1.$$

- commutation relations:



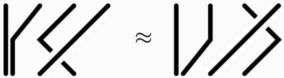
$\sigma_i \sigma_j = \sigma_j \sigma_i$



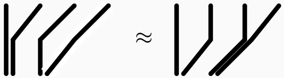
$\sigma_i a_j = a_j \sigma_i$

for $j \geqslant i+2$

- semi-commutation relations ("Thompson" relations):



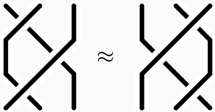
$a_i \sigma_{j-1} = \sigma_j a_i$



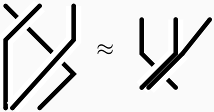
$a_i a_{j-1} = a_j a_i$

for $j \geqslant i+2$

- braid relations:



$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$



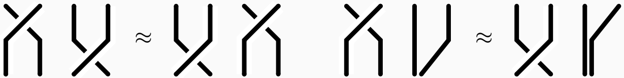
$\sigma_i \sigma_{i+1} a_i = a_{i+1} \sigma_i$

- <u>Proposition</u>: *A presentation of $B_\bullet$ in terms of the generators a_i and σ_i is*

$$\text{for } x = \sigma \text{ or } a: \quad \sigma_i x_j = x_j \sigma_i \quad \text{and} \quad a_i x_{j-1} = x_j a_i \quad \text{for } j \geqslant i+2,$$

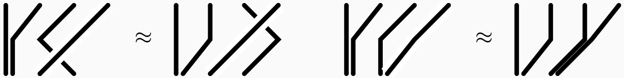
$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, \quad \sigma_i \sigma_j a_i = a_j \sigma_i, \quad \sigma_j \sigma_i a_j = a_i \sigma_i \quad \text{for } j = i+1.$$

- commutation relations:



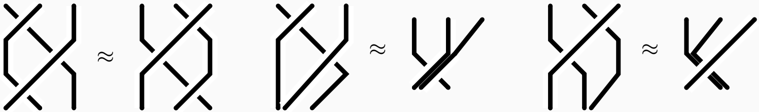
$\sigma_i \sigma_j = \sigma_j \sigma_i$ $\qquad$ $\sigma_i a_j = a_j \sigma_i$ $\qquad$ for $j \geqslant i+2$

- semi-commutation relations ("Thompson" relations):



$a_i \sigma_{j-1} = \sigma_j a_i$ $\qquad$ $a_i a_{j-1} = a_j a_i$ $\qquad$ for $j \geqslant i+2$

- braid relations:



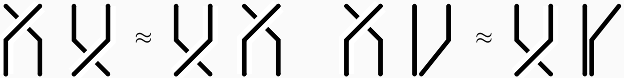
$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ $\qquad$ $\sigma_i \sigma_{i+1} a_i = a_{i+1} \sigma_i$ $\qquad$ $\sigma_{i+1} \sigma_i a_i = a_i \sigma_i$

- Proposition: *A presentation of $B_\bullet$ in terms of the generators a_i and σ_i is*

$$\text{for } x = \sigma \text{ or } a: \quad \sigma_i x_j = x_j \sigma_i \quad \text{and} \quad a_i x_{j-1} = x_j a_i \quad \text{for } j \geqslant i+2,$$

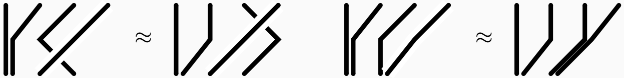
$$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, \quad \sigma_i\sigma_j a_i = a_j\sigma_i, \quad \sigma_j\sigma_i a_j = a_i\sigma_i \quad \text{for } j = i+1.$$

- commutation relations:



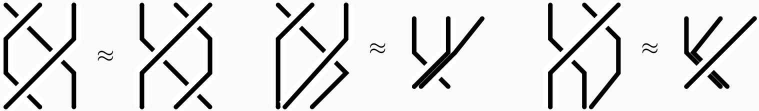
$\sigma_i\sigma_j = \sigma_j\sigma_i$ $\qquad$ $\sigma_i a_j = a_j\sigma_i$ $\qquad$ for $j \geqslant i+2$

- semi-commutation relations ("Thompson" relations):



$a_i\sigma_{j-1} = \sigma_j a_i$ $\qquad$ $a_i a_{j-1} = a_j a_i$ $\qquad$ for $j \geqslant i+2$

- braid relations:



$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ $\qquad$ $\sigma_i\sigma_{i+1}a_i = a_{i+1}\sigma_i$ $\qquad$ $\sigma_{i+1}\sigma_i a_i = a_i\sigma_i$

- Proof (beginning):
  - The relations (...) hold in $B_\bullet$ (obvious);

- Proof (beginning):
  - The relations (...) hold in $B_\bullet$ (obvious);
  - For the other direction, introduce
    - $B_\circ$:= the group presented by (...);

- Proof (beginning):
  - The relations (...) hold in $B_\bullet$ (obvious);
  - For the other direction, introduce
    - $B_\circ$:= the group presented by (...); we know $B_\circ \twoheadrightarrow B_\bullet$;

- Proof (beginning):
  - The relations (...) hold in $B_\bullet$ (obvious);
  - For the other direction, introduce
    - $B_\circ$:= the group presented by (...); we know $B_\circ \twoheadrightarrow B_\bullet$; we want $B_\circ \simeq B_\bullet$.

- Proof (beginning):
  - The relations (...) hold in $B_\bullet$ (obvious);
  - For the other direction, introduce
    - $B_\circ$:= the group presented by (...); we know $B_\circ \twoheadrightarrow B_\bullet$; we want $B_\circ \simeq B_\bullet$.
    - $B_\circ^+$:= the monoid presented by (...).

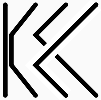
- Proof (beginning):
  - The relations (...) hold in $B_\bullet$ (obvious);
  - For the other direction, introduce
    - $B_\circ$:= the group presented by (...); we know $B_\circ \twoheadrightarrow B_\bullet$; we want $B_\circ \simeq B_\bullet$.
    - $B_\circ^+$:= the monoid presented by (...).

- Fact: *The group $B_\circ$ is generated by $\sigma_1, \sigma_2, a_1, a_2$.*

- Proof (beginning):
  - The relations (...) hold in $B_\bullet$ (obvious);
  - For the other direction, introduce
    - $B_\circ$:= the group presented by (...); we know $B_\circ \twoheadrightarrow B_\bullet$; we want $B_\circ \simeq B_\bullet$.
    - $B_\circ^+$:= the monoid presented by (...).

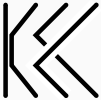
- Fact: *The group $B_\circ$ is generated by $\sigma_1, \sigma_2, a_1, a_2$.*

  Example: $\sigma_3 = a_1^{-1}\sigma_2 a_1 =$



- Proof (beginning):
  - The relations (...) hold in $B_\bullet$ (obvious);
  - For the other direction, introduce
    - $B_\circ$:= the group presented by (...); we know $B_\circ \twoheadrightarrow B_\bullet$; we want $B_\circ \simeq B_\bullet$.
    - $B_\circ^+$:= the monoid presented by (...).

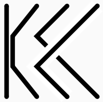
- Fact: *The group $B_\circ$ is generated by $\sigma_1, \sigma_2, a_1, a_2$.*

Example: $\sigma_3 = a_1^{-1}\sigma_2 a_1 =$ 

- Main lemma: *The monoid $B_\circ^+$ is cancellative and admits least common multiples and greatest common divisors.*

- Proof (beginning):
  - The relations (...) hold in $B_\bullet$ (obvious);
  - For the other direction, introduce
    - $B_\circ$:= the group presented by (...); we know $B_\circ \twoheadrightarrow B_\bullet$; we want $B_\circ \simeq B_\bullet$.
    - $B_\circ^+$:= the monoid presented by (...).

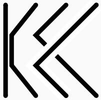
- Fact: *The group $B_\circ$ is generated by $\sigma_1, \sigma_2, a_1, a_2$.*

Example: $\sigma_3 = a_1^{-1}\sigma_2 a_1 =$ 

- Main lemma: *The monoid $B_\circ^+$ is cancellative and admits least common multiples and greatest common divisors. It is a Zappa-Szep product of B_∞^+ and F^+.*

- Proof (beginning):
  - The relations (...) hold in $B_\bullet$ (obvious);
  - For the other direction, introduce
    - $B_\circ$:= the group presented by (...); we know $B_\circ \twoheadrightarrow B_\bullet$; we want $B_\circ \simeq B_\bullet$.
    - $B_\circ^+$:= the monoid presented by (...).

- Fact: *The group $B_\circ$ is generated by $\sigma_1, \sigma_2, a_1, a_2$.*

  Example: $\sigma_3 = a_1^{-1}\sigma_2 a_1 =$ 

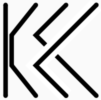
- Main lemma: *The monoid $B_\circ^+$ is cancellative and admits least common multiples and greatest common divisors. It is a Zappa-Szep product of B_∞^+ and F^+.*

$\uparrow$

$B_\infty^+ \times F^+$ with two actions:

- Proof (beginning):
  - The relations (...) hold in $B_\bullet$ (obvious);
  - For the other direction, introduce
    - $B_\circ$:= the group presented by (...); we know $B_\circ \twoheadrightarrow B_\bullet$; we want $B_\circ \simeq B_\bullet$.
    - $B_\circ^+$:= the monoid presented by (...).

- Fact: *The group $B_\circ$ is generated by $\sigma_1, \sigma_2, a_1, a_2$.*

Example: $\sigma_3 = a_1^{-1}\sigma_2 a_1 =$ 

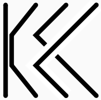
- Main lemma: *The monoid $B_\circ^+$ is cancellative and admits least common multiples and greatest common divisors. It is a Zappa-Szep product of B_∞^+ and F^+.*

$\uparrow$

$B_\infty^+ \times F^+$ with two actions: $a_i \cdot \beta = \mathrm{db}_i(\beta) \cdot a_{\beta^{-1}(i)}$

- Proof (beginning):
  - The relations (...) hold in $B_\bullet$ (obvious);
  - For the other direction, introduce
    - $B_\circ$:= the group presented by (...); we know $B_\circ \twoheadrightarrow B_\bullet$; we want $B_\circ \simeq B_\bullet$.
    - $B_\circ^+$:= the monoid presented by (...).

- Fact: *The group $B_\circ$ is generated by $\sigma_1, \sigma_2, a_1, a_2$.*

Example: $\sigma_3 = a_1^{-1} \sigma_2 a_1 =$ 

- Main lemma: *The monoid $B_\circ^+$ is cancellative and admits least common multiples and greatest common divisors. It is a Zappa-Szep product of B_∞^+ and F^+.*

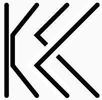
$\uparrow$

$B_\infty^+ \times F^+$ with two actions: $a_i \cdot \beta = \mathrm{db}_i(\beta) \cdot a_{\beta^{-1}(i)}$

F^+ acts on B_∞^+ by doubling strands, B_∞^+ acts on F^+ by permuting positions.

- Proof (beginning):
  - The relations (...) hold in $B_\bullet$ (obvious);
  - For the other direction, introduce
    - $B_\circ$:= the group presented by (...); we know $B_\circ \twoheadrightarrow B_\bullet$; we want $B_\circ \simeq B_\bullet$.
    - $B_\circ^+$:= the monoid presented by (...).

- Fact: *The group $B_\circ$ is generated by $\sigma_1, \sigma_2, a_1, a_2$.*

Example: $\sigma_3 = a_1^{-1}\sigma_2 a_1 =$ 

- Main lemma: *The monoid $B_\circ^+$ is cancellative and admits least common multiples and greatest common divisors. It is a Zappa-Szep product of B_∞^+ and F^+.*

$\uparrow$

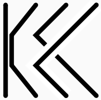
$B_\infty^+ \times F^+$ with two actions: $a_i \cdot \beta = \mathrm{db}_i(\beta) \cdot a_{\beta^{-1}(i)}$

F^+ acts on B_∞^+ by doubling strands, B_∞^+ acts on F^+ by permuting positions.

- Proposition: *The group $B_\circ$ is a group of fractions for the monoid $B_\circ^+$,*

- Proof (beginning):
  - The relations (...) hold in $B_\bullet$ (obvious);
  - For the other direction, introduce
    - $B_\circ$:= the group presented by (...); we know $B_\circ \twoheadrightarrow B_\bullet$; we want $B_\circ \simeq B_\bullet$.
    - $B_\circ^+$:= the monoid presented by (...).

- Fact: *The group $B_\circ$ is generated by $\sigma_1, \sigma_2, a_1, a_2$.*

Example: $\sigma_3 = a_1^{-1}\sigma_2 a_1 =$ 

- Main lemma: *The monoid $B_\circ^+$ is cancellative and admits least common multiples and greatest common divisors. It is a Zappa-Szep product of B_∞^+ and F^+.*

$\uparrow$

$B_\infty^+ \times F^+$ with two actions: $a_i \cdot \beta = \mathrm{db}_i(\beta) \cdot a_{\beta^{-1}(i)}$

F^+ acts on B_∞^+ by doubling strands, B_∞^+ acts on F^+ by permuting positions.

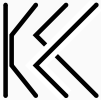
- Proposition: *The group $B_\circ$ is a group of fractions for the monoid $B_\circ^+$, and one has*

$$B_\circ = (F^+)^{-1} \cdot B_\infty \cdot F^+.$$

- Proof (beginning):
  - The relations (...) hold in $B_\bullet$ (obvious);
  - For the other direction, introduce
    - $B_\circ$:= the group presented by (...); we know $B_\circ \twoheadrightarrow B_\bullet$; we want $B_\circ \simeq B_\bullet$.
    - $B_\circ^+$:= the monoid presented by (...).

- Fact: *The group $B_\circ$ is generated by $\sigma_1, \sigma_2, a_1, a_2$.*

Example: $\sigma_3 = a_1^{-1}\sigma_2 a_1 =$



- Main lemma: *The monoid $B_\circ^+$ is cancellative and admits least common multiples and greatest common divisors. It is a Zappa-Szep product of B_∞^+ and F^+.*

$\uparrow$

$B_\infty^+ \times F^+$ with two actions: $a_i \cdot \beta = \mathrm{db}_i(\beta) \cdot a_{\beta^{-1}(i)}$

F^+ acts on B_∞^+ by doubling strands, B_∞^+ acts on F^+ by permuting positions.

- Proposition: *The group $B_\circ$ is a group of fractions for the monoid $B_\circ^+$, and one has*

$$B_\circ = (F^+)^{-1} \cdot B_\infty \cdot F^+.$$

  - Every parenthesized braid diagram can be isotoped to a diagram "dilatation + braid + contraction".

- <u>Proposition</u>: *Let* sh $: \sigma_i \mapsto \sigma_{i+1}$ *and* $a_i \mapsto a_{i+1}$ *for every* i.

- Proposition: *Let* $sh : \sigma_i \mapsto \sigma_{i+1}$ *and* $a_i \mapsto a_{i+1}$ *for every* i*. For* g, h *in* $B_\circ$*, let*

$$g * h := g \cdot sh(h) \cdot \sigma_1 \cdot sh(h)^{-1},$$

- Proposition: *Let* $sh : \sigma_i \mapsto \sigma_{i+1}$ *and* $a_i \mapsto a_{i+1}$ *for every* i. *For* g, h *in* $B_\circ$, *let*

$$g * h := g \cdot sh(h) \cdot \sigma_1 \cdot sh(h)^{-1},$$
$$g \circ h := g \cdot sh(h) \cdot a_1.$$

- Proposition: *Let* sh $: \sigma_i \mapsto \sigma_{i+1}$ *and* $a_i \mapsto a_{i+1}$ *for every* i. *For* g, h *in* $B_\circ$, *let*

$$g * h := g \cdot sh(h) \cdot \sigma_1 \cdot sh(h)^{-1},$$
$$g \circ h := g \cdot sh(h) \cdot a_1.$$

Then $(B_\circ, *, \circ)$ *is an augmented LD-system.*

• Proposition: *Let sh* : $\sigma_i \mapsto \sigma_{i+1}$ *and* $a_i \mapsto a_{i+1}$ *for every* i. *For* g, h *in* $B_\circ$, *let*

$$g * h := g \cdot sh(h) \cdot \sigma_1 \cdot sh(h)^{-1},$$
$$g \circ h := g \cdot sh(h) \cdot a_1.$$

Then $(B_\circ, *, \circ)$ *is an augmented LD-system.*

↑

$\left\{ x * (y * z) = (x * y) * (x * z) \right.$: self-distributivity, “LD”

• <u>Proposition</u>: *Let sh* : $\sigma_i \mapsto \sigma_{i+1}$ *and* $a_i \mapsto a_{i+1}$ *for every* i. *For* g, h *in* $B_\circ$, *let*

$$g * h := g \cdot sh(h) \cdot \sigma_1 \cdot sh(h)^{-1},$$
$$g \circ h := g \cdot sh(h) \cdot a_1.$$

Then $(B_\circ, *, \circ)$ *is an augmented LD-system.*

↑

$$\begin{cases} x * (y * z) = (x * y) * (x * z): \text{ self-distributivity, ``LD''} \\ x * (y * z) = (x \circ y) * z \\ x * (y \circ z) = (x * y) \circ (x * z) \end{cases}$$

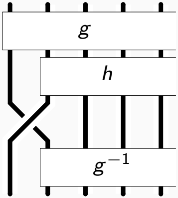
• <u>Proposition</u>: *Let sh* : $\sigma_i \mapsto \sigma_{i+1}$ *and* $a_i \mapsto a_{i+1}$ *for every* i. *For* g, h *in* $B_\circ$, *let*

$$g * h := g \cdot sh(h) \cdot \sigma_1 \cdot sh(h)^{-1},$$
$$g \circ h := g \cdot sh(h) \cdot a_1.$$

Then $(B_\circ, *, \circ)$ *is an augmented LD-system.*

↑

$$\begin{cases} x*(y*z) = (x*y)*(x*z): \text{ self-distributivity, ``LD''} \\ x*(y*z) = (x\circ y)*z \\ x*(y\circ z) = (x*y)\circ(x*z) \end{cases}$$



- Proposition: *Let sh* : $\sigma_i \mapsto \sigma_{i+1}$ *and* $a_i \mapsto a_{i+1}$ *for every* i. *For* g, h *in* $B_\circ$, *let*

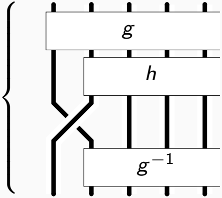
$$g * h := g \cdot sh(h) \cdot \sigma_1 \cdot sh(h)^{-1},$$
$$g \circ h := g \cdot sh(h) \cdot a_1.$$

Then $(B_\circ, *, \circ)$ *is an augmented LD-system.*

$$\uparrow$$

$$\begin{cases} x * (y * z) = (x * y) * (x * z): & \text{self-distributivity, “LD”} \\ x * (y * z) = (x \circ y) * z & \\ x * (y \circ z) = (x * y) \circ (x * z) & \end{cases}$$

$g * h$

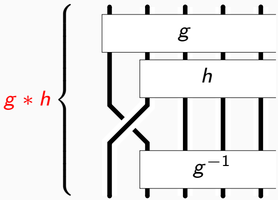


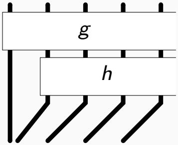
- Proposition: *Let sh* : $\sigma_i \mapsto \sigma_{i+1}$ *and* $a_i \mapsto a_{i+1}$ *for every* i. *For* g, h *in* $B_\circ$, *let*

$$g * h := g \cdot sh(h) \cdot \sigma_1 \cdot sh(h)^{-1},$$
$$g \circ h := g \cdot sh(h) \cdot a_1.$$

Then $(B_\circ, *, \circ)$ *is an augmented LD-system.*

$$\begin{cases} x * (y * z) = (x * y) * (x * z): & \text{self-distributivity, "LD"} \\ x * (y * z) = (x \circ y) * z & \\ x * (y \circ z) = (x * y) \circ (x * z) & \end{cases}$$





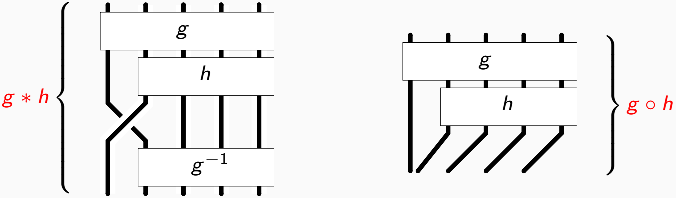
- Proposition: *Let sh* $: \sigma_i \mapsto \sigma_{i+1}$ *and* $a_i \mapsto a_{i+1}$ *for every* i. *For* g, h *in* $B_\circ$, *let*

$$g * h := g \cdot sh(h) \cdot \sigma_1 \cdot sh(h)^{-1},$$
$$g \circ h := g \cdot sh(h) \cdot a_1.$$

Then $(B_\circ, *, \circ)$ *is an augmented LD-system.*

↑

$$\begin{cases} x * (y * z) = (x * y) * (x * z): \text{ self-distributivity, ``LD''} \\ x * (y * z) = (x \circ y) * z \\ x * (y \circ z) = (x * y) \circ (x * z) \end{cases}$$



- Recall: want to prove $B_\circ \simeq B_\bullet$, i.e., $B_\bullet$ presented by the "braid-Thompson" relations (no other relation in $B_\bullet$ than those of $B_\circ$)

- Recall: want to prove $B_\circ \simeq B_\bullet$, i.e., $B_\bullet$ presented by the "braid-Thompson" relations (no other relation in $B_\bullet$ than those of $B_\circ$)
  - Need to show that, for w a parenthesized braid word, the isotopy class of the diagram $D(w)$ determines the class $[w]$ of w in $B_\circ$.

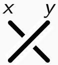
- Recall: want to prove $B_\circ \simeq B_\bullet$, i.e., $B_\bullet$ presented by the "braid-Thompson" relations (no other relation in $B_\bullet$ than those of $B_\circ$)
  - Need to show that, for w a parenthesized braid word, the isotopy class of the diagram $D(w)$ determines the class $[w]$ of w in $B_\circ$.

- Proof (sketch):
  - Use diagram colorings: Having fixed a set S with a binary operation $*$,

- Recall: want to prove $B_\circ \simeq B_\bullet$, i.e., $B_\bullet$ presented by the "braid-Thompson" relations (no other relation in $B_\bullet$ than those of $B_\circ$)
  - Need to show that, for w a parenthesized braid word, the isotopy class of the diagram $D(w)$ determines the class $[w]$ of w in $B_\circ$.

- Proof (sketch):
  - Use diagram colorings: Having fixed a set S with a binary operation $*$,
    - put colors from S on the top ends of a (parenthesized) braid diagram,

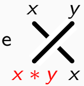
- Recall: want to prove $B_\circ \simeq B_\bullet$, i.e., $B_\bullet$ presented by the "braid-Thompson" relations (no other relation in $B_\bullet$ than those of $B_\circ$)
  - Need to show that, for w a parenthesized braid word, the isotopy class of the diagram $D(w)$ determines the class $[w]$ of w in $B_\circ$.

- Proof (sketch):
  - Use diagram colorings: Having fixed a set S with a binary operation $*$,
    - put colors from S on the top ends of a (parenthesized) braid diagram,
    - propagate using the rule 

- Recall: want to prove $B_\circ \simeq B_\bullet$, i.e., $B_\bullet$ presented by the "braid-Thompson" relations (no other relation in $B_\bullet$ than those of $B_\circ$)
  - ▶ Need to show that, for w a parenthesized braid word, the isotopy class of the diagram $D(w)$ determines the class $[w]$ of w in $B_\circ$.

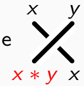
- Proof (sketch):
  - ▶ Use diagram colorings: Having fixed a set S with a binary operation $*$,
    - put colors from S on the top ends of a (parenthesized) braid diagram,
    - propagate using the rule

- Recall: want to prove $B_\circ \simeq B_\bullet$, i.e., $B_\bullet$ presented by the "braid-Thompson" relations (no other relation in $B_\bullet$ than those of $B_\circ$)
  - Need to show that, for w a parenthesized braid word, the isotopy class of the diagram $D(w)$ determines the class $[w]$ of w in $B_\circ$.

- Proof (sketch):
  - Use diagram colorings: Having fixed a set S with a binary operation $*$,
    - put colors from S on the top ends of a (parenthesized) braid diagram,
    - propagate using the rule 

- Recall: want to prove $B_\circ \simeq B_\bullet$, i.e., $B_\bullet$ presented by the "braid-Thompson" relations (no other relation in $B_\bullet$ than those of $B_\circ$)
  - Need to show that, for w a parenthesized braid word, the isotopy class of the diagram $D(w)$ determines the class $[w]$ of w in $B_\circ$.

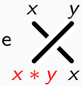
- Proof (sketch):
  - Use diagram colorings: Having fixed a set S with a binary operation $*$,
    - put colors from S on the top ends of a (parenthesized) braid diagram,
    - propagate using the rule



    - works (compatible with Reidemeister move) when $*$ obeys the LD law.

- Recall: want to prove $B_\circ \simeq B_\bullet$, i.e., $B_\bullet$ presented by the "braid-Thompson" relations (no other relation in $B_\bullet$ than those of $B_\circ$)
  - ▶ Need to show that, for w a parenthesized braid word, the isotopy class of the diagram $D(w)$ determines the class $[w]$ of w in $B_\circ$.

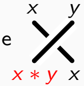
- Proof (sketch):
  - ▶ Use diagram colorings: Having fixed a set S with a binary operation $*$,
    - put colors from S on the top ends of a (parenthesized) braid diagram,
    - propagate using the rule



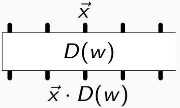
    - works (compatible with Reidemeister move) when $*$ obeys the LD law.
  - ▶ Here: use $B_\circ$ and its LD-operation $*$ for coloring diagrams

- Recall: want to prove $B_\circ \simeq B_\bullet$, i.e., $B_\bullet$ presented by the "braid-Thompson" relations (no other relation in $B_\bullet$ than those of $B_\circ$)
  - ▸ Need to show that, for w a parenthesized braid word, the isotopy class of the diagram $D(w)$ determines the class $[w]$ of w in $B_\circ$.

- Proof (sketch):
  - ▸ Use diagram colorings: Having fixed a set S with a binary operation $*$,
    - put colors from S on the top ends of a (parenthesized) braid diagram,
    - propagate using the rule



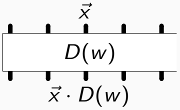
    - works (compatible with Reidemeister move) when $*$ obeys the LD law.
  - ▸ Here: use $B_\circ$ and its LD-operation $*$ for coloring diagrams
    - the point: there exists eval : $(B_\circ)^\infty \to B_\circ$ such that, in $B_\circ$,



- Recall: want to prove $B_\circ \simeq B_\bullet$, i.e., $B_\bullet$ presented by the "braid-Thompson" relations (no other relation in $B_\bullet$ than those of $B_\circ$)
  - Need to show that, for w a parenthesized braid word, the isotopy class of the diagram $D(w)$ determines the class $[w]$ of w in $B_\circ$.

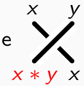
- Proof (sketch):
  - Use diagram colorings: Having fixed a set S with a binary operation $*$,
    - put colors from S on the top ends of a (parenthesized) braid diagram,
    - propagate using the rule

    - works (compatible with Reidemeister move) when $*$ obeys the LD law.
  - Here: use $B_\circ$ and its LD-operation $*$ for coloring diagrams
    - the point: there exists eval : $(B_\circ)^\infty \to B_\circ$ such that, in $B_\circ$,

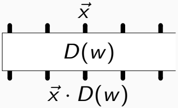


implies $\mathrm{eval}(\vec{x} \cdot D(w)) = \mathrm{eval}(\vec{x}) \cdot [w]$.

- Recall: want to prove $B_\circ \simeq B_\bullet$, i.e., $B_\bullet$ presented by the "braid-Thompson" relations (no other relation in $B_\bullet$ than those of $B_\circ$)
  - ▶ Need to show that, for w a parenthesized braid word, the isotopy class of the diagram $D(w)$ determines the class $[w]$ of w in $B_\circ$.

- Proof (sketch):
  - ▶ Use diagram colorings: Having fixed a set S with a binary operation $*$,
    - put colors from S on the top ends of a (parenthesized) braid diagram,
    - propagate using the rule 

    - works (compatible with Reidemeister move) when $*$ obeys the LD law.
  - ▶ Here: use $B_\circ$ and its LD-operation $*$ for coloring diagrams
    - the point: there exists eval : $(B_\circ)^\infty \to B_\circ$ such that, in $B_\circ$,



implies $\mathrm{eval}(\vec{x} \cdot D(w)) = \mathrm{eval}(\vec{x}) \cdot [w]$.

    - hence, $[w]$ recovered from the isotopy class of $D(w)$. □

- Proposition: *The augmented LD-system* $(B_\bullet, *, \circ)$ *is torsion-free.*

↑ every element of $B_\bullet$ generates a free subsystem

- Proposition: *The augmented LD-system* $(B_\bullet, *, \circ)$ *is torsion-free.*
  $\uparrow$
  every element of $B_\bullet$ generates a free subsystem

- Proposition: *Every element of* $B_\bullet$ *has a canonical expression in terms of special ones.*
  $\uparrow$
  the closure of 1 under $*$ and $\circ$

- Proposition: *The augmented LD-system $(B_\bullet, *, \circ)$ is torsion-free.*

  ↑ every element of $B_\bullet$ generates a free subsystem

- Proposition: *Every element of $B_\bullet$ has a canonical expression in terms of special ones.*

  ↑ the closure of 1 under $*$ and $\circ$

- Proposition: *The group $B_\bullet$ is orderable; more precisely: every element of $B_\bullet$ has an expression in which the σ_i with minimal index occurs positively only (no σ_i^{-1}).*
  - similar to B_∞, but not expected because of the a_is

• Proposition: *The augmented LD-system* $(B_\bullet, *, \circ)$ *is torsion-free.*

↑ every element of $B_\bullet$ generates a free subsystem

• Proposition: *Every element of* $B_\bullet$ *has a canonical expression in terms of special ones.*

↑ the closure of 1 under $*$ and $\circ$

• Proposition: *The group* $B_\bullet$ *is orderable; more precisely: every element of* $B_\bullet$ *has an expression in which the* σ_i *with minimal index occurs positively only (no* σ_i^{-1}*).*

▸ similar to B_∞, but not expected because of the a_is

• Proposition: The group $B_\bullet$ is (isomorphic to) M. Brin's group $\widehat{BV}$.

• Proposition: *The augmented LD-system* $(B_\bullet, *, \circ)$ *is torsion-free.*

↑ every element of $B_\bullet$ generates a free subsystem

• Proposition: *Every element of* $B_\bullet$ *has a canonical expression in terms of special ones.*

↑ the closure of 1 under $*$ and $\circ$

• Proposition: *The group* $B_\bullet$ *is orderable; more precisely: every element of* $B_\bullet$ *has an expression in which the* σ_i *with minimal index occurs positively only (no* σ_i^{-1}*).*

▶ similar to B_∞, but not expected because of the a_is

• Proposition: The group $B_\bullet$ is (isomorphic to) M. Brin's group $\widehat{BV}$.

"braid group on one strand"

• Proposition: *The augmented LD-system* $(B_\bullet, *, \circ)$ *is torsion-free.*

↑ every element of $B_\bullet$ generates a free subsystem

• Proposition: *Every element of* $B_\bullet$ *has a canonical expression in terms of special ones.*

↑ the closure of 1 under $*$ and $\circ$

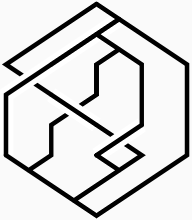
• Proposition: *The group* $B_\bullet$ *is orderable; more precisely: every element of* $B_\bullet$ *has an expression in which the* σ_i *with minimal index occurs positively only (no* σ_i^{-1}*).*

- similar to B_∞, but not expected because of the a_is

• Proposition: The group $B_\bullet$ is (isomorphic to) M. Brin's group $\widehat{BV}$.

"braid group on one strand"

- a typical element:



Plan:

- 1. Artin's braid group B_∞
- 2. Thompson's group F
- 3. The parenthesized braid group $B_\bullet$
- 4. The Artin representation of $B_\bullet$

- Recall: D_n = disk with n punctures

- Recall: D_n = disk with n punctures
  - Here: Cantor set of punctures.

- Recall: D_n = disk with n punctures
  - Here: Cantor set of punctures.
  - Technically more convenient to start with a sphere rather than with a disk.

- Recall: D_n = disk with n punctures
  - Here: Cantor set of punctures.
  - Technically more convenient to start with a sphere rather than with a disk.

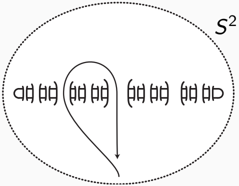
- Definition: $S_K :=$ sphere S^2 with a Cantor set of punctures removed on the equator.

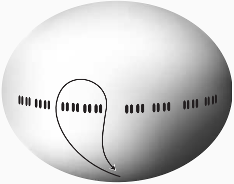
- Recall: D_n = disk with n punctures
  - Here: Cantor set of punctures.
  - Technically more convenient to start with a sphere rather than with a disk.

- Definition: S_K := sphere S^2 with a Cantor set of punctures removed on the equator.
  = two hemispheres connected by bridges indexed by dyadic numbers

- Recall: D_n = disk with n punctures
  - Here: Cantor set of punctures.
  - Technically more convenient to start with a sphere rather than with a disk.

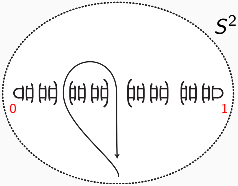
- Definition: S_K := sphere S^2 with a Cantor set of punctures removed on the equator.
  = two hemispheres connected by bridges indexed by dyadic numbers

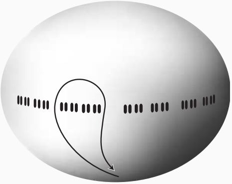




- Recall: D_n = disk with n punctures
  - Here: Cantor set of punctures.
  - Technically more convenient to start with a sphere rather than with a disk.

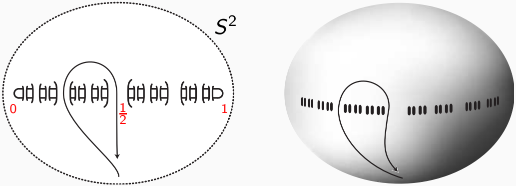
- Definition: S_K := sphere S^2 with a Cantor set of punctures removed on the equator.
  = two hemispheres connected by bridges indexed by dyadic numbers





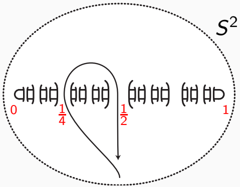
- Recall: D_n = disk with n punctures
  - Here: Cantor set of punctures.
  - Technically more convenient to start with a sphere rather than with a disk.

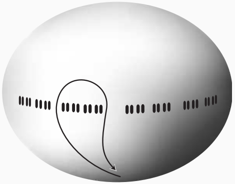
- <u>Definition</u>: $S_K :=$ sphere S^2 with a Cantor set of punctures removed on the equator.
  = two hemispheres connected by bridges indexed by dyadic numbers



- Recall: D_n = disk with n punctures
  - Here: Cantor set of punctures.
  - Technically more convenient to start with a sphere rather than with a disk.

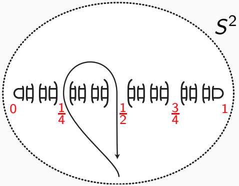
- Definition: $S_K :=$ sphere S^2 with a Cantor set of punctures removed on the equator.
  = two hemispheres connected by bridges indexed by dyadic numbers

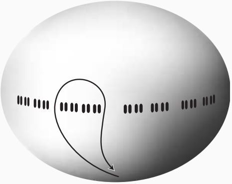




- Recall: D_n = disk with n punctures
  - Here: Cantor set of punctures.
  - Technically more convenient to start with a sphere rather than with a disk.

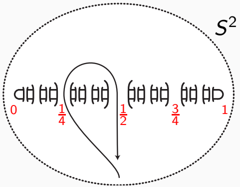
- Definition: S_K := sphere S^2 with a Cantor set of punctures removed on the equator.
  = two hemispheres connected by bridges indexed by dyadic numbers

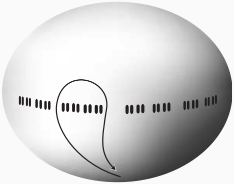




- Recall: D_n = disk with n punctures
  - Here: Cantor set of punctures.
  - Technically more convenient to start with a sphere rather than with a disk.

- Definition: S_K := sphere S^2 with a Cantor set of punctures removed on the equator.
  = two hemispheres connected by bridges indexed by dyadic numbers

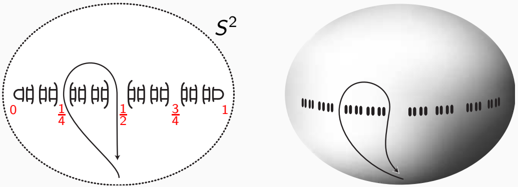




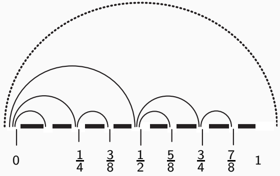
- Fact: $\pi_1(S_K)$ *is a free group with a basis indexed by finite sequences of integers.*

- Recall: D_n = disk with n punctures
  - Here: Cantor set of punctures.
  - Technically more convenient to start with a sphere rather than with a disk.

- Definition: $S_K :=$ sphere S^2 with a Cantor set of punctures removed on the equator.
  = two hemispheres connected by bridges indexed by dyadic numbers

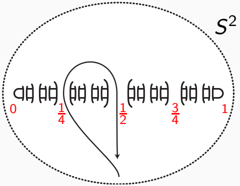


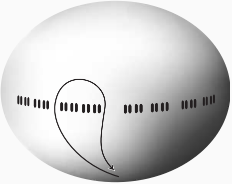
- Fact: $\pi_1(S_K)$ *is a free group with a basis indexed by finite sequences of integers.*



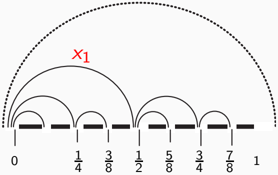
- Recall: D_n = disk with n punctures
  - Here: Cantor set of punctures.
  - Technically more convenient to start with a sphere rather than with a disk.

- Definition: S_K := sphere S^2 with a Cantor set of punctures removed on the equator.
  = two hemispheres connected by bridges indexed by dyadic numbers



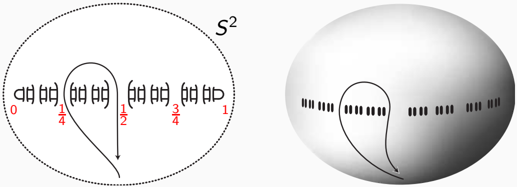


- Fact: $\pi_1(S_K)$ *is a free group with a basis indexed by finite sequences of integers.*

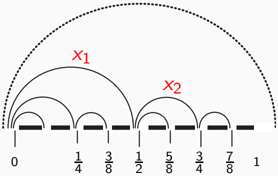


- Recall: D_n = disk with n punctures
  - Here: Cantor set of punctures.
  - Technically more convenient to start with a sphere rather than with a disk.

- Definition: S_K := sphere S^2 with a Cantor set of punctures removed on the equator.
  = two hemispheres connected by bridges indexed by dyadic numbers

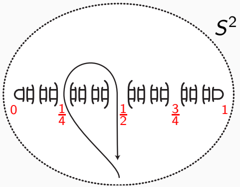


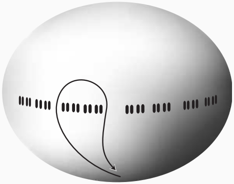
- Fact: $\pi_1(S_K)$ *is a free group with a basis indexed by finite sequences of integers.*



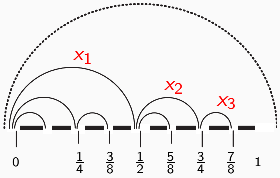
- Recall: D_n = disk with n punctures
  - Here: Cantor set of punctures.
  - Technically more convenient to start with a sphere rather than with a disk.

- Definition: S_K := sphere S^2 with a Cantor set of punctures removed on the equator.
  = two hemispheres connected by bridges indexed by dyadic numbers



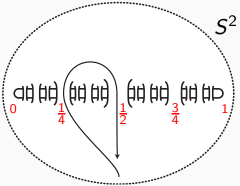


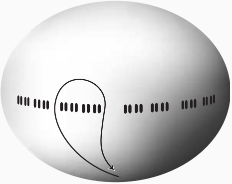
- Fact: $\pi_1(S_K)$ *is a free group with a basis indexed by finite sequences of integers.*



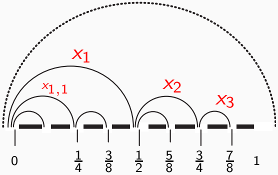
- Recall: D_n = disk with n punctures
  - Here: Cantor set of punctures.
  - Technically more convenient to start with a sphere rather than with a disk.

- Definition: S_K := sphere S^2 with a Cantor set of punctures removed on the equator.
  = two hemispheres connected by bridges indexed by dyadic numbers



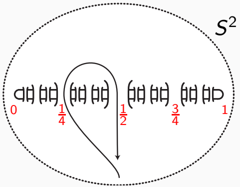


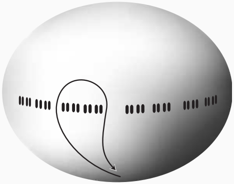
- Fact: $\pi_1(S_K)$ *is a free group with a basis indexed by finite sequences of integers.*



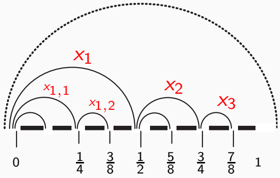
- Recall: D_n = disk with n punctures
  - Here: Cantor set of punctures.
  - Technically more convenient to start with a sphere rather than with a disk.

- Definition: $S_K :=$ sphere S^2 with a Cantor set of punctures removed on the equator.
  $=$ two hemispheres connected by bridges indexed by dyadic numbers



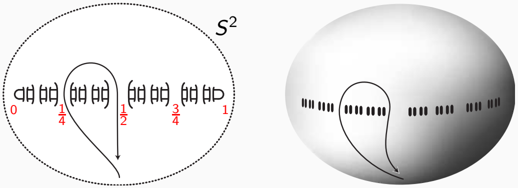


- Fact: $\pi_1(S_K)$ *is a free group with a basis indexed by finite sequences of integers.*

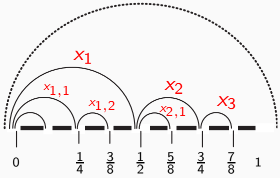


- Recall: D_n = disk with n punctures
  - Here: Cantor set of punctures.
  - Technically more convenient to start with a sphere rather than with a disk.

- Definition: $S_K :=$ sphere S^2 with a Cantor set of punctures removed on the equator.
  = two hemispheres connected by bridges indexed by dyadic numbers

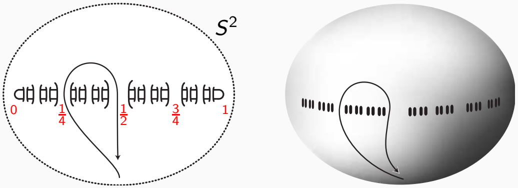


- Fact: $\pi_1(S_K)$ *is a free group with a basis indexed by finite sequences of integers.*

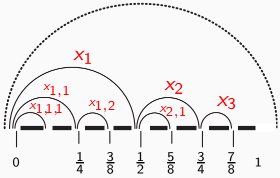


- Recall: D_n = disk with n punctures
  - Here: Cantor set of punctures.
  - Technically more convenient to start with a sphere rather than with a disk.

- Definition: $S_K :=$ sphere S^2 with a Cantor set of punctures removed on the equator.
  = two hemispheres connected by bridges indexed by dyadic numbers



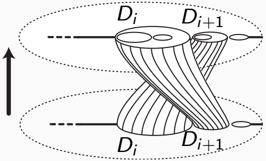
- Fact: $\pi_1(S_K)$ *is a free group with a basis indexed by finite sequences of integers.*

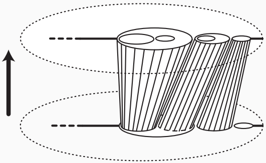


- Action of $B_\bullet$ on S_K by homeomorphisms:

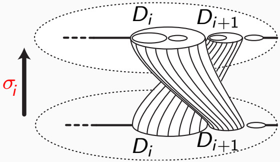
- Action of $B_\bullet$ on S_K by homeomorphisms: an embedding of $B_\bullet$ in $\mathcal{MCG}(S_K)$

- Action of $B_\bullet$ on S_K by homeomorphisms: an embedding of $B_\bullet$ in $\mathcal{MCG}(S_K)$

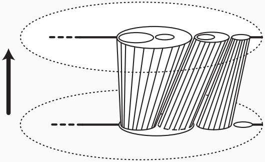




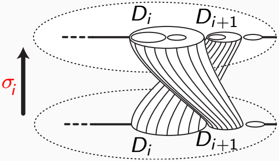
- Action of $B_\bullet$ on S_K by homeomorphisms: an embedding of $B_\bullet$ in $\mathcal{MCG}(S_K)$



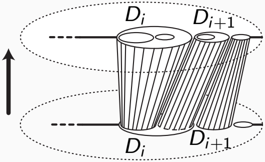
a "Dehn twist"
exchanging D_i and D_{i+1}



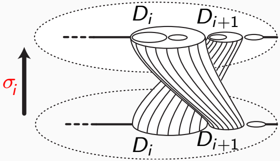
- Action of $B_\bullet$ on S_K by homeomorphisms: an embedding of $B_\bullet$ in $\mathcal{MCG}(S_K)$



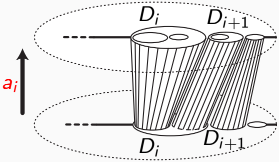
a "Dehn twist"
exchanging D_i and D_{i+1}



- Action of $B_\bullet$ on S_K by homeomorphisms: an embedding of $B_\bullet$ in $\mathcal{MCG}(S_K)$

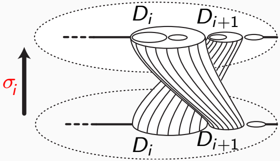


a "Dehn twist"
exchanging D_i and D_{i+1}

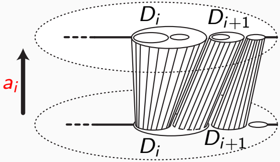


a "dilatation–contraction"
expanding D_i and contracting D_{i+1}

- Action of $B_\bullet$ on S_K by homeomorphisms: an embedding of $B_\bullet$ in $\mathcal{MCG}(S_K)$



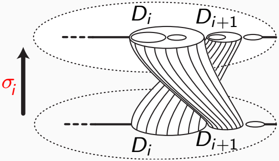
a "Dehn twist"
exchanging D_i and D_{i+1}



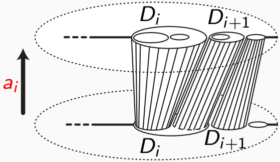
a "dilatation–contraction"
expanding D_i and contracting D_{i+1}

- Induces an action ρ on $\pi_1(S_K)$

- Action of $B_\bullet$ on S_K by homeomorphisms: an embedding of $B_\bullet$ in $\mathcal{MCG}(S_K)$

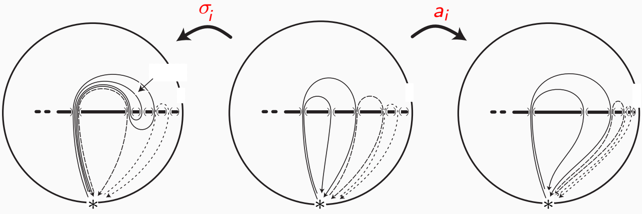


a "Dehn twist"
exchanging D_i and D_{i+1}

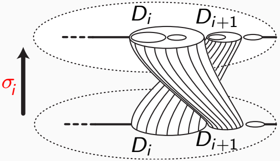


a "dilatation–contraction"
expanding D_i and contracting D_{i+1}

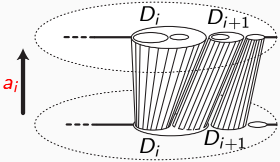
- Induces an action ρ on $\pi_1(S_K)$ ("Artin representation"):



- Action of $B_\bullet$ on S_K by homeomorphisms: an embedding of $B_\bullet$ in $\mathcal{MCG}(S_K)$

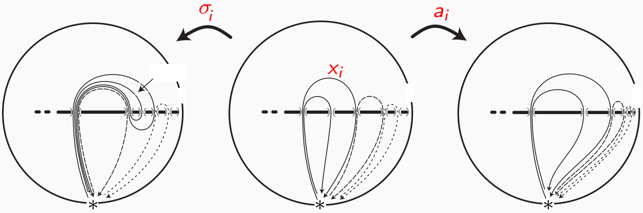


a "Dehn twist"
exchanging D_i and D_{i+1}

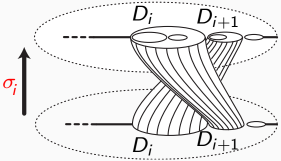


a "dilatation–contraction"
expanding D_i and contracting D_{i+1}

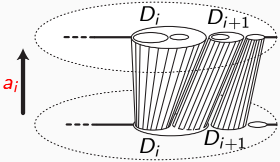
- Induces an action ρ on $\pi_1(S_K)$ ("Artin representation"):



- Action of $B_\bullet$ on S_K by homeomorphisms: an embedding of $B_\bullet$ in $\mathcal{MCG}(S_K)$

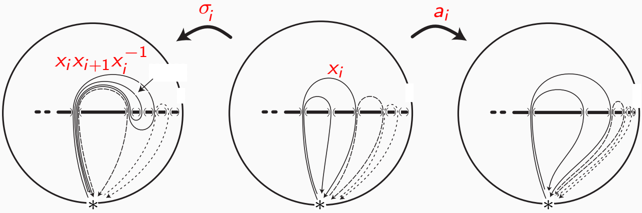


a "Dehn twist"
exchanging D_i and D_{i+1}

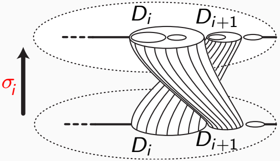


a "dilatation–contraction"
expanding D_i and contracting D_{i+1}

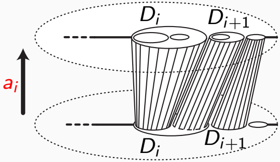
- Induces an action ρ on $\pi_1(S_K)$ ("Artin representation"):



- Action of $B_\bullet$ on S_K by homeomorphisms: an embedding of $B_\bullet$ in $\mathcal{MCG}(S_K)$

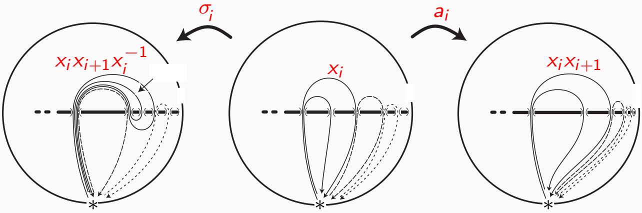


a "Dehn twist"
exchanging D_i and D_{i+1}

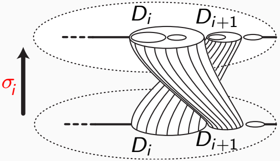


a "dilatation–contraction"
expanding D_i and contracting D_{i+1}

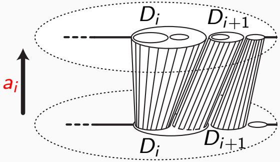
- Induces an action ρ on $\pi_1(S_K)$ ("Artin representation"):



- Action of $B_\bullet$ on S_K by homeomorphisms: an embedding of $B_\bullet$ in $\mathcal{MCG}(S_K)$

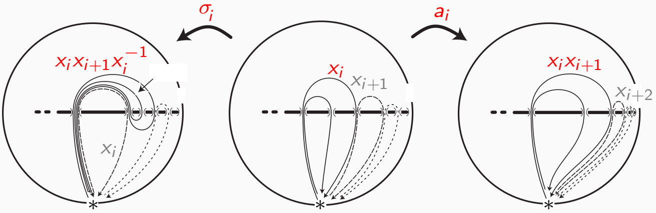


a "Dehn twist" exchanging D_i and D_{i+1}

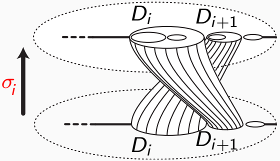


a "dilatation–contraction" expanding D_i and contracting D_{i+1}

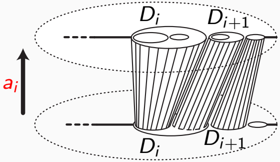
- Induces an action ρ on $\pi_1(S_K)$ ("Artin representation"):



- Action of $B_\bullet$ on S_K by homeomorphisms: an embedding of $B_\bullet$ in $\mathcal{MCG}(S_K)$

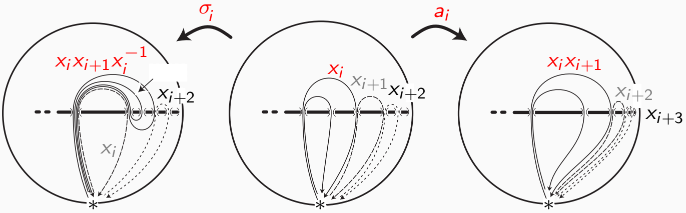


a "Dehn twist"
exchanging D_i and D_{i+1}

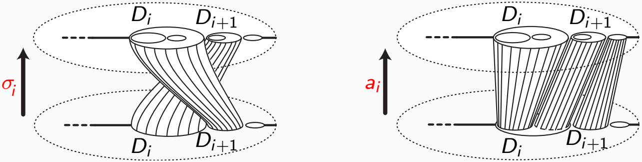


a "dilatation–contraction"
expanding D_i and contracting D_{i+1}

- Induces an action ρ on $\pi_1(S_K)$ ("Artin representation"):



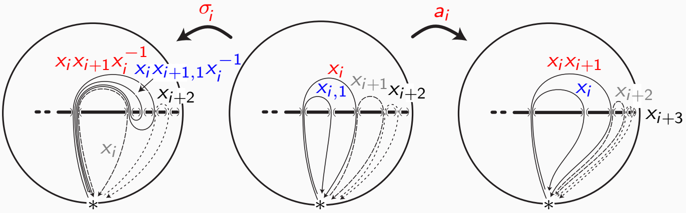
- Action of $B_\bullet$ on S_K by homeomorphisms: an embedding of $B_\bullet$ in $\mathcal{MCG}(S_K)$



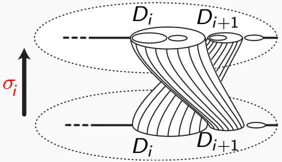
a "Dehn twist" exchanging D_i and D_{i+1}

a "dilatation–contraction" expanding D_i and contracting D_{i+1}

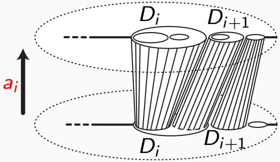
- Induces an action ρ on $\pi_1(S_K)$ ("Artin representation"):



- Action of $B_\bullet$ on S_K by homeomorphisms: an embedding of $B_\bullet$ in $\mathcal{MCG}(S_K)$

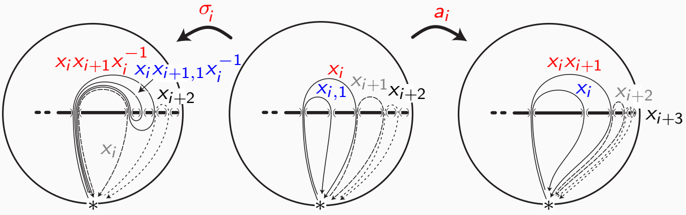


a "Dehn twist" exchanging D_i and D_{i+1}



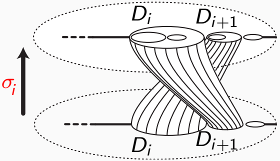
a "dilatation–contraction" expanding D_i and contracting D_{i+1}

- Induces an action ρ on $\pi_1(S_K)$ ("Artin representation"):

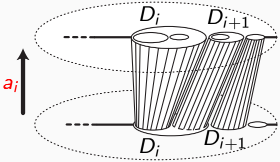


▶ read $\rho(\sigma_i)$: $\begin{cases} x_{i,s} \mapsto x_i x_{i+1,s} x_i^{-1} \\ x_{i+1,s} \mapsto x_i \\ x_{k,s} \mapsto x_{k,s} \text{ for } k \neq i, i+1 \end{cases}$

- Action of $B_\bullet$ on S_K by homeomorphisms: an embedding of $B_\bullet$ in $\mathcal{MCG}(S_K)$

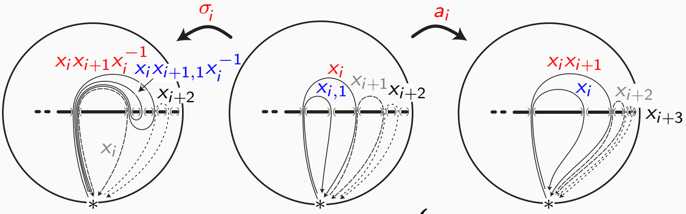


a "Dehn twist"
exchanging D_i and D_{i+1}



a "dilatation–contraction"
expanding D_i and contracting D_{i+1}

- Induces an action ρ on $\pi_1(S_K)$ ("Artin representation"):



▸ read $\rho(\sigma_i)$: $\begin{cases} x_{i,s} \mapsto x_i x_{i+1,s} x_i^{-1} \\ x_{i+1,s} \mapsto x_i \\ x_{k,s} \mapsto x_{k,s} \text{ for } k \neq i, i+1 \end{cases}$

$\rho(a_i)$: $\begin{cases} x_i \mapsto x_i x_{i+1}, \quad x_{i,1,s} \mapsto x_{i,s} \\ x_{i,j+1,s} \mapsto x_{i+1,j,s} \text{ for } j \geqslant 2 \\ x_{k,s} \mapsto x_{k,s} \text{ for } k < i \\ x_{k,s} \mapsto x_{k+1,s} \text{ for } k > i \end{cases}$

- <u>Proposition</u>: *The Artin representation* ρ *of* $B_\bullet$ *in* $Aut(F_\infty)$ *is faithful.*

- Proposition: *The Artin representation ρ of $B_\bullet$ in $Aut(F_\infty)$ is faithful.*

- Proof:
  - Uses the LD-structure again

- Proposition: *The Artin representation ρ of $B_\bullet$ in $Aut(F_\infty)$ is faithful.*

- Proof:
  - Uses the LD-structure again
  - Method: Show that, if a braid word w contains one σ_1 and no σ_1^{-1}, then $\rho(w)$ moves some x_s, hence is nontrivial.

- Proposition: *The Artin representation ρ of $B_\bullet$ in $Aut(F_\infty)$ is faithful.*

- Proof:
  - Uses the LD-structure again
  - Method: Show that, if a braid word w contains one σ_1 and no σ_1^{-1}, then $\rho(w)$ moves some x_s, hence is nontrivial.
  - Key point: $\rho(w)$ can be read from colouring trees (similar to the Hurwitz action of a braid word on a sequence)

- Proposition: *The Artin representation ρ of $B_\bullet$ in $Aut(F_\infty)$ is faithful.*

- Proof:
  - Uses the LD-structure again
  - Method: Show that, if a braid word w contains one σ_1 and no σ_1^{-1}, then $\rho(w)$ moves some x_s, hence is nontrivial.
  - Key point: $\rho(w)$ can be read from colouring trees (similar to the Hurwitz action of a braid word on a sequence)
  - If w contains one σ_1 and no σ_1^{-1},

- Proposition: *The Artin representation ρ of $B_\bullet$ in $Aut(F_\infty)$ is faithful.*

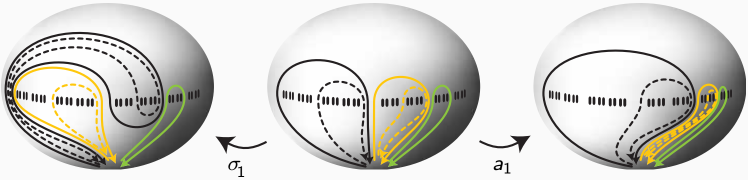
- Proof:
  - Uses the LD-structure again
  - Method: Show that, if a braid word w contains one σ_1 and no σ_1^{-1}, then $\rho(w)$ moves some x_s, hence is nontrivial.
  - Key point: $\rho(w)$ can be read from colouring trees (similar to the Hurwitz action of a braid word on a sequence)
  - If w contains one σ_1 and no σ_1^{-1}, then $\rho(w)(x_1)$ finishes with x_1^{-1},

- Proposition: *The Artin representation ρ of $B_\bullet$ in $Aut(F_\infty)$ is faithful.*

- Proof:
  - Uses the LD-structure again
  - Method: Show that, if a braid word w contains one σ_1 and no σ_1^{-1}, then $\rho(w)$ moves some x_s, hence is nontrivial.
  - Key point: $\rho(w)$ can be read from colouring trees (similar to the Hurwitz action of a braid word on a sequence)
  - If w contains one σ_1 and no σ_1^{-1}, then $\rho(w)(x_1)$ finishes with x_1^{-1}, hence $\rho(w)(x_1) \neq x_1$,

- Proposition: *The Artin representation ρ of $B_\bullet$ in $Aut(F_\infty)$ is faithful.*

- Proof:
  - Uses the LD-structure again
  - Method: Show that, if a braid word w contains one σ_1 and no σ_1^{-1}, then $\rho(w)$ moves some x_s, hence is nontrivial.
  - Key point: $\rho(w)$ can be read from colouring trees (similar to the Hurwitz action of a braid word on a sequence)
  - If w contains one σ_1 and no σ_1^{-1}, then $\rho(w)(x_1)$ finishes with x_1^{-1}, hence $\rho(w)(x_1) \neq x_1$, hence $\rho(w) \neq \text{id}$. □



- P. Dehornoy, *The parenthesized braid group*, Adv. in Math. 205 (2006) 354–409.

- P. Dehornoy, *The parenthesized braid group*, Adv. in Math. 205 (2006) 354–409.

- M. Brin, *The algebra of strand splitting. I. A braided version of Thompson's group V* J. Group Th. 10 (2007) 757–788.

- M. Brin, *The algebra of strand splitting. II. A presentation for the braid group on one strand* Intern. J. of Algebra and Comput 16 (2006) 203–219.

- P. Dehornoy, *The parenthesized braid group,* Adv. in Math. 205 (2006) 354–409.
- M. Brin, *The algebra of strand splitting. I. A braided version of Thompson's group V* J. Group Th. 10 (2007) 757–788.
- M. Brin, *The algebra of strand splitting. II. A presentation for the braid group on one strand* Intern. J. of Algebra and Comput 16 (2006) 203–219.
- P. Dehornoy, with I. Dynnikov, D. Rolfsen, B. Wiest, *Ordering braids* Math. Surveys and Monographs vol. 148, Amer. Math. Soc. (2008)

- P. Dehornoy, *The parenthesized braid group,* Adv. in Math. 205 (2006) 354–409.
- M. Brin, *The algebra of strand splitting. I. A braided version of Thompson's group V* J. Group Th. 10 (2007) 757–788.
- M. Brin, *The algebra of strand splitting. II. A presentation for the braid group on one strand* Intern. J. of Algebra and Comput 16 (2006) 203–219.
- P. Dehornoy, with I. Dynnikov, D. Rolfsen, B. Wiest, *Ordering braids* Math. Surveys and Monographs vol. 148, Amer. Math. Soc. (2008)

www.math.unicaen.fr/∼dehornoy