Coxeter-like groups for groups of set-theoretic solutions of the Yang–Baxter equation

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- 1. Set-theoretic solutions of YBE, biracks and RC-quasigroups

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- 3. Garside germs and Coxeter-like groups

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- Even for X finite, very poorly understood.

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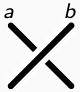

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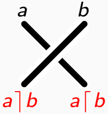

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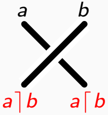

□

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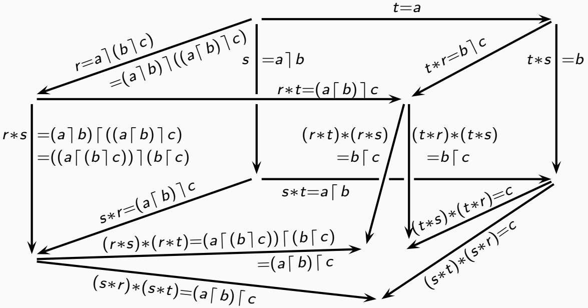

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$$G := \langle X \mid \{s(s * t) = t(t * s) \mid s, t \in X\}\rangle.$$

- Idem with monoids... $\langle ... \rangle^{+}$...

- Example: $X = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ with $x * y = f(y)$ and $f : \mathtt{a} \mapsto \mathtt{b} \mapsto \mathtt{c} \mapsto \mathtt{a}$. Then

$$G := \langle \mathtt{a}, \mathtt{b}, \mathtt{c} \mid \mathtt{ac} = \mathtt{b}^2, \mathtt{ba} = \mathtt{c}^2, \mathtt{cb} = \mathtt{a}^2\rangle.$$

- Main (open) question: Investigate "YBE-groups" and "YBE-monoids" from an algebraic and geometric viewpoint.

• Fact: The Cayley graph of an YBE-group (resp. monoid) with n atoms resembles that of $\mathbb{Z}^n$ (resp. $\mathbb{N}^n$).

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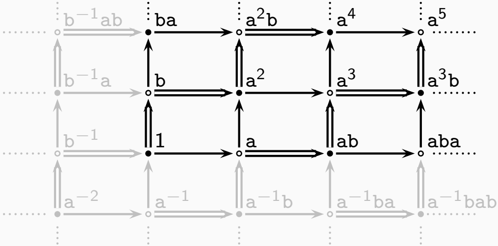

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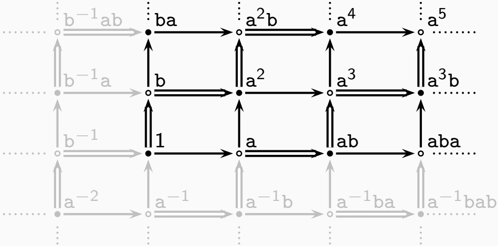

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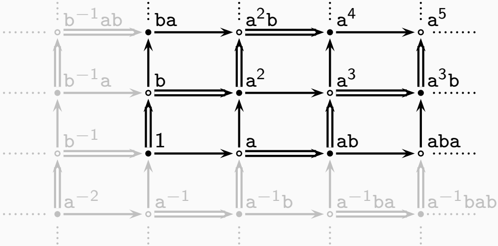

• Definition (Gateva–Van den Bergh): For M a monoid and $X \subseteq M$, an (X-based) I-structure for M is a bijection $\nu : \mathbb{N}^{(X)} \to M$ s.t. $\nu(1) = 1$, $\nu(s) = s$ for s in X, and

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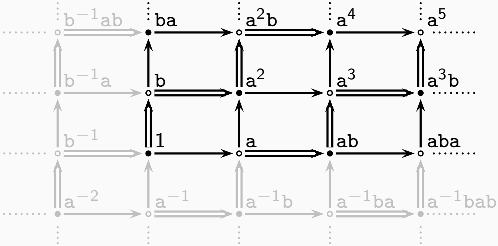

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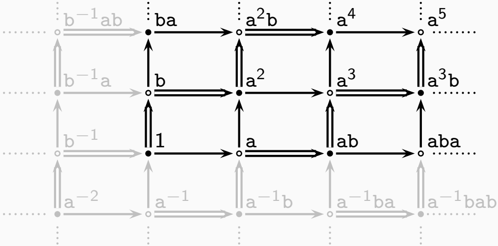

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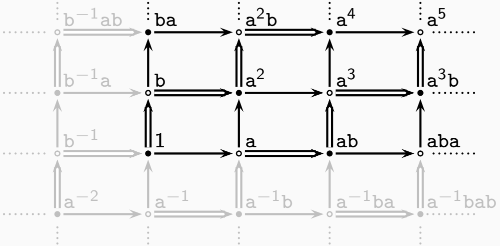

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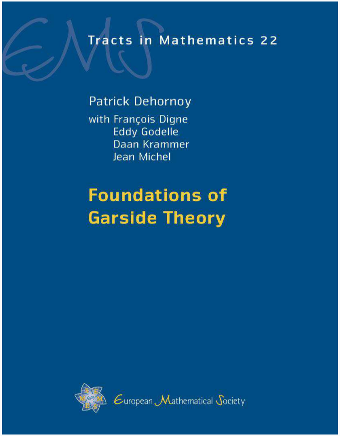

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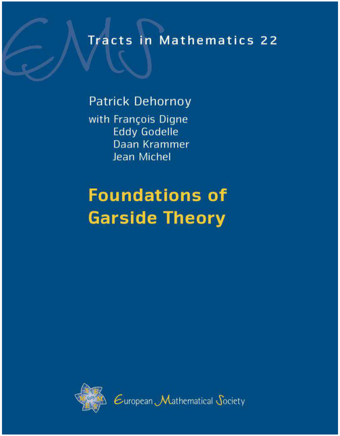

- <u>Example</u>: (D.–Dyer–Hohlweg): Every Artin–Tits monoid admits a finite Garside family.

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- Question: Does there exist such a Coxeter-like quotient for every Garside group?

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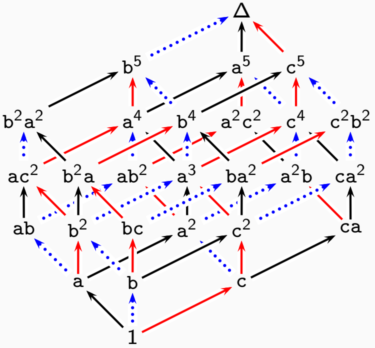

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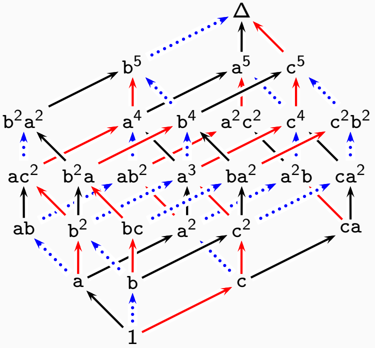

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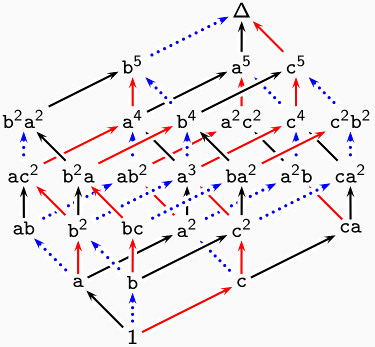

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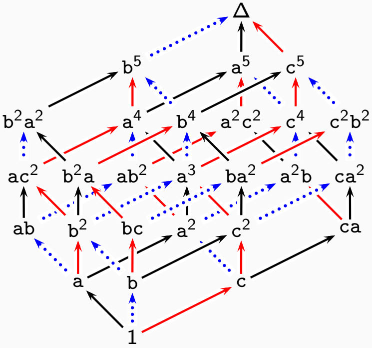

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(known: for $\#X = n$, there exists an n-dimensional unitary representation)

e.g., above: $a \mapsto \begin{pmatrix} 0 & j & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $b \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ 1 & 0 & 0 \end{pmatrix}$, $c \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ j & 0 & 0 \end{pmatrix}$

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