The group of parenthesized braids

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- A group $B_\bullet$ that extends both Artin's braid group B_∞ and Thompson's group F, occurring in various contexts:
  - "geometry group of an algebraic law",
  - subgroup of M. Brin's braided Thompson group BV,
  - quotient of a group of Greenberg–Sergiescu and Funar–Kapoudjian, ...
- Here: insist on similarity with B_∞ (use of self-distributivity) and connection with homeomorphisms of $S^2\backslash$Cantor.

Plan:

- 1. Artin's braid group B_∞
- 2. Thompson's group F
- 3. The parenthesized braid group $B_\bullet$
- 4. The Artin representation of $B_\bullet$

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- Definition (Artin): For $n \geqslant 1$, the braid group B_n

$= \{$ n-strand braid diagrams $\}$ / isotopy

$= \pi_1$(configuration space of n points of $\mathbb{C}$ mod. action of S_n)

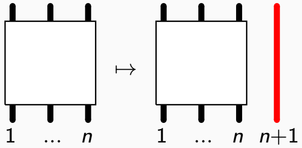
$= \mathcal{MCG}(D_n)$

$\uparrow$ {homeomorphisms of an n-punctured disk that fix ∂D_n}/isotopy

$$= \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle.$$

σ_i :

- Embedding B_n into B_{n+1}: add a trivial $(n+1)$st strand

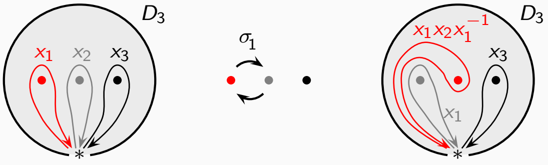


- Then $B_\infty := \varinjlim B_n = \left\langle \sigma_1, \sigma_2, \dots \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle.$

- Equivalently: Identify B_n with a subgroup of B_{n+1}, and put

$$B_\infty = \bigcup_n B_n.$$

- Viewing B_n as a group of (isotopy classes of) homeomorphisms of D_n:
  - action of B_n on the fundamental group of D_n, a free group of rank n.



- From there: homomorphism ρ from B_n to $\mathrm{Aut}(F_n)$:

$$\rho(\sigma_i) : \begin{cases} x_i & \mapsto \; x_i x_{i+1} x_i^{-1}, \\ x_{i+1} & \mapsto \; x_i, \\ x_k & \mapsto \; x_k \quad \text{for } k \neq i, i+1. \end{cases}$$

- Theorem (Artin): *The homomorphism ρ is injective.*

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- Definition (Richard Thompson, 1965):

$$F := \langle a_0, a_1, \dots \mid a_j a_i = a_i a_{j+1} \text{ for } j > i \rangle. \qquad (*)$$

  - occurs in the construction of a f.p. group with unsolvable word problem

- Fact: *The group F is a group of right fractions for the monoid F^+.*

  ↑ the monoid presented by $(*)$

- Fact: *Every element of F has a unique expression of the form*

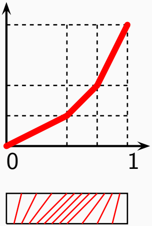
$$a_0^{p_0} a_1^{p_1} \cdots a_n^{p_n} a_n^{-q_n} \cdots a_1^{-q_1} a_0^{-q_0}$$

such that $((p_k \neq 0$ and $q_k \neq 0)$ implies $(p_{k+1} \neq 0)$ or $(q_{k+1} \neq 0))$.

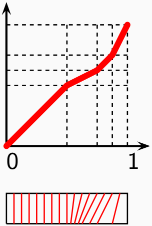
- Fact: *The group F is finitely presented:*
  - *generated by a_0 and a_1, since $a_n = a_1^{a_0^n}$ for $n \geqslant 2$;*
  - *relations: "$a_2^{a_1} = a_3$" and "$a_3^{a_1} = a_4$", that is, $a_1^{a_0 a_1} = a_1^{a_0^2}$ and $a_1^{a_0^2 a_1} = a_1^{a_0^3}$.*

- $F \simeq \{$ piecewise linear orientation preserving homeomorphisms of $[0,1]$ with discontinuities of the derivative and slopes of the form 2^k $\}$.

$a_0 \mapsto$



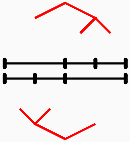
$a_1 \mapsto$



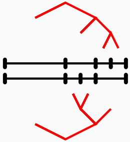
also represented as

- An element of F = a pair of dyadic decompositions of $[0,1]$:
  = a pair of finite binary rooted trees:

$a_0 \mapsto$



$a_1 \mapsto$



- <u>Fact</u>: *The center of F is trivial.*
  - Point: every homeomorphism commuting with x_1 fixes $1/2$.

- <u>Fact</u>: *Commutators in F correspond to homeomorphisms with slope 1 near 0 and 1.*
  - Hence $F/[F, F] \simeq \mathbb{Z} \oplus \mathbb{Z}$.

- <u>Proposition</u>: *Every proper quotient of F is abelian.*
  - Point: Every normal subgroup contains all commutators.

- <u>Theorem</u>: *The subgroup $[F, F]$ is simple.*
  - Point: A normal subgroup of $[F, F]$ contains all commutators.

- <u>Theorem</u> (Brin–Squier, 1985): *The group F includes no free subgroup of rank $\geqslant 2$.*
  - In fact: Every non-abelian subgroup of F includes a copy of $\mathbb{Z}^\infty$.
  - Compare with: F^+ includes a free monoid of rank 2.

- Question 1 (Gersten): Is F automatic? (F is not word hyperbolic)
  $\uparrow$
  $\exists$ finite state automaton computing a normal form for the elements

- Theorem (Guba 2005): *The Dehn function of F is quadratic.*
  $\uparrow$
  $\Phi(n) := \sup\{\text{area}(w) \mid \text{length}(w) = n \text{ and } w \text{ represents } 1 \text{ in } F\}$

- Question 2 (Geoghegan): Is F amenable?
  $\uparrow$
  $\exists$ left-invariant $[0,1]$-measure on $\mathfrak{P}(F)$

- Question 3: What is the growth rate of F^+ and F w.r.t. $\{a_0^{\pm 1}, a_1^{\pm 1}\}$?
  $\uparrow$
  $\sqrt[n]{a_n}$, with $a_n := \#\{\text{elements with length } n \text{ expression}\}$

- Theorem: - (i) (Burillo) *The growth rate of F^+ is* $\frac{1}{2\sin(\pi/14)} \approx 2.24\ldots$.
  - (ii) (Guba) *The growth rate of F lies between* $\frac{3+\sqrt{5}}{2} \approx 2.618\ldots$ *and* 3.

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- Ordinary braid diagrams:

← initial positions: • • • •

← final positions: • • • •

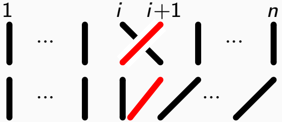
- ▸ one elementary pattern: crossing σ_i:

- Parenthesized braid diagrams: (possibly) non-equidistant positions:

← initial positions: (••) •

← final positions: • (••)

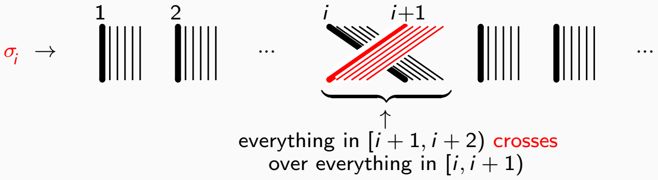
- ▸ two elementary patterns: crossing σ_i :

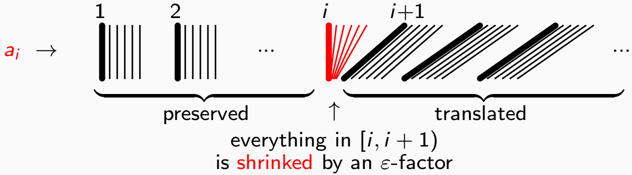


grouping a_i :

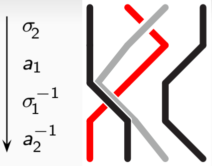
(and their inverses)

- More precisely:





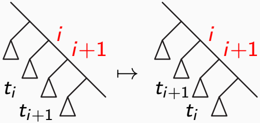
- A typical parenthesized braid diagram:



- Connection with binary trees: positions correspond to nodes in a binary tree;
  - Enumerated starting from the root and descending the right branch.

σ_i : …

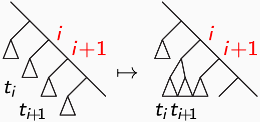
… corresponds to



σ_i = switching subtrees

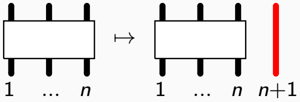
a_i : …

… corresponds to

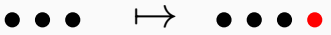


a_i = applying associativity

- For ordinary braid diagrams, only one completion:

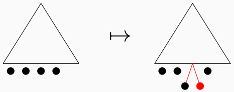


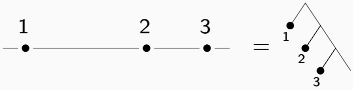
corresponding to



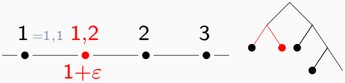
- For parenthesized braid diagrams, several completions:
  - index positions by sequences of integers

    (or, equivalently, infinitesimals)

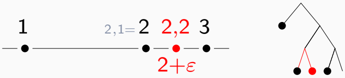




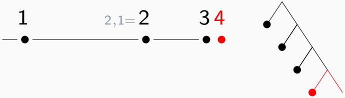
complete at 1:



complete at 2:



complete at 3:



- Parenthesized braid diagrams form a groupoid (small category with inverses):

$$\mathcal{B} := \{\text{ parenthesized braid diagrams }\}/\text{isotopy}.$$

  - Two isotopy classes D, D' can be composed when the final positions of D coincide with the initial positions of D'.

- To make a group:
  - Going from $\coprod B_n$ (groupoid) to B_∞: embed B_n into $B_{n'}$ for $n \leqslant n'$.
  - Going from $\coprod B_t$ (groupoid) to a group: embed B_t into $B_{t'}$ for $t \subseteq t'$.

  $\uparrow$ the family of (isotopy classes) of diagrams with initial positions t (a binary tree)

- Definition: The group $B_\bullet$ of parenthesized braids is

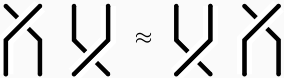
$$\varinjlim \{\text{ parenthesized braid diagrams }\}/\text{isotopy}.$$

- Proposition: *A presentation of* $B_\bullet$ *in terms of the generators* a_i *and* σ_i *is*

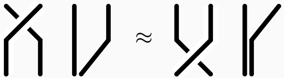
$$\text{for } x = \sigma \text{ or } a: \quad \sigma_i x_j = x_j \sigma_i \quad \text{and} \quad a_i x_{j-1} = x_j a_i \quad \text{for } j \geqslant i+2,$$

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, \quad \sigma_i \sigma_j a_i = a_j \sigma_i, \quad \sigma_j \sigma_i a_j = a_i \sigma_i \quad \text{for } j = i+1.$$

- commutation relations:



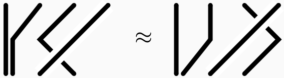
$\sigma_i \sigma_j = \sigma_j \sigma_i$



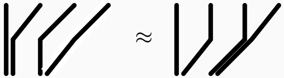
$\sigma_i a_j = a_j \sigma_i$

for $j \geqslant i+2$

- semi-commutation relations ("Thompson" relations):



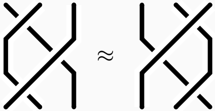
$a_i \sigma_{j-1} = \sigma_j a_i$



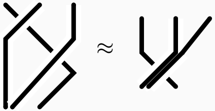
$a_i a_{j-1} = a_j a_i$

for $j \geqslant i+2$

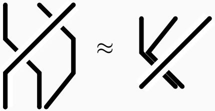
- braid relations:



$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$



$\sigma_i \sigma_{i+1} a_i = a_{i+1} \sigma_i$

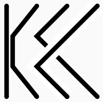


$\sigma_{i+1} \sigma_i a_i = a_i \sigma_i$

- Proof (beginning):
  - ▶ The relations (...) hold in $B_\bullet$ (obvious);
  - ▶ For the other direction, introduce
    - $B_\circ$:= the group presented by (...); we know $B_\circ \twoheadrightarrow B_\bullet$; we want $B_\circ \simeq B_\bullet$.
    - $B_\circ^+$:= the monoid presented by (...).

- Fact: *The group $B_\circ$ is generated by $\sigma_1, \sigma_2, a_1, a_2$.*

Example: $\sigma_3 = a_1^{-1}\sigma_2 a_1 =$



- Main lemma: *The monoid $B_\circ^+$ is cancellative and admits least common multiples and greatest common divisors. It is a Zappa-Szep product of B_∞^+ and F^+.*

↑

$B_\infty^+ \times F^+$ with two actions: $a_i \cdot \beta = \mathrm{db}_i(\beta) \cdot a_{\beta^{-1}(i)}$

F^+ acts on B_∞^+ by doubling strands, B_∞^+ acts on F^+ by permuting positions.

- Proposition: *The group $B_\circ$ is a group of fractions for the monoid $B_\circ^+$, and one has*

$$B_\circ = (F^+)^{-1} \cdot B_\infty \cdot F^+.$$

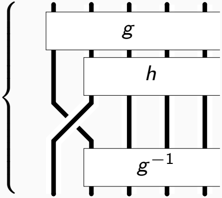
  - ▶ Every parenthesized braid diagram can be isotoped to a diagram "dilatation + braid + contraction".

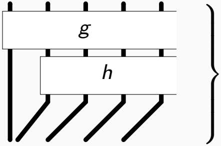
- Proposition: *Let sh* : $\sigma_i \mapsto \sigma_{i+1}$ *and* $a_i \mapsto a_{i+1}$ *for every* i. *For* g, h *in* $B_\circ$, *let*

$$g * h := g \cdot sh(h) \cdot \sigma_1 \cdot sh(h)^{-1},$$
$$g \circ h := g \cdot sh(h) \cdot a_1.$$

Then $(B_\circ, *, \circ)$ *is an augmented LD-system.*

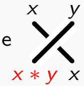
$$\uparrow$$
$$\begin{cases} x * (y * z) = (x * y) * (x * z): \text{ self-distributivity, ``LD''} \\ x * (y * z) = (x \circ y) * z \\ x * (y \circ z) = (x * y) \circ (x * z) \end{cases}$$

$g * h$


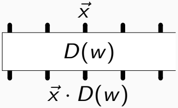


$g \circ h$

- Recall: want to prove $B_\circ \simeq B_\bullet$, i.e., $B_\bullet$ presented by the "braid-Thompson" relations (no other relation in $B_\bullet$ than those of $B_\circ$)
  - ▸ Need to show that, for w a parenthesized braid word, the isotopy class of the diagram $D(w)$ determines the class $[w]$ of w in $B_\circ$.

- Proof (sketch):
  - ▸ Use diagram colorings: Having fixed a set S with a binary operation $*$,
    - put colors from S on the top ends of a (parenthesized) braid diagram,
    - propagate using the rule 

    - works (compatible with Reidemeister move) when $*$ obeys the LD law.
  - ▸ Here: use $B_\circ$ and its LD-operation $*$ for coloring diagrams
    - the point: there exists eval : $(B_\circ)^\infty \to B_\circ$ such that, in $B_\circ$,



implies $\mathrm{eval}(\vec{x} \cdot D(w)) = \mathrm{eval}(\vec{x}) \cdot [w]$.

    - hence, $[w]$ recovered from the isotopy class of $D(w)$. □

• Proposition: *The augmented LD-system $(B_\bullet, *, \circ)$ is torsion-free.*

↑ every element of $B_\bullet$ generates a free subsystem

• Proposition: *Every element of $B_\bullet$ has a canonical expression in terms of special ones.*

↑ the closure of 1 under $*$ and $\circ$

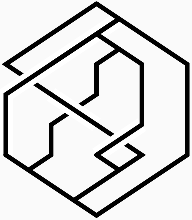
• Proposition: *The group $B_\bullet$ is orderable; more precisely: every element of $B_\bullet$ has an expression in which the σ_i with minimal index occurs positively only (no σ_i^{-1}).*

▶ similar to B_∞, but not expected because of the a_is

• Proposition: The group $B_\bullet$ is (isomorphic to) M. Brin's group $\widehat{BV}$.

"braid group on one strand"

▶ a typical element:



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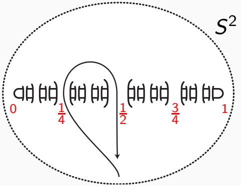
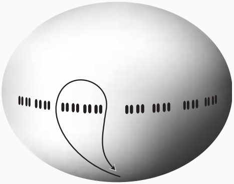
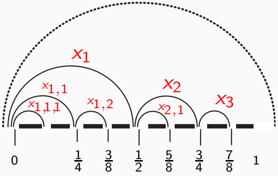
- Recall: D_n = disk with n punctures
  - Here: Cantor set of punctures.
  - Technically more convenient to start with a sphere rather than with a disk.

- Definition: $S_K :=$ sphere S^2 with a Cantor set of punctures removed on the equator.
  = two hemispheres connected by bridges indexed by dyadic numbers

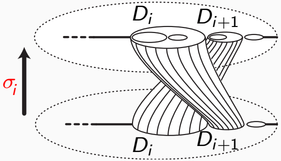




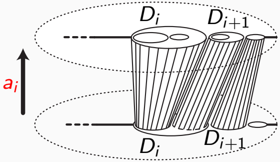
- Fact: $\pi_1(S_K)$ *is a free group with a basis indexed by finite sequences of integers.*



- Action of $B_\bullet$ on S_K by homeomorphisms: an embedding of $B_\bullet$ in $\mathcal{MCG}(S_K)$

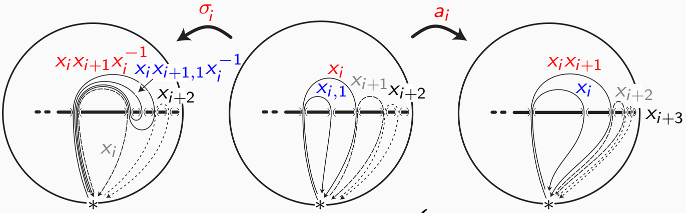


a "Dehn twist"
exchanging D_i and D_{i+1}



a "dilatation–contraction"
expanding D_i and contracting D_{i+1}

- Induces an action ρ on $\pi_1(S_K)$ ("Artin representation"):

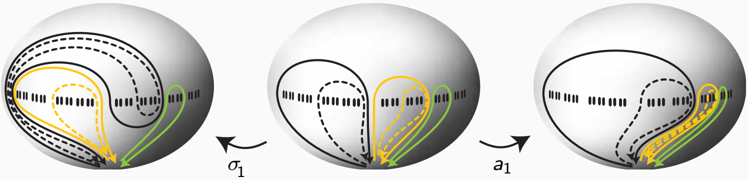


▸ read $\rho(\sigma_i)$: $\begin{cases} x_{i,s} \mapsto x_i x_{i+1,s} x_i^{-1} \\ x_{i+1,s} \mapsto x_i \\ x_{k,s} \mapsto x_{k,s} \text{ for } k \neq i, i+1 \end{cases}$

$\rho(a_i)$: $\begin{cases} x_i \mapsto x_i x_{i+1}, \quad x_{i,1,s} \mapsto x_{i,s} \\ x_{i,j+1,s} \mapsto x_{i+1,j,s} \text{ for } j \geqslant 2 \\ x_{k,s} \mapsto x_{k,s} \text{ for } k < i \\ x_{k,s} \mapsto x_{k+1,s} \text{ for } k > i \end{cases}$

- Proposition: *The Artin representation ρ of $B_\bullet$ in $Aut(F_\infty)$ is faithful.*

- Proof:
  - Uses the LD-structure again
  - Method: Show that, if a braid word w contains one σ_1 and no σ_1^{-1}, then $\rho(w)$ moves some x_s, hence is nontrivial.
  - Key point: $\rho(w)$ can be read from colouring trees (similar to the Hurwitz action of a braid word on a sequence)
  - If w contains one σ_1 and no σ_1^{-1}, then $\rho(w)(x_1)$ finishes with x_1^{-1}, hence $\rho(w)(x_1) \neq x_1$, hence $\rho(w) \neq \mathrm{id}$. □



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